Motion in One Dimension

3

Mechanics is the science that deals with motion of objects. It is **basic** to all other **branches of physics**. The branch of mechanics that *describes* the motion of objects is called **kinematics**. In this branch we answer questions like "Does the object speed up, slow down, stop, or reverse direction?" and "How is time involved in these situations?"

In this chapter, we only study motion along straight lines. The moving object of concern is either a **particle** (a point-like object) or an object that can be viewed to move like a particle.

3.1 Position and Displacement

To locate an object in one-dimensional space, we find its position with respect to some reference point, called the **origin** of an axis, such as the x-axis shown in Fig. 3.1. The **positive/negative direction** of this axis is the direction of increasing/decreasing numbers.

A change in the object's position from an initial position x_i to a final position x_f is called displacement Δx (read delta x), where:

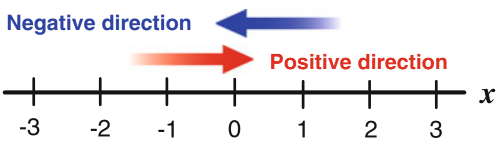


Fig. 3.1 The position of a particle that moves in one dimension is identified on an x-axis that is marked in units of length

H. A. Radi and J. O. Rasmussen, *Principles of Physics*,
Undergraduate Lecture Notes in Physics, DOI: 10.1007/978-3-642-23026-4_3,

$$\Delta x = x_f - x_i \tag{3.1}$$

The displacement is a **vector quantity** which has a magnitude and a direction. The magnitude is the distance between the initial and final positions and the direction is represented in Fig. 3.1 by a plus or minus sign for motion to the right or to the left, respectively.

3.2 Average Velocity and Average Speed

Average Velocity

Consider a particle moving along the x-axis, where its *position-time graph* is as shown in Fig. 3.2. At point P, let its position be x_i when the time was t_i and at point Q, let its position be x_f when the time was t_f (the indices i and f refer to the initial and final values for the variables under consideration). Accordingly, during the time interval $\Delta t = t_f - t_i$, the particle's displacement is $\Delta x = x_f - x_i$.

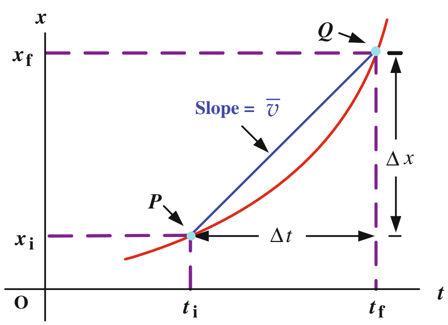


Fig. 3.2 The *position-time graph* for a particle moving along the x-axis. The slope of the line PQ measures the average velocity $\overline{v}$

One of several quantities associated with the phrase "how fast" a particle moves is the average velocity, $\overline{v}$, which is defined as follows:

Average velocity
The average velocity, $\overline{v}$, of a particle is defined as the ratio of its displacement, Δx, to the time interval, Δt. That is:

$$\overline{v} = \frac{\Delta x}{\Delta t} = \frac{x_f - x_i}{t_f - t_i} \tag{3.2}$$

From this definition, $\overline{v}$ has the dimension of length divided by time, that is m/s in SI units. The average velocity is a vector quantity which has a magnitude and direction represented by a plus or minus sign for motion to the right or to the left, respectively, see Fig. 3.1.

Average Speed

The average speed $\overline{s}$ is a different way of describing "how fast" a particle moves and it is defined as follows:

Average speed
The average speed, $\overline{s}$, of a particle is defined as the ratio of the total distance covered d to the time interval $\Delta t = t_f - t_i$. That is:

$$\overline{s} = \frac{\text{total distance}}{\Delta t} = \frac{d}{t_f - t_i} \tag{3.3}$$

So, $\overline{s}$ is different from $\overline{v}$ in that $\overline{s}$ does not depend on direction, and hence is always positive. In some cases $\overline{s}$ might be the same as $\overline{v}$.

Example 3.1

A car moving along the x-axis starts from the position $x_i = 2$ m when $t_i = 0$ and stops at $x_f = -3$ m when $t_f = 2$ s. (a) Find the displacement, the average velocity, and the average speed during this interval of time. (b) If the car goes backward and takes 3 s to reach the starting point, then repeat part (a) for the *whole* time interval.

Solution: (a) The car's displacement, see Fig. 3.3, is given by:

$$\Delta x = x_f - x_i = -3\text{ m} - 2\text{ m} = -5\text{ m}$$

The average velocity is then given by:

$$\overline{v} = \frac{\Delta x}{\Delta t} = \frac{x_f - x_i}{t_f - t_i} = \frac{-3\text{ m} - 2\text{ m}}{2\text{ s} - 0\text{ s}} = \frac{-5\text{ m}}{2\text{ s}} = -2.5\text{ m/s}$$

Since Δx and $\overline{v}$ are negative for this time interval, then the car has moved to the left, toward decreasing values of x, see Fig. 3.3. The total covered distance is $d = 5$ m and the average speed is thus:

$$\overline{s} = \frac{\text{total distance}}{\Delta t} = \frac{d}{t_f - t_i} = \frac{5 \text{ m}}{2 \text{ s} - 0 \text{ s}} = \frac{5 \text{ m}}{2 \text{ s}} = 2.5 \text{ m/s}$$

In this case, $\overline{s}$ is the same as $\overline{v}$ (except for a minus sign).

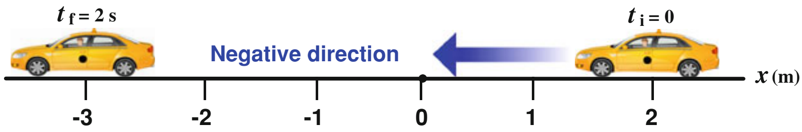


Fig. 3.3 Example 3.1

(b) After the backward movement, the final position and final time of the car are $x_f = 2$ m and $t_f = 2$ s $+ 3$ s $= 5$ s, respectively, while the total distance covered by the car is $d = 5$ m $+ 5$ m $= 10$ m. As we know, the displacement involves only the initial and final positions and will be:

$$\Delta x = x_f - x_i = 2 \text{ m} - 2 \text{ m} = 0$$

Then, the average velocity will be:

$$\overline{v} = \frac{\Delta x}{\Delta t} = \frac{x_f - x_i}{t_f - t_i} = \frac{0}{5 \text{ s} - 0 \text{ s}} = 0$$

Finally, the average speed for the whole movement of the car will be:

$$\overline{s} = \frac{\text{total distance}}{\Delta t} = \frac{d}{t_f - t_i} = \frac{10 \text{ m}}{5 \text{ s} - 0 \text{ s}} = \frac{10 \text{ m}}{5 \text{ s}} = 2 \text{ m/s}$$

As you can see, the average velocity is zero, while the average speed is 2 m/s, since the latter depends only on the total covered distance d.

3.3 Instantaneous Velocity and Speed

More commonly, we ask how fast a particle is moving at a given instant, which refers to its **instantaneous velocity** (or simply **velocity**). The velocity at any instant is obtained from the average velocity by allowing the time interval Δt to approach zero. Consider the motion of an object (for example a car). This object can be viewed as a particle for simplicity. The motion of that particle between two points P and Q on a position-time graph is shown in the right part of Fig. 3.4. As point Q is brought closer and closer to point P (through points Q_1, Q_2, ...), the time intervals (Δt_1, Δt_2, ...) get progressively smaller. The average velocity for each time interval is the slope of the dotted line in Fig. 3.4. As point Q approaches P, the time interval approaches zero, while the slope of the dotted line approaches the slope of the tangent to the curve at point P. This slope is defined to be the instantaneous velocity v at the time t_i. In short, we define:

Instantaneous velocity
The instantaneous velocity, v, of a particle is defined as the limiting value of the ratio $\Delta x/\Delta t$ as Δt approaches zero. Mathematically v can be expressed as:

$$v = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} \tag{3.4}$$

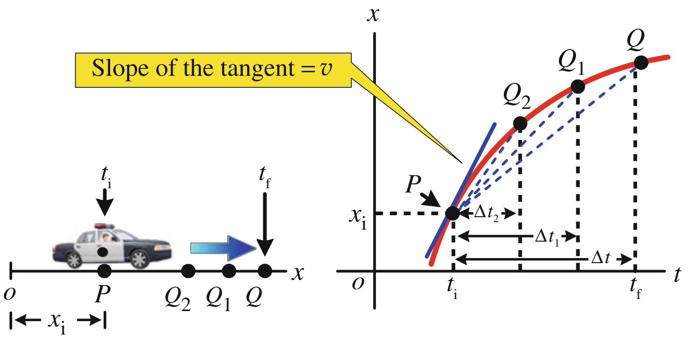


Fig. 3.4 The *left* part shows a police car (which can be considered as a particle) that moves along the x-axis. The *right* part shows the position-time graph for this motion. As Q approaches P, the average velocity $\overline{v}$ for the interval PQ approaches the slope of the tangent line at P, which is defined as the instantaneous velocity v at point P

In calculus notation, the above limit is called the derivative of x with respect to t, and written as dx/dt (abbreviated as $\dot{x}$). Thus:

$$v = \frac{dx}{dt} \quad \equiv \quad x_f - x_i = \int_{t_i}^{t_f} v\, dt \equiv \text{Area under } v\text{-}t \text{ graph} \tag{3.5}$$

The instantaneous velocity, v, can be positive, negative, or zero, depending on the slope of the position-time graph at the interval of interest in Fig. 3.5. In this figure, $v = 0$ represents the turning point, and occurs at any maximum or minimum of the x-t graph. From here on, we use the word *velocity* to denote *instantaneous velocity*.

The *speed* of a particle is defined as the magnitude of its velocity.

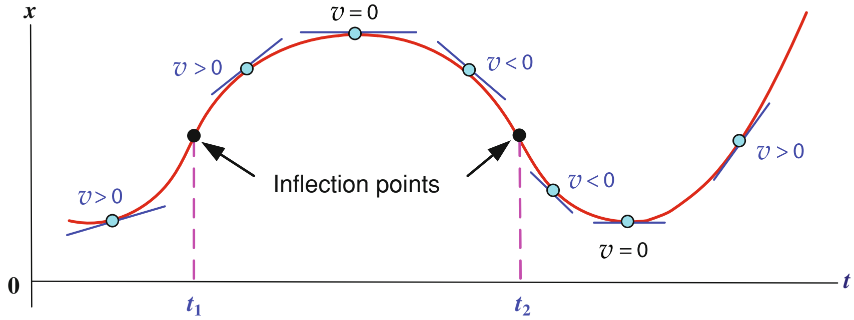


Fig. 3.5 The position-time graph for a particle moving along the x-axis. On this graph we display: (1) Positive velocities, where the slope of the tangent lines are positive, (2) Negative velocities, where the slope of the tangent lines are negative, (3) Zero velocities (turning points), where the slope of the tangent lines are zero, and (4) Inflection points at t_1 and t_2, where the increase/decrease of the velocity reaches a maximum/minimum

Example 3.2

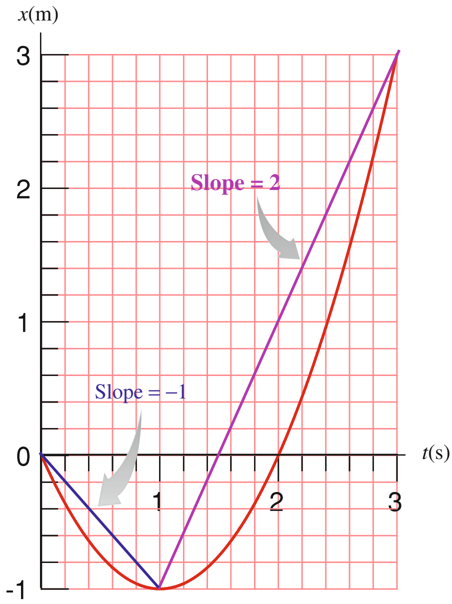
A particle moves along the x-axis and its coordinates vary with time according to the relation $x = t^2 - 2\,t$, where x is measured in meters and t is in seconds. The position-time graph for this motion is shown in Fig. 3.6. (a) Use this graph to comment about the particle's motion. (b) Find the displacement and the average velocity of the particle in the time intervals $0 \le t \le 1\,\text{s}$ and $1\,\text{s} \le t \le 3\,\text{s}$. (c) Find the velocity of the particle at $t = 2$ s.

Solution: (a) The particle starts from the origin of the x-axis and moves in the negative x direction for the first second. Its velocity is zero at $x = -1\,\text{m}$ when $t = 1\,\text{s}$ and then heads back in the positive x direction for $t > 1\,\text{s}$.

(b) In the interval $0 \le t \le 1\,\text{s}$ we have $t_i = 0$ and $t_f = 1$ s. Since $x = t^2 - 2\,t$, we get $x_i = t_i^2 - 2\,t_i = 0$ and $x_f = t_f^2 - 2\,t_f = -1$ m. Thus:

$$\Delta x = x_f - x_i = -1\text{ m} - 0\text{ m} = -1\text{ m}$$

Fig. 3.6



The average velocity is then:

$$\overline{v} = \frac{\Delta x}{\Delta t} = \frac{x_f - x_i}{t_f - t_i} = \frac{-1 \text{ m} - 0 \text{ m}}{1 \text{ s} - 0 \text{ s}} = \frac{-1 \text{ m}}{1 \text{ s}} = -1 \text{ m/s}$$

According to Fig. 3.6, this value equals the slope of the straight line drawn for this time interval.

In the interval $1\text{ s} \leq t \leq 3\text{ s}$ we have $t_i = 1$ s and $t_f = 3$ s. Again, from $x = t^2 - 2t$ we get $x_i = t_i^2 - 2t_i = -1$ m and $x_f = t_f^2 - 2t_f = 3$ m. Thus:

$$\Delta x = x_f - x_i = 3 \text{ m} - (-1 \text{ m}) = 4 \text{ m}$$

The average velocity is then:

$$\overline{v} = \frac{\Delta x}{\Delta t} = \frac{x_f - x_i}{t_f - t_i} = \frac{3 \text{ m} - (-1 \text{ m})}{3 \text{ s} - 1 \text{ s}} = \frac{4 \text{ m}}{2 \text{ s}} = 2 \text{ m/s}$$

According to Fig. 3.6, this value equals the slope of the straight line drawn for this time interval.

(c) To find the instantaneous velocity at any time t, we use Eq. 3.5 and apply the rules of differential calculus on the coordinate $x = t^2 - 2t$. That is:

$$v = \frac{dx}{dt} = \frac{d(t^2 - 2t)}{dt} = 2t - 2$$

Notice that this expression gives the velocity v at any time t and indicates that v is increasing linearly with time. It tells us that $v < 0$ during the interval $0 \leq t < 1$ s (i.e. the particle is moving in the negative x direction), and that $v = 0$ at $t = 1$ s, and finally $v > 0$ for $t > 1$ s. When $t = 2$ s we use the above expression to get:

$$v = 2 \times 2 - 2 = 2 \text{ m/s}$$

3.4 Acceleration

When the velocity of a particle changes with time, the particle is said to be accelerating. Consider the motion of a particle along the x-axis. If the particle has a velocity v_i at time t_i and a velocity v_f at time t_f as in the velocity-time graph of Fig. 3.7, then we define the *average acceleration* as:

Average acceleration
The average acceleration, $\overline{a}$, of a particle is defined as the ratio of the change in velocity $\Delta v = v_f - v_i$ to the time interval $\Delta t = t_f - t_i$. That is:

$$\overline{a} = \frac{\Delta v}{\Delta t} = \frac{v_f - v_i}{t_f - t_i} \tag{3.6}$$

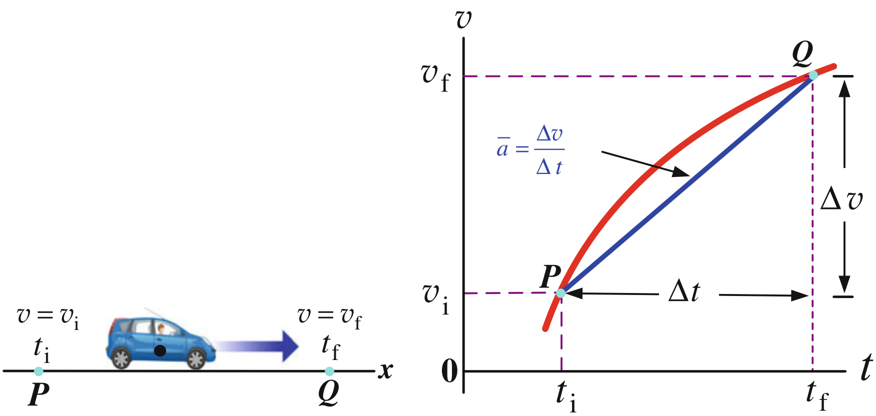


Fig. 3.7 The velocity-time graph for a car (or simply a particle) moving in a straight line. The slope of the straight line PQ is defined as the average acceleration in the time interval $\Delta t = t_f - t_i$

Acceleration is a vector quantity having dimensions of length divided by (time)^2, or L/T^2; that is m/s^2 in SI units.

It is useful to define the instantaneous acceleration as the limit of the average acceleration when Δt approaches zero. Consider the motion of a particle (for example a car that moves like a particle) between the two points P and Q on the velocity-time graph shown in the right part of Fig. 3.8. As point Q is brought closer and closer to point P (through points $Q_1, Q_2, \ldots$), the time intervals ($\Delta t_1,\ \Delta t_2, \ldots$) get progressively smaller. The average acceleration for each time interval is the slope of the dotted line in Fig. 3.8. As Q approaches P, the time interval approaches zero, while the slope of the dotted line approaches the slope of the tangent to the curve at point P. The slope of the tangent line to the curve at P is defined to be the instantaneous acceleration a at the time t_i. That is, we define the following:

Instantaneous acceleration

The instantaneous acceleration, a, of a particle is defined as the limiting value of the ratio $\Delta v/\Delta t$ when Δt approaches zero. Mathematically a can be expressed as:

$$a = \lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t} \tag{3.7}$$

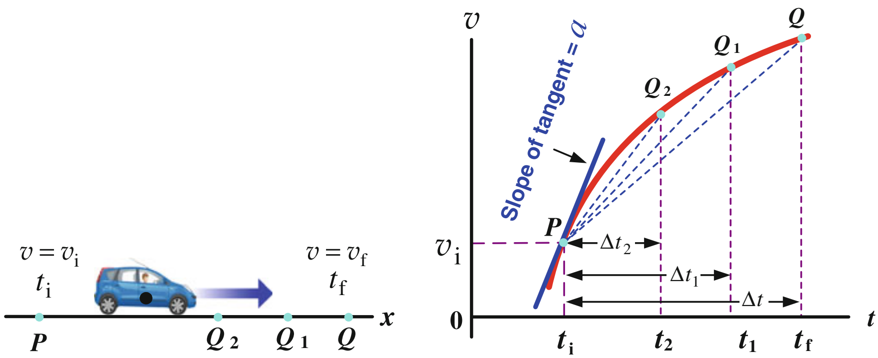


Fig. 3.8 The left part shows a car that moves along the x-axis. The right part shows the velocity-time graph that describes the car's motion. As Q approaches P, the average acceleration $\overline{a}$ for the interval PQ approaches the slope of the tangent line at P, which is defined as the instantaneous acceleration a at point P

In calculus notation, the above limit is called the *first derivative* of v with respect to t, and written as dv/dt (simplified sometimes as $\dot{v}$), or the *second derivative* of x with respect to t, and written as d^2x/dt^2 (simplified sometimes as $\ddot{x}$). Thus:

$$a = \frac{dv}{dt} = \frac{d^2x}{dt^2} \quad \equiv \quad v_f - v_i = \int_{t_i}^{t_f} a\,dt \equiv \text{Area under } a\text{-}t \text{ graph} \tag{3.8}$$

From here on, we use the word *acceleration* to designate *instantaneous acceleration*. Depending on the slope of the tangent to the velocity-time graph, acceleration a can be positive, negative (called deceleration), or zero. If $a = 0$ for a specific time interval in the $v - t$ graph, then the velocity must be a constant in this interval.

Example 3.3

The position of a particle moving along the x-axis varies with time t according to the relation $x = t^3 - 12\,t + 20$, where x is given in meters and t in seconds. (a) Find the velocity and the acceleration of the particle as a function of time. (b) Is there ever a time when $v = 0$? (c) Describe the particle's motion for $t \geq 0$.

Solution: (a) To get the velocity v as a function of time t, we differentiate the coordinate x with respect to t as follows:

$$v = \frac{dx}{dt} = \frac{d}{dt}\left(t^3 - 12\,t + 20\right) \quad \Rightarrow \quad v = 3\,t^2 - 12$$

To get the acceleration a as a function of time t, we differentiate the velocity v with respect to t as follows:

$$a = \frac{dv}{dt} = \frac{d}{dt}\left(3\,t^2 - 12\right) \quad \Rightarrow \quad a = 6\,t$$

(b) Setting $v = 0$ in the velocity relation yields:

$$0 = 3\,t^2 - 12,$$

which has the solution $t = \pm 2$ s. The negative answer has to be rejected, since time must be always positive. Thus at $t = 2$ s the velocity of the particle is zero.

(c) To describe the particle's motion for $t \geq 0$ we examine the expressions $x = t^3 - 12\,t + 20$, $v = 3\,t^2 - 12$, and $a = 6\,t$.

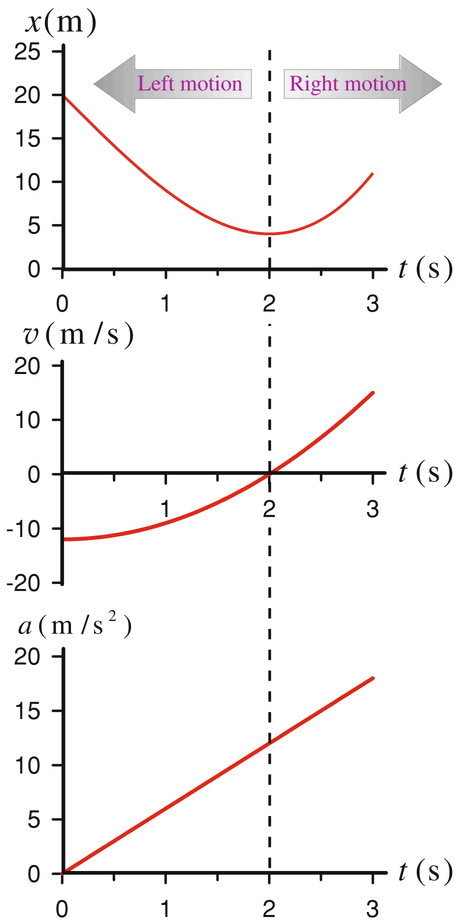
At $t = 0$, the particle is at $x = 20$ m from the origin and moving to the left with velocity $v = -12$ m/s and not accelerating since $a = 0$, see Fig. 3.9.

At $0 < t < 2$ s, the particle continues to move to the left (x decreases), but at a decreasing speed, because it is now accelerating to the right, $a =$ positive (Check the expressions of x, v, and a for $t = 1$ s and compare the results with Fig. 3.9).

At $t = 2$ s, the particle stops momentarily ($v = 0$) to reverse its direction of motion. At this moment $x = 4$ m, i.e. it will be as close as it will ever be to the origin. It will continue to accelerate to the right at an increasing rate, see Fig. 3.9.

For $t > 2$ s, the particle continues to accelerate and move to the right, and its velocity, which is now to the right, increases rapidly, see Fig. 3.9.

Fig. 3.9



3.5 Constant Acceleration

In many common types of one-dimensional motion, the acceleration is constant (or we say uniform). In this case, the average acceleration equals the instantaneous acceleration, i.e.

$$\bar{a} = a = \text{constant} \tag{3.9}$$

The shape of this relation can be displayed for positive a as shown in the left part of Fig. 3.10. Consequently, Eq. 3.6 becomes:

$$a = \frac{v_f - v_i}{t_f - t_i} \quad (\text{when } \bar{a} = a = \text{constant}) \tag{3.10}$$

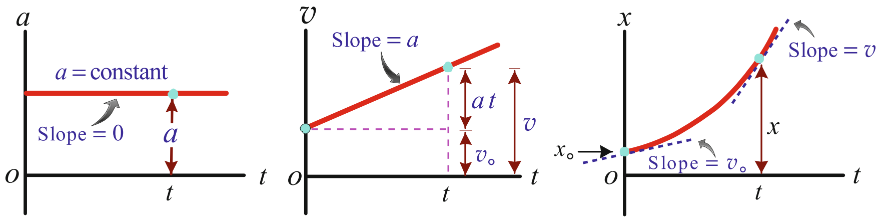


Fig. 3.10 (*Left part*) The acceleration-time graph of a particle moving along the x-axis with constant acceleration. (*Middle part*) the velocity-time graph of the particle's motion. (*Right part*) The position-time graph of the particle's motion

For convenience, we let $t_i = 0$ and $t_f = t$, where t is any arbitrary time. Also, we let $v_i = v_\circ$ (the initial velocity at time $t = 0$) and $v_f = v$ (the velocity at any time t). With this notation, we can express acceleration as:

$$a = \frac{v - v_\circ}{t}$$

Rearranging gives:

$$v = v_\circ + a\,t \quad (\text{for constant } a) \tag{3.11}$$

This linear relationship enables us to find the velocity at any time t; see the middle part of Fig. 3.10.

We can make use of the fact that when the acceleration is constant (i.e. when the velocity varies linearly with time according to Eq. 3.11 as in Fig. 3.10), the average velocity in any time interval is the arithmetic mean of the initial velocity, $v_\circ$, and the final velocity at the end of that interval, v. Thus:

$$\overline{v} = \frac{v_\circ + v}{2} \quad \text{(for constant } a\text{)} \tag{3.12}$$

To find the displacement as a function of time, we first let $x_i = x_\circ$ (the initial position at time $t = 0$) and $x_f = x$ (the position at any time t), and then use Eq. 3.2 and Eq. 3.12 to get:

$$\overline{v} = \frac{x - x_\circ}{t} = \frac{v_\circ + v}{2}$$

Rearranging gives:

$$x - x_\circ = \tfrac{1}{2}(v_\circ + v)\ t \quad \text{(for constant } a\text{)} \tag{3.13}$$

We can obtain another useful expression for the displacement by substituting Eq. 3.11 into Eq. 3.13 to get:

$$x - x_\circ = v_\circ\ t + \tfrac{1}{2}\ a\ t^2 \quad \text{(for constant } a\text{)} \tag{3.14}$$

As a first check for Eq. 3.14, one can notice that substituting $t = 0$ yields $x = x_\circ$, as it must be. A further check, taking the derivative of Eq. 3.14 with respect to time, yields Eq. 3.11. The right part of Fig. 3.10 displays the position x as a function of time t for the parabolic Eq. 3.14.

We can use Eq. 3.11 to eliminate $v_\circ$ from Eq. 3.14 to obtain the following relation:

$$x - x_\circ = v\ t - \tfrac{1}{2}\ a\ t^2 \quad \text{(for constant } a\text{)} \tag{3.15}$$

Finally, by replacing the value of t that was obtained from Eq. 3.11 into Eq. 3.13, we can obtain an expression that does not include the time variable as follows:

$$x - x_\circ = \tfrac{1}{2}(v_\circ + v)\ \frac{(v - v_\circ)}{a} = \frac{(v^2 - v_\circ^2)}{2\ a}$$

Rearranging gives:

$$v^2 = v_\circ^2 + 2\ a\ (x - x_\circ) \quad \text{(for constant } a\text{)} \tag{3.16}$$

Equations 3.11 through 3.16 are six *kinematic expressions* used to solve any one-dimensional problem with constant acceleration.

Table 3.1 lists the four kinematic equations that are used most often in solving problems for the case of constant acceleration.

Table 3.1 Equations for motion with constant acceleration

Equation	Missing quantity	Equation number
$v = v_\circ + a\,t$	$x - x_\circ$	Eq. 3.11
$x - x_\circ = \frac{1}{2}(v_\circ + v)\,t$	a	Eq. 3.13
$x - x_\circ = v_\circ\,t + \frac{1}{2}\,a\,t^2$	v	Eq. 3.14
$v^2 = v_\circ^2 + 2\,a\,(x - x_\circ)$	t	Eq. 3.16

Example 3.4

A car accelerates uniformly from rest to a speed of 100 km/h in 18 s.(a) Find the acceleration of the car. (b) Find the distance that the car travels. (c) If the car brakes to a full stop over a distance of 100 m, then find its uniform deceleration.

Solution: (a) In this problem we are given $v_\circ = 0$, $v = 100$ km/h, and $t = 18\,\text{s} = 5 \times 10^{-3}$ h and we need to find a. So, we can use $v = v_\circ + a\,t$ to find the acceleration as follows:

$$a = \frac{v - v_\circ}{t} = \frac{100\ \text{km/h} - 0}{5 \times 10^{-3}\ \text{h}} = 2 \times 10^4\ \text{km/h}^2 \equiv 2 \times 10^4\ \frac{1{,}000\ \text{m}}{(60 \times 60\ \text{s})^2} = 1.54\ \text{m/s}^2$$

(b) If the car starts from the origin of the x-axis, i.e. $x_\circ = 0$, then we are given $v_\circ = 0$, $v = 100$ km/h, $x_\circ = 0$, and $t = 5 \times 10^{-3}$ h and we need to find x, which in this case equals the distance traveled by the car. So, we use $x - x_\circ = \frac{1}{2}(v_\circ + v)\,t$ to find the position x as follows:

$$x = x_\circ + \tfrac{1}{2}(v_\circ + v)\,t = 0 + \tfrac{1}{2}(0 + 100\ \text{km/h}) \times 5 \times 10^{-3}\ \text{h} = 0.25\ \text{km} = 250\ \text{m}$$

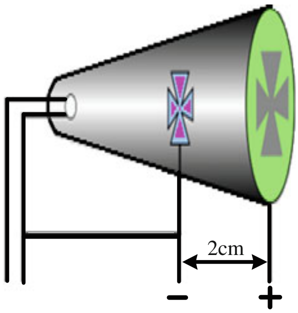
(c) We are given $v_\circ = 100$ km/h, $v = 0$, and $x - x_\circ = 0.1$ km and we need to find the deceleration a. We use $v^2 = v_\circ^2 + 2\,a\,(x - x_\circ)$ to get:

$$a = \frac{v^2 - v_\circ^2}{2\,(x - x_\circ)} = \frac{0 - (100\ \text{km/h})^2}{2 \times 0.1\ \text{km}} = -5 \times 10^4\ \text{km/h}^2 = -3.86\ \text{m/s}^2$$

Example 3.5

In a cathode ray tube of a TV set, an electron with initial velocity $v_\circ = 2 \times 10^4$ m/s enters a region 2 cm long (see Fig. 3.11) where it is electrically accelerated in a straight line. The electron emerges from this region with a velocity $v = 3 \times 10^5$ m/s. (a) What was its acceleration, assuming it was constant? (b) How long will the electron be in this region?

Fig. 3.11



Solution: (a) Taking the motion to be along the x-axis, and using $v_\circ = 2 \times 10^4$ m/s, $v = 3 \times 10^5$ m/s, and $x - x_\circ = 2\,\text{cm} = 2 \times 10^{-2}$ m, we can find the acceleration a from the relation $v^2 = v_\circ^2 + 2\,a\,(x - x_\circ)$ as follows:

$$a = \frac{v^2 - v_\circ^2}{2\,(x - x_\circ)} = \frac{(3 \times 10^5 \text{ m/s})^2 - (2 \times 10^4 \text{ m/s})^2}{2 \times 2 \times 10^{-2} \text{ m}} = 2.24 \times 10^{12} \text{ m/s}^2$$

(b) Since the displacement and velocities are known, we can use $x - x_\circ = \frac{1}{2}(v_\circ + v)\,t$ to find the time t that the electron will be electrically accelerated as follows:

$$t = \frac{2(x - x_\circ)}{v_\circ + v} = \frac{2 \times 2 \times 10^{-2} \text{ m}}{2 \times 10^4 \text{ m/s} + 3 \times 10^5 \text{ m/s}} = 1.25 \times 10^{-7}\,\text{s} = 0.125\,\mu\text{s}$$

Another way to find t is to use equation $v = v_\circ + a\,t$. In this case, $v = 3 \times 10^5$ m/s, and $a = 2.24 \times 10^{12}$ m/s^2. Thus:

$$t = \frac{v - v_\circ}{a} = \frac{3 \times 10^5 \text{ m/s} - 2 \times 10^4 \text{ m/s}}{2.24 \times 10^{12} \text{ m/s}^2} = 1.25 \times 10^{-7}\,\text{s} = 0.125\,\mu\text{s}$$

Even though a is very high in this example, but such an acceleration occurs over a very short time interval which is a typical value for such an electrically accelerated charged particle.

Example 3.6

The remote-controlled truck shown in Fig. 3.12 moves along the x-axis with a constant acceleration of $-2\,\text{m/s}^2$. As it passes the origin, i.e. $x_\circ = 0$, its initial velocity is 14 m/s. (a) At what time t' and position x' does $v' = 0$ (i.e. when the truck stops momentarily)? (b) At what times t_1 and t_2 is the truck at $x = 24$ m, and what is its velocity then?

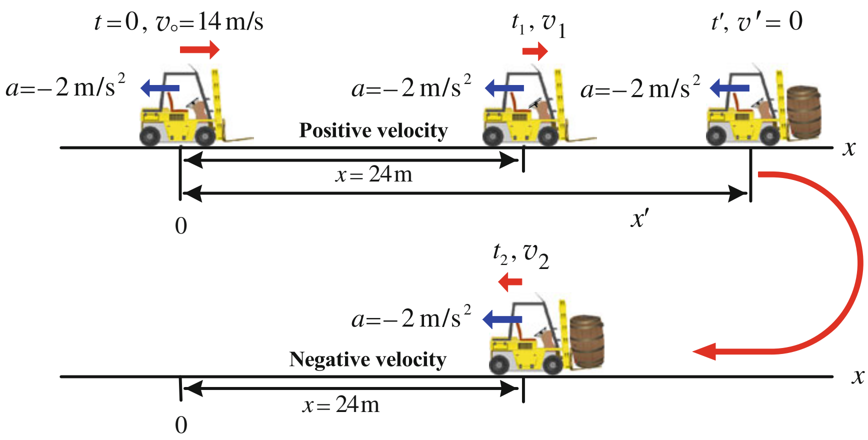


Fig. 3.12

Solution: (a) Given $v_\circ = 14$ m/s, $v' = 0$, and $a = -2$ m/s^2, we can find t' by using $v' = v_\circ + a\,t'$ as follows:

$$t' = \frac{v' - v_\circ}{a} = \frac{0 - 14\text{ m/s}}{-2\text{ m/s}^2} = 7\text{ s}$$

To find the position x' we can use $v'^2 = v_\circ^2 + 2\,a\,(x' - x_\circ)$, since we are given $v_\circ = 14$ m/s, $v' = 0$, $x_\circ = 0$, and $a = -2$ m/s^2. Thus:

$$x' = x_\circ + \frac{v'^2 - v_\circ^2}{2\,a} = 0 + \frac{0 - (14\text{ m/s})^2}{2 \times (-2\text{ m/s}^2)} = 49\text{ m}$$

(b) Using $x = 24$ m, $x_\circ = 0$, $v_\circ = 14$ m/s, and $a = -2$ m/s^2 in $x - x_\circ = v_\circ\,t + \frac{1}{2}\,a\,t^2$, we find, after omitting the units temporarily, that:

$$24 - 0 = 14\,t + \tfrac{1}{2}\,(-2)\,t^2 \quad \Rightarrow \quad t^2 - 14\,t + 24 = 0$$

Solving this quadratic equation yields:

$$t = \frac{14 \pm \sqrt{(-14)^2 - 4 \times 1 \times 24}}{2 \times 1} = \frac{14 \pm 10}{2} \quad \Rightarrow \quad t = \begin{cases} t_1 = 2\text{ s} \\ t_2 = 12\text{ s} \end{cases}$$

Thus, $t_1 = 2$ s is the time the truck takes from the origin to the position $x = 24$ m. Furthermore, $t_2 = 12$ s is the time the truck takes from O, passing the point $x = 24$ m, reaching the point $x' = 49$ m and returning back to $x = 24$ m.

For $x = 24$ m, $v_\circ = 14$ m/s, $a = -2$ m/s^2, and $t_1 = 2$ s, we use the formula $v = v_\circ + a\,t$ to get v_1 as follows:

$$v_1 = v_\circ + a\,t_1 = 14 \text{ m/s} + (-2 \text{ m/s}^2) \times (2 \text{ s}) = 10 \text{ m/s}$$

Also, for $x = 24$ m, $v_\circ = 14$ m/s, $a = -2$ m/s^2, and $t_2 = 12$ s, we use the formula $v = v_\circ + a\,t$ to get v_2 as follows:

$$v_2 = v_\circ + a\,t_2 = 14 \text{ m/s} + (-2 \text{ m/s}^2) \times (12 \text{ s}) = -10 \text{ m/s}$$

Observe that the two speeds are equal, i.e. $|v_1| = |v_2| = 10$ m/s.

In this example, we do not pay any attention to the cause of this constant acceleration, but this will be clarified later on when we study the dynamical aspect of mechanics.

3.6 Free Fall

Due to gravity, it is well known that all dropped objects near the Earth's surface will accelerate downward with a nearly constant acceleration when the effect of air resistance is very small and can be neglected. We use the term "free fall" for this motion and the same will be applied to objects that are either thrown up or down. We shall denote the magnitude of the *acceleration due to gravity* by the symbol g, which is very close to 9.8 m/s^2 near the Earth's surface.

Therefore, for free falls near the Earth's surface, the constant acceleration equations of motion Eqs. 3.11 through 3.16, and hence equations of Table 3.1, can be applied. However, we can make them simpler to use with the following minor changes:

(1) The motion is along the vertical y-axis.

(2) The free-fall acceleration is negative if the y-axis is chosen to be upward, and hence we replace the acceleration a with $-g$.

(3) The free-fall acceleration is positive if the y-axis is chosen to be downward, and hence we replace the acceleration a with $+g$.

Table 3.2 lists the four kinematic equations that are frequently used in solving free-fall problems with constant acceleration, where always $|a| = g = 9.8$ m/s^2 for motions near the Earth's surface.

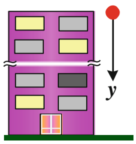
Table 3.2 Equations for free-fall motion with constant acceleration

y (up), $a = -g$	Equation	y (down), $a = g$	Equation
$y\uparrow$	$v = v_\circ - g\,t$	$y\downarrow$	$v = v_\circ + g\,t$
	$y - y_\circ = \frac{1}{2}(v_\circ + v)\,t$		$y - y_\circ = \frac{1}{2}(v_\circ + v)\,t$
	$y - y_\circ = v_\circ\,t - \frac{1}{2}\,g\,t^2$		$y - y_\circ = v_\circ\,t + \frac{1}{2}\,g\,t^2$
	$v^2 = v_\circ^2 - 2\,g(y - y_\circ)$		$v^2 = v_\circ^2 + 2\,g(y - y_\circ)$

Example 3.7

A ball is dropped from a tall building, as shown in Fig. 3.13. Choose the positive y to be downward with its origin at the top of the building, i.e. $y_\circ = 0$. Find the following for the ball's motion: (a) its acceleration, (b) the distance it falls in 2 s, (c) its velocity after falling 15 m, (d) the time it takes to fall 25 m, and (e) the time it takes to reach a velocity of 29.4 m/s.

Fig. 3.13



Solution: (a) Since the positive y is downward, then the ball's acceleration is positive (downward) and will be given by $a = g = 9.8$ m/s^2. Also, the ball's velocity will be always positive.

(b) We are given $v_\circ = 0$, $y_\circ = 0$, $a = g = 9.8$ m/s^2, and $t = 2$ s. To find y, we use $y - y_\circ = v_\circ\,t + \frac{1}{2}\,g\,t^2$ as follows:

$$y = 0 + \tfrac{1}{2}\,(9.8\text{ m/s}^2) \times (2\text{ s})^2 = 19.6\text{ m}$$

(c) We are given $v_\circ = 0$, $y_\circ = 0$, $a = g = 9.8$ m/s^2, and $y = 15$ m. To find v, we use $v^2 = v_\circ^2 + 2\,g(y - y_\circ)$ as follows:

$$v^2 = 0 + 2 \times (9.8\text{ m/s}^2) \times 15\text{ m} \quad\Rightarrow\quad v = \pm\sqrt{294}\text{ m/s} \quad\Rightarrow\quad v = 17.2\text{ m/s}$$

(d) We are given $v_\circ = 0$, $y_\circ = 0$, $y = 25$ m, and $a = g = 9.8$ m/s^2. To find t, we use $y - y_\circ = v_\circ\,t + \frac{1}{2}\,g\,t^2$ as follows:

$$25 = 0 + \tfrac{1}{2}\,(9.8\text{ m/s}^2) \times t^2 \quad\Rightarrow\quad t = \pm\sqrt{5.1}\text{ s} \quad\Rightarrow\quad t = 2.3\text{ s}$$

(e) We are given $v_\circ = 0$, $v = 29.4$ m/s, and $a = g = 9.8$ m/s^2. To find t, we use $v = v_\circ + g\,t$ as follows:

$$t = \frac{v - v_\circ}{g} = \frac{29.4 \text{ m/s} - 0 \text{ m/s}}{9.8 \text{ m/s}^2} = 3 \text{ s}$$

Example 3.8

A boy throws a ball upwards, giving it an initial speed $v_\circ = 15$ m/s. Neglect air resistance. (a) How long does the ball take to return to the boy's hand? (b) What will be its velocity then?

Solution: (a) We choose the positive y upward with its origin at the boy's hand, i.e. $y_\circ = 0$, see Fig. 3.14. Then, the ball's acceleration is negative (downward) during the ascending and descending motions, i.e. $a = -g = -9.8$ m/s^2. When the ball returns to the boy's hand its position y is zero. Since $v_\circ = 15$ m/s, $y_\circ = 0$, $y = 0$, and $a = -g$, then we can find t from $y - y_\circ = v_\circ\, t - \frac{1}{2}\, g\, t^2$ as follows:

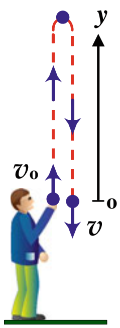
$$0 = (15 \text{ m/s})\, t - \tfrac{1}{2}\,(9.8 \text{ m/s}^2) \times t^2 \quad \Rightarrow \quad t = \frac{2 \times (15\text{m/s})}{9.8 \text{ m/s}^2} = 3.1 \text{ s}$$

(b) We are given $v_\circ = 15$ m/s, $y_\circ = 0$, $y = 0$, and $a = -g = -9.8$ m/s^2. To find v, we use $v^2 = v_\circ^2 - 2\, g(y - y_\circ)$ as follows:

$$v^2 = v_\circ^2 - 0 \quad \Rightarrow \quad v = \pm\sqrt{v_\circ^2} = \pm v_\circ = \pm 15 \text{ m/s}$$

We should select the negative sign, because the ball is moving downward just before returning to the boy's hand, i.e. $v = -15$ m/s.

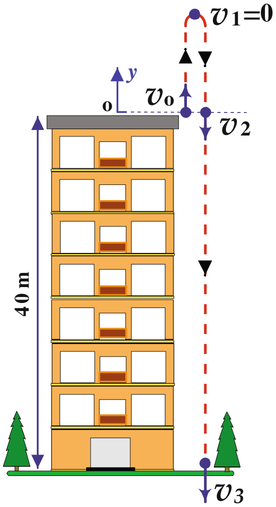
Fig. 3.14



Example 3.9

A ball is thrown upward from the top of a building with an initial velocity $v_\circ = 20$ m/s. The building is 40 m high and the ball just misses the edge of the building roof on its way down; see Fig. 3.15 and take $g = 10$ m/s^2. Neglecting air resistance, find: (a) the time t_1 for the ball to reach its highest point, (b) how high will it rise, (c) how long will it take to return to its starting point, (d) the velocity v_2 of the ball at this instant, and (e) the velocity v_3 and the total time of flight t_3 just before the ball hits the ground.

Fig. 3.15



Solution: (a) We choose upward as positive, i.e. $a = -g = -10$ m/s^2 during ascending and descending motions. Also, we choose the origin at the top of the building, i.e. $y_\circ = 0$, see Fig. 3.15. Since at the maximum height the ball stops momentarily, we use $v_\circ = 20$ m/s and $v_1 = 0$ in $v_1 = v_\circ - g\,t_1$ to find t_1 as follows:

$$0 = 20 \text{ m/s} - (10 \text{ m/s}^2)\ t_1$$

$$t_1 = \frac{20 \text{ m/s}}{10 \text{ m/s}^2} = 2 \text{ s}$$

(b) For the maximum height, we use the notation $y_1 \equiv y_{\max}$. To find the maximum height from the position of the thrower, we use the formula $y_{\max} - y_\circ = v_\circ\ t_1 - \frac{1}{2}\ g\ t_1^2$ as follows:

$$y_{\max} = (20 \text{ m/s}) \times (2 \text{ s}) - \tfrac{1}{2}(10 \text{ m/s}^2) \times (2\text{s})^2 = 20 \text{ m}$$

(c) When the ball returns to its starting point, the y coordinate is zero again, i.e. $y_2 = 0$. To find t_2 we use $y_2 - y_\circ = v_\circ\, t_2 - \frac{1}{2}\, g\, t_2^2$ as follows (after omitting the units temporarily, since they are consistent):

$$0 = 20\, t_2 - \tfrac{1}{2} \times 10 \times t_2^2$$

This equation can be factored to give:

$$t_2[20 - 5\, t_2] = 0$$

One solution is $t_2 = 0$, which corresponds to the time that the ball starts its motion. The other solution is $t_2 = 4\,\text{s}$, which is the solution we are after. Thus:

$$t_2 = 4\ \text{s}$$

(d) The value $t_2 = 4$ s found in part (c) can be inserted into the formula $v_2 = v_\circ - g\, t_2$ as follows:

$$v_2 = 20\ \text{m/s} - (10\ \text{m/s}^2) \times (4\ \text{s}) = -20\ \text{m/s}$$

Note that the velocity of the ball when it returns to its starting point is equal in magnitude to its initial velocity but opposite in direction. This indicates that the motion is symmetric, and generally we have:

$$v_2 = -v_\circ$$

(e) When the ball reaches the ground, its position is $y_3 = -40$ m. We can insert this value in $v_3^2 = v_\circ^2 - 2\, g(y_3 - y_\circ)$ to find v_3 as follows:

$$v_3^2 = (20\ \text{m/s})^2 - 2 \times (10\ \text{m/s}^2)[(-40\ \text{m}) - 0] = 1{,}200\,\text{m}^2/\text{s}^2$$

Thus:

$$v_3 = \pm\sqrt{1{,}200\,\text{m}^2/\text{s}^2} = \pm 34.64\ \text{m/s}$$

Since the ball is moving downward, we choose the negative value. Thus:

$$v_3 = -34.64\ \text{m/s}$$

To find the total time of flight t_3, we use $v_3 = v_\circ - g\, t_3$ as follows:

$$t_3 = \frac{v_\circ - v_3}{g} = \frac{(20\ \text{m/s}) - (-34.64\ \text{m/s})}{10\ \text{m/s}^2} = \frac{54.64\ \text{m/s}}{10\ \text{m/s}^2} = 5.5\ \text{s}$$

3.7 Exercises

Section 3.2 Average Velocity and Average Speed

(1) A runner on a straight track covers 1 km in 4 minutes. What is his average velocity in: (a) km/min, (b) km/s, and (c) km/h?

(2) A car travels in the positive x direction for 20 km at 40 km/h. It then continues in the same direction for another 20 km at 80 km/h. (a) What is the average velocity of the car during this 40 km trip? (b) What is its average speed?

(3) Suppose the motion of the particle in Fig. 3.2 is described by the equation $x = a + b\,t^2$, where $a = 10$ m and $b = 2$ m/s^2. (a) Find the displacement of the particle in the time interval between $t_i = 2$ s and $t_f = 4$ s. (b) Find the average velocity and the average speed during this interval of time.

(4) On an average, an eye blink lasts 100 ms. How far does a rocket moving with an average speed of $\bar{s} = 3{,}600$ km/h, see Fig. 3.16, travel during a pilot's blink?

Fig. 3.16 See Exercise (4)

(5) A graph of position (in meters) versus time (in seconds) for a boy traveling in the positive x direction is displayed in the Fig. 3.17. Find the average velocity for the following cases: (a) $t_i = 2$ s and $t_f = 4$ s, (b) $t_i = 2$ s and $t_f = 6$ s, and (c) $x_i = 12$ m and $x_f = 30$ m.

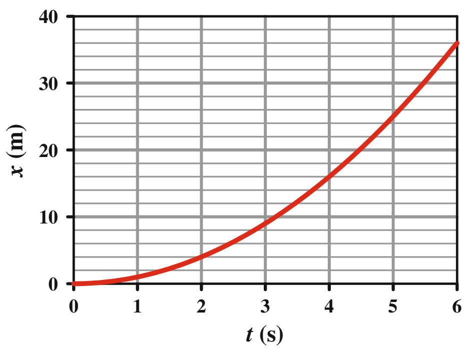


Fig. 3.17 See Exercise (5)

(6) A body moves along a straight line with position given by $x = 8\,t - 2\,t^2$, where x is in meters and t is in seconds. Find the average velocity and average speed of the body in the intervals: (a) from $t_i = 0$ to $t_f = 2$ s, and (b) from $t_i = 0$ to $t_f = 5$ s.

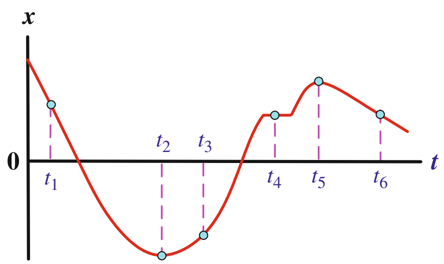
Section 3.3 Instantaneous Velocity and Speed

(7) The position of a plane during take-off along a straight runway is given by $x = k\,t^2$, where $k = 1.2\,\text{m/s}^2$,is measured in meters, and t is in seconds. (a) Find the displacement and the average velocity of the plane in the time intervals $0 \le t \le 4$ s and $4\text{ s} \le t \le 10$ s. (b) Find the velocity of the plane at $t = 4$ s and at $t = 10$ s.

(8) A particle moves along the x-axis according to the relation $x = 6 - 6\,t + t^2$, where x is measured in meters and t is measured in seconds. (a) Find the values of x for $t = 1, 2, 3, 4,$ and 5 s. (b) Find the values of the velocity v for $t = 1, 2, 3, 4,$ and 5 s. (c) For each value of t indicate whether the particle is moving toward an increasing or decreasing x. (d) Is there ever an instant when the velocity is zero? (e) Is there a time after $t = 5$ s when the particle is moving toward decreasing x?

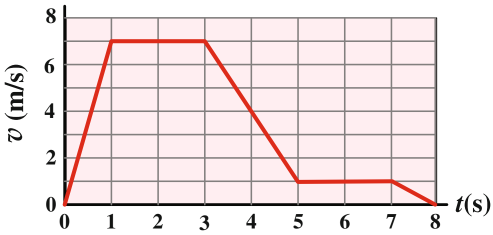
(9) The position-time graph for a particle moving along the x-axis is shown in the Fig. 3.18. Determine whether the velocity is positive, negative, or zero at the times t_1, t_2, t_3, t_4, t_5, and t_6.

Fig. 3.18 See Exercise (9)



(10) The graph of Fig. 3.19 shows the velocity of a runner plotted as a function of time. What is the interval where the velocity of the runner: (a) increases rapidly, (b) decreases rapidly, and (c) stays constant?

Fig. 3.19 See Exercise (10)



(11) How far does the runner whose $v - t$ graph is shown in the previous exercise travel in 8 s if at $t = 0$ the runner is at $x = 0$?

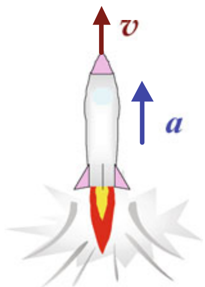
Section 3.4 Acceleration

(12) A particle is moving along the x-axis with velocity $v_{\mathrm{i}} = 50$ m/s at $t_{\mathrm{i}} = 0$. Its velocity decreases uniformly and reaches $v_{\mathrm{f}} = 0$ at $t_{\mathrm{f}} = 10$ s. What was the average acceleration during this 10 s interval?

(13) A car moving along the x-axis has a position given by the formula $x = 6 + 8\,t + 2\,t^2$, where x is measured in meters and t is in seconds. (a) Find the car's instantaneous velocity as a function of time. (b) Find its instantaneous acceleration as a function of time. (c) What will its velocity and acceleration be at $t = 5$ s?

(14) The velocity of a rocket during the first 6 s of its initial launch stage, see the Fig. 3.20, is given by $v = 20\,t - 0.4\,t^2$, where v is measured in meter/second and t is measured in seconds. (a) Find the average acceleration of the rocket from $t_{\mathrm{i}} = 0$ to $t_{\mathrm{f}} = 1$ s, and from $t_{\mathrm{i}} = 5$ s to $t_{\mathrm{f}} = 6$ s. (b) Find the acceleration a of the rocket at any time t during the interval $0 \leq t \leq 6$ s.

Fig. 3.20 See Exercise (14)



(15) Using the formula for the velocity given in the previous exercise, find the position x of the rocket at any time t during the interval $0 \le t \le 6$ s. Then, find the values of the position, velocity, and acceleration at $t = 0,\ t = 3$ s, and $t = 6$ s.

(16) A particle has $x = 0$ at $t = 0$ and its velocity as a function of time is shown in the Fig. 3.21. (a) Sketch the acceleration as a function of time. (b) Find the average acceleration of the particle in the time interval $t_i = 0$ to $t_f = 5$ s. (c) Find the acceleration of the particle at $t = 4$ s.

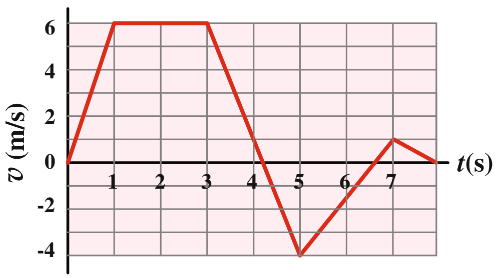


Fig. 3.21 See Exercise (16)

(17) A particle in one-dimensional motion has a velocity at any instant of time t given by $v = 6 + 4\,t + 3\,t^2$. (a) Find the initial velocity when $t = 0$. (b) Find the velocity when 2 s have passed. (c) Find the expression for the acceleration, and then its value when 2 s have elapsed. (d) Find the expression for the displacement $\Delta x = x - x_\circ$.

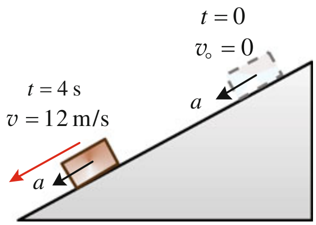
Section 3.5 Constant Acceleration

(18) An object starts from rest and moves with constant acceleration of 4 m/s^2. Find its speed and the distance it has traveled after 5 s have elapsed.

(19) A box slides down an incline with a uniform acceleration, see Fig. 3.22. It starts from rest and attains a speed of 12 m/s in 4 s. Find: (a) the acceleration, and (b) the distance moved in the first 4 s.

(20) A plane starts from rest and accelerates uniformly along a straight runway before takeoff. If the plane moves 1 km in 10 s, then find: (a) the acceleration, (b) the speed at the end of the 10 s period, (c) the distance moved in the first 20 s.

Fig. 3.22 See Exercise (19)



(21) A particle moving at 25 m/s in a straight line slows uniformly at a rate of 2 m/s every second. In an interval of 10 s, find: (a) the acceleration, (b) the final velocity, (c) the distance moved.

Section 3.6 Free Fall

(22) A stone strikes the ground with a speed of 25 m/s. (a) From what height was it released? (b) How long was it falling? (c) If the stone is thrown down with a speed of 10 m/s from the same height, then what will be its speed just before hitting the ground?

(23) A ball is thrown upward with a speed of 19.6 m/s. (a) How high does it go until its upward speed decreases to zero? (b) How long does the ball take in this upward trip? (c) How long does the ball take to return to the initial position? (d) What will be its velocity then?

(24) A bottle is dropped from a bridge and strikes the water after 5 s. (a) Find the speed of the bottle when it strikes the water. (b) Find how high the bridge is located above the water level.

(25) A sandbag dropped from a balloon reaches the ground in 5 s, see Fig. 3.23. Find the height of the balloon if: (a) it was at rest in the air, (b) it was ascending with a speed of 10 m/s when the sandbag was dropped, (c) it was descending with a speed of 10 m/s when the sandbag was dropped.

(26) A ball is thrown vertically *downward* from the edge of a cliff with an initial speed of 23 m/s. After a period of 1.4 s has elapsed, find: (a) how fast is it moving? and (b) how far has it moved?

(27) A ball is thrown vertically *upward* from the edge of a building with an initial velocity of 23 m/s. After a period of 1.4 s has elapsed, find: (a) how fast is it moving? and (b) how far has it moved?

(28) A ball is thrown vertically upward with a speed of 50 m/s from a building 20 m high, see Fig. 3.24. Find:(a) the time t_1 for the ball to reach the highest point,

(b) how high it will rise, (c) how long it will take to return to the starting point, (d) the velocity v_2 of the ball at this instant, (e) the velocity v_3 with which the ball strikes the ground, and (f) the total time of flight t_3.

Fig. 3.23 See Exercise (25)

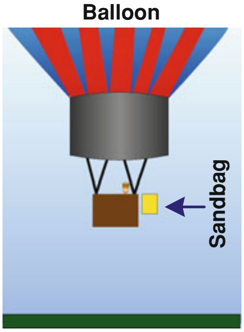
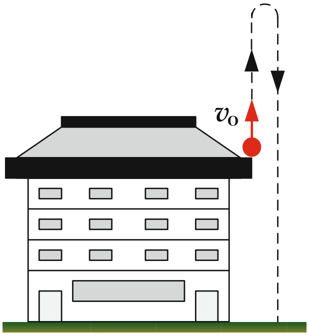


Fig. 3.24 See Exercise (28)



(29) A child drops balls from a bridge at regular intervals of 1 s, see Fig. 3.25. At the moment the fourth ball is released, the first strikes the water. (a) How high is the bridge? (b) How far above the water are each of the falling balls at this moment: (c) If the child decided to drop a ball once the previous one has reached the water surface, how long should he wait between every ball drop?

(30) A rubber ball is released from a height of 2 m above the floor, see the Fig. 3.26. The ball bounces repeatedly, always rising to 1/2 of the height through which it falls. Treat the ball as a particle that bounces an infinite number of times and take $g = 10$ m/s^2. (a) What is the average speed of the ball during the first fall? (b) Show that its total distance traveled during an infinite number of bounces is 6 m? (c) Show that the total elapsed time for an infinite number of bounces

is $2[\sqrt{2}+1]^2/\sqrt{10}$ s. (d) Find the average speed from the time of release to the end of the infinite number of bounces.
[Hint: Use the binomial series $(1-x)^{-1} = 1 + x + x^2 + x^3 + \ldots, \ |x| < 1$]

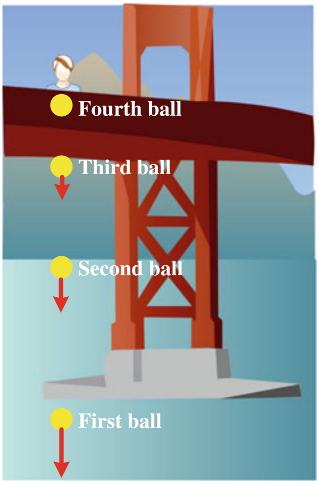


Fig. 3.25 See Exercise (29)

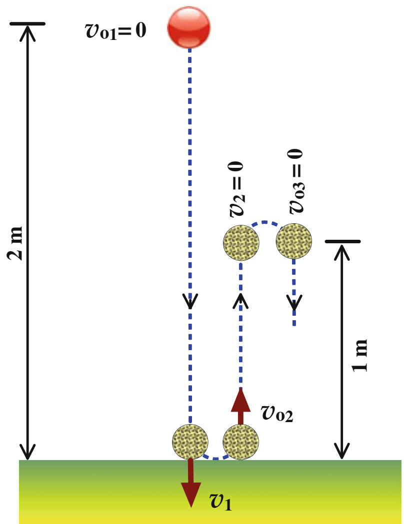


Fig. 3.26 See Exercise (30)

(31) An acrobat jumps straight up in the air and his center of mass (CM) took 0.2 s from the moment he just left the ground to the moment he just reached the

highest point, see the Fig. 3.27. Neglecting air resistance, (a) what was his initial vertical velocity just before his legs left the ground, (b) how high did his CM rise above the ground, and (c) what will be his velocity just before touching the ground in his way back?

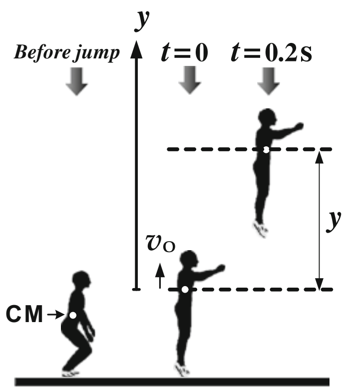


Fig. 3.27 See Exercise (31)

(32) Show that the vertical trajectory of a particle thrown upward is symmetric about its maximum when we neglect air resistance. That is, its height above the ground at time Δt before reaching its maximum equals its height above the ground after the same time interval Δt measured after reaching its maximum.

(33) A student drops a dartboard to the ground from a window, i.e. $v_{ob} = 0$. One second after dropping the board, he throws a dart at the board with initial speed of 20 m/s in order to score just before the board reaches the ground. See Fig. 3.28 and take $g = 10$ m/s^2. (a) Find the time of flight T of the dartboard. (b) Find the height of the window. (c) Find the velocity of both the dart and the dartboard just before hitting the ground.

(34) The remote-controlled truck shown in Fig. 3.12 is used to pick up a package from a shelf in a factory. From rest and at $t = 0$, the truck accelerate at a_1 for a time interval t_1, then travels with constant speed for a time interval t_2, and finally decelerate at $-a_3$ for a time interval t_3. Show that the total distance traveled by the truck is $a_1 t_1 (t_2 + t_3) + \frac{1}{2}(a_1 t_1^2 - a_3 t_3^2)$.

(35) A diver drops his body from a diving board at a distance H above the water's surface into a deep swimming pool. The diver's motion stops at a distance h below the surface of the water. By choosing the downward direction to be

positive, see Fig. 3.29, prove that the average acceleration of the diver while he is under the water is $\overline{a} = -(H/h)g$.

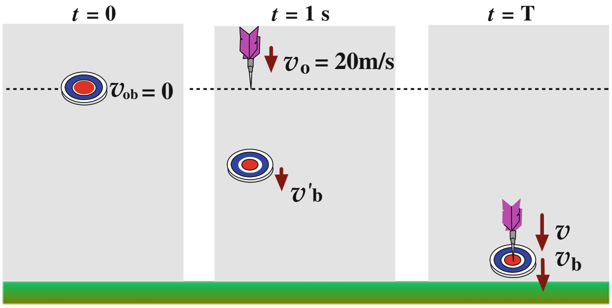


Fig. 3.28 See Exercise (33)

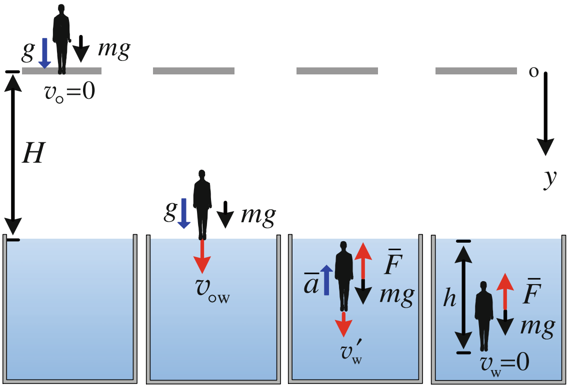


Fig. 3.29 See Exercise (35)