Oscillations and Wave Motion

14

Any object that repeats its motion at regular time intervals is said to perform a **periodic** or **harmonic motion**. If the motion is a sinusoidal function of time, we call it **simple harmonic motion**.

14.1 Simple Harmonic Motion

Assume the motion of a particle moving back and forth about the origin o of the x-axis between the limits $x = +A$ and $x = -A$, as shown in Fig. 14.1.

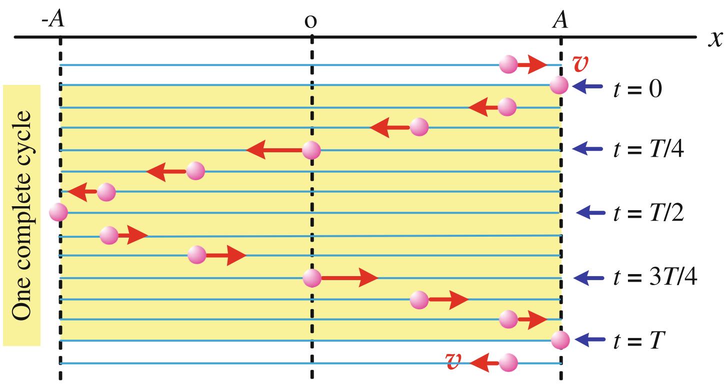

Fig. 14.1 Multiple snapshots of a particle oscillating about the origin of the x-axis between the two limits $x = +A$ and $x = -A$. If the time $t = 0$ is chosen to be when the particle is at $x = +A$, then the particle returns to $x = +A$ when $t = T$, where T is the period of the motion

In this figure, we chose $t = 0$ at the point where the particle is at $x = +A$ and $t = T$ when the particle returns to the same point $x = +A$ after one complete

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cycle. In this case, T represents the **period** of the motion. The **frequency** f of the simple harmonic motion is equal to the number of oscillations per unit time. Therefore, the frequency is related to the period T by the following relation:

$$f = \frac{1}{T} \quad \text{(Harmonic motion)} \tag{14.1}$$

and has the SI unit s^{-1}, cycle/s, or hertz (Hz). Additionally, we define the **angular frequency** of the motion by the relation:

$$\omega = \frac{2\pi}{T} \quad \text{(Harmonic motion)} \tag{14.2}$$

Accordingly, this relation can be written in terms of the frequency f as follows:

$$\omega = 2\pi f \quad \text{(Harmonic motion)} \tag{14.3}$$

where the SI unit of ω is rad/s. For such a motion, the displacement x of the particle from o is given generally as a function of time as:

$$x(t) = A\cos(\omega t + \phi) \tag{14.4}$$

where A is the amplitude of the motion and ϕ is the **phase angle** or **phase constant** (ϕ is zero in Fig. 14.1). The cosine function in Eq. 14.4 varies between the limits ± 1, so the displacement $x(t)$ varies between the limits $\pm A$, as shown in Fig. 14.2.

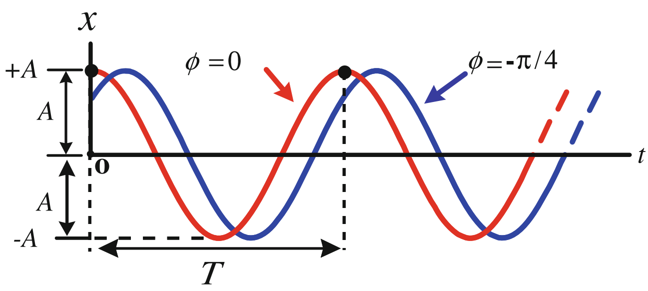


Fig. 14.2 A sketch of the relation $x(t) = A\cos(\omega t + \phi)$ for $\phi = 0$ and $\phi = -\pi/4$

14.1.1 Velocity and Acceleration of SHM

We can find an expression for the velocity v of a particle moving with a harmonic motion by differentiating Eq. 14.4 as follows:

$$v(t) = \frac{dx}{dt} = \frac{d[A\cos(\omega t + \phi)]}{dt}$$

Thus:

$$v(t) = -\omega A \sin(\omega t + \phi) = -v_{max} \sin(\omega t + \phi)\ , \quad v_{max} = \omega A \qquad (14.5)$$

By differentiating this expression, we generate the following expression for the acceleration of the oscillating particle:

$$a(t) = \frac{dv}{dt} = \frac{d[-\omega A \sin(\omega t + \phi)]}{dt}$$

Thus:

$$a(t) = -\omega^2 A \cos(\omega t + \phi) = -a_{max} \cos(\omega t + \phi)\ , \quad a_{max} = \omega^2 A \qquad (14.6)$$

Figure 14.3 shows a plot of Eqs. 14.4–14.6 for $\phi = 0$.

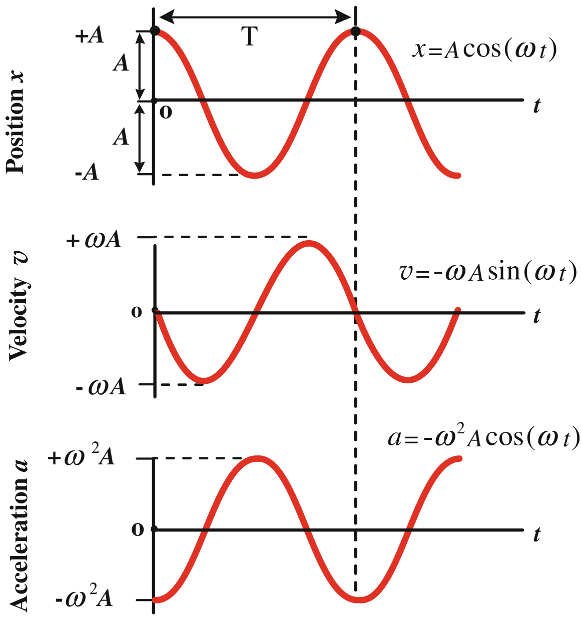


Fig. 14.3 The *upper part* of the figure shows the variation of the displacement $x(t)$ with time t of a particle oscillating with a SHM when the phase angle ϕ is equal to zero. The middle and lower parts display the variation of $v(t)$ and $a(t)$ with time. In all parts of the figure, the period T marks one complete oscillation

Example 14.1

A particle oscillates with a simple harmonic motion along the x axis. Its displacement from the origin varies with time according to the equation:

$$x = (2\text{ m})\cos(0.5\pi\, t + \pi/3)$$

where t is in seconds and the argument of the cosine is in radians. (a) Find the amplitude, frequency, and period of the motion. (b) Find the velocity and acceleration of the particle at any time. (c) Find both the maximum speed and acceleration of the particle. (d) Find the displacement of the particle between $t = 0$ and $t = 2$ s.

Solution: (a) By comparing the given equation to the general form $x(t) = A\cos(\omega t + \phi)$, we find that: the amplitude $A = 2$ m, the angular frequency $\omega = 0.5\pi$ rad/s, and the phase constant $\phi = \pi/3$ rad.

Therefore, the frequency will be:

$$f = \omega/2\pi = (0.5\pi\text{ rad/s})/(2\pi\text{ rad}) = 0.25\text{ s}^{-1} = 0.25\text{ Hz}$$

Hence the period will be given by:

$$T = 1/f = 1/0.25\text{ s}^{-1} = 4\text{ s}$$

(b) We differentiate x to find v, and then v to find a, as follows:

$$\begin{aligned}
v &= \frac{dx}{dt} = \frac{d[(2\text{ m})\cos(0.5\pi\, t + \pi/3)]}{dt} \\
&= (2\text{ m})[-\sin(0.5\pi\, t + \pi/3)] \times (0.5\pi\text{ rad/s}) \\
&= -(\pi\text{ m/s})\sin(0.5\pi\, t + \pi/3) \\
a &= \frac{dv}{dt} = \frac{d[(-\pi\text{ m/s})\sin(0.5\pi\, t + \pi/3)]}{dt} \\
&= (-\pi\text{ m/s})[\cos(0.5\pi\, t + \pi/3)] \times (0.5\pi\text{ rad/s}) \\
&= -(0.5\pi^2\text{ m/s}^2)\cos(0.5\pi\, t + \pi/3)
\end{aligned}$$

Remember, the rad is a dimensionless quantity and can be removed from or inserted into any calculation without altering the dimension of the result.

(c) Since the maximum values of the sine and cosine are unity, then v of part (b) varies between $\pm\pi$ m/s, and a of part (b) varies between $\pm 0.5\pi^2$ m/s^2. Thus, the maximum speed and maximum acceleration are as follows:

$$v_{max} = \pi \text{ m/s}$$
$$a_{max} = 0.5\pi^2 \text{ m/s}^2$$

We can also use Eqs. 14.5 and 14.6 to find v_{max} and a_{max} as follows:

$$v_{max} = \omega A = (0.5\pi \text{ rad/s})(2\text{ m}) = \pi \text{ m/s}$$
$$a_{max} = \omega^2 A = (0.5\pi \text{ rad/s})^2(2\text{ m}) = 0.5\pi^2 \text{ m/s}^2$$

(d) The position of the particle at $t = 0$ is denoted by x_i and is given by:

$$x_i = (2 \text{ m}) \cos(0 + \pi/3) = (2 \text{ m})(0.5) = 1 \text{ m}$$

The position of the particle at $t = 2$ s is denoted by x_f and is given by:

$$x_f = (2 \text{ m}) \cos(\pi + \pi/3) = (2 \text{ m})(-0.5) = -1 \text{ m}$$

Therefore, the displacement between $t = 0$ and $t = 2$ s is:

$$\Delta x = x_f - x_i = -1 \text{ m} - 1 \text{ m} = -2 \text{ m}$$

Figure 14.4a shows the plot of x versus t for the given function, while Fig. 14.4b depicts snapshots of the oscillating particle about the origin of the x-axis between the two limits $x = +2$ m and $x = -2$ m.

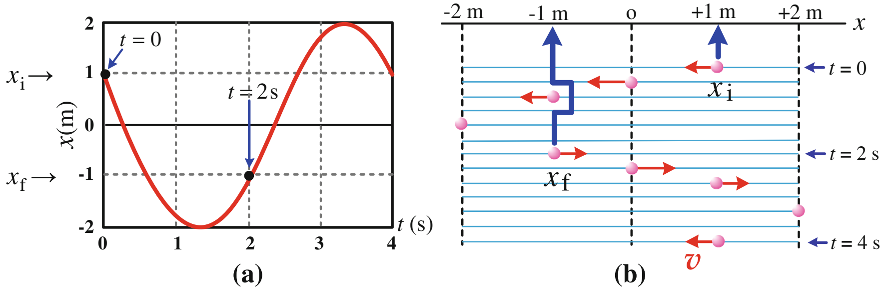


Fig. 14.4

14.1.2 The Force Law for SHM

We can combine Eqs. 14.4 and 14.6 to yield:

$$a(t) = -\omega^2 x(t) \tag{14.7}$$

This equation is the hallmark of simple harmonic motion. It states that the acceleration is proportional to the displacement but opposite in sign, and they are related by the square of the angular frequency, ω^2.

Once we know the acceleration as a function of time, we can use Newton's second law to find what force must act on the particle to produce this acceleration. Now, we combine Newton's second law with Eq. 14.7 as follows:

$$F = m\,a = -(m\,\omega^2)x \quad \text{or} \quad \frac{d^2x}{dt^2} + \omega^2\,x = 0 \tag{14.8}$$

This form (a force proportional to the displacement but opposite in sign) is familiar to us. It is Hooke's law for a spring, which was introduced in Sect. 6.3. That is:

$$F = -k_{\text{H}}\,x \qquad \text{(Hooke's law)} \tag{14.9}$$

By comparison, the equivalent spring constant in SHM is:

$$k_{\text{H}} = m\,\omega^2 \tag{14.10}$$

We can take Eq. 14.9 as an alternative definition of SHM.

Simple Harmonic Motion
SHM is the motion executed by a particle of mass m subject to a force proportional to its displacement but opposite in sign.

The block-spring system of Fig. 14.5 forms a linear simple harmonic oscillator. The angular frequency ω of the simple harmonic motion is related to the spring constant k_{H} and the mass of the block by Eq. 14.10, which gives:

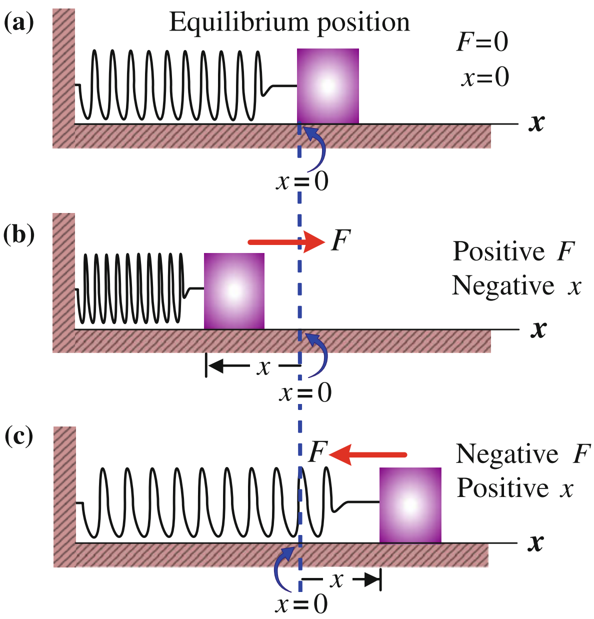
$$\omega = \sqrt{\frac{k_{\text{H}}}{m}} \tag{14.11}$$

Therefore, using Eqs. 14.5, 14.6, and 14.11, we can find the maximum values for the velocity and acceleration of the oscillations as follows:

$$v_{\text{max}} = \omega\,A = \sqrt{\frac{k_{\text{H}}}{m}}\;A \tag{14.12}$$

$$a_{\text{max}} = \omega^2 A = \frac{k_{\text{H}}}{m} A \tag{14.13}$$

Fig. 14.5 The variation of the force of a spring on a block. (a) When $x = 0$, the force is zero (*equilibrium position*). (b) When x is negative, the force is positive (*compressed spring*). (c) When x is positive, the force is negative (*stretched spring*)



By combining Eqs. 14.2 and 14.11, we can find the period T of the oscillations as follows:

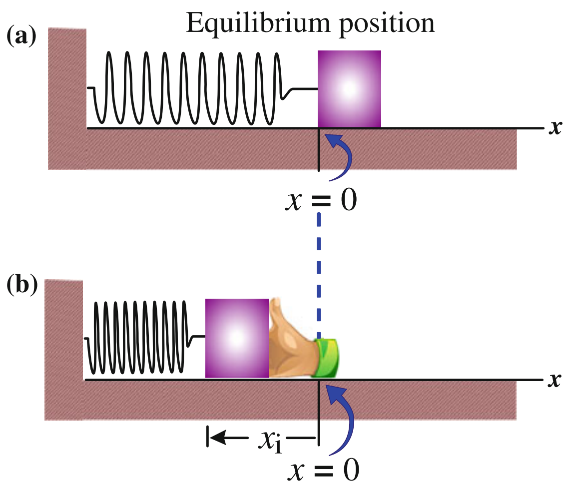
$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k_H}} \tag{14.14}$$

That is, the period T and hence the frequency $f = 1/T$ depend only on the mass of the particle m and the spring constant k_H, and not on the parameters of the motion, such as A or ϕ.

Example 14.2

A block of mass $m = 400\,\text{g}$ is attached to a light spring of force constant $k_H = 10\,\text{N/m}$, see Fig. 14.6a. The block is pushed against the spring from $x = 0$ to $x_i = -10\,\text{cm}$, see Fig. 14.6b, and then released to oscillate on a horizontal frictionless surface. (a) Find the angular frequency and the period of the block-spring system. (b) Find the maximum speed and maximum acceleration of the block. (c) Find the position, speed, and acceleration of the block at any time. (d) Repeat the above parts when the block is projected with initial velocity $v_i = -0.5$ m/s from another initial position $x_i = +10$ cm.

Fig. 14.6



Solution: (a) Using Eqs. 14.11 and 14.14 we find the angular frequency and the period of motion as follows:

$$\omega = \sqrt{\frac{k_H}{m}} = \sqrt{\frac{10 \text{ N/m}}{400 \times 10^{-3} \text{ kg}}} = 5 \text{ rad/s}$$

$$T = \frac{2\pi}{\omega} = \frac{2 \times 3.1416 \text{ rad}}{5 \text{ rad/s}} = 1.26 \text{ s}$$

(b) Since $A = |x_i| = 10$ cm, then Eqs. 14.12 and 14.13 will give:

$$v_{max} = \omega A = (5 \text{ rad/s})(0.1 \text{ m}) = 0.5 \text{ m/s}$$

$$a_{max} = \omega^2 A = (5 \text{ rad/s})^2 (0.1 \text{ m}) = 2.5 \text{ m/s}^2$$

(c) We can find the phase constant ϕ when we apply the initial condition $x(0) = -A$ at $t = 0$ to the form $x(t) = A\cos(\omega t + \phi)$. Thus:

$$x(t) = A\cos(\omega t + \phi) \;\Rightarrow\; x(0) = A\cos(\phi) \;\Rightarrow\; -A = A\cos(\phi) \;\Rightarrow\; \phi = \pi$$

Therefore, $x(t) = A\cos(\omega t + \pi)$. Using this expression and the results of parts (a) and (b), we get:

$$x(t) = A\cos(\omega t + \pi) = (0.1 \text{ m})\cos(5t + \pi)$$

$$v(t) = -\omega A \sin(\omega t + \pi) = -(0.5 \text{ m/s})\sin(5t + \pi)$$

$$a(t) = -\omega^2 A\cos(\omega t + \pi) = -(2.5 \text{ m/s}^2)\cos(5t + \pi)$$

(d) In this case, we start with the general form $x(t) = A\cos(\omega t + \phi)$, where A and ϕ are not known, but ω does not change because it is independent of how the oscillation is set into motion. Thus:

$$(1) \quad x(0) = A\cos(0 + \phi) = x_i$$
$$(2) \quad v(0) = -\omega A\sin(0 + \phi) = v_i$$

Dividing Eq. (2) by Eq. (1) gives a phase-constant relation:

$$\frac{-\omega A\sin(\phi)}{A\cos(\phi)} = \frac{v_i}{x_i} \quad \Rightarrow \quad \tan(\phi) = -\frac{v_i}{\omega x_i} = -\frac{-0.5\ \text{m/s}}{(5\ \text{rad/s})(0.1\ \text{m})} = 1$$

Thus: $$\phi = \tan^{-1}(1) = 0.785\,\text{rad} = 0.25\,\pi\ \text{rad}$$

Now, Eq. (1) allows us to find the new amplitude A as follows:

$$A = \frac{x_i}{\cos(\phi)} = \frac{(0.1\ \text{m})}{\cos(0.25\pi)} = 0.14\,\text{m} = 14\ \text{cm}$$

The new maximum speed and acceleration will be:

$$v_{max} = \omega A = (5\ \text{rad/s})(0.14\,\text{m}) = 0.7\ \text{m/s}$$
$$a_{max} = \omega^2 A = (5\ \text{rad/s})^2(0.14\,\text{m}) = 3.5\ \text{m/s}^2$$

Finally, the new expressions for x, v, and a are as follows:

$$x(t) = A\cos(\omega t + \phi) = (0.14\ \text{m})\cos(5\,t + 0.25\pi)$$
$$v(t) = -\omega A\sin(\omega t + \phi) = -(0.7\,\text{m/s})\sin(5\,t + 0.25\pi)$$
$$a(t) = -\omega^2 A\cos(\omega t + \phi) = -(3.5\,\text{m/s}^2)\cos(5\,t + 0.25\pi)$$

14.1.3 Energy of the Simple Harmonic Oscillator

Consider the block-spring system shown in Fig. 14.6 when the spring is massless and the horizontal surface is frictionless (known as the linear oscillator). In such a situation, the kinetic energy of the system is associated entirely with the block. Its value depends only on the velocity v given by Eq. 14.5. Thus:

$$K = \tfrac{1}{2}mv^2 = \tfrac{1}{2}m\omega^2 A^2\sin^2(\omega t + \phi) \tag{14.15}$$

When using Eq. 14.11 to substitute k_H/m for ω^2, we find:

$$K = \tfrac{1}{2}mv^2 = \tfrac{1}{2}k_H A^2 \sin^2(\omega t + \phi) \tag{14.16}$$

On the other hand, the potential energy of the linear oscillator of Fig. 14.6 is associated entirely with the spring. Its value depends only on the position x given by Eq. 14.4. Thus:

$$U = \tfrac{1}{2}k_H x^2 = \tfrac{1}{2}k_H A^2 \cos^2(\omega t + \phi) \tag{14.17}$$

The **mechanical energy** E of the simple harmonic oscillator is thus:

$$\begin{aligned} E = K + U &= \tfrac{1}{2}k_H A^2 \sin^2(\omega t + \phi) + \tfrac{1}{2}k_H A^2 \cos^2(\omega t + \phi) \\ &= \tfrac{1}{2}k_H A^2[\sin^2(\omega t + \phi) + \cos^2(\omega t + \phi)] \end{aligned} \tag{14.18}$$

When we use the identity $\sin^2\theta + \cos^2\theta = 1$, we find:

$$E = K + U = \tfrac{1}{2}k_H A^2 \tag{14.19}$$

That is, the mechanical energy (or the **total energy**) of a simple harmonic oscillator is constant, independent of time, and is proportional to the square of the amplitude. Because $v = 0$ at $x = \pm A$, i.e. $K = 0$, the mechanical energy equals the maximum potential energy, i.e. $E = U_{max} = \tfrac{1}{2}k_H A^2$. At the equilibrium position $x = 0$ we have $U = 0$, and the mechanical energy equals the maximum kinetic energy, i.e. $E = K_{max} = \tfrac{1}{2}mv_{max}^2 = \tfrac{1}{2}k_H A^2$.

Since the potential energy U is expressed as a function of the position x through the relation $U = \tfrac{1}{2}k_H x^2$, then Eq. 14.19 allows us of to express the kinetic energy as a function of x as follows:

$$K = E - U = \tfrac{1}{2}k_H A^2 - \tfrac{1}{2}k_H x^2 = \tfrac{1}{2}k_H(A^2 - x^2) \tag{14.20}$$

Figure 14.7a displays both the kinetic energy K and potential energy U as a function of time t, while Fig. 14.7b displays the variation of K and U as a function of position x.

Finally, by using Eq. 14.20 we can find the velocity of the block at any arbitrary position x as follows:

$$K = \tfrac{1}{2}mv^2 = \tfrac{1}{2}k_H(A^2 - x^2)$$

$$v = \pm\sqrt{\frac{k_H}{m}(A^2 - x^2)} \tag{14.21}$$

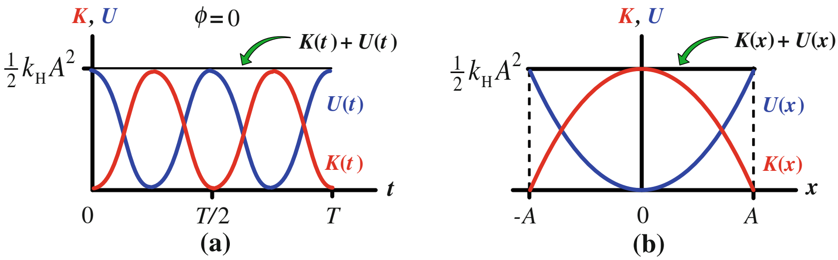


Fig. 14.7 (a) The kinetic energy $K(t)$ and the potential energy $U(t)$ as a function of time when $\phi = 0$ for a simple harmonic oscillator. Note that $K(t)$ and $U(t)$ peak twice during every period. (b) The kinetic energy $K(x)$ and the potential energy $U(x)$ as a function of x. For $x = 0$ the energy is entirely kinetic, and for $x = \pm A$ it is entirely potential

When using Eq. 14.11 to substitute k_H/m for ω^2, we find:

$$v = \pm\omega\sqrt{(A^2 - x^2)} \tag{14.22}$$

This relation verifies the fact that the speed is a maximum when $x = 0$ and is zero at both of the turning points $x = \pm A$.

Example 14.3

A block of mass $m = 320\,\text{g}$ is fastened to a light spring whose force constant k_H is 72 N/m, see Fig. 14.8a. The block is pulled a distance $x_i = 50\,\text{cm}$ from its equilibrium position at $x = 0$ on a horizontal frictionless surface, see Fig. 14.8b, and released at $t = 0$. (a) What is the mechanical energy of the oscillating block? (b) What is the maximum speed of the oscillating block? (c) Find the velocity, kinetic energy, and potential energy of the block when its position is 30 cm?

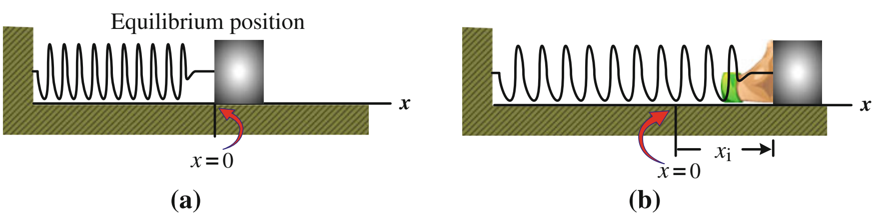


Fig. 14.8

Solution: (a) Since $A = x_i = 50\,\text{cm} = 0.5\,\text{m}$, then Eq. 14.19 gives:

$$E = \tfrac{1}{2}k_H A^2 = \tfrac{1}{2}(72\ \text{N/m})(0.5\ \text{m})^2 = 9\ \text{J}$$

(b) At $x = 0$, we know that $U = 0$ and $E = \frac{1}{2}mv_{max}^2$; therefore:

$$\tfrac{1}{2}mv_{max}^2 = E = 9\ \text{J}$$

$$v_{max} = \sqrt{\frac{2E}{m}} = \sqrt{\frac{2(9\ \text{J})}{0.32\ \text{kg}}} = 7.5\ \text{m/s}$$

(c) We use Eq. 14.21 to find the velocity at $x = 30\,\text{cm}$ as follows:

$$v = \pm\sqrt{\frac{k_H}{m}(A^2 - x^2)} = \pm\sqrt{\frac{72\ \text{N/m}}{0.32\ \text{kg}}[(0.5\ \text{m})^2 - (0.3\ \text{m})^2]} = \pm 6\ \text{m/s}$$

Now, we can find K and U when $x = 30\,\text{cm} = 0.3\,\text{m}$ as follows:

$$K = \tfrac{1}{2}mv^2 = \tfrac{1}{2}(0.32\ \text{kg})(\pm 6\ \text{m/s})^2 = 5.76\ \text{J}$$

$$U = \tfrac{1}{2}k_H x^2 = \tfrac{1}{2}(72\ \text{N/m})(0.3\ \text{m})^2 = 3.24\ \text{J}$$

14.2 * Damped Simple Harmonic Motion

When non-conservative forces, such as friction, oppose the motion of an oscillator, its mechanical energy diminishes with time, and the motion is said to be **damped**. One such system is a block of mass m attached to a spring and immersed in a viscous liquid, see Fig. 14.9a. Let us assume that the liquid exerts a damping force F_d that is proportional to the velocity v_x of the oscillator. If v_x is small, then:

$$F_d = -bv_x \tag{14.23}$$

where b is a damping constant. The total force acting on the block is:

$$\Sigma F_x = -k_H\, x - bv_x \tag{14.24}$$

If we set $v_x = dx/dt$ and substitute with ΣF_x in Newton's second law, $\Sigma F_x = m\, d^2x/dt^2$, we find the following differential equation:

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + k_H\, x = 0 \tag{14.25}$$

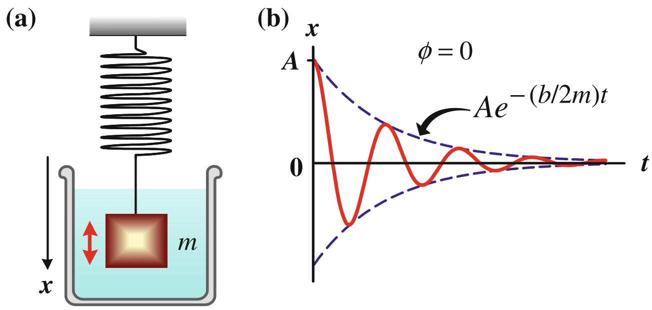


Fig. 14.9 (a) A damped oscillator consisting of a block immersed in a viscous liquid. (b) Graph of x versus t for a damped oscillator

This equation has a solution displayed in Fig. 14.9b, and is given by:

$$x(t) = Ae^{-bt/2m}\cos(\omega_d\, t + \phi) \tag{14.26}$$

where the angular frequency of the damped oscillator ω_d is given by:

$$\omega_d = \sqrt{\frac{k_H}{m} - \frac{b^2}{4m^2}} \qquad \left(\omega_d \xrightarrow[\text{or } b\to 0]{b\ll 2\sqrt{k_H m}} \sqrt{\frac{k_H}{m}} = \omega\right) \tag{14.27}$$

14.3 Sinusoidal Waves

Waves are of three types: mechanical, electromagnetic, and matter waves. This chapter deals only with mechanical waves. We encounter mechanical waves almost constantly every day in our lives. For such waves, information and energy move from one point to another, but matter does not. Common examples of such waves are sound, water, and seismic waves. These waves require the following:

(1) Some source of disturbance (or vibration),
(2) A medium that can be disturbed,
(3) Some physical mechanism responsible for allowing adjacent portions of the medium to influence each other.

14.3.1 Transverse and Longitudinal Waves

Figure 14.10 displays a single pulse wave sent from one end of a long stretched string toward the other fixed end. As the wave passes the point P on the string, the y

coordinate of this point will increase, reach a maximum, and then decrease to zero. In the case of an *ideal string* (that is when no frictional forces within the string cause the pulse to die out as it travels), the wave pulse in the string moves along the string with some constant velocity v.

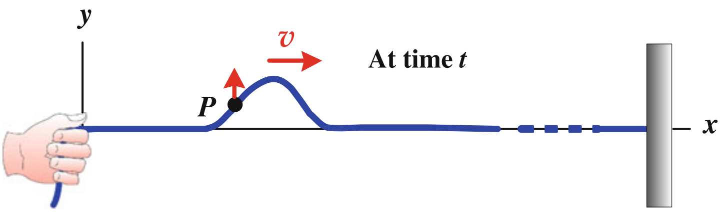


Fig. 14.10 Sending a single pulse through a long stretched string

Figure 14.11 displays a continuous sinusoidal wave sent from one end of a long stretched string toward the other fixed end. The wave has a sinusoidal shape at any time. That is, the wave has the shape of a sine curve (or a cosine curve) at any location x and time t. As the sinusoidal wave travels along the string with some constant velocity $\vec{v}$, the y coordinate of point P on the string will oscillate up and down continuously.

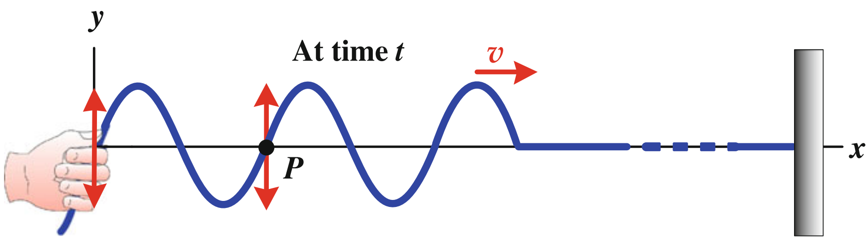


Fig. 14.11 Sending a continuous sinusoidal wave through a long stretched string. Any string element (represented by point P) oscillates perpendicular to the direction of the wave's velocity

From Figs. 1.10 and 1.11 we find that the displacement of every oscillating element on the string is perpendicular to the direction of the wave's velocity. This motion is called a **transverse** motion, and the generated wave is called a **transverse wave**.

Figure 14.12 shows how we can produce a sound wave by using a movable piston fitted in a long open pipe filled with air. A sound wave can be generated by pushing the piston toward the right or toward the left. The rightward motion of the piston compresses the air in the region next to it, i.e. increasing its pressure. Accordingly, the compressed air (or the change in pressure) travels as a pulse from one region to another toward the right along the pipe. If the push and pull of the piston is sinusoidal as in Fig. 14.12, a sinusoidal wave will travel along the pipe.

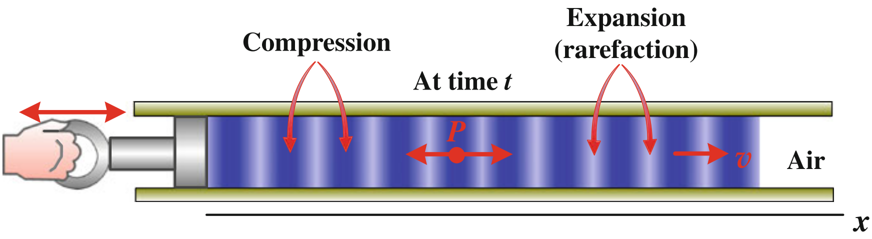


Fig. 14.12 Sending a continuous sinusoidal wave through an air-filled pipe by moving a piston back and forth in a sinusoidal manner. Any air element (represented by point P) oscillates back and forth parallel to the direction of the wave's velocity v

From Fig. 14.12, we find that the displacement of every air element in the pipe is parallel to the direction of the wave's velocity. This motion is **longitudinal**, and the generated wave is called a **longitudinal wave**.

Both transverse and longitudinal waves are **traveling waves**, because they travel from one point to another.

14.3.2 Wavelength and Frequency

Two physical characteristics are important in describing periodic waves: the **wavelength** (denoted by λ) and the **frequency** (denoted by f). Both are defined below:

Wavelength λ:
One wavelength λ is the *minimum distance between any two points on a wave where both points behave identically*.

Frequency f:
The frequency f of a wave is the *rate at which the disturbance repeats itself.*

A third important physical characteristic of waves is the **wave velocity** (denoted by v). In fact, mechanical waves travel, or propagate, with a specific velocity that is determined by the properties of the medium being disturbed.

14.3.3 Harmonic Waves: Simple Harmonic Motion

A harmonic wave that is traveling toward increasing x has a sinusoidal shape like the transverse wave of the string in Fig. 14.11. The displacement $y = y(x, t)$ of a harmonic wave can be written in terms of a sine (or a cosine) function of the position x at time t as follows:

$$y = A\sin(kx - \omega t) \quad \text{or} \quad y = A\cos(kx - \omega t)\,, \tag{14.28}$$

where A is the magnitude of the maximum displacement, known as the **amplitude** of the wave. The quantities k and ω are constants whose meanings will be discussed shortly. The quantity $kx - \omega t$ is called the **phase** of the wave. From now on, we will use only the sine form.

Figure 14.13 shows the transverse displacement y as a function of the position x at $t = 0$, i.e. the figure is a snapshot of the wave at that instant. With $t = 0$, the sine form of Eq. 14.28 becomes:

$$y = A\sin kx \quad (t = 0) \tag{14.29}$$

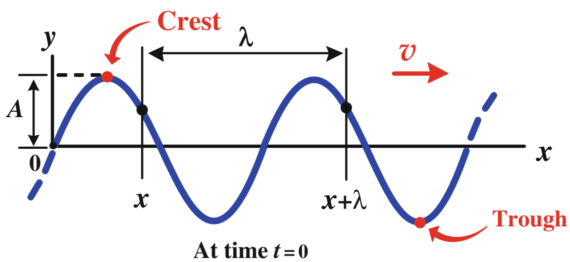


Fig. 14.13 A snapshot at $t = 0$ of a harmonic wave traveling to the right with a speed v in a taut string. A typical wavelength λ is shown, which is the minimum distance between any two points on the wave where both points behave identically

The **wavelength** λ of a wave is the distance between two successive crests, or identically behaving two points on the x axis having the same displacement y and slope dy/dx. If this occurs in Fig. 14.13 at x and $x + \lambda$, then Eq. 14.29 gives:

$$y = A\sin kx = A\sin k(x+\lambda) \tag{14.30}$$

The sine function repeats itself when its angle is increased by 2π rad. Thus, Eq. 14.30 is satisfied only if $k\lambda = 2\pi$, i.e.:

$$k = \frac{2\pi}{\lambda} \quad \text{(Harmonic wave)} \tag{14.31}$$

where k is called the **Angular wave number** (or simply the **wave number**) of the wave and has the SI unit rad/m (Not to be confused with the ***spring constant*** k_H in Hooke's law).

Figure 14.14 shows the transverse displacement y as a function of time t at an arbitrary location taken to be at $x = 0$. Thus, when monitoring the string, one would see point $x = 0$ moving up and down in a motion given by Eq. 14.28 with $x = 0$. That point is said to perform **simple harmonic motion**. So, Eq. 14.28 becomes:

$$y = A\sin(-\omega t) = -A\sin(\omega t) \quad (x = 0) \tag{14.32}$$

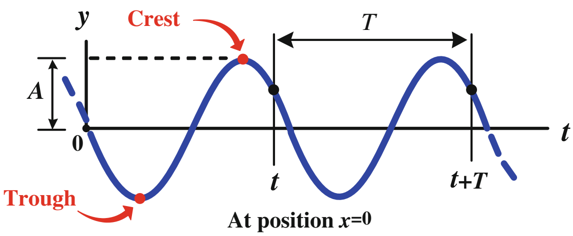


Fig. 14.14 A graph showing the displacement of the string at $x = 0$ as a function of t when the sinusoidal harmonic wave passes through that point. A typical period T is shown, which is the minimum time between any two points on the wave when both behave identically

The **period** T of the wave is the time between two successive points behaving identically on the t axis having the same displacement y. If this occurs in Fig. 14.14 at t and $t + T$, then Eq. 14.32 gives:

$$y = -A\sin\omega t = -A\sin\omega(t+T) \tag{14.33}$$

Again, the sine function repeats itself when its angle is increased by 2π rad. Thus, Eq. 14.33 is satisfied only if $\omega T = 2\pi$, i.e.:

$$\omega = \frac{2\pi}{T} \quad \text{(Harmonic wave)} \tag{14.34}$$

where ω is defined previously in Eq. 14.2 and called the **angular frequency** of the wave which has the SI unit rad/s.

The **frequency** f of a harmonic wave is equal to the number of crests (or troughs), or any point on the wave, that passes any point in a unit time interval. The following relation relates the frequency f to the period T and the angular frequency ω:

$$f = \frac{1}{T} = \frac{\omega}{2\pi} \quad \text{(Harmonic wave)} \tag{14.35}$$

As defined previously, it has the SI unit s^{-1}, or cycle/s, or hertz (Hz).

Figure 14.15 shows a snapshot of the wave at $t = 0$ and $t = \Delta t$. The ratio $\Delta x/\Delta t$ (or, in the differential limit dx/dt) is the **wave speed** v, i.e. $v = dx/dt$. As the wave moves, each point (such as point P) retains its displacement y. For each such point, although x and t are changing, Eq. 14.28 tells us that the argument of the sine function must be constant. That is:

$$k\,x - \omega\,t = \text{constant} \tag{14.36}$$

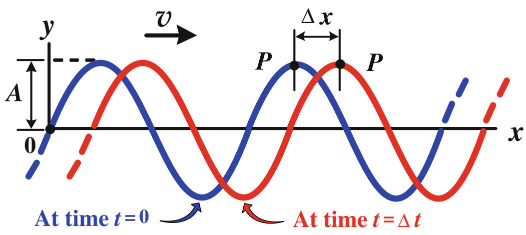


Fig. 14.15 Snapshots of a traveling wave at $t = 0$ and $t = \Delta t$. During this time the entire curve shifts a distance Δx with a speed $v = \Delta x/\Delta t$.

To find the speed of the wave v, we differentiate Eq. 14.36 with respect to t to get:

$$k\frac{dx}{dt} - \omega = 0$$

Thus:

$$v = \frac{dx}{dt} = \frac{\omega}{k} \tag{14.37}$$

Using $k = 2\pi/\lambda$ and $\omega = 2\pi/T$, we can rewrite the speed as:

$$v = \frac{\omega}{k} = \frac{\lambda}{T} \quad \text{or} \quad v = \lambda\,f \tag{14.38}$$

Therefore, we can rewrite Eq. 14.28 in different forms such as:

$$y = A\sin\left[2\pi\left(\frac{x}{\lambda} - \frac{t}{T}\right)\right] \quad \text{or} \quad y = A\sin\left[\frac{2\pi}{\lambda}(x - vt)\right] \tag{14.39}$$

The harmonic wave given by Eq. 14.28 assumes that the displacement y is zero at $x = 0$ and $t = 0$. If the transverse wave is not zero, we generally express the harmonic wave in the form:

$$y = \begin{cases} A\sin(kx - \omega t + \phi) \\ \text{or} \\ A\cos(kx - \omega t + \phi) \end{cases}, \quad (k = 2\pi/\lambda,\ \omega = 2\pi f) \tag{14.40}$$

where ϕ is a constant, called the **phase constant**, that can be determined from the wave's initial conditions.

Example 14.4

A harmonic wave traveling along a string in the direction of increasing x has the following form $y = 0.4\sin(0.2x - 5t)$, where all the numerical constants are in SI units. (a) Find the amplitude, wave number, angular frequency, and speed of the wave. (b) Find the wavelength, period, and frequency of the wave.

Solution: (a) Comparing this wave with $y = A\sin(kx - \omega t + \phi)$, we find the amplitude, wave number, angular frequency, and phase to be:

$$A = 0.4\,\text{m}, \quad k = 0.2\,\text{rad/m}, \quad \omega = 5\,\text{rad/s}, \quad \text{and} \quad \phi = 0$$

From Eq. 14.37 we find the speed of the wave to be:

$$v = \frac{\omega}{k} = \frac{5\ \text{rad/s}}{0.2\ \text{rad/m}} = 25\ \text{m/s}$$

(b) Equation 14.31 gives the wavelength of the wave as follows:

$$\lambda = \frac{2\pi}{k} = \frac{2\pi\ \text{rad}}{0.2\ \text{rad/m}} = 31.4\ \text{m}$$

From Eq. 14.34 we can find the period of the wave as follows:

$$T = \frac{2\pi}{\omega} = \frac{2\pi\ \text{rad}}{5\ \text{rad/s}} = 1.26\ \text{s}$$

Equation 14.35 gives the frequency of the wave as follows:

$$f = \frac{1}{T} = \frac{1}{1.26\ \text{s}} = 0.8\ \text{s}^{-1} = 0.8\ \text{cycle/s} = 0.8\ \text{Hz}$$

Note that the quantities calculated are independent of the amplitude A.

14.4 The Speed of Waves on Strings

String waves are the most common examples of transverse waves. Let us consider a single symmetrical pulse traveling with a speed v in a stretched string that is under a tensional force of magnitude τ (we use the symbol τ to represent tension, which avoids confusion with the symbol T used to represent the period of oscillation), see Fig. 14.16. We assume that the string has a linear mass density $\mu = m/L$, where m is the mass of the string and L is its length.

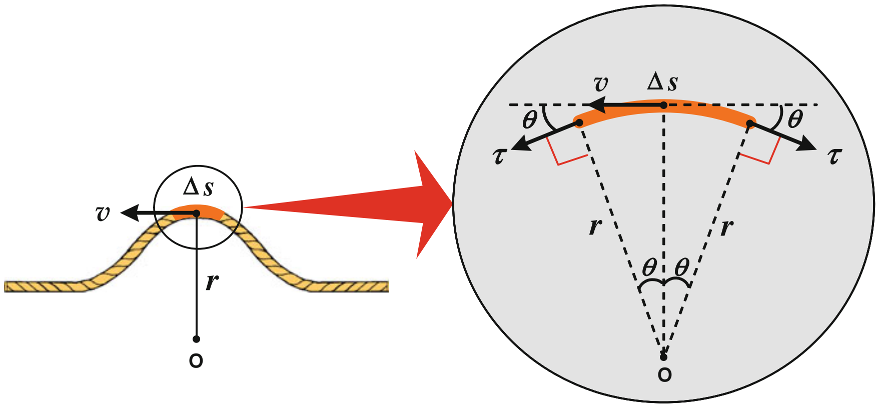


Fig. 14.16 A symmetrical pulse moving to the left on a stretched string with speed v. To find this speed we apply Newton's second law on a small segment of length Δs located at the top of the pulse.

*Consider a small segment at the top of the pulse, of length Δs, forming an arc of a circle of radius r, see Fig. 14.16. A force equal in magnitude to the string tension τ pulls tangentially on this segment at each end. The horizontal components of these forces cancel, but the vertical components add to form a radial restoring force of magnitude:

$$F_r = 2\tau \sin\theta \approx 2\tau\theta = \tau 2\theta = \tau \frac{\Delta s}{r} \tag{14.41}$$

where we have used the approximation $\sin\theta \approx \theta$ when Δs is very small and also used the relation $\Delta s = r \times (2\theta)$.

The mass Δm of the segment Δs is given by:

$$\Delta m = \mu \Delta s \tag{14.42}$$

According to Fig. 14.16, the string segment Δs is moving radially toward the center of a circle of radius r with a centripetal acceleration of magnitude given by:

$$a_r = \frac{v^2}{r} \tag{14.43}$$

When we apply Newton's second law *force* = *mass* × *acceleration*, i.e. $F_r = \Delta m\, a_r$, and also apply Eqs. 14.41–14.43, we get the following relation:

$$\tau \frac{\Delta s}{r} = \mu \Delta s \times \frac{v^2}{r}$$

Solving this equation for the speed v yields:

$$\boxed{v = \sqrt{\frac{\tau}{\mu}}} \tag{14.44}$$

This equation tells us that the speed of a wave along an ideal stretched string depends only on the characteristics of the string (the magnitude of the tension τ and the mass per unit length μ) and not on the frequency f of the wave. Actually, the frequency f is fixed by whatever generates the wave, while the wavelength is fixed by Eq. 14.38, i.e. by the relation $\lambda = v/f$.

Example 14.5

A uniform string has a linear mass density of 0.2 kg/m. The string passes over a massless frictionless pulley to a block of mass $m = 4$ kg, see Fig. 14.17. Find the speed of a single pulse sent from one end of the string toward the pulley.

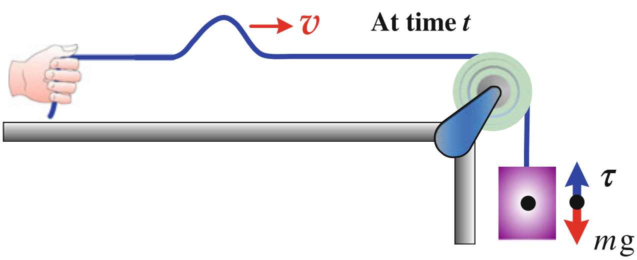


Fig. 14.17

Solution: The magnitude of the tension τ in the string is equal to the magnitude of the weight of the suspended block. Thus:

$$\tau = mg = (4 \text{ kg}) \times (9.8 \text{ m/s}^2)$$
$$= 39.2 \text{ N}$$

Using this result and the linear density $\mu = 0.2$ kg/m in Eq. 14.44, we find the value of the speed of the wave to be:

$$v = \sqrt{\frac{\tau}{\mu}}$$
$$= \sqrt{\frac{39.2 \text{ N}}{0.2 \text{ kg/m}}} = 14 \text{ m/s}$$

14.5 Energy Transfer by Sinusoidal Waves on Strings

Waves transport kinetic and potential energy when they propagate through a medium.

This can be easily demonstrated by hanging an object on a stretched string and then sending a pulse through it, see Fig. 14.18. As the pulse meets the object, the object will move up and hence acquire kinetic and potential energy.

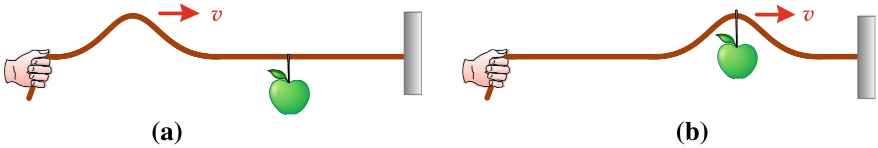


Fig. 14.18 (a) A pulse traveling on a stretched string over which an object is hung. (b) Kinetic energy and potential energy are transferred to the object when the pulse arrives

Consider a string of mass per unit length μ and tension of magnitude τ that is connected to a source of vibration as shown in Fig. 14.19a. When the source vibrates, it does work to produce a sinusoidal wave that travels to the right as shown in Fig. 14.19b. Now, let us focus our attention on an element of the string of mass Δm and length Δx located at a particular point x. This element will move up and down in a simple harmonic motion, see Fig. 14.19b.

Assume the oscillation of this element in the y direction has an amplitude A, wave number k, and angular frequency ω. Then, according to Eq. 14.28, the transverse velocity v_y (not to be confused with the wave velocity v) at a particular position x will be:

$$v_y = \left.\frac{dy}{dt}\right|_{x=\text{constant}} = \frac{\partial y}{\partial t} = \frac{\partial\,[A\sin(kx-\omega t)]}{\partial t}$$
$$= -\omega A\cos(kx-\omega t) \tag{14.45}$$

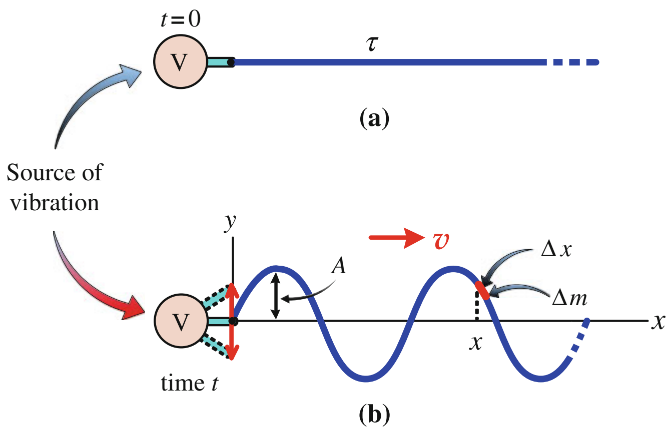


Fig. 14.19 (a) A source of vibration connected to a stretched string under tension τ. (b) A snapshot of a traveling harmonic wave on the string at a time chosen to be at time t

The kinetic energy ΔK associated with a string element of mass $\Delta m = \mu\Delta x$ will be given by:

$$\Delta K = \tfrac{1}{2}\Delta m\, v_y^2 = \tfrac{1}{2}\mu\Delta x\, v_y^2 \tag{14.46}$$

When allowing Δx to approach zero, this relation becomes a differential relationship and will take the following form:

$$dK = \tfrac{1}{2}\mu\, dx\, v_y^2 = \tfrac{1}{2}\mu\omega^2 A^2\cos^2(k\,x-\omega\,t)\,dx \tag{14.47}$$

At a given instant, let us integrate this expression over all the string elements of a complete wavelength, which will give us the total kinetic energy K_λ in one wavelength:

$$K_\lambda = \int dK = \tfrac{1}{2}\mu\omega^2 A^2\int_0^\lambda \cos^2(k\,x-\omega\,t)\,dx \tag{14.48}$$

*If we take a snapshot at time $t = 0$, then we can evaluate the above integral by performing the following steps:

$$\begin{aligned}\int_{x=0}^{x=\lambda} \cos^2(k\,x)\;dx &= \tfrac{1}{k}\int_{z=0}^{z=k\lambda=2\pi} \cos^2 z\; dz \\ &= \tfrac{1}{k}\int_0^{2\pi} \tfrac{1}{2}[1+\cos 2z]\; dz \\ &= \tfrac{1}{2k}\left[z+\tfrac{1}{2}\sin 2z\right]_0^{2\pi} \\ &= \tfrac{1}{2k}\left[(2\pi+\tfrac{1}{2}\sin 4\pi)-0\right] = \frac{\lambda}{4\pi}2\pi \\ &= \frac{\lambda}{2}\end{aligned} \tag{14.49}$$

where we have used $z = kx$, $\cos^2 z = (1+\cos 2z)/2$ and $k = 2\pi/\lambda$ to arrive to the above result. Of course, we get the same answer if we perform the above steps at any other time different from zero. When we substitute the above result into Eq. 14.48, we get:

$$K_\lambda = \tfrac{1}{4}\mu\omega^2 A^2\,\lambda \tag{14.50}$$

A similar analysis to the total potential energy U_λ in one wavelength will give exactly the same result. Thus:

$$U_\lambda = \tfrac{1}{4}\mu\omega^2 A^2\lambda \tag{14.51}$$

The total energy in one wavelength of the wave is the sum of the obtained kinetic and potential energies:

$$E_\lambda = K_\lambda + U_\lambda = \tfrac{1}{2}\mu\omega^2 A^2\,\lambda \tag{14.52}$$

As the sinusoidal wave travels along the string, that amount of energy (E_λ) will cross any given point on the string during a time interval equal to one period of the oscillation. Thus, the rate of energy (or power) transferred by the wave through the string is:

$$P = \frac{\Delta E}{\Delta t} = \frac{E_\lambda}{T}$$

Therefore:

$$P = \tfrac{1}{2}\mu\omega^2 A^2\frac{\lambda}{T}$$

Using the relation $v = \lambda/T$ given by Eq. 14.38, we finally attain the following form:

$$P = \tfrac{1}{2}\mu v \omega^2 A^2 \tag{14.53}$$

In this expression the factors μ and v depend on the material and tension of the string. On the other hand, the factors ω and A depend on the source that generates the sinusoidal wave. The dependence of the power of a wave on the square of its angular frequency and on the square of its amplitude is a general result, i.e. true for all wave types.

Example 14.6

A string that is taut under tension of magnitude $\tau = 40$ N has a linear density μ of 64 g/m. A wave is traveling along the string with a frequency f of 120 Hz and amplitude A of 8 mm. (a) Find the speed of the wave. (b) What is the rate of energy that must be supplied by a generator to produce this wave in the string? (c) If the string is to transfer energy at a rate of 500 W, what must be the required wave amplitude when all other parameters remain the same?

Solution: (a) Equation 14.44 gives the speed of the wave as follows:

$$v = \sqrt{\frac{\tau}{\mu}} = \sqrt{\frac{40\ \text{N}}{0.064\ \text{kg/m}}} = 25\ \text{m/s}$$

(b) First we calculate the angular frequency ω as follows:

$$\begin{aligned}\omega = 2\pi f &= 2 \times (3.1416\ \text{rad}) \times (120\ \text{s}^{-1})\\ &= 754\ \text{rad/s}\end{aligned}$$

The power supplied to the string is calculated by using the obtained values and the given information in Eq. 14.53 as follows:

$$\begin{aligned}P &= \tfrac{1}{2}\mu v \omega^2 A^2\\ &= \tfrac{1}{2}(0.064\ \text{kg/m})(25\ \text{m/s})(754\ \text{rad/s})^2(0.008\ \text{m})^2\\ &= 29.1\ \text{W}\end{aligned}$$

(c) The ratio between the new power P' and the old power P is:

$$\frac{P'}{P} = \frac{\frac{1}{2}\mu\, v\, \omega^2 A'^2}{\frac{1}{2}\mu\, v\, \omega^2 A^2} = \frac{A'^2}{A^2}$$

Thus: $$A' = A\sqrt{\frac{P'}{P}} = 0.008\ \text{m}\sqrt{\frac{500\ \text{W}}{29.1\ \text{W}}} = 0.033\ \text{m} = 3.3\ \text{cm}$$

14.6 The Linear Wave Equation

In Sect. 14.3.3 we introduced the wave function $y = y(x, t)$ to represent waves traveling on strings. Actually, all these wave functions represent solutions of a differential equation called the **linear wave equation**. This equation is basic to many forms of wave motions, such as waves on strings.

We consider a single symmetrical transverse pulse that is traveling with a speed v in a stretched ideal string under tensional force of magnitude τ and has a linear density μ, see Fig. 14.20.

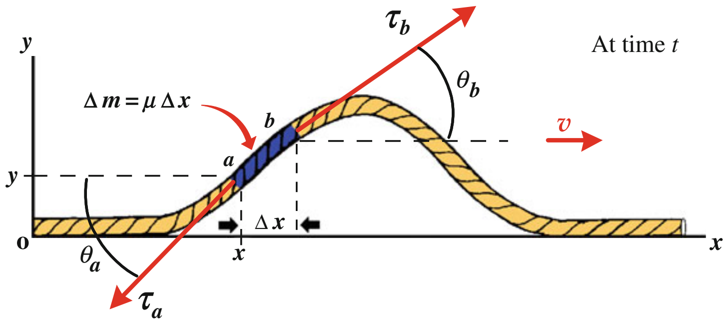


Fig. 14.20 A pulse traveling with a speed v in a string under tension τ. The figure shows an element of length Δx at the point (x, y)

In this figure we consider a small element $a\,b$ of length Δx with ends at angles θ_a and θ_b with the x axis. Also, for an ideal string we consider $\tau_b \cos\theta_b = \tau_a \cos\theta_a = \tau$. Thus, with the use of this result, the net vertical force acting on the string element can be written as:

$$\begin{aligned} \Sigma F_y &= \tau_b \sin\theta_b - \tau_a \sin\theta_a \\ &= \tau \tan\theta_b - \tau \tan\theta_a = \tau(\tan\theta_b - \tan\theta_a) \end{aligned} \tag{14.54}$$

The tangent of an angle is represented by dy/dx when y depends only on x. Since we are evaluating this tangent at a particular instant of time t, we need to express this tangent in partial form as $\partial y/\partial x$. Substituting this form of tangents into Eq. 14.54 gives:

$$\Sigma F_y = \tau \left[\left(\frac{\partial y}{\partial x} \right)_b - \left(\frac{\partial y}{\partial x} \right)_a \right] \tag{14.55}$$

When we apply Newton's second law to the vertical motion of an element of mass $\Delta m = \mu \; \Delta x$, we get:

$$\Sigma F_y = \Delta m \; a_y = \mu \; \Delta x \left(\frac{\partial^2 y}{\partial t^2} \right) \tag{14.56}$$

Combining Eqs. 14.55 with Eq. 14.56, we get:

$$\begin{aligned} \mu \; \Delta x \left(\frac{\partial^2 y}{\partial t^2} \right) &= \tau \left[\left(\frac{\partial y}{\partial x} \right)_b - \left(\frac{\partial y}{\partial x} \right)_a \right] \\ \frac{\mu}{\tau} \left(\frac{\partial^2 y}{\partial t^2} \right) &= \frac{\left(\frac{\partial y}{\partial x} \right)_b - \left(\frac{\partial y}{\partial x} \right)_a}{\Delta x} = \frac{\frac{\partial y(x + \Delta x, t)}{\partial x} - \frac{\partial y(x, t)}{\partial x}}{\Delta x} \end{aligned} \tag{14.57}$$

From the definition of partial differentiation, we know that:

$$\frac{\partial}{\partial x} f(x, t) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x, t) - f(x, t)}{\Delta x}$$

Thus, if we associate $f(x + \Delta x, t)$ with $(\partial y / \partial x)_b$ and $f(x, t)$ with $(\partial y / \partial x)_a$, we see that, in the limit $\Delta x \to 0$, the right-hand side of Eq. 14.57 can be expressed as a partial derivative as follows:

$$\frac{\partial}{\partial x} \left(\frac{\partial y}{\partial x} \right) = \lim_{\Delta x \to 0} \frac{\left(\frac{\partial y}{\partial x} \right)_b - \left(\frac{\partial y}{\partial x} \right)_a}{\Delta x}$$

Then, with the use of this result and Eq. 14.44, namely $v = \sqrt{\tau / \mu}$, we can write Eq. 14.57 as a partial differential equation in the following general form:

$$\frac{\partial^2 y}{\partial x^2} - \frac{1}{v^2} \frac{\partial^2 y}{\partial \; t^2} = 0 \tag{14.58}$$

This is the linear wave equation as it applies to waves on strings and generally applies to various types of traveling waves. We can prove that the sinusoidal wave $y(x, t) = A \sin(kx - \omega \, t)$ satisfies this equation.

14.7 Standing Waves

We consider two identical waves of the same wavelength and amplitude traveling simultaneously in opposite directions in a stretched string. The resultant wave in the string will be the algebraic sum of the two waves. This is one of the examples of a principle known as the **superposition principle**. Generally, this principle says that when several effects occur simultaneously, their net effect is the sum of the individual effects. The superposition principle will be introduced in more detail in Chap. 15 when we study the properties of standing sound waves.

To analyze this situation, we assume that the two string waves have the same frequency f (the same $\omega = 2\pi f$), wavelength λ (the same $k = 2\pi/\lambda$), and amplitude A but travel in opposite directions. Therefore, we can write these two waves in the following form:

$$\begin{aligned} y_1 &= A\sin(k\,x - \omega\,t), \\ y_2 &= A\sin(k\,x + \omega\,t) \end{aligned} \tag{14.59}$$

where y_1 represents a wave traveling in the positive x-direction and y_2 represents a wave traveling in the negative x-direction. The superposition of y_1 and y_2 gives the following resultant:

$$y = y_1 + y_2 = A\,[\sin(k\,x - \omega\,t) + \sin(k\,x + \omega\,t)] \tag{14.60}$$

To simplify this expression, we use the trigonometric identity:

$$\sin(a \pm b) = \sin a \cos b \pm \cos a \sin b \tag{14.61}$$

If we substitute $a = kx$ and $b = \omega\,t$ in this identity, then the resultant wave y reduces to:

$$y = (2\,A\sin k\,x)\cos\omega\,t \tag{14.62}$$

The resultant wave y represented by Eq. 14.62 gives a special kind of simple harmonic motion. Here, every element of the medium oscillates in simple harmonic motion with the same angular frequency ω (through the factor $\cos\omega\,t$) with an amplitude (given by the factor $2\,A\sin k\,x$) that varies with position x. This wave is called a **standing wave** because there is no motion of the disturbance along the x-direction.

A standing wave is distinguished by stationary positions with *zero amplitudes* called *nodes* (see Fig. 14.21). This happens when x satisfies the condition $\sin kx = 0$, that is, when:

$$k\,x = 0,\ \pi,\ 2\pi,\ 3\pi, \ldots$$

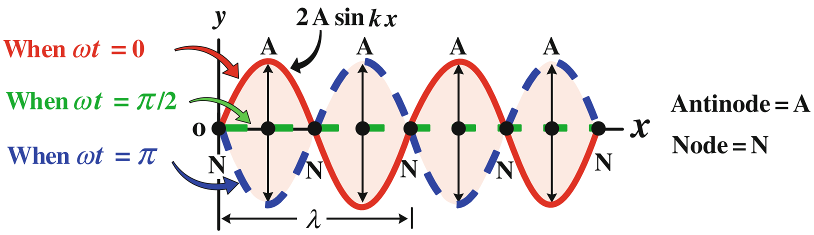


Fig. 14.21 The time dependence of the vertical displacement (from equilibrium) of any individual element in the standing wave y is governed by $\cos\omega t$. Each element vibrates within the confines of the envelope $2A\sin kx$. The nodes (N) are points of zero displacement, and the antinodes (A) are points of maximum displacement

When using $k = 2\pi/\lambda$, these values give $x = 0, \frac{\lambda}{2}, \lambda, \frac{3\lambda}{2}, \ldots$, that is:

$$x = 0,\ \frac{\lambda}{2},\ \lambda,\ \frac{3\lambda}{2}, \ldots = n\frac{\lambda}{2}, \quad (n = 0, 1, 2, \ldots) \quad \text{(Nodes)} \tag{14.63}$$

Also, a standing wave is distinguished by elements with the *greatest possible displacements* called *antinodes* (see Fig. 14.21). This happens when x satisfies the condition $\sin kx = \pm 1$, that is, when:

$$kx = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots$$

Also, using $k = 2\pi/\lambda$, these values give $x = \frac{\lambda}{4}, \frac{3\lambda}{4}, \frac{5\lambda}{4}, \ldots$, that is:

$$x = \frac{\lambda}{4}, \frac{3\lambda}{4}, \frac{5\lambda}{4}, \ldots = (n + \tfrac{1}{2})\frac{\lambda}{2}, \quad (n = 0, 1, 2, \ldots) \quad \text{(Antinodes)} \tag{14.64}$$

Equations 14.63 and 14.64 indicate the following general features of nodes and antinodes (see Fig. 14.21):

Spotlight
(1) The distance between adjacent nodes is $\lambda/2$.
(2) The distance between adjacent antinodes is $\lambda/2$.
(3) The distance between a node and adjacent antinode is $\lambda/4$.

At $t = 0$ $(\omega t = 0)$, the two oppositely traveling waves are in phase, producing a wave pattern in which each element of the medium is experiencing its maximum displacement from equilibrium, see Fig. 14.22a. At $t = T/4$, $(\omega t = \pi/2)$, the traveling waves have moved one quarter of a wavelength (one to the right and the other to the left). At this time, each element of the medium is passing through the equilibrium position in its simple harmonic motion. The result is zero displacement for each element at all values of x, see Fig. 14.22b. At $t = T/2$ ($\omega t = \pi$), the traveling waves are again in phase, producing a wave pattern that is inverted relative to the $t = 0$ pattern, see Fig. 14.22c. The pattern at $t = 3T/4$ (Fig. 14.22d) resembles that at $t = T/2$. Also, the pattern at $t = T$ (Fig. 14.22e) resembles that at $t = 0$.

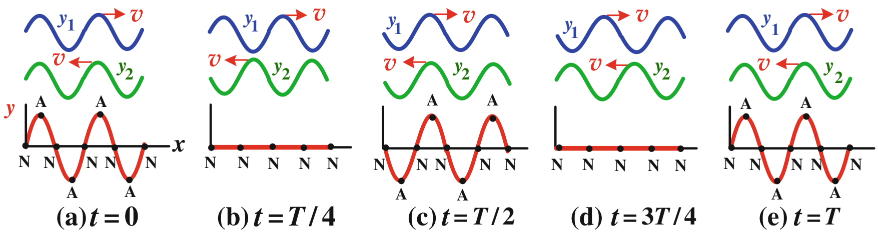


Fig. 14.22 Standing-wave patterns y at different times for the two oppositely traveling identical waves y_1 and y_2. Nodes (N) have no displacements while antinodes (A) have maximum displacements

Example 14.7

A standing wave is produced by two identical sinusoidal waves traveling in opposite directions in a taut string. The two waves are given by:

$$y_1 = (0.02\,\text{m})\sin(5x - 10t)$$
$$y_2 = (0.02\,\text{m})\sin(5x + 10t)$$

where x and y are in meters, t is in seconds, and the argument of the sine is in radians. (a) Find the amplitude of the simple harmonic motion of the element on the string located at $x = 10\,\text{cm}$. (b) Find the positions of the nodes and antinodes in the string. (c) Find the maximum and minimum y values of the simple harmonic motion of a string element located at any antinode.

Solution: (a) Equation 14.62 gives the standing wave produced from y_1 and y_2 with $A = 0.02\,\text{m}$, $k = 5\,\text{rad/m}$, and $\omega = 10\,\text{rad/s}$. Thus:

$$y = (2A\sin kx)\cos\omega t = [(0.04\,\text{m})\sin 5x]\cos 10\,t$$

The coefficient of the cosine at $x = 10\,\text{cm} = 0.1$ m will be:

$$y_{\max} = (0.04\,\text{m})\sin 5\,x|_{x=0.1} = (0.04\,\text{m})\sin(0.5\,\text{rad}) = 0.019\,\text{m} = 1.9\,\text{cm}$$

(b) When $k = 5\,\text{rad/m} = 2\pi/\lambda$, we find the wavelength to be $\lambda = 0.4\pi$ m. Therefore, from Eq. 14.63 we find the nodes to be located at:

$$x = n\frac{\lambda}{2} = (0.2\pi n)\ \text{m}, \quad (n = 0, 1, 2, \ldots)$$

From Eq. 14.64, the antinodes will be located at:

$$x = (n + \tfrac{1}{2})\frac{\lambda}{2} = [0.2\pi(n + \tfrac{1}{2})]\ \text{m}, \quad (n = 0, 1, 2, \ldots)$$

(c) The maximum and minimum y values of the simple harmonic motion of a string element located at any antinode are:

$$y_{\max/\min} = 2A\,(\sin 5\,x)|_{\max/\min} = 2A\,(\pm 1) = \pm\,0.04\,\text{m} = \pm\,0.04\,\text{m} = \pm\,4\,\text{cm}$$

14.7.1 Reflection at a Boundary

A wave moving along a stretched string can be reflected from one of its ends in two different ways, as shown in Fig. 14.23. The first way is to fix the far end of the string, and the second way is to allow the far end to move freely up and down.

When the incident pulse in Fig. 14.23a reaches the fixed end, it exerts an upward force on the wall through the support. By Newton's third law, the support at the wall exerts an opposite force on the string. This reaction force generates an inverted reflected pulse that travels in a direction opposite to the incident pulse. In a reflection of this kind, there must be no displacement of the string at the right end, which is referred to as a **node** at the support, because the string is fixed there.

In Fig. 14.23b, the right end of the string is tied to a weightless ring that is free to slide without friction along a vertical rod. When the incident pulse reaches the ring, the ring moves up along the rod. The ring rises as high as the incoming pulse, and then the downward component of the tension pulls the ring back down. This movement of the ring produces a non-inverted reflected pulse of the same amplitude as the incident pulse. In a reflection of this kind, there must be a maximum displacement of the string at the right end, which is referred to as an **antinode**, because the string is not fixed there.

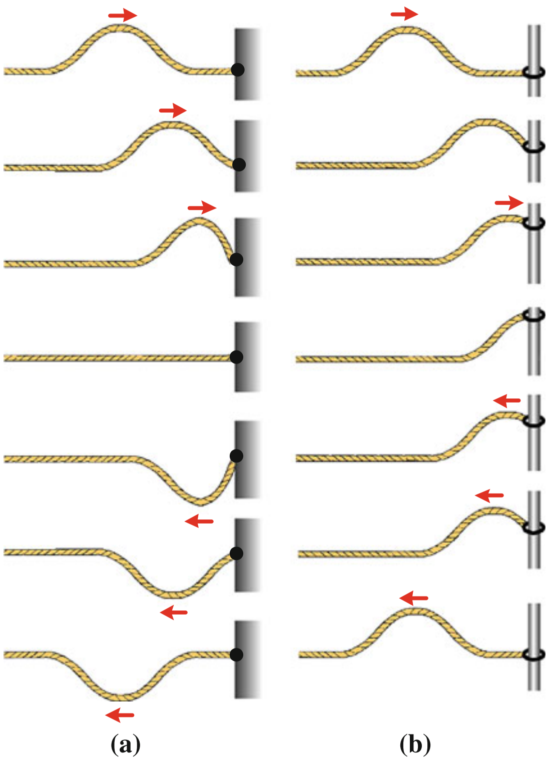


Fig. 14.23 (a) An incident pulse from the left is inversely reflected when the right side of the string is fixed to a wall. (b) The same incident pulse is reflected unchanged in sign when the right side of the string is tied to a ring that can slide without friction on a vertical rod

14.7.2 Standing Waves and Resonance

When one end of a stretched string is oscillating in a sinusoidal fashion while the other end is fixed, the incident wave and the reflected wave interfere with each other.

For certain frequencies, this interference produces a standing wave with nodes and antinodes like those shown in Figs. 14.21 and 14.22. Such a standing wave is said to be produced at **resonance**, and the string *resonates* at these **resonant frequencies**. If the string is oscillating at some other frequency, a standing wave is not set up.

Generally, an imposed boundary condition on a string sets up a number of natural patterns of oscillation called **normal modes**.

Consider a stretched string between two points separated by a distance L, see Fig. 14.24a. Visualize that the string is somehow made to oscillate at a resonance frequency to set up a specific standing wave pattern. Since both ends are fixed, then for this boundary condition there must be at least two nodes and one antinode for the standing wave pattern. The normal modes of oscillation for the string can be explained by considering the following three patterns:

(1) The first normal mode (the first harmonic, or the fundamental):
The simplest pattern that can meet the boundary condition of two fixed ends is shown in Fig. 14.24b. Note that there are two imposed nodes at both ends and only one antinode, which is at the center of the string. There is only half a wavelength in the length L. Thus, for this pattern, $\lambda_1/2 = L$, i.e. $\lambda_1 = 2\,L$.

(2) The second normal mode (the second harmonic):
The second pattern that can meet the boundary condition of two fixed ends is shown in Fig. 14.24c. This pattern has three nodes and two antinodes. This standing wave must have $\lambda_2 = L$.

(3) The third normal mode (the third harmonic):
The third pattern that can meet the boundary condition of two fixed ends is shown in Fig. 14.24d. This pattern has four nodes and three antinodes . This standing wave must have $\lambda_3 = 2L/3$.

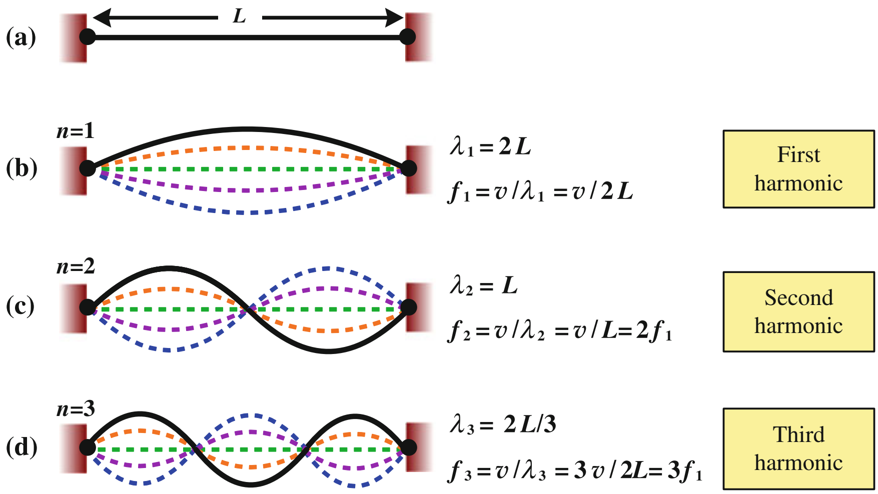


Fig. 14.24 (a) A string of length L that is fixed at both ends. The normal modes of vibration are shown for; (b) the first harmonic (or the fundamental), (c) the second harmonic, and (d) the third harmonic

In general, the relation between the wavelength λ of the various normal modes for a string of length L fixed at both ends is given by:

$$\lambda_n = \frac{2L}{n}, \quad (n = 1,\ 2,\ 3, \ldots) \quad \text{(String, fixed ends)} \tag{14.65}$$

where the index n refers to the n^{th} normal mode of the possible oscillation of the string (or the number of **loops** in the string).

The resonance frequencies associated with these modes are obtained from the relation $f = v/\lambda$, where the speed of the wave is the same for all the frequencies. Using Eq. 14.65, we find the resonance frequencies f_n of the normal modes to be (see Fig. 14.24):

$$f_n = \frac{v}{\lambda_n} = n\frac{v}{2L}, \quad (n = 1,\ 2,\ 3, \ldots) \quad \text{(String, fixed ends)} \tag{14.66}$$

According to Eq. 14.44, the speed of the wave v is related to the tension in the string τ and the linear mass density μ by the relation $v = \sqrt{\tau/\mu}$. Substituting with this relation into Eq. 14.66 we get:

$$f_n = \frac{n}{2L}\sqrt{\frac{\tau}{\mu}}, \quad (n = 1,\ 2,\ 3, \ldots) \quad \text{(String, fixed ends)} \tag{14.67}$$

The lowest resonance frequency f_1, which corresponds to $n = 1$, is called the **fundamental frequency** and is given by:

$$f_1 = \frac{1}{2L}\sqrt{\frac{\tau}{\mu}} \quad \text{(String, fixed ends)} \tag{14.68}$$

The resonance frequencies of the remaining normal modes are integer multiples of the fundamental frequency (Fig. 14.24), that is:

$$f_n = n f_1, \quad (n = 1,\ 2,\ 3, \ldots) \quad \text{(String, fixed ends)} \tag{14.69}$$

Example 14.8

The middle-C key on a piano (key No. 40) has a fundamental frequency of 262 Hz, and the A key above the middle C in frequency has a fundamental frequency of 440 Hz, see Fig. 14.25. (a) Find the frequencies of the next two harmonics of the C string. (b) The strings of the keys A and C have the same linear mass density but the length L_A of the string A is 65% of the length L_C of string C. What will be the ratio of the tensions τ_A/τ_C in the two strings?

Solution: (a) Equation 14.69 gives the higher harmonics in terms of the fundamental frequency. Thus, for $f_1 = 262$ Hz we get:

$$f_2 = 2f_1 = 2 \times 262\ \text{Hz} = 524\ \text{Hz}$$
$$f_3 = 3f_1 = 3 \times 262\ \text{Hz} = 786\ \text{Hz}$$

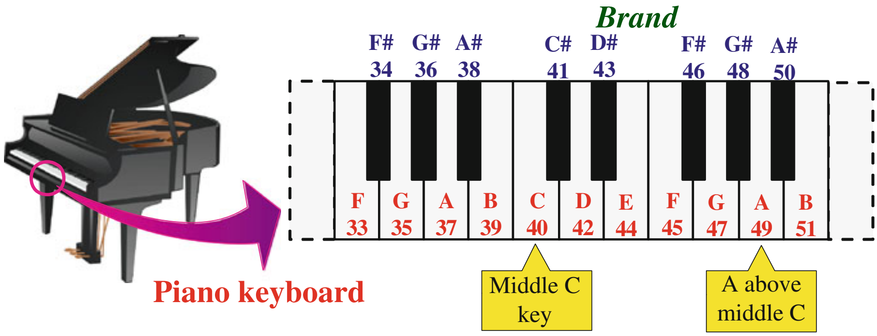


Fig. 14.25

(b) When the two strings vibrate at their fundamental frequencies, we can use Eq. 14.68 to write down the following relations:

$$f_{1A} = \frac{1}{2L_A}\sqrt{\frac{\tau_A}{\mu}} \quad \text{and} \quad f_{1C} = \frac{1}{2L_C}\sqrt{\frac{\tau_C}{\mu}}$$

Thus, the ratio of the two frequencies is $f_{1A}/f_{1C} = (L_C/L_A)\sqrt{\tau_A/\tau_C}$. When we square this relation, we get the ratio of the magnitude of the two tensions as follows:

$$\frac{\tau_A}{\tau_C} = \left(\frac{L_A}{L_C}\right)^2\left(\frac{f_{1A}}{f_{1C}}\right)^2 = \left(\frac{65}{100}\right)^2\left(\frac{440\ \text{Hz}}{262\ \text{Hz}}\right)^2 = 1.19$$

Example 14.9

The one end A of a string is attached to a vibrator of frequency 100 Hz, while the other end passes over a pulley at point B to a block of mass m, see Fig. 14.26. The separation L between A and B is 1.5 m and the linear mass density of the string is 1.5 g/m. (a) Find the mass m needed to allow the vibrator to set up the third harmonic on the string. (b) What standing-wave mode is set up if $m = 0.5$ kg?

Solution: (a) The tension τ in the string must equal to the weight of the mass m, i.e. $\tau = mg$. Substitution with this tension into Eq. 14.67 gives the resonance frequencies in a general form as follows:

$$f_n = \frac{n}{2L}\sqrt{\frac{mg}{\mu}}, \quad (n = 1,\ 2,\ 3, \ldots)$$

We need to set the tension in the string (by the mass m) so that the vibrator frequency is equal to the frequency of the third harmonic, i.e.:

$$f_3 = \frac{3}{2L}\sqrt{\frac{mg}{\mu}}$$

Thus:

$$m = \frac{4L^2\mu f_3^2}{9\,g} = \frac{4 \times (1.5\ \text{m})^2(1.5 \times 10^{-3}\ \text{kg/m})(100\ \text{Hz})^2}{9 \times (9.8\ \text{m/s}^2)} = 1.5306\ \text{kg}$$

(b) If we insert $m = 0.5$ kg and $f_n = 100$ Hz into the first equation, we get:

$$n = 2Lf_n\sqrt{\frac{\mu}{mg}} = 2 \times (1.5\ \text{m})(100\ \text{Hz})\sqrt{\frac{1.5 \times 10^{-3}\ \text{kg/m}}{(0.5\ \text{kg})(9.8\ \text{m/s}^2)}} = 5.25$$

With $m = 0.5$ kg, we get $n = 5.25$. Because n has to be an integer, then this vibrator cannot set up a standing wave on the string.

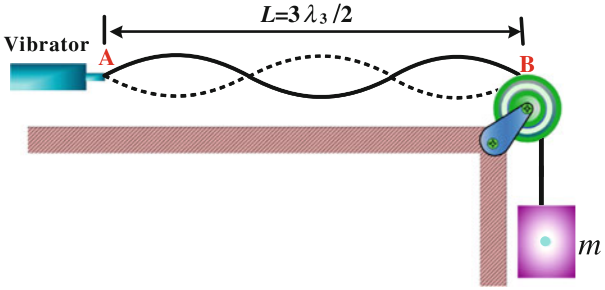


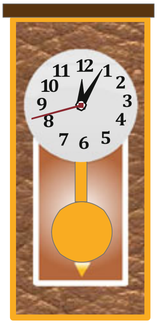
Fig. 14.26

14.8 Exercises

Section 14.1 Simple Harmonic Motion

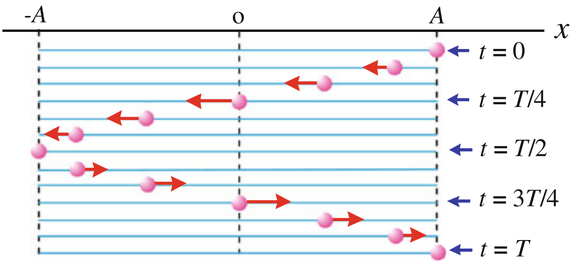
(1) Some clocks use a pendulum to keep time, see Fig. 14.27. The bob of a clock requires 1 s for a single small-amplitude swing. (a) What is the period of the pendulum? (b) What is frequency of the pendulum? (c) What is the angular frequency of the pendulum's oscillations?

Fig. 14.27 See Exercise (1)



(2) A particle executes a simple harmonic motion along the x-axis with amplitude A. The particle returns to its starting position every $T = 0.25$ s, see Fig. 14.28. (a) Find the period, frequency, and angular frequency of this motion. (b) Find the particle's displacement as a function of time.

Fig. 14.28 See Exercise (2)



(3) A particle oscillates with a simple harmonic motion along the x axis. Its displacement from the origin varies with time according to the equation $x = (1.5\text{ m})\cos(2\pi\, t + \phi)$, where $\phi = -\pi/4$ rad, t is in seconds and the argument of the cosine is in radians, see the blue curve of Fig. 14.29. (a) Find the value of the amplitude, frequency, and period of the motion. (b) Find the velocity and acceleration of the particle as a function of time. (c) Find both the maximum speed and acceleration of the particle. (d) Find the displacement of the particle between $t = 0$ and $t = 1$ s.

(4) When the mechanical energy of one oscillation of a spring-block system is doubled, what is the ratio of their amplitudes?

(5) A block of mass $m = 0.8$ kg oscillates freely with period $T = 0.9$ s when attached to a linear spring that obeys Hooke's law, see Fig. 14.30. An unknown mass M attached to the same spring is observed to have a period of oscillation

of 1.2 s. (a) Find the spring constant k_H of the spring. (b) Find the value of the unknown mass M.

Fig. 14.29 See Exercise (3)

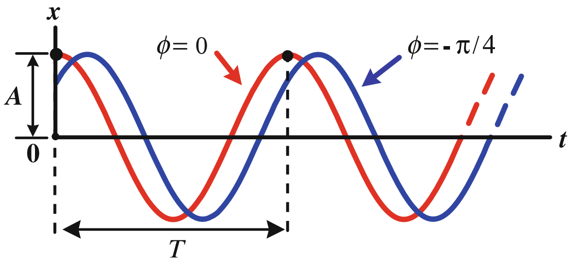
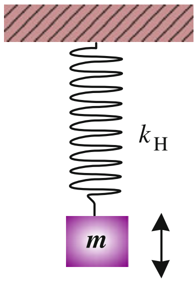


Fig. 14.30 See Exercise (5)



(6) A block of mass $m = 0.5\,\text{kg}$ rests on a horizontal frictionless surface and is connected to a spring, as shown in Fig. 14.31. When the system is set into motion with amplitude $A = 0.35\,\text{m}$, it repeats its motion every 0.5 s. (a) Find the block's period, frequency, and angular frequency. (b) Find the spring constant, the maximum speed of the block, and the maximum force exerted by the spring on the block.

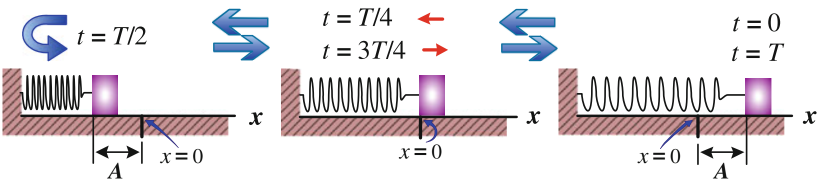


Fig. 14.31 See Exercise (6)

(7) Two springs 1 and 2 have the same un-stretched length but different force constants $k_{H_1} \equiv k_1$ and $k_{H_2} \equiv k_2$, respectively. The springs are connected to a

block of mass m that rests on a horizontal frictionless surface as shown in Fig. 14.32. Calculate the effective force constant k_{eff} in each of the three cases (a), (b), and (c) of the figure.

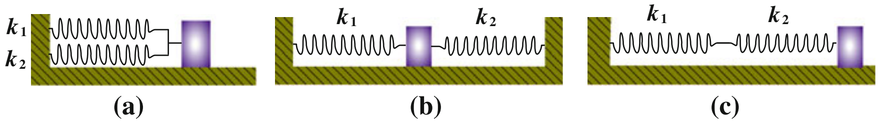


Fig. 14.32 See Exercise (7)

(8) When $k_1 = k_2 = k$ in Exercise 7, find the frequency of oscillation of the block in each of the three cases (a), (b), and (c).

(9) When a group of four persons, each of mass 60 kg, steps into a small car of mass 936 kg, the four springs of the car are compressed by 4 cm. Take $g = 10\,\text{m/s}^2$. (a) What is the effective spring constant k_H of the springs? (b) Find the period and frequency of the car after hitting a road bump that causes the car to oscillate up and down, assuming the oscillations of the four springs are in phase.

(10) In Exercise 7, show that the frequencies f of oscillation of the block in the two cases (b) and (c) are given respectively by:

$$f = \sqrt{f_1^2 + f_2^2}, \quad f = \sqrt{f_1^2 f_2^2/(f_1^2 + f_2^2)}$$

where f_1 and f_2 are the frequencies when the block is connected to only spring 1 or spring 2, respectively.

(11) The velocity of a 0.5 kg mass attached to the end of a spring is represented by $v = -(4\,\text{m/s})\sin(2\,t)$. Find the total energy E.

(12) A block of mass $m = 0.2$ kg is fastened to a light spring whose spring constant k_H is 5 N/m, see part (a) of Fig. 14.33. The block is pulled a distance $x_i = 5$ cm from its equilibrium position at $x = 0$ on a horizontal frictionless surface, see part (b) of Fig. 14.33, and then released at $t = 0$. (a) What is the mechanical energy of the oscillator? (b) What is the maximum speed of the oscillator? (c) Find the speed, kinetic energy, and potential energy of the block when its position is 2 cm.

(13) Assume that the mass in Exercise 12 is 0.025 kg, the force constant k_H is 0.4 N/m, and that the motion starts by imparting to the block at $x_i = 0.1$m a velocity toward the right of 0.4 m/s. (a) Find the period T, frequency f, and angular frequency ω of the oscillator. (b) Find the total energy E, amplitude A,

the phase angle ϕ, the maximum speed $v_{\max}$, and the maximum acceleration $a_{\max}$. (c) Write down the position, velocity, and acceleration in terms of time t, then substitute with $t = \pi/8$ s and find their values.

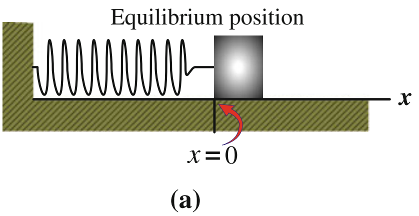


Fig. 14.33 See Exercise (12)

(14) A bullet of mass $m = 10$ g is fired horizontally with a speed v into a stationary wooden block of mass $M = 4$ kg. The block is resting on a horizontal smooth surface and attached to a massless spring with spring constant $k_H = 150$ N/m, where the other end of the spring is fixed through a wall, as shown in Fig. 14.34a. In a very short time, the bullet penetrates the block and remains embedded before compressing the spring, as shown in Fig. 14.34b. The maximum distance that the block compresses the spring is 8 cm, as shown in Fig. 14.34c. (a) What is the speed of the bullet? (b) Find the period T and frequency f of the oscillating system.

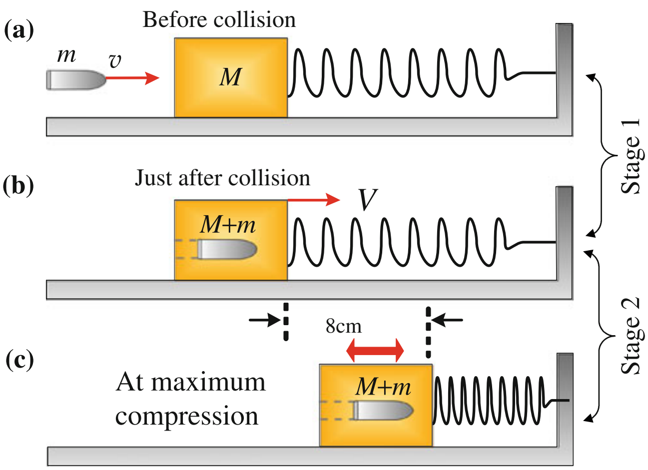


Fig. 14.34 See Exercise (14)

Section 14.2 Damped Simple Harmonic Motion

(15) An object of mass $m = 0.25$kg oscillates in a fluid at the end of a vertical spring of spring constant $k_H = 85$ N/m, see Fig. 14.35. The effect of the fluid resistance is governed by the damping constant $b = 0.07$kg/s. (a) Find the period of the damped oscillation. (b) By what percentage does the amplitude of the oscillation decrease in each cycle? (c) How long does it take for the amplitude of the damped oscillation to drop to half of its initial value?

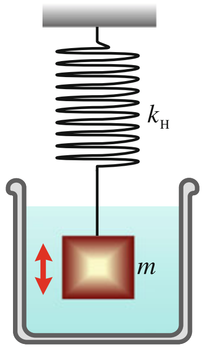


Fig. 14.35 See Exercise (15)

(16) A simple pendulum has a length L and a mass m. Let the arc length s and the angle θ measure the position of m at any time t, see Fig. 14.36. (a) When a damped force $F_d = -bv_s$ exists, show that the equation of motion of the pendulum is given for small angles by:

$$m\frac{d^2\theta}{dt^2} + b\frac{d\theta}{dt} + \frac{mg}{L}\theta = 0$$

(b) By comparison with Eq. 14.25, show that the above differential equation has a solution given by:

$$\theta = \theta_\circ\, e^{-bt/2m} \cos(\omega_d\, t), \quad \omega_d = \sqrt{\frac{g}{L} - \frac{b^2}{4m^2}}$$

where $\theta_\circ$ is initial angular amplitude at $t = 0$ and $\omega_d = 2\pi f_d$ is damped angular frequency, see Fig. 14.9b. (c) When the pendulum has $L = 1$ m, $m = 0.1$ kg, and the angular amplitude θ becomes 0.5 $\theta_\circ$ after 1 minute, find the damping constant b and the ratio $(f - f_d)/f$, where f is the undamped frequency.

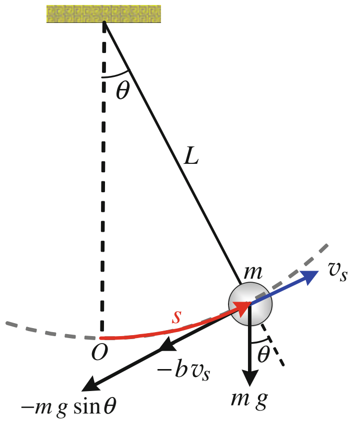


Fig. 14.36 See Exercise (16)

Section 14.3 Sinusoidal Waves

(17) Given a sinusoidal wave represented by $y = (0.2\,\text{m})\sin(k\,x - \omega\,t)$, where $k = 4$ rad/m, and $\omega = 8$ rad/s, determine the amplitude, wavelength, frequency, and speed of this wave.

(18) A harmonic wave traveling along a string has the form $y = (0.25\,\text{m})\sin(3\,x - 40\,t)$, where x is in meters and t is in seconds. (a) Find the amplitude, wave number, angular frequency, and speed of this wave. (b) Find the wavelength, period, and frequency of this wave?

Section 14.4 The Speed of Waves on Strings

(19) A uniform string has a mass per unit length of 5×10^{-3} kg/m. The string passes over a massless, frictionless pulley to a block of mass $m = 135$ kg, see Fig. 14.37, and take $g = 10\,\text{m/s}^2$. Find the speed of a pulse that is sent from one end of the string toward the pulley. Does the value of the speed change when the pulse is replaced by a sinusoidal wave?

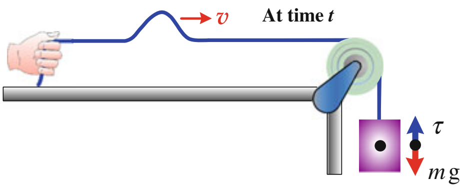


Fig. 14.37 See Exercise (19)

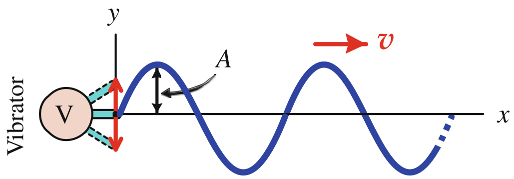
(20) Assume a transverse wave traveling on a uniform taut string of mass per unit length $\mu = 4 \times 10^{-3}$ kg/m. The wave has an amplitude of 5 cm, frequency of 50 Hz, and speed of 20 m/s. (a) Write an equation in SI units of the form $y = A\sin(kx - \omega t)$ for this wave. (b) Find the magnitude of the tension in the string.

Section 14.5 Energy Transfer by Sinusoidal Waves on Strings

(21) A sinusoidal wave of amplitude 0.05 m is transmitted along a string that has a linear density of 40 g/m and is under 100 N of tension. If the wave source has a maximum power of 300 W, what is the highest frequency at which the source can operate?

(22) A long string has a mass per unit length μ of 125 g/m and is taut under tension τ of 32 N. A wave is supplied by a generator as shown in Fig. 14.38. This wave travels along the string with a frequency f of 100 Hz and amplitude A of 2 cm. (a) Find the speed and the angular frequency of the wave. (b) What is the rate of energy that must be supplied by a generator to produce this wave in the string? (c) If the string is to transfer energy at a rate of 100 W, what must be the required wave amplitude when all other parameters remain the same?

Fig. 14.38 See Exercise (22)



(23) A sinusoidal wave is traveling along a string of linear mass density $\mu = 75$ g/m and is described by the equation:

$$y = (0.25 \text{ m})\sin(2x - 40t)$$

where x is in meters and t in seconds. (a) Find the speed, wavelength, and frequency of the wave. (b) Find the power transmitted by the wave.

Section 14.6 The Linear Wave Equation

(24) A one-dimensional wave traveling with velocity v is found to satisfy the partial differential equation [see Eq. 14.58]:

$$\frac{\partial^2 y}{\partial x^2} - \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2} = 0$$

Show that the following functions are the solutions to this linear wave equation: (a) $y = A\sin(k\,x - \omega\,t)$. (b) $y = A\cos(k\,x - \omega\,t)$. (c) $y = \exp[b(x - v\,t)]$, where b is a constant. (d) $y = \ln[b(x - v\,t)]$, where b is a constant. (e) Any function y having the form $y = f(x - v\,t)$.

(25) If the linear wave functions $y_1 = f_1(x, t)$ and $y_2 = f_2(x, t)$ satisfy the wave Eq. 14.58, then show that the combination $y = C_1\,f_1(x, t) + C_2\,f_2(x, t)$ also satisfies the same equation, where C_1 and C_2 are constants.

Section 14.7 Standing Waves

(26) A standing wave having a frequency of 20 Hz is established on a rope 1.5 m long that has fixed ends. Its wavelength is observed to be twice the rope's length. Determine the wave's speed.

(27) A stretched string of length 0.6 m and mass 30 g is observed to vibrate with a fundamental frequency of 30 Hz. The amplitude of any antinodes in the standing wave is 0.04 m. (a) What is the amplitude of a transverse wave in the string? (b) What is the speed of a transverse wave in the string? (c) Find the magnitude of the tension in the string.

(28) A student wants to establish a standing wave with a speed 200 m/s on a string that is fixed at both ends and is 2.5 m long. (a) What is the minimum frequency that should be applied? (b) Find the next three frequencies that cause standing wave patterns on the string.

(29) Two identical waves traveling in opposite directions in a string interfere to produce a standing wave of the form:

$$y = [(2\,\mathrm{m})\sin(2\,x)]\cos(20\,t)$$

where x is in centimeters, t is in seconds, and the arguments of the sine and cosine are in radians. Find the amplitude, wavelength, frequency, and speed of the interfering waves.

(30) A standing wave is produced by two identical sinusoidal waves traveling in opposite directions in a taut string. The two waves are given by:

$$y_1 = (2\,\text{cm})\sin(2.3\,x - 4\,t) \quad \text{and} \quad y_2 = (2\,\text{cm})\sin(2.3\,x + 4\,t)$$

where x and y are in centimeters, t is in seconds, and the argument of the sine is in radians. (a) Find the amplitude of the simple harmonic motion of an element on the string located at $x = 3$ cm. (b) Find the position of the nodes and antinodes on the string. (c) Find the maximum and minimum y values of the simple harmonic motion of a string element located at any antinode.

(31) A guitar string has a length $L = 64$ cm and fundamental frequency $f_1 = 330$ Hz, see part (a) of Fig. 14.39. By pressing down with your finger on the string, it is found that the string is shortened in a way so that it plays an F note with a fundamental frequency $f_1' = 350$ Hz, see part (b) of Fig. 14.39. [Assume the speed of the wave remains constant before and after pressing] How far is your finger from the near end of the string?

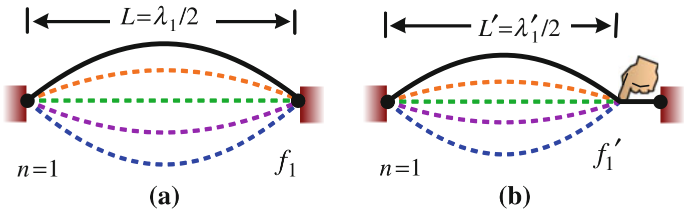


Fig. 14.39 See Exercise (31)

(32) A violin string oscillates at a fundamental frequency of 262 Hz when unfingered. At what frequency will it vibrate if it is fingered two-fifths of the length from its end?

(33) A string that has a length $L = 1$ m, mass per unit length $\mu = 0.1$ kg/m, and tension $\tau = 250$ N is vibrating at its fundamental frequency. What effect on the fundamental frequency occurs when only: (a) The length of the spring is doubled. (b) The mass per unit length of the spring is doubled. (c) The tension of the spring is doubled.

(34) Show that the resonance frequency f_n of standing waves on a string of length L and linear density μ, which is under a tensional force of magnitude τ, is given by $f_n = n\sqrt{\tau/\mu}/2L$, where n is an integer.

(35) Show by direct substitution that the standing wave given by Eq. 14.62,

$$y = (2\,A \sin k\,x)\ \cos\omega\,t$$

is a solution of the general linear wave Eq. 14.58:

$$\frac{\partial^2 y}{\partial x^2} - \frac{1}{v^2}\frac{\partial^2 y}{\partial\, t^2} = 0$$

(36) End A of a string is attached to a vibrator that vibrates with a constant frequency f, while the other end B passes over a pulley to a block of mass m, see Fig. 14.40. The separation L between points A and B is 2.5 m and the linear mass density of the string is 0.1 kg/m. When the mass m of the block is either 16 or 25 kg, standing waves are observed; however, standing waves are not observed for masses between these two values. Take $g = 10\,\text{m/s}^2$. (Hint: The greater the tension in the string, the smaller the number of nodes in the standing wave) (a) What is the frequency of the vibrator? (b) Find the largest m at which a standing wave could be observed.

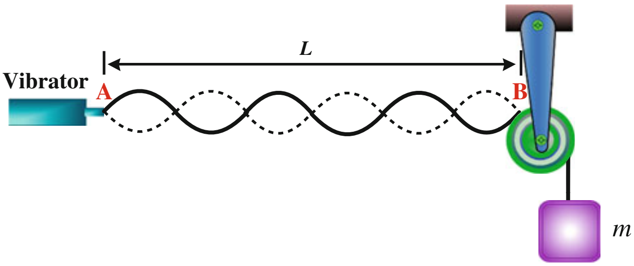


Fig. 14.40 See Exercise (36)

(37) Two identical sinusoidal waves traveling in opposite directions on a string of length $L = 3$ m interfere to produce a standing wave pattern of the form:

$$y = [(0.2\,\text{m})\sin(2\pi x)]\ \cos(20\pi\,t)$$

where x is in meters, t in seconds, and the arguments of the sine and cosine are in radians. (a) How many loops are there in this pattern? (b) What is the fundamental frequency of vibration of the string?

(38) Two strings 1 and 2, each of length $L = 0.5$ m, but different mass densities μ_1 and μ_2, are joined together with a knot and then stretched between two

fixed walls as shown in Fig. 14.41. For a particular frequency, a standing wave is established with a node at the knot, as shown in the figure. (a) What is the relation between the two mass densities? (b) Answer part (a) when the frequency is changed so that the next harmonic in each string is established.

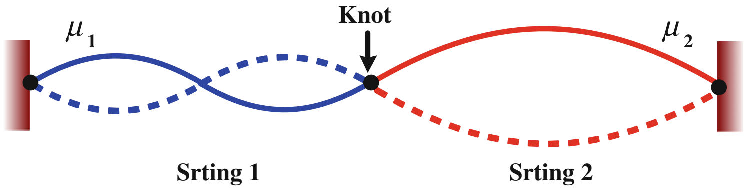


Fig. 14.41 See Exercise (38)

(39) The strings 1 and 2 of exercise 38 have $L_1 = 0.64$ m, $\mu_1 = 1.8$ g/m, $L_2 = 0.8$ m, and $\mu_2 = 7.2$ g/m, respectively, and both are held at a uniform tension $\tau = 115.2$ N. Find the smallest number of loops in each string and the corresponding standing wave frequency.

(40) In the case of the smallest number of loops in exercise 39, determine the total number of nodes and the position of the nodes measured from the left end of string 1.