7 Linear Momentum, Collisions, and Center of Mass

In this chapter, we introduce the linear momentum of a particle and the law of *conservation of linear momentum* of a system of particles under certain conditions. We use this law and the conservation of energy to analyze translational motion when particles collide. For a system of isolated particles, or an extended object, we introduce the concept of *center of mass* to show that conservation of linear momentum applies under certain conditions, as it does for isolated particles. At the end of this chapter, we treat systems with variable mass. We first consider cases where the mass increases with time and then we consider cases where the mass decreases with time.

7.1 Linear Momentum and Impulse

First, let us consider Newton's second law, when a net force $\vec{F}$ acts on a particle of mass m, $\vec{F} = m\vec{a}$. After replacing $\vec{a}$ with $d\vec{v}/dt$, we get:

$$\vec{F} = m\frac{d\vec{v}}{dt} = \frac{d(m\vec{v})}{dt} \tag{7.1}$$

According to this equation, the net force $\vec{F}$ (abbreviation of $\sum\vec{F}$) acting on a particle is equal to the change in the product $m\vec{v}$ per unit time. This product is called the **linear momentum** (or the *momentum*) of a particle having a mass m and velocity $\vec{v}$, and it is assigned the symbol $\vec{p}$, that is:

$$\vec{p} = m\vec{v} \tag{7.2}$$

In the SI system, $\vec{p}$ has the units kg.m/s. In Cartesian coordinates, this equation is equivalent to the following component equations:

$$p_x = mv_x, \quad p_y = mv_y, \quad p_z = mv_z. \tag{7.3}$$

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We can therefore rewrite Eq. 7.1 in a new form as follows:

$$\vec{F} = \frac{d\vec{p}}{dt} \quad \text{(Newton's second law)} \tag{7.4}$$

The two forms of Newton's second law $\vec{F} = m\vec{a}$ and $\vec{F} = d\vec{p}/dt$ are equivalent if the mass m is constant.

Next, to derive the linear impulse-momentum theorem, we rewrite Eq. 7.4 in a differential form as follows:

$$d\vec{p} = \vec{F}\, dt \tag{7.5}$$

If the momentum of the particle changes from $\vec{p}_\mathrm{i}$ at time t_i to $\vec{p}_\mathrm{f}$ at time t_f, we can then integrate this expression to find the change in momentum as follows:

$$\int_{\vec{p}_\mathrm{i}}^{\vec{p}_\mathrm{f}} d\vec{p} = \int_{t_\mathrm{i}}^{t_\mathrm{f}} \vec{F}\, dt \tag{7.6}$$

or

$$\Delta\vec{p} = \vec{p}_\mathrm{f} - \vec{p}_\mathrm{i} = \int_{t_\mathrm{i}}^{t_\mathrm{f}} \vec{F}\, dt \tag{7.7}$$

The right-hand side of this equation is called the impulse $\vec{J}$ (kg.m/s or N.s) of the net force $\vec{F}$ for the time interval $\Delta t = t_\mathrm{f} - t_\mathrm{i}$. Thus:

$$\vec{J} = \int_{t_\mathrm{i}}^{t_\mathrm{f}} \vec{F}\, dt = \Delta\vec{p} \tag{7.8}$$

This is known as the impulse-momentum theorem. During collisions, $\vec{F}$ jumps from zero to a large value and abruptly returns to zero again, all in a very short time interval $\Delta t = t_\mathrm{f} - t_\mathrm{i}$, see Fig. 7.1a. The integral in Eq. 7.8 can be represented by $\overline{\vec{F}}\, \Delta t$, where $\overline{\vec{F}}$ is the average force exerted on the particle during the time interval Δt, see Fig. 7.1b. Therefore, the impulse-momentum theorem reduces to the following:

$$\vec{J} = \int_{t_\mathrm{i}}^{t_\mathrm{f}} \vec{F}\, dt = \overline{\vec{F}}\, \Delta t = \Delta\vec{p} \quad \text{and} \quad \overline{\vec{F}} = \frac{\Delta\vec{p}}{\Delta t} \tag{7.9}$$

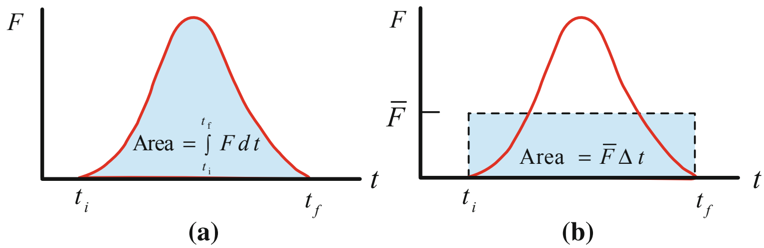
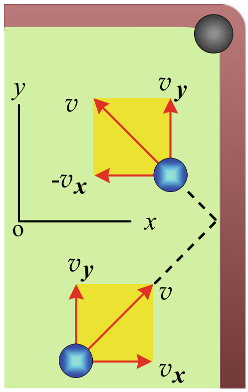


Fig. 7.1 (a) Variation of the force F with time t during a collision. (b) The average force $\overline{F}$ acting over a time interval $\Delta t = t_f - t_i$ gives the same impulse as the actual force F during the same time interval Δt

Example 7.1

A billiard ball of mass $m = 170\,\mathrm{g}$ has velocity components $v_x = v_y = 4\,\mathrm{m/s}$, see Fig. 7.2. The ball bounces back from a table's edge with the same speed and angle after being in contact with the edge for 0.2 s. Assume that friction and rotational motion are negligible. (a) What is the change in the horizontal and vertical components of the ball's momentum? (b) What is the average force exerted on the ball by the wall?

Fig. 7.2



Solution: (a) Bouncing with the same speed and angle means that the x component of the velocity is reversed, while the y component remains unchanged (this is known as an elastic collision). Since the x component of the ball's momentum is mv_x before the collision and $-mv_x$ afterward, the change in the ball's momentum will be:

$$\Delta p_x = (p_x)_f - (p_x)_i = -mv_x - mv_x$$
$$= -2mv_x = -2(0.17\,\mathrm{kg})(4\,\mathrm{m/s}) = -1.36\,\mathrm{kg.m/s}$$

Because of the unchanged y component of the velocity, we get

$$\Delta p_y = (p_y)_\mathrm{f} - (p_y)_\mathrm{i} = mv_y - mv_y = 0$$

(b) According to part (a), we have $\Delta\vec{p} = \Delta p_x \vec{\mathrm{i}} = -2mv_x \vec{\mathrm{i}}$, which by Eq. 7.9 means that the force exerted by the wall on the ball will be in the negative x direction. Thus:

$$\overline{\vec{F}} = \Delta\vec{p}/\Delta t = -2mv_x/\Delta t\,\vec{\mathrm{i}} = (-1.36\,\mathrm{kg.m/s})/(0.2\,\mathrm{s})\,\vec{\mathrm{i}} = -(6.8\,\mathrm{N})\,\vec{\mathrm{i}}$$

7.2 Conservation of Linear Momentum

Consider a system of n particles with linear momenta $\vec{p}_1, \vec{p}_2, \ldots,$ and $\vec{p}_n$. Some forces on these particles are external to the system, and others are internal. These forces may be of any type, including gravitational, electric, or magnetic.

Let $\vec{P}$ be the total linear momentum of the system, which is the vector sum of all individual momenta. Thus:

$$\vec{p}_1 + \vec{p}_2 + \cdots + \vec{p}_n = \sum \vec{p}_i = \vec{P} \tag{7.10}$$

When differentiating this equation with respect to time, we get:

$$\sum \frac{d\vec{p}_i}{dt} = \sum \vec{F}_i = \frac{d\vec{P}}{dt} \tag{7.11}$$

where $\sum \vec{F}_i$ represents the sum of all *forces* (internal plus external) exerted on the particles of the system. Then we can write the sum $\sum \vec{F}_i$ as follows:

$$\sum \vec{F}_i = \sum \vec{F}_\mathrm{ext} + \sum \vec{F}_\mathrm{int} \tag{7.12}$$

where $\sum \vec{F}_\mathrm{ext}$ is the vector sum of all external forces acting on the particles of the system. By Newton's third law, the internal forces form action-reaction pairs and their sum cancel each other out, i.e., $\sum \vec{F}_\mathrm{int} = 0$. Therefore, Eq. 7.11 reduces to:

$$\sum \vec{F}_\mathrm{ext} = \frac{d\vec{P}}{dt} \quad \text{(System of particles)} \tag{7.13}$$

This equation represents a generalization of the single-particle equation $\sum \vec{F} = d\vec{p}/dt$ that is deduced for a single particle.

For an isolated system, the sum of the external forces is zero. Setting $\sum \vec{F}_{ext} = 0$ in Eq. 7.13 yields $d\vec{P}/dt = 0$, or:

$$\vec{P} = \text{constant} \qquad \text{(Isolated system)} \tag{7.14}$$

Spotlight
Thus, the total linear momentum of an isolated system of particles remains constant

This is the **law of conservation of momentum**, which can be written as:

$$\vec{P}_i = \vec{P}_f \qquad \text{(Isolated system)} \tag{7.15}$$

where the subscripts refer to the total momentum of the system at initial time i and final time f.

Example 7.2

Two trams, 1 and 2, have an equal mass of $m = 5{,}000$ kg each. Tram 1 is traveling with a speed $v_1 = 15$ m/s before striking tram 2, which was at rest. If the two trams lock together as the result of the collision as shown in Fig. 7.3, what is their common speed immediately after collision?

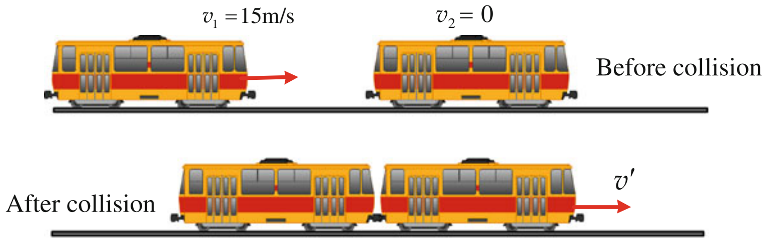


Fig. 7.3

Solution: We consider a short time interval after the collision so that heat and external forces such as friction can be ignored. Then we can apply the conservation of the total horizontal momentum:

$$P_i = P_f$$

The initial total momentum of the two trams before collision is:

$$P_i = m_1 v_1 + m_2 v_2 = m_1 v_1 + 0 = (5{,}000\,\text{kg})(15\,\text{m/s}) = 75{,}000\,\text{kg. m/s}$$

The final total momentum of the two trams after collision is:

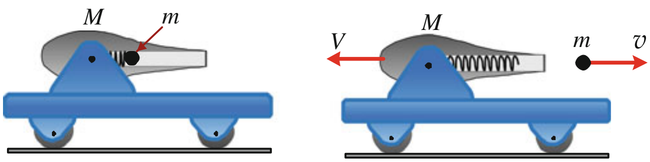
$$P_f = m_1 v' + m_2 v' = (m_1 + m_2)v' = (5{,}000\,\text{kg} + 5{,}000\,\text{kg})v' = (10{,}000\,\text{kg})v'$$

Applying the conservation of total momentum $P_i = P_f$, we get:

$$75{,}000\,\text{kg.m/s} = (10{,}000\,\text{kg})v' \quad \Rightarrow \quad v' = \frac{75{,}000\,\text{kg.m/s}}{10{,}000\,\text{kg}} = 7.5\,\text{m/s}$$

Example 7.3

A cannon of mass $M = 1{,}500\,\text{kg}$ shoots a projectile of mass $m = 100\,\text{kg}$ with a horizontal speed $v = 30\,\text{m/s}$, as shown in Fig. 7.4. If the cannon can *recoil freely* on a horizontal ground, what is its recoil speed V just after shooting the projectile?



Before shooting After shooting

Fig. 7.4

Solution: We take our system to be the cannon and the projectile, which both are at rest initially before shooting. When the trigger is pulled, the forces involved in the shooting are internal and hence cancel. During the very short time of shooting, we can assume that the external forces such as friction are very small compared to the forces exerted by the shooting. In addition, the external gravitational forces acting on the system have no components in the horizontal direction. Then the momentum conservation along the horizontal direction is:

$$P_i = P_f$$

The initial total horizontal momentum before the shooting is:

$$P_i = m \times 0 + M \times 0 = 0$$

The final total horizontal momentum after the shooting is:

$$P_f = mv + MV$$

Applying the conservation of total momentum $P_i = P_f$, we get:

$$V = -\frac{mv}{M} = -\frac{(100\,\text{kg})(30\,\text{m/s})}{1{,}500\,\text{kg}} = -2\,\text{m/s}$$

The minus sign indicates that the velocity and momentum of the cannon is opposite to that of the projectile. Since the cannon has a much larger mass than the projectile, its recoil speed is much less than that of the projectile.

7.3 Conservation of Momentum and Energy in Collisions

During most types of collisions, forces are usually unknown. Nevertheless, by using the conservation laws of momentum and energy we can determine much information about the motion after collision in terms of information before collision. When objects are very hard, so that no heat or other forms of energy are produced during collisions, the kinetic energy is conserved before and after collision. Such a collision is referred to as an **elastic collision**. Thus, in elastic collisions we have the following for a system of particles:

$$\left\{\begin{array}{c}\text{Total kinetic energy before} = \text{Total kinetic energy after}\\ \sum \frac{1}{2}mv^2_{before} = \sum \frac{1}{2}mv^2_{after}\end{array}\right\}\left\{\begin{array}{c}\text{Elastic}\\ \text{collision}\end{array}\right\} \tag{7.16}$$

Collisions in which kinetic energy is not conserved are said to be **inelastic collisions**. However, we should remember that the total energy is conserved even if kinetic energy is not. Thus:

$$\left\{\begin{array}{c}\text{Total energy before} = \text{Total energy after}\\ \sum \frac{1}{2}mv^2_{before} = \sum \frac{1}{2}mv^2_{after} + \text{other forms of energy}\end{array}\right\}\left\{\begin{array}{c}\text{Inelastic}\\ \text{collision}\end{array}\right\} \tag{7.17}$$

7.3.1 Elastic Collisions in One and Two Dimensions

First, we apply the conservation laws of momentum and kinetic energy in an elastic collision of two small objects that collide head-on. Figure 7.5 shows two objects of masses m_1 and m_2 (treated as particles) moving along the x-axis with velocities v_1 and v_2, respectively. Usually the object of mass m_1 is called the projectile while the object of mass m_2 is called the target. After collision their velocities are v'_1 and v'_2, respectively. If the sign of any velocity is positive, then the object is moving in the direction of increasing x, whereas if the sign of the velocity is negative, then the object is moving in the direction of decreasing x.

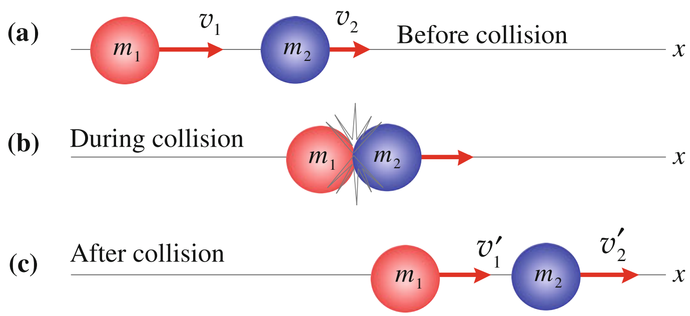


Fig. 7.5 Two small objects of masses m_1 and m_2, (a) approaching each other before collision, (b) colliding head-on, and (c) moving away from each other after collision

From the conservation of momentum, $P_i = P_f$, we have:

$$m_1 v_1 + m_2 v_2 = m_1 v'_1 + m_2 v'_2$$

From the conservation of kinetic energy of elastic collisions, we have:

$$\tfrac{1}{2} m_1 v_1^2 + \tfrac{1}{2} m_2 v_2^2 = \tfrac{1}{2} m_1 v'^2_1 + \tfrac{1}{2} m_2 v'^2_2$$

If we know the masses and the velocities before collision, we can solve the above two equations for the two unknowns v'_1 and v'_2. We rewrite the momentum and kinetic-energy equations as follows:

$$m_1(v_1 - v'_1) = m_2(v'_2 - v_2) \tag{7.18}$$

$$m_1(v_1^2 - v'^2_1) = m_2(v'^2_2 - v_2^2) \tag{7.19}$$

Using the identity $a^2 - b^2 = (a-b)(a+b)$, we write the last equation as:

$$m_1(v_1 - v'_1)(v_1 + v'_1) = m_2(v'_2 - v_2)(v'_2 + v_2) \tag{7.20}$$

When dividing Eq. 7.20 by Eq. 7.18, we get:

$$v_1 + v'_1 = v'_2 + v_2 \tag{7.21}$$

We can rewrite this equation as:

$$v_1 - v_2 = -(v_1' - v_2') \tag{7.22}$$

This shows that for any elastic head-on collisions, the *relative velocity* of two objects before collision equals the negative of their *relative velocity* after collision, regardless of the masses of the objects.

In addition, Eqs. 7.18 and 7.21 can be used to find the final velocities (normally the unknown quantities) in terms of the initial velocities (normally the known quantities) as follows:

$$v_1' = \frac{m_1 - m_2}{m_1 + m_2} v_1 + \frac{2m_2}{m_1 + m_2} v_2 \tag{7.23}$$

$$v_2' = \frac{2m_1}{m_1 + m_2} v_1 + \frac{m_2 - m_1}{m_1 + m_2} v_2 \tag{7.24}$$

We can apply these equations to some very important special cases:

- Equal masses ($m_1 = m_2$). Equations 7.23 and 7.24 show that:

$$\boxed{v_1' = v_2} \quad \text{and} \quad \boxed{v_2' = v_1} \quad \text{(The objects exchange velocities)}$$

- Object 2 (the target) is initially at rest ($v_2 = 0$). Equations 7.23 and 7.24 becomes:

$$\boxed{v_1' = \frac{m_1 - m_2}{m_1 - m_2} v_1} \quad \text{and} \quad \boxed{v_2' = \frac{2m_1}{m_1 + m_2} v_1} \tag{7.25}$$

(a) If $m_1 \gg m_2$, i.e., the projectile is *heavier* than the target, then:

$$\boxed{v_1' \approx v_1} \quad \text{and} \quad \boxed{v_2' \approx 2v_1}$$

The much heavier object (projectile) continues with unaltered velocity, while the light object (target) takes off with twice the velocity of the heavy object

(b) If $m_1 \ll m_2$, i.e., the projectile is much *lighter* than the target, then:

$$\boxed{v_1' \approx -v_1} \quad \text{and} \quad \boxed{v_2' \approx 0}$$

The light object (projectile) has its velocity reversed while the heavy object (target) remains approximately at rest

The general Eqs. 7.23 and 7.24 should not be memorized. In each different problem we can easily start from scratch by applying the conservation of momentum and kinetic energy to solve questions in any elastic head-on collision.

Example 7.4

A tennis ball of mass $m_1 = 0.04\,\text{kg}$, moving with a speed of 5 m/s, has an elastic head-on collision with a target ball of mass $m_2 = 0.06\,\text{kg}$ that was moving at a speed of 3 m/s. What is the velocity of each ball after the collision if the two balls are moving: (a) in the same direction as shown in Fig. 7.6a? (b) in opposite direction as shown in Fig. 7.6b?

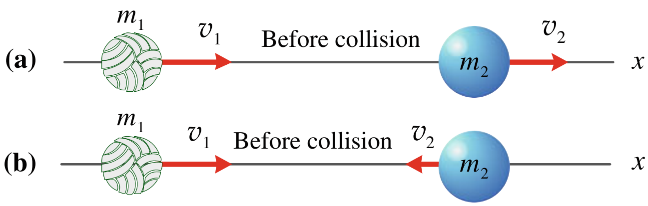


Fig. 7.6

Solution: (a) In Fig. 7.6a, we have $v_1 = +5$ m/s and $v_2 = +3$ m/s. Using Eq. 7.22, we find a relationship between the velocities as:

$$v_1 - v_2 = -(v_1' - v_2') \quad \Rightarrow \quad 5\,\text{m/s} - 3\,\text{m/s} = v_2' - v_1' \quad \Rightarrow \quad v_2' = 2\,\text{m/s} + v_1'$$

Using this result in the conservation of momentum, we have:

$$m_1 v_1 + m_2 v_2 = m_1 v_1' + m_2 v_2'$$

$$m_1 v_1 + m_2 v_2 = m_1 v_1' + m_2 (2\,\text{m/s} + v_1')$$

$$v_1' = \frac{m_1 v_1 + m_2 (v_2 - 2\,\text{m/s})}{m_1 + m_2} = \frac{(0.04\,\text{kg})(5\,\text{m/s}) + (0.06\,\text{kg})(3\,\text{m/s} - 2\,\text{m/s})}{0.04\,\text{kg} + 0.06\,\text{kg}}$$
$$= \frac{(0.2\,\text{kg.m/s}) + (0.06\,\text{kg.m/s})}{0.1\,\text{kg}} = \frac{0.26\,\text{kg.m/s}}{0.1\,\text{kg}} = 2.6\,\text{m/s}$$

The other unknown velocity is v_2', which can now be obtained from:

$$v_2' = 2\,\text{m/s} + v_1' = 2\,\text{m/s} + 2.6\,\text{m/s} = 4.6\,\text{m/s}$$

After the collision, the plus signs of v_1' and v_2' tell us that the tennis ball and the target will move in the same positive x direction, but the tennis ball will slow down, while the target will speed up; see Fig. 7.7.

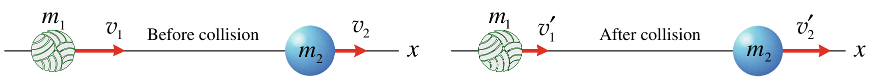


Fig. 7.7

(b) In Fig. 7.6b, we have $v_1 = +5$ m/s and $v_2 = -3$ m/s. Using Eq. 7.22, we find the relationship between the velocities as:

$$v_1 - v_2 = -(v_1' - v_2') \quad \Rightarrow \quad 5\,\text{m/s} - (-3\,\text{m/s}) = v_2' - v_1' \quad \Rightarrow \quad v_2' = 8\,\text{m/s} + v_1'$$

Similarly, using this result in the conservation of momentum, we get:

$$m_1 v_1 + m_2 v_2 = m_1 v_1' + m_2 v_2'$$

$$m_1 v_1 + m_2 v_2 = m_1 v_1' + m_2 (8\,\text{m/s} + v_1')$$

$$v_1' = \frac{m_1 v_1 + m_2 (v_2 - 8\,\text{m/s})}{m_1 + m_2} = \frac{(0.04\,\text{kg})(5\,\text{m/s}) + (0.06\,\text{kg})(-3\,\text{m/s} - 8\,\text{m/s})}{0.04\,\text{kg} + 0.06\,\text{kg}}$$
$$= \frac{(0.2\,\text{kg.m/s}) - (0.66\,\text{kg.m/s})}{0.1\,\text{kg}} = -\frac{0.46\,\text{kg.m/s}}{0.1\,\text{kg}} = -4.6\,\text{m/s}$$

The other unknown velocity can now be obtained from:

$$v_2' = 8\,\text{m/s} + v_1' = 8\,\text{m/s} - 4.6\,\text{m/s} = 3.4\,\text{m/s}$$

After the collision the minus sign of v_1' tells us that the tennis ball reverses its motion and moves in the negative x direction, while the positive sign of v_2' tells us that the target also reverses its motion and moves in the positive x direction, see Fig. 7.8 with proper arrows.

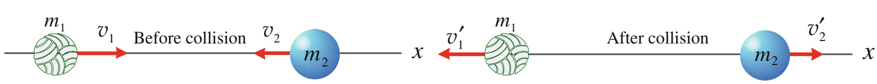


Fig. 7.8

Now, let us apply the conservation laws of momentum and kinetic energy to an elastic collision of two objects that are not colliding head-on. Figure 7.9 shows one common type of non-head-on collision at which one object (the "projectile") of mass m_1 moves along the x-axis with a speed v_1 and strikes a second stationary object (the "target") of mass m_2. After the collision, the two masses m_1 and m_2 go off at the angles θ_1 and θ_2, respectively, which are measured relative to the projectile's initial direction. We see this type of collision in nuclear experiments, or more commonly in billiard games.

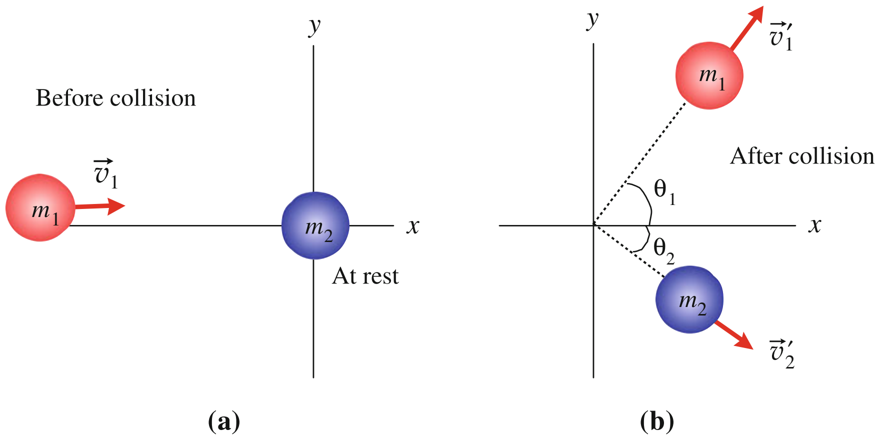


Fig. 7.9 (a) A projectile of mass m_1 moving in the x direction with velocity $\vec{v}_1$ toward a stationary target of mass m_2. (b) After collision, the projectile and target move away with velocities $\vec{v}'_1$ and $\vec{v}'_2$, respectively

We apply the law of conservation of momentum along the x and y axes, and in cases of elastic collisions we also apply the law of conservation of kinetic energy as follows:

$$\text{Momentum long } x\text{-axis}: \quad m_1 v_1 = m_1 v'_1 \cos\theta_1 + m_2 v'_2 \cos\theta_2 \tag{7.26}$$

$$\text{Momentum long } y\text{-axis}: \quad 0 = m_1 v'_1 \sin\theta_1 - m_2 v'_2 \sin\theta_2 \tag{7.27}$$

$$\text{Kinetic energy}: \quad \tfrac{1}{2} m_1 v_1^2 = \tfrac{1}{2} m_1 v_1'^2 + \tfrac{1}{2} m_2 v_2'^2 \tag{7.28}$$

If m_1, m_2, and v_1 are known quantities, then we are left with the four unknowns v'_1, θ_1, v'_2, and θ_2. Since we only have three equations, one of the four unknowns must be provided; otherwise, we cannot solve the problem.

Example 7.5

A projectile of mass $m_1 = m$ moving along the x direction with a speed $v_1 = 10\sqrt{3}$ m/s collides elastically with a stationary target of mass $m_2 = 2m$. After the collision, the projectile is deflected at an angle of 90°, as shown in Fig. 7.10. (a) What is the speed and angle of the target after collision? (b) What is the final speed of the projectile and the fraction of kinetic energy transferred to the target?

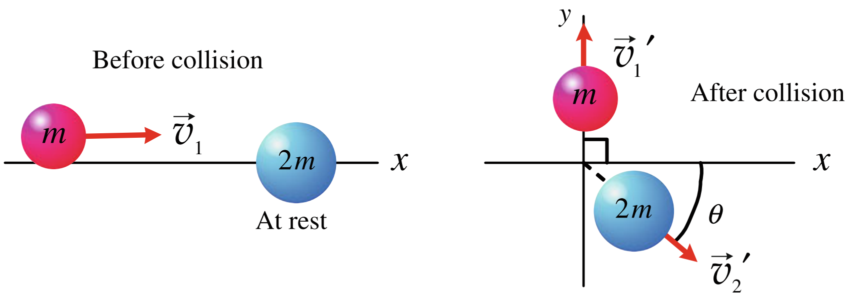


Fig. 7.10

Solution: (a) From the conservation of momentum in two dimensions and conservation of kinetic energy, we get the following relationships:

$$\begin{aligned}
&\text{Momentum along } x: && mv_1 = 2mv_2' \cos\theta && \Rightarrow && v_1 = 2v_2' \cos\theta \\
&\text{Momentum along } y: && 0 = mv_1' - 2mv_2' \sin\theta && \Rightarrow && v_1' = 2v_2' \sin\theta \\
&\text{Kinetic energy}: && \tfrac{1}{2}mv_1^2 = \tfrac{1}{2}mv_1'^2 + \tfrac{1}{2}2mv_2'^2 && \Rightarrow && v_1^2 - v_1'^2 = 2v_2'^2
\end{aligned}$$

Squaring and adding the two momentum equations together, we get:

$$v_1^2 + v_1'^2 = 4v_2'^2$$

Adding this result to the one obtained from the conservation of kinetic energy, we get:

$$2v_1^2 = 6v_2'^2 \quad \Rightarrow \quad v_2'^2 = \frac{1}{3}v_1^2 \quad \Rightarrow \quad v_2' = \frac{1}{\sqrt{3}}v_1 = \frac{1}{\sqrt{3}}(10\sqrt{3}\,\text{m/s}) = 10\,\text{m/s}$$

Using this result in the x-momentum component, we find the angle:

$$v_1 = 2v_2' \cos\theta \quad \Rightarrow \quad v_1 = \frac{2}{\sqrt{3}}v_1 \cos\theta \quad \Rightarrow \quad \cos\theta = \frac{\sqrt{3}}{2} \quad \Rightarrow \quad \theta = 30^\circ$$

(b) We can substitute $v_2' = 10$ m/s and $\theta = 30°$ in the y-momentum component to find the speed v_1' as follows:

$$v_1' = 2(10\,\text{m/s})(\sin 30°) = 10\,\text{m/s}$$

The fraction transferred is the final energy of the target divided by the initial kinetic energy of the projectile.

$$\frac{K_{\text{target}}}{K_{\text{projectile}}} = \frac{\frac{1}{2}(2m)v_2'^2}{\frac{1}{2}mv_1^2} = \frac{\frac{1}{2}(2m)\left(v_1^2/3\right)}{\frac{1}{2}mv_1^2} = \frac{2}{3} \equiv 66.67\%$$

7.3.2 Inelastic Collisions

In some collisions, part of the initial kinetic energy is transferred to other types of energy (such as thermal or potential energy), or part of the internal energy (such as chemical or nuclear) is released as a form of kinetic energy. These types of collisions are called **inelastic collisions** because the total final kinetic energy can be less than or greater than the total initial kinetic energy (i.e., the kinetic energy is not conserved). If two objects stick together after collision, the collision is called a **completely inelastic collision**. Even though kinetic energy is not conserved in those collisions, total energy is conserved.

Example 7.6

A bullet of mass $m = 10$ g is fired horizontally with a speed v into a large wooden stationary block of mass $M = 2$ kg that is suspended vertically by two cords. This arrangement is called the ballistic pendulum, see Fig. 7.11. In a very short time, the bullet penetrates the pendulum and remains embedded. The entire system starts to swing through a maximum height $h = 10$ cm. Find the relation that gives the speed v in terms of the height h, and then find its value.

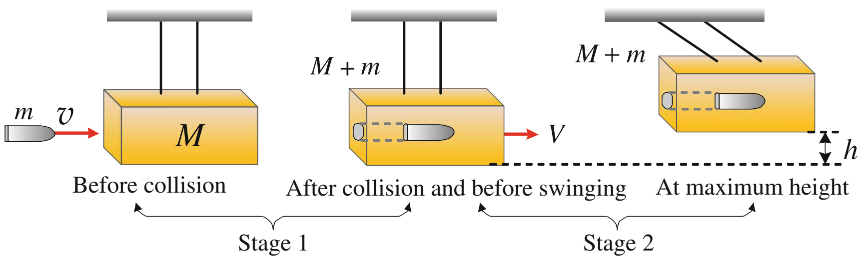


Fig. 7.11

Solution: In stage 1, momentum is conserved. Thus:

$$mv = (M + m)V \quad \Rightarrow \quad V = \frac{m}{M + m}v$$

In stage 2, the mechanical energy, $K + U$, is conserved. Thus:

$$\tfrac{1}{2}(M + m)V^2 + 0 = 0 + (M + m)gh \quad \Rightarrow \quad V^2 = 2gh \quad \Rightarrow \quad V = \sqrt{2gh}$$

Inserting this result into the previous relation gives v in terms of h as:

$$v = \frac{M + m}{m}\sqrt{2gh} \quad \Rightarrow \quad v = \frac{2.01\,\text{kg}}{0.01\,\text{kg}}\sqrt{2(10\,\text{m/s}^2)(0.1\,\text{m})} = 284.3\,\text{m/s}$$

7.4 Center of Mass (CM)

Until now, we have dealt with translation motion of an object that can be approximated by a point particle. In fact, real objects can undergo both translational and rotational motions. From general practical observations, it is found that when an applied resultant force $\Sigma\vec{F}_{ext}$ acts on an extended object (or a system of particles) of total mass M, the translation motion of the object moves as if the resultant force were applied on a single point at which the mass of the object were concentrated. This behavior is independent of other motion, such as rotational or vibrational motion. This special point is called the **center of mass** (abbreviated by CM) of the object.

As an example, consider the motion of the center of mass of the wrench over a horizontal surface shown in Fig. 7.12a. The CM follows a straight line under a zero net force. In Fig. 7.12b, the CM follows a straight line even when the wrench rotates about the CM.

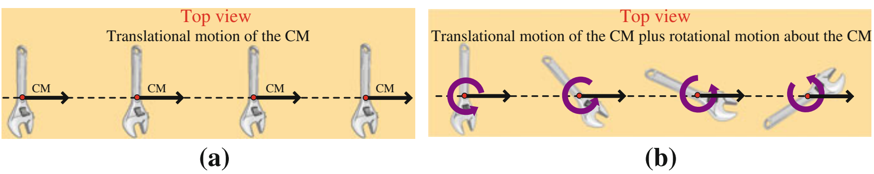


Fig. 7.12 (a) A *top view* of the translational motion of the CM of a wrench over a horizontal surface (the *red dot* represents the wrench's CM at different moments). (b) A *top view* of the translational motion of the CM plus the rotational motion about the CM

Figure 7.13 depicts a system of two masses m_1 and m_2 located on the x-axis at positions x_1 and x_2, respectively. The center of mass of this system of particles is at the position x_{CM} and defined as follows:

$$x_{CM} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2} \tag{7.29}$$

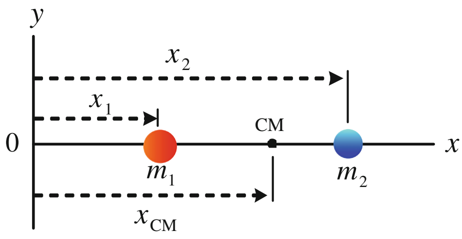


Fig. 7.13 The coordinate of the center of mass (x_{CM}) of a system of two particles is a point located between the particles

For a system consisting of n particles, where n could be very large, Eq. 7.29 becomes:

$$x_{CM} = \frac{m_1 x_1 + m_2 x_2 + \cdots + m_n x_n}{m_1 + m_2 + \cdots + m_n} = \frac{\sum_{i=1}^{n} m_i x_i}{\sum_{i=1}^{n} m_i} = \frac{\sum_i m_i x_i}{M} \tag{7.30}$$

The symbol $\sum_{i=1}^{n}$ indicates the sum over all particles, where i takes an integer values from 1 to n. Often the symbol $\sum_{i=1}^{n}$ is replaced by the symbol $\sum_i$ (or even $\sum$). The total mass of the system is $M = \sum m_i$.

If the particles are spread out in three dimensions and x_i, y_i, and z_i are the coordinates of the i^{th} particle of mass m_i and position vector $\vec{r}_i = x_i \vec{i} + y_i \vec{j} + z_i \vec{k}$, then we define the coordinates of the CM as:

$$x_{CM} = \frac{\sum m_i x_i}{M}, \quad y_{CM} = \frac{\sum m_i y_i}{M}, \quad z_{CM} = \frac{\sum m_i z_i}{M}, \tag{7.31}$$

where $M = \sum m_i$ is the total mass of the system. The position vector of the CM is thus:

$$\vec{r}_{CM} = x_{CM}\vec{i} + y_{CM}\vec{j} + z_{CM}\vec{k} = \frac{\sum m_i x_i \vec{i} + \sum m_i y_i \vec{j} + \sum m_i z_i \vec{k}}{M} \tag{7.32}$$

The position vector of the CM can be simplified as:

$$\vec{r}_{CM} = \frac{\sum m_i \vec{r}_i}{M} \tag{7.33}$$

For an extended object, we divide the object into tiny elements, each of mass Δm_i around a point with coordinates x_i, y_i, and z_i. When we take the limit as $n \to \infty$, then

Δm_i becomes an infinitesimal mass dm with coordinates x, y, and z. The summations in Eq. 7.31 become integrals and we get:

$$\boxed{x_{CM} = \frac{1}{M}\int x\,dm}, \quad \boxed{y_{CM} = \frac{1}{M}\int y\,dm}, \quad \boxed{z_{CM} = \frac{1}{M}\int z\,dm} \tag{7.34}$$

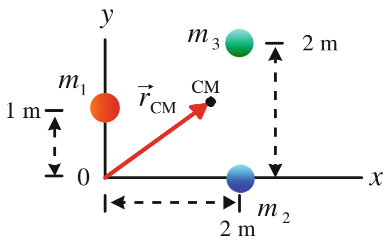
where $M = \int dm$ is the total mass of the system, and in vector notation, Eq. 7.33 becomes:

$$\vec{r}_{CM} = \frac{1}{M}\int \vec{r}\,dm \tag{7.35}$$

Example 7.7

A system of three particles of masses $m_1 = 0.5\,\text{kg}$, $m_2 = 1\,\text{kg}$, and $m_3 = 1.5\,\text{kg}$ are spread out in two dimensions and located as shown in Fig. 7.14. Find the center of mass of the system.

Fig. 7.14



Solution: According to Fig. 7.14, m_1, m_1, and m_1 have coordinates (0, 1 m), (2 m, 0), and (2 m, 2 m), respectively. Thus, we use the x and y components of Eq. 7.31 with only three terms as follows:

$$x_{CM} = \frac{\sum_{i=1}^{3} m_i x_i}{\sum_{i=1}^{3} m_i} = \frac{(0.5\,\text{kg})(0) + (1\,\text{kg})(2\,\text{m}) + (1.5\,\text{kg})(2\,\text{m})}{0.5\,\text{kg} + 1.0\,\text{kg} + 1.5\,\text{kg}}$$
$$= \frac{5\,\text{kg.m}}{3\,\text{kg}} = 1.67\,\text{m}$$

$$y_{CM} = \frac{\sum_{i=1}^{3} m_i x_i}{\sum_{i=1}^{3} m_i} = \frac{(0.5\,\text{kg})(1\,\text{m}) + (1\,\text{kg})(0) + (1.5\,\text{kg})(2\,\text{m})}{0.5\,\text{kg} + 1.0\,\text{kg} + 1.5\,\text{kg}}$$
$$= \frac{3.5\,\text{kg.m}}{3\,\text{kg}} = 1.17\,\text{m}$$

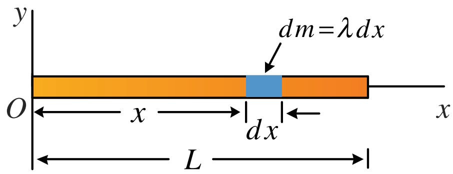
The center-of-mass position vector is thus $\vec{r}_{CM} = (1.67\,\text{m})\vec{i} + (1.17\,\text{m})\vec{j}$.

Example 7.8

A horizontal rod has a mass M and length L. Find the location of its center of mass from its left end: (a) if the rod has a uniform mass per unit length λ, and (b) if the rod has a mass per unit length λ that increases linearly from its left end according to the relation $\lambda = \alpha x$, where α is a constant.

Solution: (a) According to the geometry of Fig. 7.15, $y_{CM} = z_{CM} = 0$. For a uniform rod $\lambda = M/L$. If we divide the rod into infinitesimal elements of length dx, then the mass of each element is $dm = \lambda\, dx$.

Fig. 7.15



Accordingly, Eq. 7.34 gives:

$$x_{CM} = \frac{1}{M}\int x\,dm = \frac{1}{M}\int_0^L x\,\lambda\,dx = \frac{\lambda}{M}\int_0^L x\,dx = \frac{1}{L}\,\frac{x^2}{2}\bigg|_0^L = \frac{1}{L}\frac{L^2}{2} = \frac{L}{2}$$

where we used $\lambda = M/L$. Thus, as expected, the center of mass of a uniform rod is at its center.

(b) In this case, λ is not a constant. Therefore, Eq. 7.34 gives:

$$x_{CM} = \frac{1}{M}\int x\,dm = \frac{1}{M}\int_0^L x\,\lambda\,dx = \frac{\alpha}{M}\int_0^L x^2\,dx = \frac{\alpha}{M}\,\frac{x^3}{3}\bigg|_0^L = \frac{\alpha L^3}{3M}$$

We can eliminate α by writing M in terms of α and L as follows:

$$M = \int dm = \int_0^L \lambda\,dx = \int_0^L \alpha\,x\,dx = \alpha\,\frac{x^2}{2}\bigg|_0^L = \alpha\frac{L^2}{2}$$

Substituting this result into the expression of x_{CM}, we get:

$$x_{CM} = \frac{\alpha L^3}{3M} = \frac{\alpha L^3}{3\alpha L^2/2} = \frac{2}{3}L$$

7.5 Dynamics of the Center of Mass

In some cases, it is desirable to ignore rotational and vibrational motion in a system. In these cases, the center-of-mass concept greatly simplifies the analysis of the motion because the system of many-particles or an extended object can be treated as a single particle located at the CM of the system. To do this, we examine the motion of a system of n particles when the *total mass M of the system remains constant*. We begin by rewriting Eq. 7.33 as follows:

$$M\,\vec{r}_{\rm CM} = \sum m_i \vec{r}_i \tag{7.36}$$

Differentiating this equation with respect to time gives:

$$M\frac{d\vec{r}_{\rm CM}}{dt} = \sum m_i \frac{d\vec{r}_i}{dt}$$

or

$$M\,\vec{v}_{\rm CM} = \sum m_i \vec{v}_i \tag{7.37}$$

where $\vec{v}_{\rm CM}$ is the velocity of the center of mass and $\vec{v}_i$ is the velocity of the $i^{\rm th}$ particle that has a mass m_i. We differentiate again with respect to time to obtain:

$$M\frac{d\vec{v}_{\rm CM}}{dt} = \sum m_i \frac{d\vec{v}_i}{dt}$$

or

$$M\,\vec{a}_{\rm CM} = \sum m_i \vec{a}_i \tag{7.38}$$

where now $\vec{a}_{\rm CM}$ is the acceleration of the center of mass and $\vec{a}_i$ is the acceleration of the $i^{\rm th}$ particle. Although the center of mass is just a geometrical point in space, it has a position vector $\vec{r}_{\rm CM}$, a velocity $\vec{v}_{\rm CM}$, and an acceleration $\vec{a}_{\rm CM}$.

From Newton's second law, $m_i\vec{a}_i$ must equal the *net force* $\vec{F}_i$ that acts on the $i^{\rm th}$ particle of the system. Therefore, Eq. 7.38 takes the form:

$$M\,\vec{a}_{\rm CM} = \sum m_i \vec{a}_i = \sum \vec{F}_i \tag{7.39}$$

The sum of the net forces, $\sum \vec{F}_i$, that are exerted on the particles of the system can be divided into *external forces* (exerted on the particles from outside the system) and *internal forces* (exerted on the particles from within the system). By Newton's third law, as in Sect. 7.2, the internal forces cancel out in the sum $\sum \vec{F}_i$. Consequently, Eq. 7.39 can be written as follows:

$$\sum \vec{F}_{\text{ext}} = M\,\vec{a}_{CM} \tag{7.40}$$

Thus, for a system composed of a *group of particles* or formed out of an *extended object*, we conclude that:

Spotlight
The net external force on a system equals the total mass of the system times the acceleration of its center of mass.

If we compare Eq. 7.40 with Newton's second law for a single particle [see Eq. 5.2], we see that the point-particle model that has been used for all problems can be described in terms of the center of mass. Thus, we conclude that:

Spotlight
For a system of particles (or an extended object) of a total mass M, the center of mass point exists as if all the mass M were concentrated at that point and all the external forces acted on the same point.

Thus, the translational motion of any object or system of particles is known from the motion of the center of mass, as in Figs. 7.12 and 7.16.

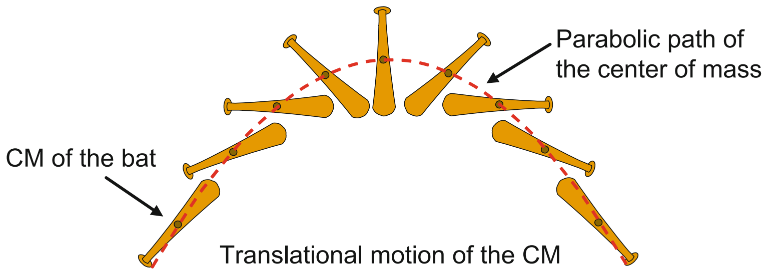


Fig. 7.16 When a bat is thrown into the air, the center of mass of the bat follows a parabolic path, but all other points of the bat follow complicated paths

Since $m_i\vec{v}_i$ is the linear momentum $\vec{p}_i$ of the i^{th} particle and $\sum \vec{p}_i = \vec{P}$ is the total linear momentum of the system, then we can rewrite Eq. 7.37 as follows:

$$M\,\vec{v}_{\text{CM}} = \vec{P} \tag{7.41}$$

Therefore, we conclude that:

For a system of particles:
The total linear momentum of a system of particles equals the total mass multiplied by the velocity of the center of mass.

For an extended object:
The linear momentum of an extended object equals its total mass multiplied by the velocity of its center of mass.

Now we differentiate Eq. 7.41 with respect to time to get:

$$M\frac{d\vec{v}_{\mathrm{CM}}}{dt} = \frac{d\vec{P}}{dt} \quad \text{(System of particles or objects)} \tag{7.42}$$

We can use Eq. 7.40, $\sum \vec{F}_{\mathrm{ext}} = M\,\vec{a}_{\mathrm{CM}} = M\,d\vec{v}_{\mathrm{CM}}/dt$, to get:

$$\sum \vec{F}_{\mathrm{ext}} = \frac{d\vec{P}}{dt} \quad \text{(System of particles or objects)} \tag{7.43}$$

Equations 7.43 and 7.42 lead to the following conclusion:

$$\text{If} \sum \vec{F}_{\mathrm{ext}} = 0, \text{ then } \begin{cases} \vec{P} = \text{constant} \\ \qquad \text{and} \\ \vec{v}_{\mathrm{CM}} = \text{constant} \end{cases} \tag{7.44}$$

That is, if the net force acting on a system is zero (which is true for any isolated system), then the total linear momentum as well as the velocity of the center of mass are both conserved. This is a generalization to the law of conservation of momentum discussed in Sect. 7.2. In fact, this result greatly simplifies the analysis of the motion of complex systems and extended objects.

Example 7.9

Two particles of masses $m_1 = 30$ g and $m_2 = 70$ g undergo an elastic head-on collision. Particle m_1 has an initial velocity of 2 m/s along the positive x-direction, while m_2 is initially at rest. (a) What are the velocities of the particles after the collision? (b) What is the velocity of the center of mass? Sketch the velocities of m_1, m_2, and CM at different times before and after the collision.

Solution: (a) From Eq. 7.23 we have:

$$v_1' = \frac{m_1 - m_2}{m_1 + m_2} v_1 = \frac{30\,\text{g} - 70\,\text{g}}{30\,\text{g} + 70\,\text{g}} (2\,\text{m/s}) = -0.8\,\text{m/s}$$

The negative sign indicates that m_1 rebounds after the collision and moves along the negative x-direction. From Eq. 7.24, we have:

$$v_2' = \frac{2m_1}{m_1 + m_2} v_1 = \frac{(2)(30\,\text{g})}{30\,\text{g} + 70\,\text{g}} (2\,\text{m/s}) = +1.2\,\text{m/s}$$

Thus, the relatively heavy target m_2 moves along the positive x-direction, but with a slower speed than the incoming particle m_1.

(b) Since $\sum \vec{F}_{\text{ext}} = 0$, then $P_{\text{before}} = P_{\text{after}}$ and Eq. 7.41 gives:

$$v_{\text{CM}} = \frac{P}{M} = \frac{m_1 v_1 + 0}{m_1 + m_2} = \frac{m_1}{m_1 + m_2} v_1 = \frac{30\,\text{g}}{30\,\text{g} + 70\,\text{g}} (2\,\text{m/s}) = +0.6\,\text{m/s}$$

Figure 7.17 displays v_1, v_2, v_1', v_2', and v_{CM} at different times. Notice that the velocity of the center of mass is unaffected by the collision.

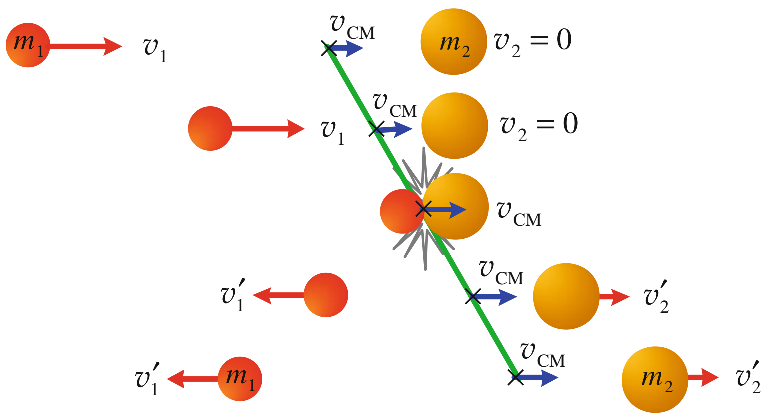


Fig. 7.17

Example 7.10

After the rocket of Fig. 7.18a is fired, the CM of the system continues to follow a parabolic trajectory from a constant downward gravitational force. When the system has a total mass M and speed $v_1 = 216\,\text{m/s}$, a prearranged explosion separates the system into two parts, a space capsule of mass $m_1 = M/4$ and a rocket of mass $m_2 = 3M/4$. The velocities of the two parts are perpendicular and the capsule has an upward initial speed $v_1' = 571$ m/s, see Fig. 7.18b. Describe the motion of the CM and find the initial speed v_2' of the rocket just after the separation of the space capsule and the rocket.

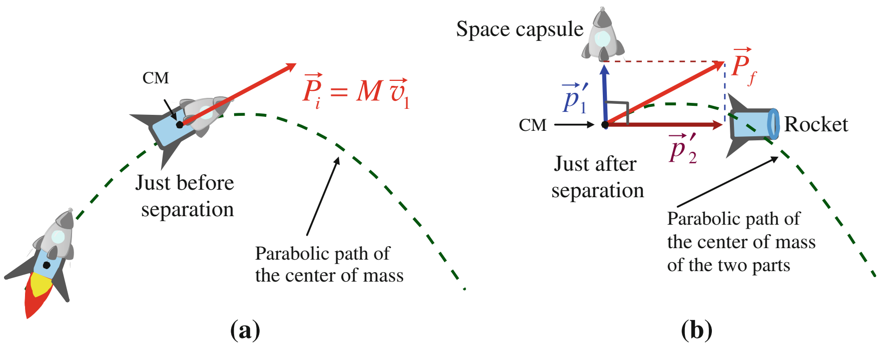


Fig. 7.18

Solution: Since the forces of the explosion are internal to the system composed of the rocket and the capsule, the initial momentum $\vec{P}_i$ just before the separation must equal the final total momentum $\vec{P}_f$ right after the separation. In addition, the center of mass of the two parts continues to follow the original parabolic path, until the rocket hits the ground. Conservation of total momentum gives:

$$\vec{P}_i = \vec{P}_f \quad \Rightarrow \quad \vec{P}_i = \vec{p}\,'_1 + \vec{p}\,'_2 \quad \Rightarrow \quad (Mv_1)^2 = \left(\frac{M}{4}v_1'\right)^2 + \left(\frac{3M}{4}v_2'\right)^2$$

Eliminating M from the last result and solving for v_2', we find:

$$v_2' = \sqrt{\frac{16v_1^2 - v_1'^2}{9}} = \sqrt{\frac{16(216\,\text{m/s})^2 - (571\,\text{m/s})^2}{9}} = 216\,\text{m/s}$$

7.6 Systems of Variable Mass

For systems with a variable mass M, we can use Eq. 7.43, $\sum \vec{F}_{\text{ext}} = d\vec{P}/dt$, whether the mass M increases (as in dropping material onto a conveyer belt, where $dM/dt > 0$) or the mass M decreases (as in rockets, where $dM/dt < 0$).

7.6.1 Systems of Increasing Mass

For the general treatment of systems of increasing mass, we use Fig. 7.19 that depicts the following:

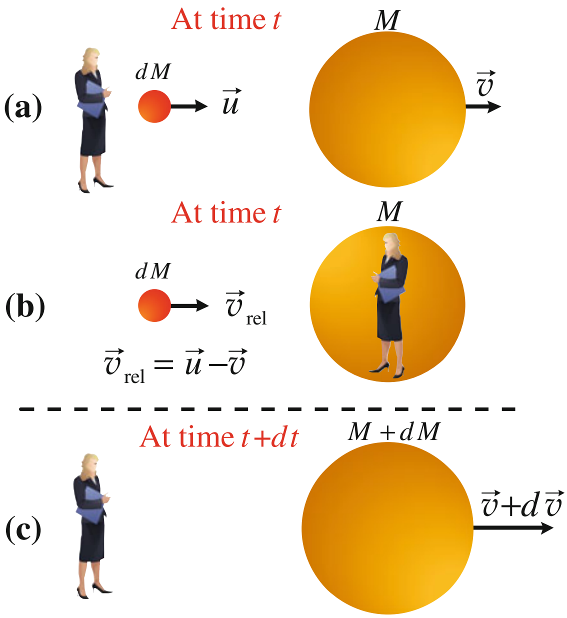


Fig. 7.19 (a) At time t, the differential mass dM is about to combine with the mass M. (b) The velocity of dM as seen by an observer on M at the same time t. (c) At time $t + dt$, the mass dM has combined with M

- At time t
  We have a system consisting of mass M moving with velocity $\vec{v}$ and momentum $M\vec{v}$. Also, we have an infinitesimal mass dM moving with velocity $\vec{u}$ and momentum $dM\,\vec{u}$, see Fig. 7.19a. The initial total momentum of the system can be expressed as:

$$\vec{P}_i = M\,\vec{v} + dM\,\vec{u}$$

Relative to an observer sitting on the mass M, see Fig. 7.19b, the observer will view the infinitesimal mass dM moving with a relative velocity $\vec{v}_{rel}$ where:

$$\vec{v}_{rel} = \vec{u} - \vec{v}$$

- At time $t + dt$

  The infinitesimal mass dM combines with the mass M forming a system of mass $M + dM$ moving with velocity $\vec{v} + d\vec{v}$, see Fig. 7.19c. Then, the final total momentum of the system is:

$$\vec{P}_f = (M + dM)(\vec{v} + d\vec{v})$$

Note that dM can be positive (when momentum is being transferred *into* the mass M) or negative (when momentum is being transferred *out* of the mass M). The change in momentum of the system is thus:

$$\begin{aligned} d\vec{P} &= \vec{P}_f - \vec{P}_i = [(M + dM)(\vec{v} + d\vec{v})] - [M\,\vec{v} + dM\,\vec{u}] \\ &= M\,d\vec{v} - dM(\vec{u} - \vec{v}) \end{aligned} \qquad (7.45)$$

where the term $dM\,d\vec{v}$ is dropped because it is the product of two differential quantities.

When we substitute Eq. 7.45 into $\sum \vec{F}_{\text{ext}} = d\vec{P}/dt$, we get:

$$\sum \vec{F}_{\text{ext}} + (\vec{u} - \vec{v})\frac{dM}{dt} = M\frac{d\vec{v}}{dt} \qquad (7.46)$$

This can be simplified by using the relative velocity $\vec{v}_{\text{rel}} = \vec{u} - \vec{v}$, such as:

$$\sum \vec{F}_{\text{ext}} + \vec{v}_{\text{rel}}\frac{dM}{dt} = M\frac{d\vec{v}}{dt} \quad \Rightarrow \quad \vec{F}_{\text{net}} = M\frac{d\vec{v}}{dt} \qquad (7.47)$$

The right-hand side of this equation, $M\,d\vec{v}/dt$, refers to the mass times the acceleration. The first term on the left-hand side of the equation, $\sum \vec{F}_{\text{ext}}$, refers to the external force on the mass M. The second term on the left-hand side, $\vec{v}_{\text{rel}}\,dM/dt$, refers to the force exerted on M, in terms of the rate at which the momentum is being transferred into M (due to the addition of mass).

7.6.2 Systems of Decreasing Mass; Rocket Propulsion

Now we treat systems with decreasing mass by considering the case of rocket propulsion. Figure 7.20a represents the following:

- At time t

  We have a system boundary consisting of a rocket of mass M moving with velocity $\vec{v}$ and momentum $M\,\vec{v}$, see Fig. 7.20a. The initial total momentum of the system can be expressed as: $\vec{P}_i = M\vec{v}$

- At time $t + dt$

  We have a system boundary consisting of a rocket of mass $M - dM$ moving with velocity $\vec{v} + d\vec{v}$ and an ejected exhaust of mass dM moving with velocity $\vec{u}$, see Fig. 7.20b. The final total momentum of the system boundary is:

$$\vec{P}_f = (M - dM)(\vec{v} + d\vec{v}) + dM\,\vec{u} \tag{7.48}$$

Relative to an observer sitting on the rocket, see Fig. 7.20c, that observer will view the exhaust of mass dM moving with a relative velocity $\vec{v}_{\text{rel}}$ where: $\vec{v}_{\text{rel}} = (\vec{v} + d\vec{v}) - \vec{u}$.

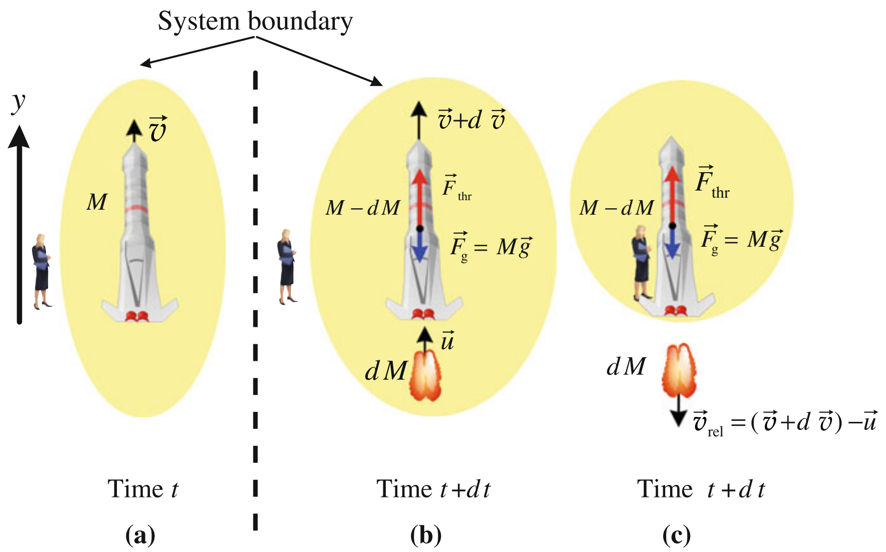


Fig. 7.20 (a) At time t, the rocket has a mass M. (b) At time $t + dt$, the mass of the exhaust dM has been ejected from M. (c) The velocity of the exhaust dM as seen by an observer on the rocket at time $t + dt$

The change in momentum between the system boundaries is thus:

$$\begin{aligned} d\vec{P} &= \vec{P}_f - \vec{P}_i = [(M - dM)(\vec{v} + d\vec{v}) + dM(\vec{v} + d\vec{v}) - \vec{v}_{\text{rel}}] - M\vec{v} \\ &= M\,d\vec{v} - dM\,\vec{v}_{\text{rel}} \end{aligned} \tag{7.49}$$

When we substitute with Eq. 7.49 into $\sum \vec{F}_{\text{ext}} = d\vec{P}/dt$, we get:

$$\sum \vec{F}_{\text{ext}} + \vec{v}_{\text{rel}} \frac{dM}{dt} = M \frac{d\vec{v}}{dt} \tag{7.50}$$

This is identical to Eq. 7.47 except that $\vec{v}_{rel}$ is against $\vec{v}$ and dM/dt is negative. The term $\vec{v}_{rel}\, dM/dt$ refers to the force exerted on M in terms of the rate at which the momentum is being transferred out of M (due to the ejection of mass). For rockets, this term is positive since dM/dt is negative and $\vec{v}_{rel}$ is negative (opposite to $\vec{v}$). This term is called the **thrust**, $\vec{F}_{thr}$, and represents the force exerted on the rocket by the ejected gasses. Thrust is defined as follows:

$$\vec{F}_{thr} = \vec{v}_{rel}\frac{dM}{dt} \tag{7.51}$$

In one-dimensional vertical motion under a constant gravitational force, where $\sum F_{ext} = -Mg$, we can find the speed of the rocket at any time t, by rewriting Eq. 7.50 as:

$$dv = -g dt + v_{rel}\frac{dM}{M} \tag{7.52}$$

Since v_{rel} is constant, we can integrate this equation from an initial speed $v_\circ$ (when the mass was $M_\circ$) to a any speed v (when the mass becomes M). This gives:

$$\int_{v_\circ}^{v} dv = -g\int_{0}^{t} dt + v_{rel}\int_{M_\circ}^{M}\frac{dM}{M}$$

or

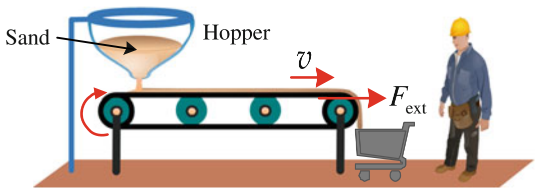
$$v - v_\circ = -gt + v_{rel}\ln\frac{M}{M_\circ} \tag{7.53}$$

Note that v_{rel} is negative because it is opposite to the rocket's motion and $\ln M/M_\circ$ is also negative because $M_\circ > M$.

Example 7.11

Figure 7.21 shows a stationary hopper that drops sand at a rate $dM/dt = 80$ kg/s onto a conveyer belt. The belt is supported by frictionless rollers and moves at a constant speed $v = 1.5$ m/s under the action of a constant external force $\vec{F}_{ext}$. (a) Find the value of the external force $\vec{F}_{ext}$ that is needed to keep the belt moving with a constant speed. (b) Find the power delivered by the external force $\vec{F}_{ext}$. (c) Find the rate of the kinetic energy acquired by the falling sand due to the change in its horizontal motion.

Fig. 7.21



Solution: (a) We use the one-dimensional form of Eq. 7.46 by considering $u = 0$ to represent the stationary hopper. We also take $dv/dt = 0$ because the belt is moving with constant speed. Thus:

$$F_{ext} + (u - v)\frac{dM}{dt} = M\frac{dv}{dt} \quad \Rightarrow \quad F_{ext} + (0 - v)\frac{dM}{dt} = 0 \quad \Rightarrow \quad F_{ext} = v\frac{dM}{dt}$$

$$\therefore \qquad F_{ext} = (1.2\,\text{m/s})(5\,\text{kg/s}) = 6\,\text{N}$$

The only horizontal force on the sand is the friction of the belt f_s. Thus, $f_s = F_{ext}$.

(b) The power delivered by $\vec{F}_{ext}$ is work done by this force in 1s. Thus:

$$P = \frac{dW}{dt} = \vec{F}_{ext} \bullet \vec{v} = F_{ext}\, v = v^2\frac{dM}{dt} = (1.2\,\text{m/s})^2(5\,\text{kg/s}) = 7.2\,\text{W}$$

This work per unit time is the power output required by the motor.

(c) The rate of the kinetic energy acquired by the falling sand is:

$$\frac{dK}{dt} = \frac{d}{dt}\left(\frac{1}{2}Mv^2\right) = \frac{1}{2}\frac{dM}{dt}v^2 = \frac{1}{2}(5\,\text{kg/s})(1.2\,\text{m/s})^2 = 3.6\,\text{W}$$

This is only half the power delivered by $\vec{F}_{ext}$. The other half goes into thermal energy produced by friction between the sand and the belt.

Example 7.12

A rocket has a mass 2×10^4 kg of which 10^4 kg is fuel. When the rocket is lunched vertically from the ground, it consumes fuel from its rear at a rate of 1.5×10^3 kg/s with an exhaust speed of 2.5×10^3 m/s relative to the rocket. Neglect air resistance and take the acceleration due to gravity to be $g = 9.8\,\text{m/s}^2$. (a) Find the thrust on the rocket. (b) Find the net force on the rocket, once when it is full of fuel and once when it is empty. (c) Find the final speed of the rocket when the fuel burns completely.

Solution: (a) Since the motion is in one dimension and we can take upward as positive, then v_{rel} is negative because it is downward and dM/dt is negative because the rocket's mass is decreasing. Therefore, the thrust is:

$$F_{\text{thr}} = v_{\text{rel}}\frac{dM}{dt} = (-2.5\times 10^3\ \text{m/s})(-1.5\times 10^3\ \text{kg/s}) = 3.75\times 10^6\ \text{N}$$

(b) Initially the net force on the rocket is:

$$F_{\text{net}} = F_{\text{thr}} - M_\circ\, g = 3.75\times 10^6\ \text{N} - (2\times 10^4\ \text{kg})(9.8\ \text{m/s}^2) = 3.554\times 10^6\ \text{N}$$

The net force just before the rocket is out of fuel is:

$$F_{\text{net}} = F_{\text{thr}} - M_\circ\, g = 3.75\times 10^7\ \text{N} - (1\times 10^4\ \text{kg})(9.8\ \text{m/s}^2) = 3.652\times 10^6\ \text{N}$$

(c) The time required to reach fuel burnout is the time needed to use all the fuel (10^4 kg) at rate of 1.5×10^3 kg/s. Thus:

$$t = \frac{10^4\ \text{kg}}{1.5\times 10^3\ \text{kg/s}} = 6.67\ \text{s}$$

By taking $v_\circ = 0$ and using Eq. 7.53, we find that:

$$v - v_\circ = -gt + v_{\text{rel}} \ln\frac{M}{M_\circ}$$

$$v = -(9.8\ \text{m/s}^2)(6.67\ \text{s}) + (-2.5\times 10^3\ \text{m/s}) \times \left(\ln\frac{1\times 10^4\ \text{kg}}{2\times 10^4\ \text{kg}}\right) = 1667.5\ \text{m/s}$$

7.7 Exercises

Section 7.1 Linear Momentum and Impulse

(1) What is the momentum of an electron of speed $v = 0.99\,c$, if the rest mass of the electron is $m = 9.11\times 10^{-31}$ kg and the speed of light is $c = 3\times 10^8$ m/s?

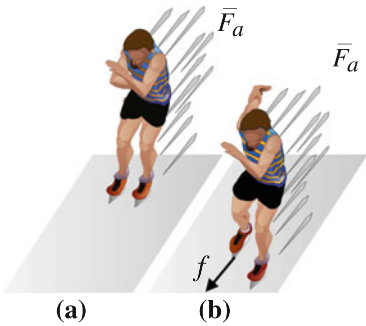
(2) (a) What is the momentum of an 8,000-kg truck when its speed is 20 m/s? What speed must a 2,000-kg car attain in order to have: (b) the same momentum as the truck, (c) the same kinetic energy as the truck?

(3) A ball of mass $m = 0.4$ kg is moving horizontally with a speed 6 m/s when it strikes a vertical obstacle. The ball rebounds with a speed 2 m/s. What is the change in momentum of the ball?

(4) A baseball has a mass of 0.2-kg and a speed of 30 m/s. After the baseball is struck by the batter, its velocity changed to 50 m/s in the opposite direction. (a) Find the change in momentum of the ball and the impulse of the strike. (b) Find the average force exerted by the bat on the ball if remains in contact for 0.002 s.

(5) A 70-kg ice skater experiences a constant air frictional force of magnitude $\overline{F}_a = 30\,\text{N}$ for 7 s, see Fig. 7.22a. (a) What is the change in the velocity of the skier? (b) What constant forward frictional force f must the skater apply in order to reduce the velocity of part (a) by half, see Fig. 7.22b?

Fig. 7.22 See Exercise (5)



(6) A 4-kg particle has a velocity $\vec{v} = (4\vec{i} - 3\vec{j})$ m/s. (a) What are the x and y components of its momentum? (b) Find the magnitude and direction of the momentum.

(7) Rain is falling on an object at time t with a force of $\vec{F} = (8\,t\,\vec{i} - 3\,t^2\vec{j})$N. Find the change in the object's momentum between $t_i = 0$ and $t_f = 2$ s.

(8) In a training session, water with a horizontal speed of 25 m/s leaves a fireman's hose at a rate of 12 kg/s and comes to rest after striking a firewall, see Fig. 7.23. Ignoring the water splashes, what is the average force exerted by the water on the wall?

(9) The force-time graph for a ball struck by a bat is approximated as shown in Fig. 7.24. From this graph, find (a) the impulse delivered by the ball, (b) the average force exerted on the ball, and (c) the maximum force exerted on the ball.

(10) A mass m undergoes a free fall with a constant acceleration g. What is its momentum after it has been dropped (i.e., released from rest) and falls a distance h?

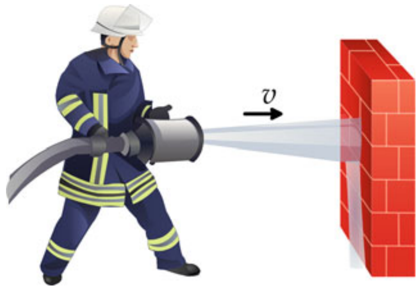


Fig. 7.23 See Exercise (8)

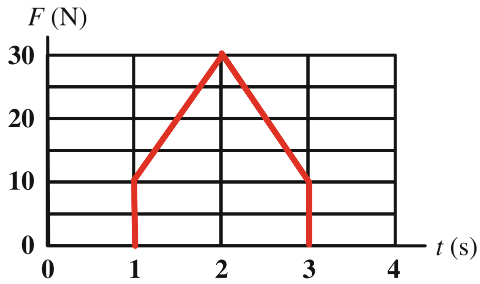


Fig. 7.24 See Exercise (9)

(11) A ball of mass 0.4 kg is dropped from a height $y_1 = 0.8$ m. The ball rebounds from the floor and reaches a maximum height $y_2 = 0.2$ m, see Fig. 7.25. Ignore air resistance and take $g = 10\,\text{m/s}^2$. (a) What is the impulse exerted by the floor on the ball? (b) What fraction of the ball's kinetic energy is lost in the Impact?

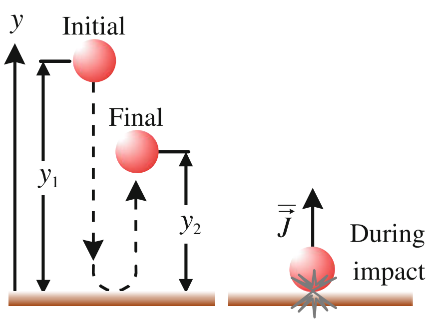
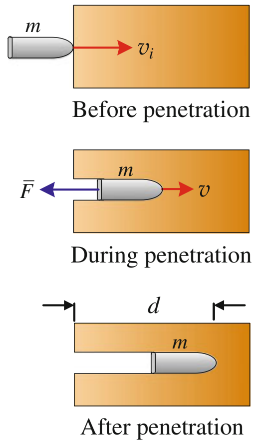


Fig. 7.25 See Exercise (11)

(12) A bullet of mass $m = 6$ g moving with $v_i = 80$ m/s strikes a wooden block and stops after penetrating a distance $d = 5$ cm, see Fig. 7.26. Assume that the bullet undergoes a constant deceleration due to an average resistive force $\overline{F}$.

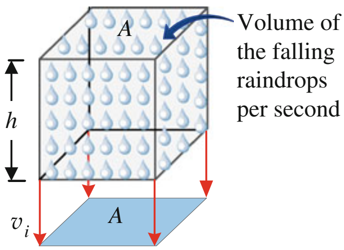
Find: (a) the penetration time, (b) the impulse on the wooden block, and (c) the average force $\bar{F}$ exerted on the bullet.

Fig. 7.26 See Exercise (12)



(13) Rain is falling vertically with a speed $v_i = 6$ m/s and can fill a container to a height of 18 cm in one hour. Water has a mass density $\rho = 10^3$ kg/m^3. (a) Find the height h that the rain will fill the container in one second (neglect air volume between raindrops). (b) Estimate the mass of water that falls per unit time on a flat surface of an area $A = 2\,\text{m}^2$, see Fig. 7.27. (b) If the raindrops do not rebound, find the average force exerted by the rain on that surface.

Fig. 7.27 See Exercise (13)



(14) A 5-kg steel ball strikes a wall with a speed of 10 m/s at an angle $\theta = 60°$ with the wall's surface, see Fig. 7.28. The ball bounces off with the same speed and angle and is in contact with the wall for 0.01 s. Choose the x-axis to be toward the wall. (a) What is the change in momentum of the ball? (b) What is the average force exerted on the ball by the wall?

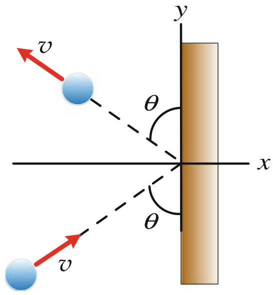


Fig. 7.28 See Exercise (14)

(15) Redo Exercise (14) with a value of θ that produce: (a) the smallest change in momentum and the average force, (b) the largest change in momentum and the average force.

Section 7.2 Conservation of Linear Momentum

(16) A locomotive of mass $m_1 = 40{,}000\,\text{kg}$ rolls at the speed $v_1 = 2\,\text{m/s}$ along a level track. It collides and couples with a stationary fully loaded freight car of mass $m_2 = 60{,}000\,\text{kg}$, see Fig. 7.29. (a) What is the speed after the collision? (b) Find the decrease in kinetic energy that results from the collision. (c) With what velocity should the freight be moving toward the locomotive in order for both objects to stay at rest after the collision?

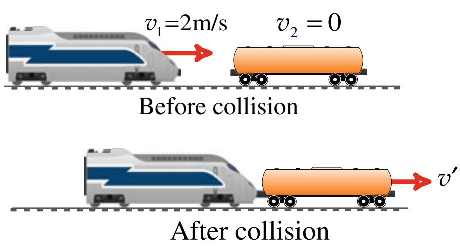


Fig. 7.29 See Exercise (16)

(17) An object at rest explodes into two fragments. One fragment of mass m_1 acquires twice the kinetic energy of the second fragment of mass m_2, see Fig. 7.30. What is the ratio of their masses?

(18) A parent atomic nucleus at rest decays radioactively into an alpha particle of mass m_1 and a residual nucleus of mass $m_2 = 232m_1$. What will be the speed of this recoiling nucleus if the speed of the alpha particle is $v'_1 = 1.5 \times 10^5$ m/s?

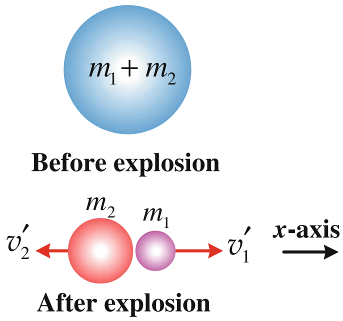


Fig. 7.30 See Exercise (17)

(19) A 60-kg boy holding a 4-kg package is sitting on a stationary boat of 100-kg mass, see Fig. 7.31. The boy throws the package horizontally with a velocity $v_2' = -5$ m/s. The boy and boat move together after the package is thrown and the boat moves without friction on the water surface. What is the speed of the boat?

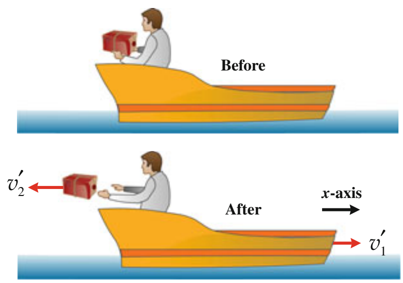


Fig. 7.31 See Exercise (19)

(20) A railroad flatcar of mass M can roll without friction along a horizontal track. Initially, a man of mass m is standing on the car when it is at rest. The man starts to run on the car with a constant speed v, as measured with respect to an observer on the ground, see Fig. 7.32. (a) Find the speed V of the car with respect to the ground. (b) What is the relative speed v_{rel} of the man with respect to the car?

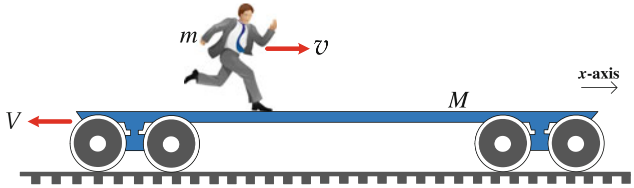
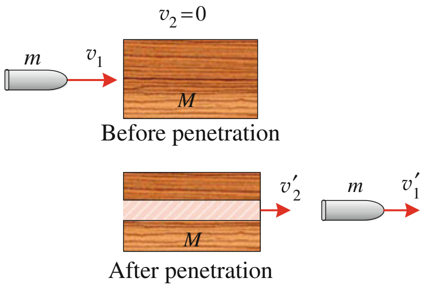


Fig. 7.32 See Exercise (20)

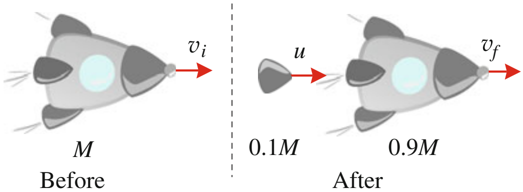
(21) A bullet of mass $m = 10\,\text{g}$ travels with velocity $v_1 = +250\,\text{m/s}$ toward a stationary wooden block of mass $M = 2\,\text{kg}$ that is resting on a horizontal frictionless surface, see Fig. 7.33. The bullet penetrates the block and emerges from the other side with velocity $v_1' = +150\,\text{m/s}$. Neglect the mass removed from the block by the bullet. (a) How fast does the block move after the bullet emerges from the other side of the block? (b) What fraction of the bullet's kinetic energy is lost in the penetration? (c) What fraction of the bullet's energy goes to heat?

Fig. 7.33 See Exercise (21)



(22) A spaceship of mass M is traveling along the x-axis with a speed $v_i = 580\,\text{m/s}$ with respect to an observer on the Earth. The ship ejects a cargo module of mass 0.1 M and then travels relative to the cargo with a speed $v_{\text{rel}} = 140\,\text{m/s}$, see Fig. 7.34. What is the velocity v_f of the ship with respect to the observer?

Fig. 7.34 See Exercise (22)



Section 7.3 Conservation of Momentum and Energy in Collisions

Subsection 7.3.1 Elastic Collisions in One and Two Dimensions

(23) A tennis ball of mass $m_1 = 0.06\,\text{kg}$, moving with a speed of 4 m/s, has an elastic head-on collision with a target ball of mass $m_2 = 0.09\,\text{kg}$ initially moving in the same direction at a speed of 3 m/s. What is the velocity of each ball after the collision?

(24) A ball of mass $m_1 = 0.5$ kg, moving along the x-axis with a speed of 5 m/s, has an elastic head-on collision with a target ball of mass $m_2 = 1$ kg initially at rest. What is the velocity of each ball after the collision?

(25) A ball of mass m_1 and velocity v_1 undergoes an elastic head-on collision with a second ball of mass m_2 initially at rest. Then m_1 rebounds with a speed $v_1' = -0.5v_1$. Find the value of m_2 in terms of m_1.

(26) A croquet ball of mass $m_1 = 1$ kg and velocity v_1 undergoes an elastic head-on collision with a second ball of mass m_2 that is initially at rest. Then m_1 moves with a velocity v_1' and m_2 moves with a velocity $v_2' = (4/5)v_1$. (a) Find the value of m_2. (b) Find the relation between v_1' and v_1. (c) What fraction of the original kinetic energy goes to the second ball?

(27) Find the fraction of kinetic energy lost by a neutron of mass $m_1 = 1.01$u when it undergoes an elastic head-on collision with a stationary nucleus of: (a) a hydrogen atom ($^1_1$H) of mass $m_2 = 1.01$ u, (b) a heavy hydrogen atom ($^2_1$H) of mass $m_2 = 2.01$ u, (c) a carbon ($^{12}_6$C) atom of mass $m_2 = 12.00$ u, and (d) a lead atom ($^{208}_{82}$Pb) of mass $m_2 = 208$ u.

(28) A block of mass $m_1 = 1$ kg slides along a frictionless horizontal surface with a speed $v_1 = 4$ m/s toward a stationary second block of mass $m_2 = 0.5$ kg. The second block is connected to an elastic spring that is not stretched and has a spring constant $k_H = 100$ N/m. The other end of that spring is fixed to a wall, see Fig. 7.35. (a) Is the collision elastic? Explain your answer. (b) What will be the maximum compression of the spring?

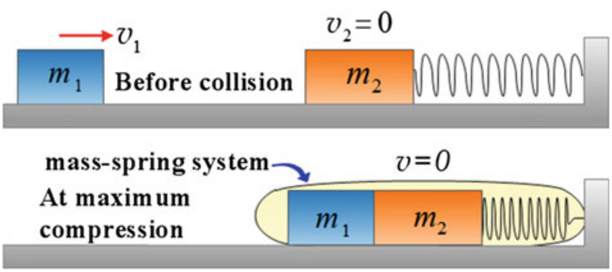


Fig. 7.35 See Exercise (28)

(29) A block of mass $m_1 = 2.5$ kg slides along a frictionless horizontal surface with a speed $v_1 = 8$ m/s toward a stationary second block of mass $m_2 = 7.5$ kg. A massless spring with spring constant $k_H = 1{,}920$ N/m is attached to the near

side of m_2, as shown in Fig. 7.36. (a) Is the collision elastic? Explain. (b) What is the speed of the mass-spring system at the maximum compression? (c) What will be the maximum compression of the spring? (d) What will be the final velocities of the two blocks?

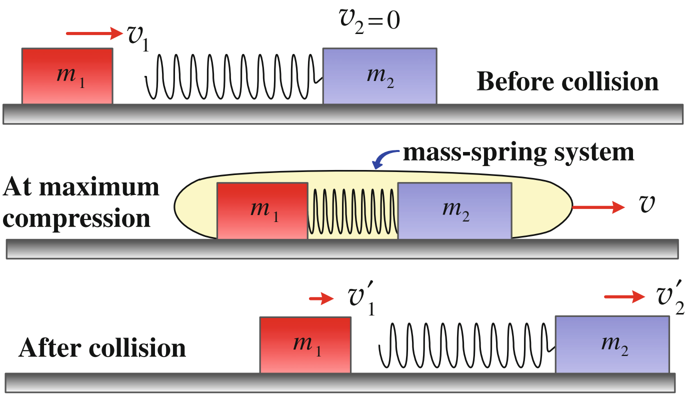


Fig. 7.36 See Exercise (29)

(30) Repeat Exercise (29), this time with $m_1 = 7.5$ kg and $m_2 = 2.5$ kg.

(31) Show that the fraction of kinetic energy transferred to the target in Example 7.5 is independent of the value of the speed of the projectile, v_1. Then, redo this example when $m_2 = 3m_1$.

(32) A hockey puck traveling at 30 m/s on a smooth ice surface is deflected by $\theta_1 = 30°$ from its original direction when it collides elastically with a second stationary identical puck. The second puck acquires a velocity at $\theta_2 = 60°$ from the original velocity of the first puck, see Fig. 7.37. Find the speed of the pucks after the collision.

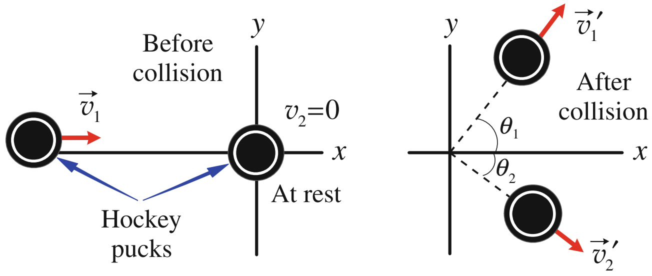


Fig. 7.37 See Exercise (32)

(33) If each angle in Exercise (32) is equal to 45°, then show that only the application of conservation of momentum is enough to find the speed of the pucks after the collision. Also, show that any other two equal angles (say 30°) are physically unacceptable.

(34) A ball of momentum $\vec{p}_1$ collides with an identical stationary ball. The first ball deflects by an angle θ_1 from its original direction with a momentum $\vec{p}\,'_1$ while the second ball deflects by an angle θ_2 from the original direction of the first ball with a momentum $\vec{p}\,'_2$. If the two balls have an elastic collision, then use the conservation of momentum vector diagram of Fig. 7.38 and conservation of kinetic energy to show that the two balls will always move off perpendicular to each other, i.e. $\theta_1 + \theta_2 = 90°$.

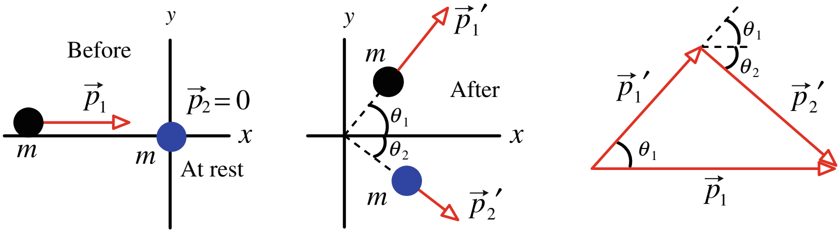


Fig. 7.38 See Exercise (34)

Subsection 7.3.2 Inelastic Collisions

(35) In a ballistic experiment like the one shown in Example 7.6, a bullet causes the pendulum to rise to a maximum height $h_1 = 1.3$ cm. A second bullet of the same mass causes the pendulum to rise to a maximum height $h_2 = 5.2$ cm. Express the speed of the second bullet as a multiple of the first bullet.

(36) Find a formula that gives the fractional change of kinetic energy, $(K_f - K_i)/K_i$, in terms of m and M in the first stage of the ballistic pendulum of Example 7.6. Calculate this fraction for $m = 10$ g and $M = 0.5$ kg.

(37) A 15-kg mass is moving along the positive x-axis at 30 m/s, and a 5-kg mass is moving along the negative x-axis at 50 m/s. The two masses collide head-on and stick together. (a) Find their velocity after the collision. (b) Find the fractional change of kinetic energy.

(38) A car of mass $m_1 = 1{,}000$ kg moving with a speed v_1 collides with another stationary car of mass $m_2 = 2{,}200$ kg. The two cars stick together after the

collision. Both drivers had their brakes locked throughout the incident, see Fig. 7.39. The police officer measures the skidding distance d to be 2.25 m and estimated the coefficient of kinetic friction between the tires and the road to be 0.8. (a) What was the speed of the oncoming car? (b) Find the fractional change of kinetic energy lost during the impact period.

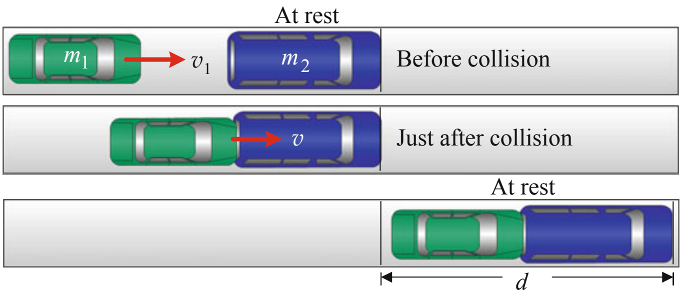


Fig. 7.39 See Exercise (38)

(39) A nucleus at rest spontaneously disintegrates into two nuclei, one of which has double the mass of the other. Assume that the total mass is conserved before and after disintegration. (a) Find the relation between the speeds of the two fragments. (b) If 6×10^{-17} J of energy is released in this disintegration, how much kinetic energy does each nucleus acquire?

(40) Two objects having the same mass $m = 5$ kg collide. Their velocities before collision are $\vec{v}_1 = (3\vec{i} + 6\vec{j})$(m/s) and $\vec{v}_2 = (-2\vec{i} + 1\vec{j})$(m/s). After collision, the first object acquires a velocity $\vec{v}'_1 = (-1\vec{i} + 4\vec{j})$(m/s). (a) What is the final velocity of the second object? (b) How much kinetic energy is lost or gained in this collision?

(41) A stationary radioactive nucleus decays into three fragments. Two of these fragments are emitted perpendicularly to each other and have momenta $|\vec{p}'_1| = 5 \times 10^{-23}$ kg.m/s and $|\vec{p}'_2| = 1.2 \times 10^{-22}$ kg.m/s. Find the magnitude and direction of the third fragment.

(42) A ball of mass $m_1 = 2.4$ kg moving horizontally with a speed of $v_1 = 3$ m/s collides (not head-on) with a second ball of $m_2 = 1.5$ kg moving in the opposite direction with a speed $v_2 = 5$ m/s, see Fig. 7.40. The first ball bounces off the second ball with an angle $\theta_1 = 60°$ and speed $v'_1 = 1.5$ m/s. (a) What is the

final velocity of the second ball? (b) How much kinetic energy is lost in this collision?

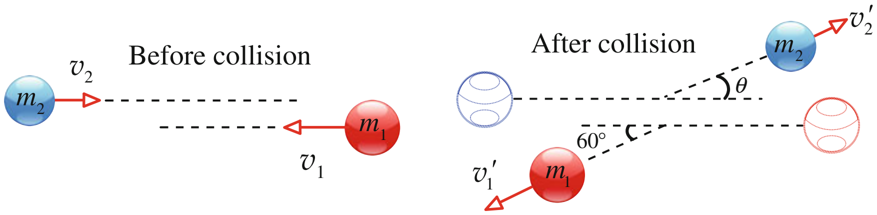


Fig. 7.40 See Exercise (42)

(43) Two identical putty balls move along a frictionless floor, as shown in Fig. 7.41. Their velocity vectors $\vec{v}_1$ and $\vec{v}_2$ make an angle θ and move with the same speed, i.e. $v_1 = v_2$. The two balls stick together after collision. (a) Use the momentum vector diagram shown in the figure to prove that the magnitude v and the direction ϕ of their common velocity $\vec{v}$ are given by:

$$v = \tfrac{1}{2} v_1 \sqrt{2 + 2\cos\theta}, \quad \phi = \sin^{-1}\left[\sin\theta / \sqrt{2 + 2\cos\theta}\right]$$

(b) Taking $v_1 = v_2 = 20$ m/s and $\theta = 45°$ to calculate v, ϕ, and the fractional change in the kinetic energy.

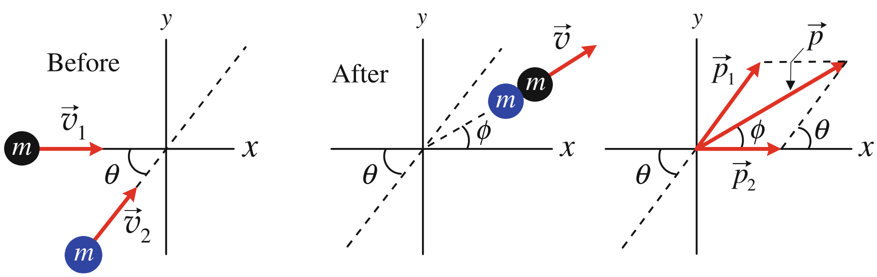


Fig. 7.41 See Exercise (43)

Section 7.4 Center of Mass (CM)

(44) An oxygen atom ($^{16}_{8}$O) has a mass $m_O = 16$ u, and a carbon atom ($^{12}_{6}$C) has a mass $m_C = 12$ u. The average distance between their nuclei in the CO molecule

is $d = 0.113$ nm, see Fig. 7.42. How far from the oxygen nucleus is the center of mass of the molecule?

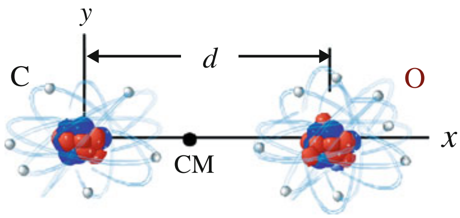


Fig. 7.42 See Exercise (44)

(45) For the system of particles shown in Fig. 7.43 and when $d = 1$ m, find the location of the x and y components of the center of mass. Does your answer depend on the value of m? Explain.

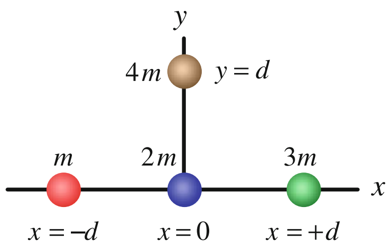


Fig. 7.43 See Exercise (45)

(46) Three particles, each of mass m, are located at the corners of an equilateral triangle of side a, as shown in the Fig. 7.44. Show that the center of mass of the system lies on a common point on the three lines that connect each vertex with the midpoint of the opposite side (the medians).

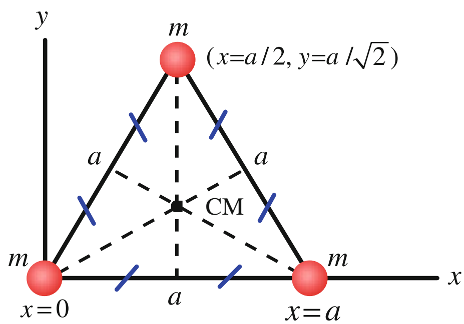
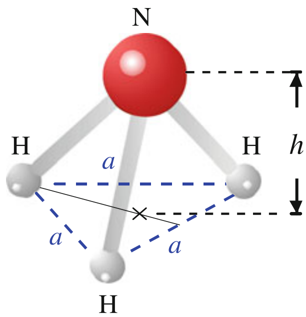


Fig. 7.44 See Exercise (46)

(47) In an ammonia molecule (NH_3), the three hydrogen (${}^1_1H$) atoms are at the corners of an equilateral triangle of side $a = 0.16$ nm that forms the base of a pyramid, with nitrogen atom (${}^{14}_6N$) at the apex above the center of this triangle by $h = 0.037$ nm, see Fig. 7.45. Find the distance of the center of mass of the ammonia molecule above the plane of the hydrogen atoms.

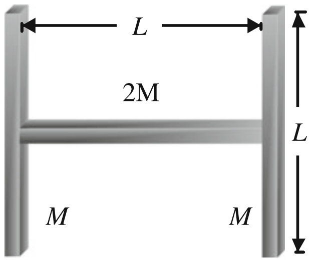
Fig. 7.45 See Exercise (47)



(48) A mass $m_1 = 2$ kg is connected to a mass $m_2 = 3$ kg by a massless rod. The location of m_1 and m_2 are given by the position vectors $\vec{r}_1 = (4\vec{i} + 5\vec{j})$(m) and $\vec{r}_2 = (2\vec{i} + 3\vec{j})$(m), respectively. Find the coordinates of the center of mass.

(49) Three uniform thin rods, each of length L, are arranged to form the shape shown in Fig. 7.46. The vertical arms have mass M and the horizontal arm has a mass *2M*. Find the center of mass of the assembly.

Fig. 7.46 See Exercise (49)



(50) Find the center of mass of a uniform cone of radius R and height h, see Fig. 7.47. (Hint: Divide the cone into an infinite number of disks, each of thickness dx.)

(51) A pyramid has a height H and square base area of side L, see Fig. 7.48. Find the center of mass of the pyramid above its base. Calculate z_{CM} for the Great

Pyramid of Khufu at Giza, Egypt, which has height $H = 138.8$ m and base square area of side $L = 230.4$ m. (Hint: Divide the pyramid into an infinite number of squares, each of height dz.)

Fig. 7.47 See Exercise (50)

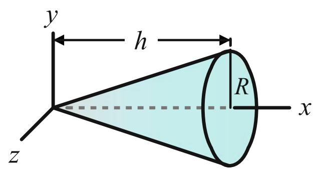
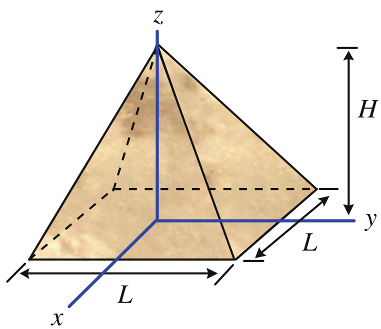


Fig. 7.48 See Exercise (51)

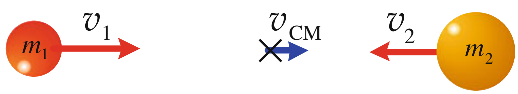


Section 7.5 Dynamics of the Center of Mass

(52) The velocities of two particles of masses $m_1 = 2$ kg and $m_2 = 3$ kg are given by the position vectors $\vec{v}_1 = (4\vec{i} + 5\vec{j})$(m/s) and $\vec{v}_2 = (2\vec{i} - 3\vec{j})$(m/s), respectively. Find the velocity of the center of mass of that system.

(53) A ball of mass $m_1 = 2$ kg traveling with velocity $\vec{v}_1 = 15\vec{i}$ m/s collides head-on and elastically with a second ball of mass $m_1 = 3$ kg traveling with velocity $\vec{v}_2 = -4\vec{i}$ m/s, see Fig. 7.49. (a) Find the velocities of the two balls after the collision. (b) Find the velocity of the center of mass before and after the collision.

Fig. 7.49 See Exercise (53)



(54) Two particles of masses $m_1 = 0.2$ kg and $m_1 = 0.3$ kg are initially at rest 2 m apart. The two particles form an isolated system. Each particle starts to attract the other with an equal internal constant force of magnitude 0.12 N. (a) What is the speed of their center of mass before and after the start of the attractive force? (b) What distance does m_1 move before colliding with m_2? (c) What will be the speed of m_1 and m_2 just before the collision?

(55) A man of mass $m = 70$ kg stands on one end of a flat boat, which is always moving horizontally without friction at a speed of $v_\circ = 5$ m/s over water. The boat has a mass $M = 210$ kg and length $L = 20$ m, see Fig. 7.50. The man starts to walk to the other end in the direction of the boat's motion with a relative speed $v_{rel} = 2$ m/s. (a) What is the location and velocity of the center of mass before and after the motion of the man? (b) What is the velocity v_B of the boat once the man starts to move? (c) What time does the man take to reach the other end, and how far has the boat moved?

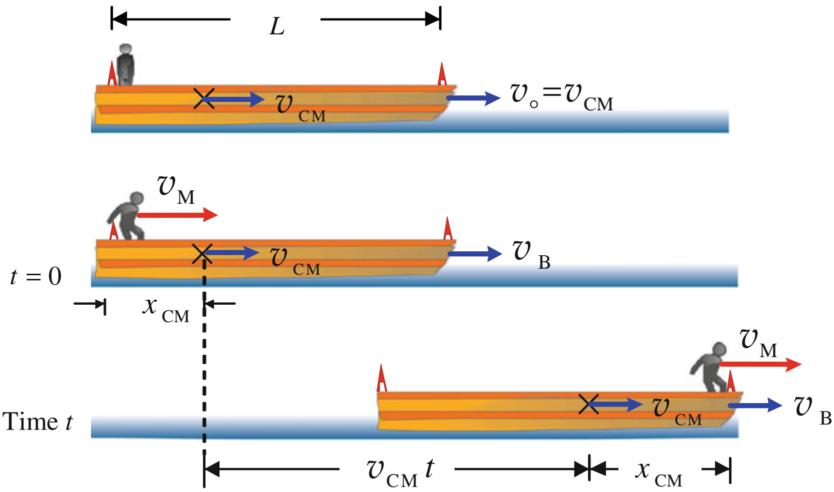


Fig. 7.50 See Exercise (55)

(56) A projectile is fired from the ground with an initial speed $v_\circ$ of 40 m/s at an angle $\theta_\circ$ of 15° m/s above the horizontal direction. At the maximum height, the projectile explodes into two fragments of equal mass, see Fig. 7.51. One fragment stops momentarily and falls vertically, while the second one flies

initially in a horizontal direction. How far from the ground do the center of mass and the second fragment land? Take $g = 10\,\text{m/s}^2$.

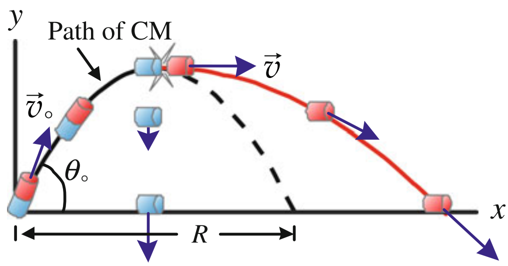


Fig. 7.51 See Exercise (56)

Section 7.6 Systems of Variable Mass

(57) A stationary grain funnel drops grain at a rate $dM/dt = 840\,\text{kg/min}$ onto a railroad car moving with a constant speed $v = 3.5\,\text{m/s}$, see Fig. 7.52. (a) What external force must be applied to the car to keep it moving at constant speed (in the absence of friction)? (b) Find the power delivered by this force. (c) Find the rate of the kinetic energy acquired by the falling grains.

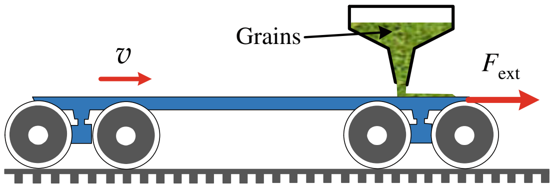


Fig. 7.52 See Exercise (57)

(58) During the first second of its flight in free space, a rocket ejects exhaust that is 1/50 of its mass with a relative speed $v_{rel} = 2.5 \times 10^3\,\text{m/s}$. What is the acceleration of the rocket?

(59) A rocket of mass $M = 3{,}000\,\text{kg}$ ejects fuel and oxidizer at a rate of 150 kg/s in order to acquire an acceleration $a = 4g$. What is the relative speed of the exhaust and the thrust on the rocket?

(60) A rocket of mass $M = 3{,}000\,\text{kg}$ is moving in free space with a speed $v = 3 \times 10^2\,\text{m/s}$ relative to the Earth. The rocket ejects fuel at a rate of 15 kg/s with a relative speed of $2.5 \times 10^3\,\text{m/s}$. (a) Find the thrust on the rocket. (b) Find the final speed of the rocket when its fuel burns completely after 20 s.