Rotational Motion

8

In this chapter, we first treat the rotation of an extended object about a fixed axis. This is commonly known as pure rotational motion. The analysis is greatly simplified when the object is rigid. To perform this analysis, we first ignore the cause of rotation and describe the rotational motion in terms of angular variables and time. This is known as ***rotational kinematics***. We then discuss the causes of rotation. This is known as ***rotational dynamics***, and through the study of this topic we introduce the concept of ***torque***. After that we treat some general cases where the axis of rotation is not fixed in space. In these cases, rigid bodies can undergo both rotational and translational motion, as in the rolling of objects.

8.1 Radian Measures

One **radian** (1 rad) is the angle subtended at the center of a circle of radius r by an arc of length s equal to the radius of the circle, i.e. $s = r$, see Fig. 8.1a. Since the circumference of a circle of radius r is $s = 2\pi r$, where $\pi \simeq 3.14$, then 360° (or one revolution) corresponds to an angle of $(2\pi r)/r = 2\pi$ rad, see also Appendix B. Thus:

$$1\,\text{rev} = 360^\circ = 2\pi\ \text{rad} \quad \Rightarrow \quad 180^\circ = \pi\ \text{rad} \tag{8.1}$$

Therefore:

$$1^\circ = (\pi/180)\ \text{rad} \simeq 0.02\,\text{rad}$$
$$1\,\text{rad} = 180^\circ/\pi \simeq 57.3^\circ.$$

H. A. Radi and J. O. Rasmussen, *Principles of Physics*,
Undergraduate Lecture Notes in Physics, DOI: 10.1007/978-3-642-23026-4_8,

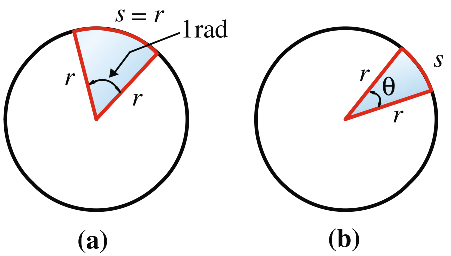


Fig. 8.1 (a) The definition of one radian (1 rad). (b) The definition of angle θ as the ratio of the arc length s to the radius r

Generally, if θ (in radians) represents any arbitrary angle subtended by an arc of length s on the circumference of a circle of radius r, see Fig. 8.1b, then the following relation must be satisfied:

$$\theta = \frac{s}{r} \quad \text{(Radian measure)} \tag{8.2}$$

8.2 Rotational Kinematics; Angular Quantities

Angular Position

The rotational motion of a rigid body (or a particle) about an axis is completely specified by an angle θ that a fixed line in the rigid body (or the particle) makes with some reference fixed line in the space, usually chosen as the x-axis. Additionally, the rotational motion is greatly simplified if θ is expressed in radians. This angle θ is defined as the **angular position** of the rigid body (or the particle).

Figure 8.2 represents a rigid body that is rotating about a fixed axis passing through point O, where that axis is perpendicular to the plane of the figure. Line OP is fixed in the body and completely specified at time t by the angular position θ which the line OP makes with respect to the x-axis. Therefore, the angular position of the rigid body, or the particle at point P which has polar coordinates (r, θ), is:

$$\text{Angular position} = \theta \tag{8.3}$$

Angular Displacement

When the rigid body rotates as shown in Fig. 8.3, the angular position of the line OP changes from θ_1 at time t_1 to θ_2 at a later time t_2. The quantity $\Delta\theta = \theta_2 - \theta_1$ is defined as the **angular displacement** of the a rigid body:

$$\text{Angular displacement} \equiv \Delta\theta = \theta_2 - \theta_1 \tag{8.4}$$

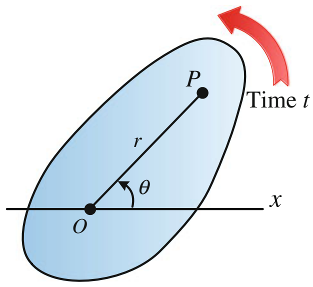


Fig. 8.2 The definition of the angular position θ of a rigid body, or a particle at point P with polar coordinates (r, θ), with respect to the x-axis

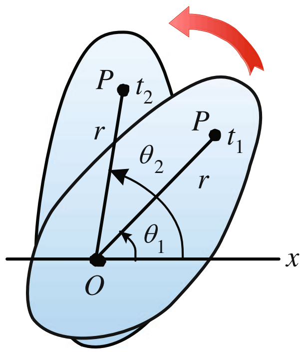


Fig. 8.3 The angular displacement $\Delta\theta = \theta_2 - \theta_1$ of a rigid body that occurs during the time interval $\Delta t = t_2 - t_1$

Angular velocity

In analogy to the average linear (translational) speed, we define the **average angular speed** $\overline{\omega}$ as the rate of change of angular displacement (ω is the lowercase Greek letter omega). That is:

$$\overline{\omega} = \frac{\Delta\theta}{\Delta t} = \frac{\theta_2 - \theta_1}{t_2 - t_1} \tag{8.5}$$

The **instantaneous angular velocity** ω is defined as the limiting value of the ratio $\Delta\theta/\Delta t$ when Δt approaches zero. Thus:

$$\omega = \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t} = \frac{d\theta}{dt} \qquad \equiv \qquad \theta_f - \theta_i = \int_{t_i}^{t_f} \omega \, dt \tag{8.6}$$

In SI units the angular velocity has the unit of radians per second (rad/s) and can be written as second^{-1} (s^{-1}) because radians are not dimensional quantities.

The last two equations hold for every point on the rigid body. That is, all points of the rigid body rotate through the same angular displacement in the same time. As in linear motion in one-dimension (where the linear velocity can be positive or negative), we take ω to be positive if θ increases (in a counterclockwise sense) and ω to be negative if θ decreases (in a clockwise sense).

Angular Acceleration

When the angular velocity of the rotating body is not constant, the body has an angular acceleration. Assume that ω_1 and ω_2 are the angular velocities at times t_1 and t_2, respectively, as shown in Fig. 8.4. Then we define the **average angular acceleration** $\overline{\alpha}$ (Greek alpha "α") as the rate of change of angular velocity as follows:

$$\overline{\alpha} = \frac{\Delta\omega}{\Delta t} = \frac{\omega_2 - \omega_1}{t_2 - t_1} \tag{8.7}$$

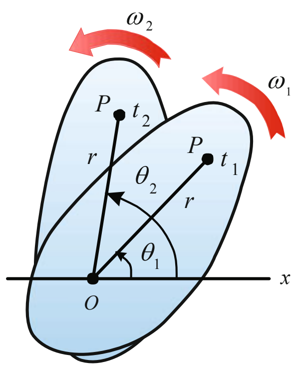


Fig. 8.4 The change in the angular velocity $\Delta\omega = \omega_2 - \omega_1$ of a rigid body which occurs during the time interval $\Delta t = t_2 - t_1$

Therefore, the **instantaneous angular acceleration** α is defined as the limiting value of the ratio $\Delta\omega/\Delta t$ when Δt approaches zero. Thus:

$$\alpha = \lim_{\Delta t \to 0} \frac{\Delta\omega}{\Delta t} = \frac{d\omega}{dt} \quad \equiv \quad \omega_f - \omega_i = \int_{t_i}^{t_f} \alpha \, dt \tag{8.8}$$

Angular acceleration has the unit of radians per second square (rad/s^2) and can be written as second^{-2} (s^{-2}).

Example 8.1

A reference line in a spinning disk has an angular position given by $\theta = 3\,t^2 - 12\,t + 9$, where θ is in radians and t is in seconds. (a) Find ω and α as a function of time. (b) Find the times when the angular position θ and the angular velocity ω become zero. (c) Describe the rotational motion of the disk for $t \geq 0$.

Solution: (a) To find ω, we differentiate θ with respect to time:

$$\omega = \frac{d\theta}{dt} = \frac{d}{dt}(3\,t^2 - 12\,t + 9) = (6\,t - 12)\,\text{rad/s}$$

Thus, ω could be negative or positive depending on t. To find the angular acceleration α, we differentiate ω with respect to time:

$$\alpha = \frac{d\omega}{dt} = \frac{d}{dt}(6\,t - 12) = 6\,\text{rad/s}^2$$

(b) Setting $\theta = 0$, we get:

$$3\,t^2 - 12\,t + 9 = 0 \quad \Rightarrow \quad t = \frac{12 \pm \sqrt{12^2 - 4 \times 3 \times 9}}{2 \times 3} \quad \Rightarrow \quad t = 1\,\text{s and } t = 3\,\text{s}$$

Thus, θ reaches zero at both $t = 1$ s and $t = 3$ s. Setting $\omega = 0$ gives:

$$6\,t - 12 = 0 \quad \Rightarrow \quad t = 2\,\text{s (when } \omega = 0)$$

(c) We can describe the rotation as follows:

- At $t = 0$ the reference line is at $\theta = 9$ rad and the disk's initial angular velocity is $\omega = -12$ rad/s.
- As time increases during the interval $0 < t < 2$ s, we have $\omega < 0$. That is, the disk is rotating in the clockwise sense, but with a decreasing angular speed, since $\alpha > 0$. In addition, θ reaches the value $\theta = 0$ at $t = 1$ s, and then attains negative values.
- At $t = 2$ s, the disk stops momentarily when $\theta = -3$ rad.
- As time increases during the interval $t > 2$ s, we have $\omega > 0$. In addition, θ goes back to zero again when $t = 3$ s. Afterward, both ω and θ will increase indefinitely.

8.3 Constant Angular Acceleration

The definitions of angular quantities are similar to those of linear quantities, except that θ, ω, and α replace the linear variables x, v, and a, respectively. Therefore, the angular equations for **constant angular acceleration** will be analogous to those presented in Chap. 3 and can be derived in exactly the same way, see Table 3.1. Table 8.1 summarizes the angular kinematic equations and their linear equivalents.

Table 8.1 Equations for motion with constant linear and angular accelerations

Linear equations		Angular equations
$v = v_\circ + a\,t$	$x \Leftrightarrow \theta$	$\omega = \omega_\circ + \alpha\,t$
$x - x_\circ = \frac{1}{2}(v_\circ + v)\,t$	$v \Leftrightarrow \omega$	$\theta - \theta_\circ = \frac{1}{2}(\omega_\circ + \omega)\,t$
$x - x_\circ = v_\circ\,t + \frac{1}{2}\,a\,t^2$	$a \Leftrightarrow \alpha$	$\theta - \theta_\circ = \omega_\circ\,t + \frac{1}{2}\,\alpha\,t^2$
$v^2 = v_\circ^2 + 2\,a\,(x - x_\circ)$		$\omega^2 = \omega_\circ^2 + 2\,\alpha\,(\theta - \theta_\circ)$

Example 8.2

A wheel accelerates uniformly from rest to an angular speed of 25 rad/s in 10 s. (a) Find the angular acceleration of the wheel. (b) Find the tangential and radial acceleration of a point 10 cm from the wheel's center. (c) How many revolutions has the wheel turned during this time interval? (d) Then, find the wheel's angular deceleration if it comes to a full stop after 5 rev.

Solution: (a) We are given $\omega_\circ = 0$, $\omega = 25$ rad/s, and $t = 10$ s. To find the angular acceleration a, we can use $\omega = \omega_\circ + \alpha\,t$ as follows:

$$\alpha = \frac{\omega - \omega_\circ}{t} = \frac{25\text{ rad/s} - 0}{10\text{ s}} = 2.5\text{ rad/s}^2$$

(b) Using Eqs. 8.13 and 8.14 (See Sect. 8.5), we get:

$$a_t = r\,\alpha = (10 \times 10^{-2}\text{ m})(2.5\text{ rad/s}^2) = 0.25\text{ m/s}^2$$

$$a_r = r\,\omega^2 = (10 \times 10^{-2}\text{ m})(25\text{ rad/s})^2 = 62.5\text{ m/s}^2$$

(c) If we assume that the wheel starts from $\theta_\circ = 0$, then we are given $\omega_\circ = 0$, $\omega = 25$ rad/s, $\theta_\circ = 0$, and $t = 10$ s. To find θ, which in this case equals the angle traveled by a certain reference line in the wheel, we use $\theta - \theta_\circ = \frac{1}{2}(\omega_\circ + \omega)\,t$ as follows:

$$\theta = \theta_\circ + \tfrac{1}{2}(\omega_\circ + \omega)\, t = 0 + \tfrac{1}{2}(0 + 25\,\text{rad/s}) \times 10\,\text{s} = 125\,\text{rad}$$

Thus: $$\theta = 125\,\text{rad} \times \left(\frac{1\,\text{rev}}{2\pi\,\text{rad}}\right) \simeq 20\,\text{rev}$$

(d) We are given $\omega_\circ = 25$ rad/s, $\omega = 0$, and, $\theta - \theta_\circ = 5\,\text{rev} = 10\pi$ rad. To find the angular deceleration α, we use $\omega^2 = \omega_\circ^2 + 2\,\alpha\,(\theta - \theta_\circ)$ to get:

$$\alpha = \frac{\omega^2 - \omega_\circ^2}{2\,(\theta - \theta_\circ)} = \frac{0 - (25\,\text{rad/s})^2}{2\ \times 10\pi\,\text{rad}} = -9.95\,\text{rad/s}^2$$

8.4 Angular Vectors

We can treat angular velocity as a vector by choosing the axis of rotation to be the direction of the angular-velocity vector. By convention, the right-hand rule is used to determine $\vec{\omega}$. To apply this rule, we curl the four fingers of the right hand around the rotation axis and point in the direction of rotation; then the thumb would point in the direction of $\vec{\omega}$. The angular acceleration vector $\vec{\alpha} = d\vec{\omega}/dt$ will be along $\vec{\omega}$ if $|\omega|$ increases with time, and will be opposite to $\vec{\omega}$ if $|\omega|$ decreases with time, see Fig. 8.5.

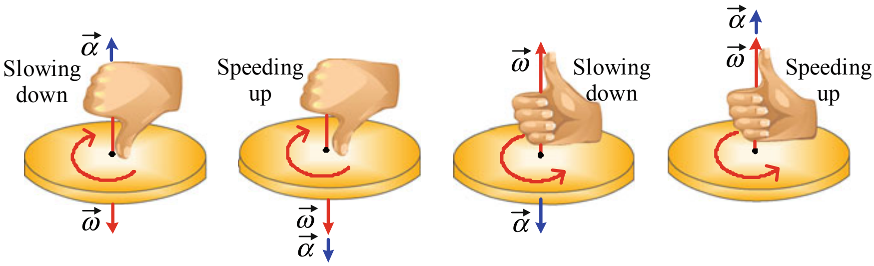


Fig. 8.5 Using the right-hand rule to obtain the direction of the vectors ω and α in cases of increasing and decreasing ω

8.5 Relating Angular and Linear Quantities

When a rigid body rotates with angular velocity ω, every point on the body moves in a circle with its center on the rotational axis, see Fig. 8.6. Because point P in the figure moves in a circle of radius r, this point defines a linear vector $\vec{v}$ whose direction is always tangent to its circular path. This tangential velocity has a magnitude defined by the tangential speed $v = ds/dt$, where s is the arc length traveled by point P along the circular path. Recalling from Eq. 8.2 that $s = r\,\theta$ and noting that r is constant, we find:

$$v = \frac{ds}{dt} = r\frac{d\theta}{dt} \tag{8.9}$$

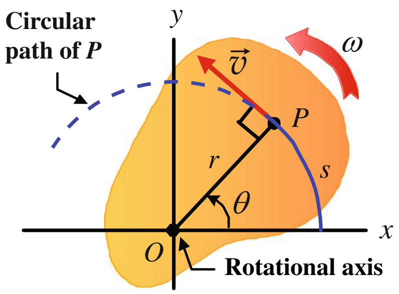


Fig. 8.6 The Point P on a rotating rigid body has a tangential velocity $\vec{v}$ which is always tangent to the circular path of this point

Using $\omega = d\theta/dt$, see Eq. 8.6, we get:

$$\boxed{v = r\,\omega} \quad \text{(Radian measure)} \tag{8.10}$$

This relation indicates that, although ω is the same for every point on the rigid body, points on the object with different radii have different tangential speeds v. In fact, the tangential speed v increases as one moves outward from the center of rotation.

When the angular speed ω is constant, then the linear speed v of any point on the rigid body is constant and hence undergoes uniform circular motion. The period of one revolution $T = 2\pi r/v$ and the frequency f (rev/s or Hz) can be written in terms of ω as follows:

$$T = \frac{2\pi}{\omega}, \quad f = \frac{1}{T} = \frac{\omega}{2\pi} \quad \text{(Radian measure)} \tag{8.11}$$

We can find the magnitude of the tangential acceleration a_t of point P by differentiating Eq. 8.10 with respect to time as follows:

$$a_t = \frac{dv}{dt} = r\frac{d\omega}{dt} \tag{8.12}$$

Using $\alpha = d\omega/dt$, see Eq. 8.8, we get:

$$\boxed{a_t = r\,\alpha} \quad \text{(Radian measure)} \tag{8.13}$$

In addition, we know that a point (or a particle) moving in a circular path of radius r with speed v undergoes a radial acceleration $\vec{a}_r$ of magnitude $a_r = v^2/r$ directed toward the axis of rotation. Thus, by using $v = r\,\omega$, the magnitude of the radial acceleration becomes:

$$a_r = r\,\omega^2 \quad \text{(Radian measure)} \tag{8.14}$$

As shown in Fig. 8.7, the total linear acceleration $\vec{a}$ at point P is:

$$\vec{a} = \vec{a}_t + \vec{a}_r \tag{8.15}$$

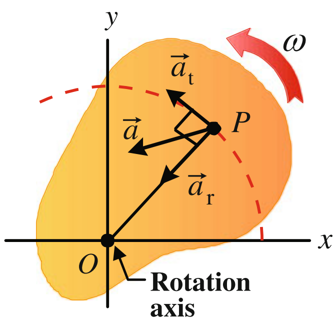


Fig. 8.7 The total linear acceleration $\vec{a}$ of point P on a rotating rigid body has two perpendicular components, the tangential component a_t and the radial component a_r

The magnitude of this acceleration is thus:

$$a = \sqrt{a_t^2 + a_r^2} = \sqrt{r^2\alpha^2 + r^2\omega^4} = r\sqrt{\alpha^2 + \omega^4} \tag{8.16}$$

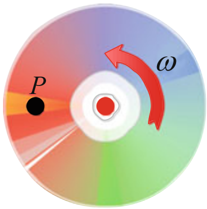
Example 8.3

A typical compact disk (CD) rotates at 300 rev/min. (a) What is the angular velocity of the disk? (b) What is the linear speed of a point P that lies 4 cm from its axis, see Fig. 8.8? (c) If one bit of data is represented at point P and has a length of $L = 0.5$ μm, find the number of bits N that the reading head can read per second.

Solution: (a) First, we write the frequency f in SI unit as follows:

$$f = 300\left(\frac{\text{rev}}{\text{min}}\right)\left(\frac{\text{min}}{60\text{ s}}\right) = 5\text{ rev/s} = 5\text{ Hz}$$

Fig. 8.8



Then, using Eq. 8.11, the angular speed ω is:

$$\omega = 2\pi f = 31.4\,\text{rad/s}$$

(b) The linear speed of a point 4 cm from the axis of the disk is:

$$v = r\,\omega = (4 \times 10^{-2}\,\text{m})(31.4\,\text{rad/s}) = 1.26\,\text{m/s}$$

(c) Using $NL = v$, we get the number per second N to be:

$$N = \frac{v}{L} = \frac{1.26\,\text{m/s}}{0.5 \times 10^{-6}\,\text{m}} = 2.5 \times 10^{6}\,\text{bits/s} = 2.5\,\text{Mbps}.$$

Example 8.4

A grindstone of radius $r = 2$ m rotates with an angular position $\theta = t^3 + 2\,t^2 - 2$, where θ is in radians and t is in seconds. (a) Find ω and α as a function of time and find their values at $t = 2$ s. (b) Find the speed v and the components of the acceleration a at $t = 2$ s for a point on the rim of the grindstone.

Solution: (a) To find ω, we differentiate θ with respect to time:

$$\omega = \frac{d\theta}{dt} = \frac{d}{dt}(t^3 + 2\,t^2 - 2) = (3\,t^2 + 4\,t)\,\text{rad/s} \quad \Rightarrow \quad \omega|_{t=2\,\text{s}} = 20\,\text{rad/s}$$

To find α, we differentiate ω with respect to time:

$$\alpha = \frac{d\omega}{dt} = \frac{d}{dt}(3\,t^2 + 4\,t) = (6\,t + 4)\,\text{rad/s}^2 \quad \Rightarrow \quad \alpha|_{t=2\,\text{s}} = 16\,\text{rad/s}^2$$

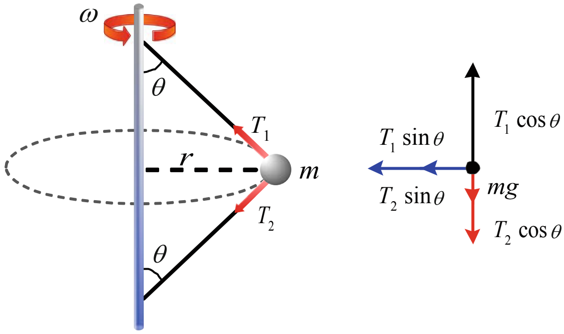
(b) Using Eqs. 8.10, 8.13, and 8.14, we get:

$$\begin{aligned}
v &= r\,\omega = r(3\,t^2 + 4\,t)\,\text{rad/s} &\Rightarrow\quad v|_{t=2\,\text{s}} &= 40\,\text{m/s}\\
a_\text{t} &= r\,\alpha = r(6\,t + 4)\,\text{m/s}^2 &\Rightarrow\quad a_\text{t}|_{t=2\text{s}} &= 32\,\text{m/s}^2\\
a_\text{r} &= r\,\omega^2 = r(3\,t^2 + 4\,t)^2\,\text{m/s}^2 &\Rightarrow\quad a_\text{r}|_{t=2\,\text{s}} &= 800\,\text{m/s}^2
\end{aligned}$$

Example 8.5

A ball of mass $m = 0.1$ kg rotates in a circular path of radius $r = 0.2$ m with an angular speed $\omega = 8$ rad/s while being attached to two strings of equal length, each making an angle $\theta = 30°$ with a vertical rod as shown in Fig. 8.9. Find the magnitude of the tension in the two strings.

Fig. 8.9



Solution: From the free-body diagram shown above, the vertical forces must balance. That is:

$$T_1 \cos\theta - T_2 \cos\theta = mg$$

According to Eq. 8.14, the magnitude of the radial acceleration is given in terms of the angular speed ω as $a_r = r\,\omega^2$. Therefore:

$$m\,r\,\omega^2 = T_1 \sin\theta + T_2 \sin\theta$$

Multiplying both sides of the first equation by $\sin\theta$ and both sides of the second equation by $\cos\theta$, then adding (or subtracting) the results, we can get the magnitude of the tension in the two strings as follows:

$$\begin{aligned} T_1 &= \frac{m}{2\sin\theta}(r\,\omega^2 + g\tan\theta) \\ &= \frac{0.1\,\text{kg}}{2\sin 30°}[(0.2\,\text{m})(8\,\text{rad/s})^2 + (10\,\text{m/s}^2)(\tan 30°)] = 1.86\,\text{N} \end{aligned}$$

$$\begin{aligned} T_2 &= \frac{m}{2\sin\theta}(r\,\omega^2 - g\tan\theta) \\ &= \frac{0.1\,\text{kg}}{2\sin 30°}[(0.2\,\text{m})(8\,\text{rad/s})^2 - (10\,\text{m/s}^2)(\tan 30°)] = 0.70\,\text{N} \end{aligned}$$

8.6 Rotational Dynamics; Torque

Rotational dynamics is the study of rotational motion and the causes of changes in motion. Just as linear motion is analogous to rotational motion from a kinematics perspective, we will see that this analogy applies also from a dynamics perspective.

We know from our everyday experience that, when an object rotates about an axis, the rate of this rotation depends on the magnitude and direction of the exerted force and how far this force is applied away from the rotation axis. This dependence is measured by a vector quantity called **torque (or moment)** $\vec{\tau}$ (Greek tau "τ").

Figure 8.10a depicts a cross section of a rigid body that is free to rotate about a fixed axis at O. A force $\vec{F}$ perpendicular to the axis of rotation acts on the body at point P whose position vector from O is $\vec{r}$. The smaller angle between the two vectors $\vec{F}$ and $\vec{r}$ is θ. The ability of $\vec{F}$ to rotate the body about O from point P depends on the torque $\vec{\tau}$ as follows:

$$\vec{\tau} = \vec{r} \times \vec{F} \tag{8.17}$$

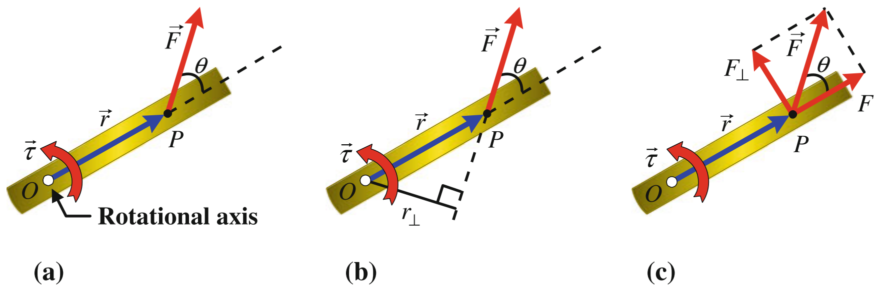


Fig. 8.10 (a) The torque $\vec{\tau}$ produced by a force $\vec{F}$ that acts at point P on a rigid body which can rotate freely about an axis passing through point O. (b) The torque can be written as $r_\perp F$, where $r_\perp$ is the moment arm of the force $\vec{F}$. (c) The torque can also be written as $r F_\perp$, where $F_\perp$ is the perpendicular component of the force to $\vec{r}$

Accordingly, its magnitude (see Chap. 2) is:

$$\tau = r F \sin\theta \tag{8.18}$$

The SI unit of the torque is m.N [not to be confused with the unit of energy (1 J = 1 N.m)]. By convention, torque is positive if the force has the tendency to rotate the object in a counterclockwise sense; and is negative if it has the tendency to rotate the object in a clockwise sense. Also, the reverse of this convention can be used.

Based on Fig. 8.10b and c, the magnitude τ can be written as:

$$\tau = r_{\perp} F \quad (\text{with } r_{\perp} = r \sin\theta) \tag{8.19}$$

$$\tau = r F_{\perp} \quad (\text{with } F_{\perp} = F \sin\theta) \tag{8.20}$$

where the distance $r_{\perp}$ is the perpendicular distance from the axis of rotation O to the line along which the force acts (also called the **lever arm**, or the **moment arm**). In addition, $F_{\perp}$ is the component of the force perpendicular to $\vec{r}$. This component is what causes that rotation. The other component, $F_{\parallel}$, is parallel to the position vector $\vec{r}$, passes through O and causes no rotation.

If two or more forces act on a rigid body, where each force tends to produce rotation about an axis passing through some point, the net torque on the body will be the sum of all torques:

$$\Sigma\vec{\tau} = \vec{\tau}_1 + \vec{\tau}_2 + \ldots \tag{8.21}$$

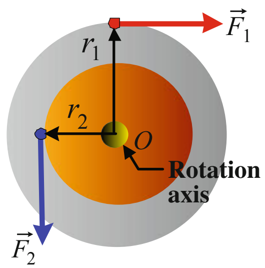
Using the sign convention introduced previously for torques, we can omit the vector notation and write the net torque as follows:

$$\Sigma\tau = \tau_1 + \tau_2 + \ldots \tag{8.22}$$

Example 8.6

Two wheels of radii $r_1 = 20\,\text{cm}$ and $r_2 = 30\,\text{cm}$ are fastened together as shown in Fig. 8.11. Together, they can rotate freely about an axle O perpendicular to the page. Two forces of magnitudes $F_1 = 20\,\text{N}$ and $F_2 = 40\,\text{N}$ are applied as shown in the figure. Find the net torque on the wheel.

Fig. 8.11



Solution: Designate counterclockwise torque as positive. The force $\vec{F}_1$ produces a torque $\vec{\tau}_1$ that tends to rotate the wheel in a clockwise sense. Thus, the sign of τ_1 is negative and equal to $-F_1 r_1$. The force $\vec{F}_2$ produces a torque $\vec{\tau}_2$ that tends to rotate the wheel in a counterclockwise sense. Thus, the sign of τ_2 is positive and equal to $+F_2 r_2$. By using Eq. 8.22, the net torque is:

$$\begin{aligned}\Sigma\tau &= \tau_1 + \tau_2 = -F_1 r_1 + F_2 r_2 \\ &= -(20\text{ N})(20 \times 10^{-2}\text{ m}) + (40\text{ N})(30 \times 10^{-2}\text{ m}) \\ &= 8\text{ m.N}\end{aligned}$$

The net torque acts to rotate the wheel in the counterclockwise sense.

8.7 Newton's Second Law for Rotation

We will show that Newton's second law $\Sigma F \propto a$ for translational motion corresponds to $\Sigma\tau \propto \alpha$ for rotational motion about a fixed axis.

First, we consider a particle of mass m attached to one end of a rod of negligible mass while the other end can rotate freely at point O. The mass rotates in a circle of radius r under the influence of a tangential force $\vec{F}_t$, as shown in Fig. 8.12. In this figure we do not display the radial force $\vec{F}_r$.

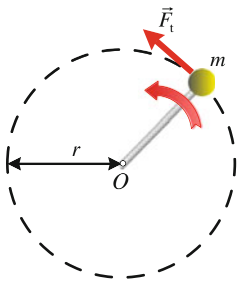


Fig. 8.12 A particle of mass m is rotating in a circle of radius r under the influence of a tangential force $\vec{F}_t$

According to Newton's second law, the tangential force $\vec{F}_t$ produces a tangential acceleration $\vec{a}_t$. Then:

$$F_t = m\, a_t$$

The tangential acceleration is related to the angular acceleration through the relationship $a_t = r\,\alpha$, see Eq. 8.13. Thus,

$$F_t = m\,r\,\alpha \tag{8.23}$$

Since $\vec{F}_t$ produces a torque $\vec{\tau}$ about the origin, this torque tends to rotate the particle in a counterclockwise sense. The magnitude of $\vec{\tau}$ is:

$$\tau = r\,F_t \tag{8.24}$$

Substituting with Eq. 8.23 into Eq. 8.24, we get:

$$\tau = m\,r^2\,\alpha \tag{8.25}$$

which can be written as:

$$\left.\begin{aligned} \tau &= I\,\alpha, \\ I &= m\,r^2 \end{aligned}\right\} \quad \text{(Single particle)} \tag{8.26}$$

That is, the applied torque is proportional to the angular acceleration, and represents the rotational equivalent of Newton's second law. The quantity $I = m\,r^2$ represents the *rotational inertia* of the particle about O and is called the **moment of inertia**. The SI units of I is kg.m^2.

We can apply this result to a system of particles located at various distances from a certain axis of rotation. For the i^{th} particle, we apply Eq. 8.25 to get $\tau_i = (m_i\,r_i^2)\alpha$. Then, the total torque about that axis will be $\sum\tau = (\sum m_i\,r_i^2)\,\alpha = I\alpha$. Thus:

$$\left.\begin{aligned} \Sigma\tau &= I\alpha, \\ I &= \sum m_i\,r_i^2 \end{aligned}\right\} \quad \text{(System of particles)} \tag{8.27}$$

Notice the analogy between the translational relation $\Sigma F = m\ a$ and the rotational relation $\Sigma\tau = I\ \alpha$, where $F \Leftrightarrow \tau$ and $m \Leftrightarrow I$.

Now we consider a rigid body rotating about a fixed axis at O. We can think of this body as an infinite number of mass elements dm of infinitesimal size, see Fig. 8.13. Each mass element rotates in a circular path about the origin with an angular acceleration $\vec{a}_t$ produced by an external tangential force $\vec{F}_t$.

By applying Newton's second law to a given mass element, we get:

$$dF_t = (dm)a_t$$

All elements of the rigid body have the same angular acceleration α. Since $a_t = r\,\alpha$ is the angular acceleration of each element, then:

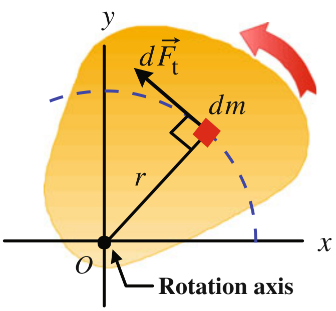


Fig. 8.13 Each element of mass *dm* is rotating about *O* in a circle of radius *r* under the influence of a tangential force $d\vec{F}_t$

$$dF_t = \alpha(dm)r \tag{8.28}$$

The magnitude of the differential torque $d\tau$ produced by dF_t is:

$$d\tau = r\,dF_t \tag{8.29}$$

Using Eq. 8.28, the expression for $d\tau$ becomes:

$$d\tau = \alpha r^2 dm \tag{8.30}$$

Now we can integrate both sides of this differential relation to find the net torque $\Sigma\tau$ about O due to external forces as follows:

$$\Sigma\tau = \alpha \int r^2 dm \tag{8.31}$$

which can be written as:

$$\left.\begin{aligned} \Sigma\tau &= I\,\alpha, \\ I &= \int r^2 dm \end{aligned}\right\} \quad \text{(Rigid body)} \tag{8.32}$$

In this case, $I = \int r^2 dm$ is the moment of inertia of the rigid body about the rotation axis through O. All equations of the form $\Sigma\tau = I\,\alpha$ hold even if the external forces have radial components, since the action of these components passes through the axis of rotation.

Parallel-Axis Theorem

If we calculate the moment of inertia of a body about any axis that passes through its center of mass, then we can prove that the moment of inertia about any axis parallel to that axis is given by:

$$I = I_{CM} + M\,h^2 \tag{8.33}$$

where M is the total mass of the body and h is the perpendicular distance between the two parallel axes. Figure 8.15 shows this for the case of a rod.

Example 8.7

A horizontal rod of uniform mass per unit length λ has a mass M and length L. Use the relation $I = \int r^2 dm$ to calculate the moment of inertia of the rod about: (a) an axis passing through its center, and (b) an axis passing through its end. (c) Check your result by using the parallel-axis theorem.

Solution: (a) For a uniform rod, $\lambda = M/L$. If we divide the rod into infinitesimal elements of length dx, then the mass of each element is $dm = \lambda\, dx$. Figure 8.14 shows an axis through CM and the left end.

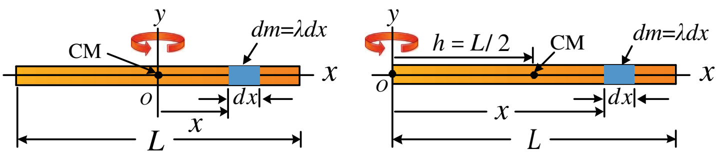


Fig. 8.14

For an axis passing through the CM, I in Eq. 8.32 leads to:

$$I_{CM} = \int r^2 dm = \int_{-L/2}^{+L/2} x^2 \lambda\, dx = \frac{M}{L}\int_{-L/2}^{+L/2} x^2 dx = \frac{M}{L}\int_{-L/2}^{+L/2} x^2 dx$$

$$= \frac{M}{L}\left[\frac{x^3}{3}\right]_{-L/2}^{+L/2} = \frac{1}{12} M\,L^2$$

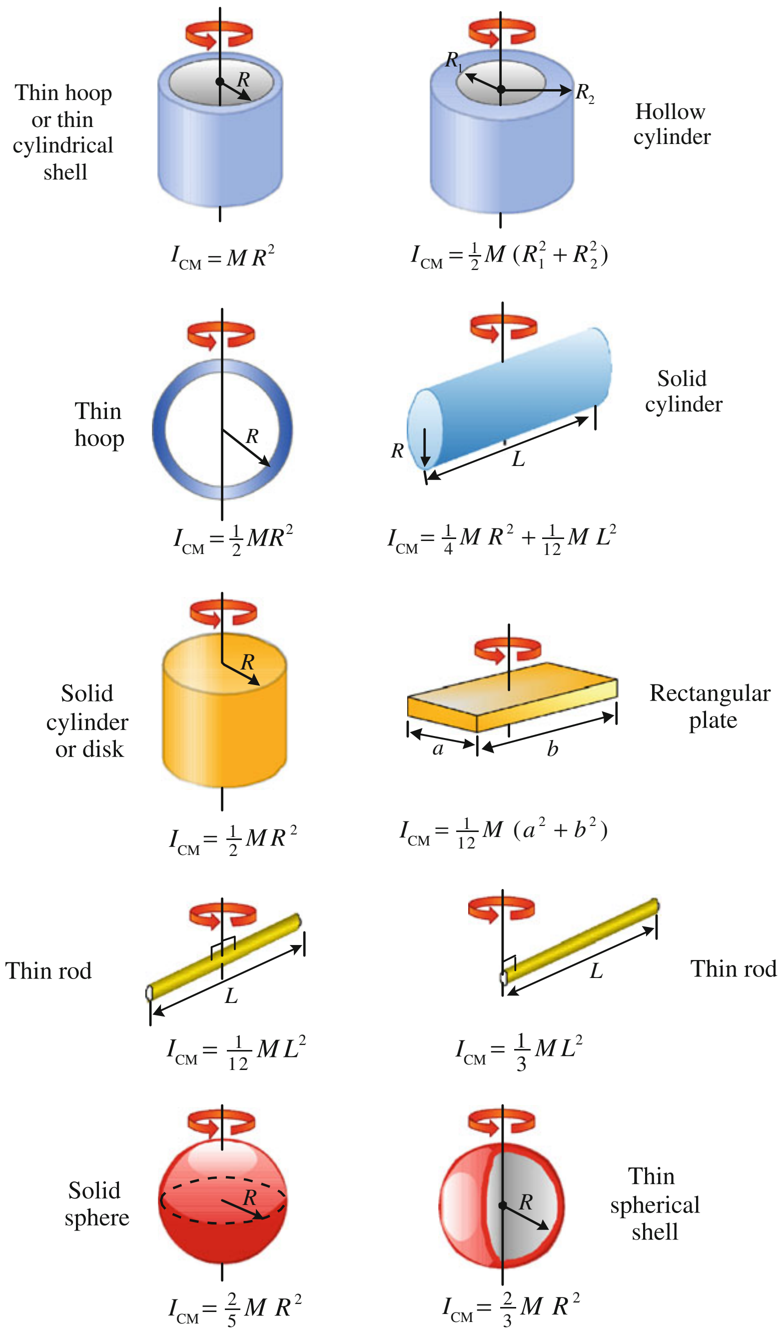


Fig. 8.15 Moments of inertia for some objects about specific axes

(b) For an axis passing through one end, I in Eq. 8.32 leads to:

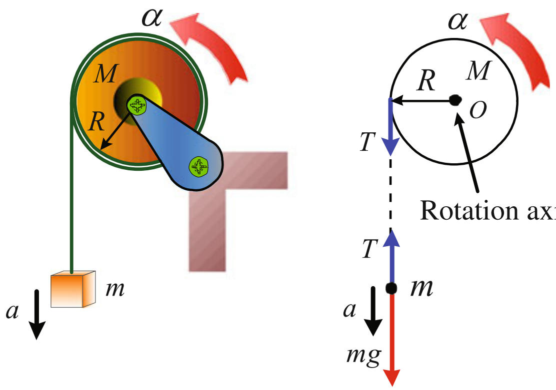
$$I = \int r^2 dm = \int_0^L x^2 \lambda\, dx = \frac{M}{L}\int_0^L x^2\, dx = \frac{M}{L}\int_0^L x^2\, dx = \frac{M}{L}\left[\frac{x^3}{3}\right]_0^L = \frac{1}{3} M L^2$$

(c) Applying the theorem $I = I_{CM} + M h^2$, one can obtain the same result.

Example 8.8

A pulley of mass $M = 6$ kg and radius $R = 20$ cm is mounted on a frictionless axis, as shown in Fig. 8.16. A massless cord is wrapped around the pulley while its other end supports a block of mass $m = 3$ kg. If the cord does not slip, find the linear acceleration of the block, the angular acceleration of the pulley, and the tension in the cord. Take $g = 10\,\text{m/s}^2$.

Fig. 8.16



Solution: For a downward motion of the block with acceleration a, the weight $m\,g$ must be greater than the tension T, see the free-body diagram of Fig. 8.16. Therefore, from Newton's second law of linear motion, we get:

$$(1) \quad m\,g - T = m\,a$$

From the free-body diagram of Fig. 8.16, we see that the torque τ that acts on the pulley is $R\,T$. Applying Newton's second law in angular form, Eq. 8.32, we obtain:

$$\Sigma\tau = I\,\alpha \quad \Rightarrow \quad R\,T = \left(\tfrac{1}{2} M R^2\right)\alpha \quad \Rightarrow \quad T = \tfrac{1}{2} M R\,\alpha$$

where the moment of inertia of the pulley $I = \frac{1}{2}M R^2$ is taken from Fig. 8.15. The linear acceleration of the block is equal to the tangential acceleration of the pulley, i.e., $a_t = a$. Since $a_t = R\,\alpha$, then the last equation reduces to:

$$(2) \quad T = \tfrac{1}{2}M\,a$$

Eliminating the tension from Eqs. (1) and (2), we get:

$$a = \frac{2m}{2m + M}g = \frac{2 \times (3\,\text{kg})}{2 \times (3\,\text{kg}) + 6\,\text{kg}} \times (10\,\text{m/s}^2) = 5\,\text{m/s}^2$$

The angular acceleration of the pulley is thus:

$$\alpha = \frac{a_t}{R} = \frac{a}{R} = \frac{5\text{m/s}^2}{0.2\,\text{m}} = 25\,\text{rad/s}^2$$

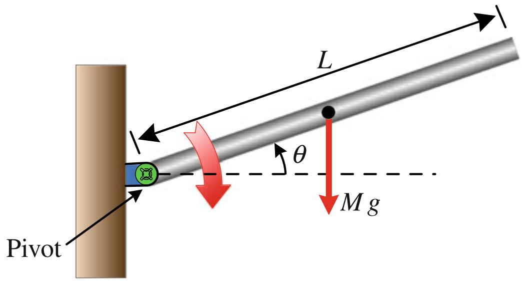
We use Eq. (2) to find the tension in the cord as follows:

$$T = \tfrac{1}{2}M\,a = \tfrac{1}{2} \times (6\,\text{kg})(5\ \text{m/s}^2) = 15\ \text{N}$$

Example 8.9

A uniform thin rod of mass $M = 2$ kg and length $L = 20$ cm is attached from one end to a frictionless pivot. The rod is free to rotate in a vertical plane. The rod is released when it is in the vertical position. Figure 8.17 shows the situation when the angle between the rod and the horizontal is θ. (a) Determine the angular acceleration of the rod as a function of θ for $-90° \le \theta \le +90°$ and find its maximum value. (b) Find the angle where the tangential acceleration of the free end of the rod equals g. Take $g = 10\,\text{m/s}^2$.

Fig. 8.17



Solution: (a) The moment arm of the force exerted by the pivot on the rod is zero. Therefore, the only force that contributes to the torque is the gravitational force $M\vec{g}$ with moment arm $\frac{1}{2}L\cos\theta$. Consequently, the angular acceleration is not constant because the torque exerted on the rod varies with θ. Call clockwise torques positive. Then the magnitude of this clockwise torque is:

$$\tau = \left(\tfrac{1}{2}L\cos\theta\right) M\, g$$

By applying Newton's second law in its angular form, $\sum\tau = I\,\alpha$, and taking $I = \frac{1}{3}M\,L^2$ from Fig. 8.15 for the axis of rotation at one end, we obtain:

$$\left(\tfrac{1}{2}L\cos\theta\right) M\, g = \left(\tfrac{1}{3}M\,L^2\right)\alpha$$

Thus: $$\alpha = \frac{3g}{2L}\cos\theta$$

At any angle θ, all points on the rod have this angular acceleration and the maximum value of α occurs at $\theta = 0$. Thus:

$$\alpha_{\text{max}} = \frac{3g}{2L}\cos 0^\circ = \frac{3(10\,\text{m/s}^2)}{2(20\times 10^{-2}\,\text{m})} = 75\,\text{rad/s}^2$$

The dependence of α on the angle θ indicates that the angular acceleration starts from zero when $\theta = 90^\circ$, then increases with decreasing θ, becomes maximum of 75 rad/s at $\theta = 0$, then decreases for negative values of θ, and reaches zero again at $\theta = -90^\circ$.

(b) To find the tangential acceleration of the free end of the rod at any angle θ, we use the relation $a_{\text{t}} = L\,\alpha$ and substitute with α to get:

$$a_{\text{t}} = L\,\alpha = \tfrac{3}{2}g\cos\theta$$

Note that a_{t} does not depend on the length of the rod L. Now, setting $a_{\text{t}} = g$ in the previous relation, we find the value of θ to be:

$$\cos\theta = \tfrac{2}{3} \quad\Rightarrow\quad \theta = \cos^{-1}\tfrac{2}{3} = 48.2^\circ$$

8.8 Kinetic Energy, Work, and Power in Rotation

Rotational Kinetic Energy

Analogous to translational kinetic energy $\left(\frac{1}{2}mv^2\right)$, an object that rotates about an axis is said to have **rotational kinetic energy**. Using this analogy between translational and rotational motions, where $m \Leftrightarrow I$ and $v \Leftrightarrow \omega$, one would expect that the rotational kinetic energy will be given by the expression $\frac{1}{2}I\omega^2$. We can show that this expression is indeed true.

Consider the rigid body of Fig. 8.13 to be a collection of tiny particles rotating about a fixed axis with angular speed ω. If the i^{th} particle has a mass m_i, distance r_i from the axis of rotation, and tangential speed $v_\text{i} = r_\text{i}\omega$, then its kinetic energy is:

$$K_i = \tfrac{1}{2}m_\text{i}v_\text{i}^2 = \tfrac{1}{2}m_\text{i}\, r_\text{i}^2\omega^2 \qquad (8.34)$$

The total kinetic energy of the rotating body will be:

$$K = \sum K_\text{i} = \tfrac{1}{2}\left(\sum m_\text{i}\, r_\text{i}^2\right)\omega^2 \qquad (8.35)$$

Since $\sum m_\text{i}\, r_\text{i}^2$ is the moment of inertial of the rigid body and tends to $\int r^2 dm$ for a continuous mass distribution, then as expected we get:

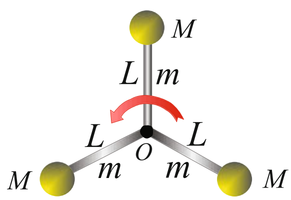
$$K_\text{R} = \tfrac{1}{2}I\,\omega^2 \quad \text{(Rotational kinetic energy)} \qquad (8.36)$$

We refer to K_R as rotational kinetic energy, which has the units of energy.

Example 8.10

Figure 8.18 shows three tiny spheres, each of mass M, are fastened by three identical rods each of mass m and of length L. The system is allowed to rotate with an angular speed ω about an axis that is perpendicular to the page and passes through O. Find the moment of inertia and the rotational kinetic energy about this axis.

Fig. 8.18



Solution: Using I from Eq. 8.27 and taking $\frac{1}{3}m\,L^2$ as the moment of inertia of each rod about O, the system's moment of inertia will be:

$$I = 3(M\,L^2) + 3(\tfrac{1}{3}m\,L^2) = (3M + m)L^2$$

Therefore, the rotational kinetic energy of the system about O will be:

$$K_{\mathrm{R}} = \tfrac{1}{2}I\,\omega^2 = \tfrac{1}{2}(3M + m)L^2\omega^2$$

Example 8.11

A block of mass $m = 2$ kg rests on an inclined plane of angle $\theta = 30°$. The block is connected by a cord of negligible mass that is wrapped around a pulley of mass $M = 2.5$ kg and radius $R = 0.8$ m, see Fig. 8.19. The block slides on the incline against a frictional force f of 0.5 N, and the pulley rotates without friction about its axis. How fast will the block be moving after sliding a distance $d = 1.5$ m along the incline?

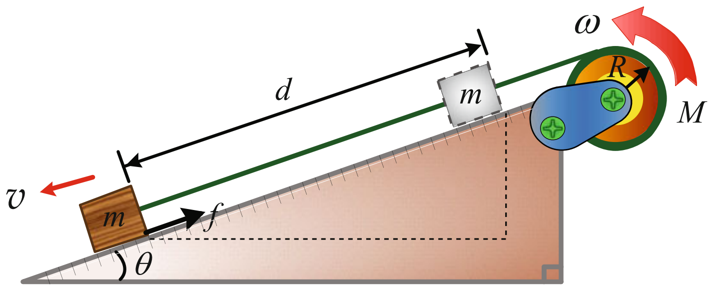


Fig. 8.19

Solution: The work done by the frictional force W should be equal to the change in the total energy ΔE of the block-pulley system. Thus:

$$W = \Delta E = E_{\mathrm{f}} - E_{\mathrm{i}}$$

where $E_{\mathrm{i}} = K_{\mathrm{i}} + (K_{\mathrm{R}})_{\mathrm{i}} + U_{\mathrm{i}}$ and $E_{\mathrm{f}} = K_{\mathrm{f}} + (K_{\mathrm{R}})_{\mathrm{f}} + U_{\mathrm{f}}$ are the initial and final total energies of the system, respectively. If we assign a zero value for the gravitational potential of the block at the final position, then $U_{\mathrm{i}} = m\,g(d\sin\theta)$ and $U_{\mathrm{f}} = 0$. Also, $(K_{\mathrm{R}})_{\mathrm{i}} = 0$ and $(K_{\mathrm{R}})_{\mathrm{f}} = \tfrac{1}{2}I\omega^2$, where $I = \tfrac{1}{2}M\,R^2$ for a disk rotating about its central axis. In addition, $K_{\mathrm{i}} = 0$ and $K_{\mathrm{f}} = \tfrac{1}{2}mv^2$. Using these relations and substituting with $W = -f\,d$ and $\omega = v/R$ into the last equation, we get:

$$-f\,d = [\tfrac{1}{2}m\,v^2 + \tfrac{1}{2}(\tfrac{1}{2}M\,R^2)(v/R)^2 + 0] - [0 + 0 + m\,g\,d\sin\theta]$$

By rearranging the terms, we have:

$$v = 2\sqrt{\frac{(m\,g\sin\theta - f)d}{2m + M}} = 2\sqrt{\frac{[(2\,\text{kg})(9.8\,\text{m/s}^2)\sin 30° - 0.5\,\text{N}](1.5\,\text{m})}{2\times(2\,\text{kg}) + (2.5\,\text{kg})}}$$
$$= 2.93\,\text{m/s}$$

Work done in Rotational Motion

We assume that the rotation of the rigid body in Fig. 8.20 is produced by an *external force* $\vec{F}$ that acts at a point P, which is at a distance r from the rotational axis through O. The radial component of $\vec{F}$ does not cause rotation, because it has a zero moment arm, while the tangential component $F_t = F\sin\phi$ does cause rotation. The differential work done by $\vec{F}$ on the rigid body as it rotates through an infinitesimal distance $ds = r\,d\theta$ about O is:

$$dW = \vec{F} \bullet d\vec{s} = F_t\,d\,s = F\sin\phi\,d\,s = F\sin\phi\,r\,d\theta \tag{8.37}$$

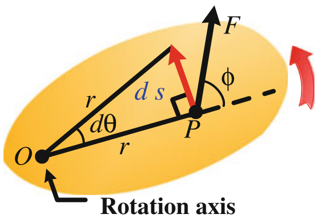


Fig. 8.20 A rigid body rotates about an axis through O under the action of a single external force $\vec{F}$ acting at point P

Since the magnitude of the torque due to $\vec{F}$ about O is $\tau = F_t\,r$, then:

$$dW = \tau\,d\theta \tag{8.38}$$

This is the rotational version of the one-dimensional relation $dW = F\,ds$, namely $F \Leftrightarrow \tau$ and $s \Leftrightarrow \theta$. For a single force, we use $\tau = I\,\alpha = I\,d\omega/dt$ and the chain rule of differentiation to get:

$$dW = \tau d\theta = I\frac{d\omega}{dt}d\theta = I\frac{d\omega}{d\theta}\frac{d\theta}{dt}d\theta = I\frac{d\omega}{d\theta}\omega\,d\theta = I\omega\,d\omega \tag{8.39}$$

By integrating Eq. 8.39, we obtain the total work as follows:

$$W = \int_{\theta_i}^{\theta_f} \tau \, d\theta \quad \text{or} \quad W = \int_{\omega_i}^{\omega_f} I\omega \, d\omega = \tfrac{1}{2} I\omega_f^2 - \tfrac{1}{2} I\omega_i^2 = \Delta K_{\mathrm{R}} \tag{8.40}$$

The relation $W = \Delta K_{\mathrm{R}}$ is the work-energy principle for rotational motion of a rigid body about a fixed axis.

Power in Rotational Motion

The rate of work done at time t, dW/dt, or the instantaneous power P, is obtained from Eq. 8.38 as follows:

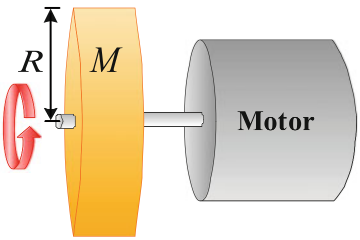
$$P = \frac{dW}{dt} = \tau\omega \tag{8.41}$$

The right-hand side of this expression is the rotational version of the linear-motion equation $P = Fv$, where $F \Leftrightarrow \tau$ and $v \Leftrightarrow \omega$.

Example 8.12

A disk of mass $M = 0.2$ kg and radius $R = 5$ cm is attached coaxially to the massless shaft of an electric motor, see Fig. 8.21. The motor runs steadily at 900 rpm and delivers 2 hp. (a) What is the angular speed of the disk in SI units? (b) What is the rotational kinetic energy of the disk? (c) How much torque does the motor deliver?

Fig. 8.21



Solution: (a) The angular speed of the motor of the disk is:

$$\omega = \left(900 \frac{\text{rev}}{\text{min}}\right)\left(\frac{\text{min}}{60\,\text{s}}\right)\left(\frac{2\pi\,\text{rad}}{\text{rev}}\right) = 94.2\,\text{rad/s}$$

(b) The rotational kinetic energy of the disk is:

$$K_R = \tfrac{1}{2} I\omega^2 = \tfrac{1}{2}\left(\tfrac{1}{2} M R^2\right)\omega^2$$
$$= \tfrac{1}{4}(0.2\,\text{kg}) \times (0.05\,\text{m})^2 (94.2\,\text{rad/s})^2 = 1.11\ \text{J}$$

This is the amount of energy needed to bring the disk from rest to the angular speed $\omega = 94.2$ rad/s.

(c) The power delivered by the motor to maintain a constant angular speed $\omega = 94.2$ rad/s for the disk and to oppose all kinds of friction is:

$$P = 2 \times (746\ \text{W}) = 1{,}492\ \text{W}$$

Using Eq. 8.41, $P = \tau\omega$, we can find the torque as follows:

$$\tau = \frac{P}{\omega} = \frac{1{,}492\ \text{W}}{94.2\,\text{rad/s}} = 15.8\,\text{m.N}$$

8.9 Rolling Motion

Rolling as Rotation and Translation Combined

Assume that the wheel of Fig. 8.22 is rolling on a flat surface without slipping, and that its axes of rotation always remain parallel. In this figure, point Q on the rim of the wheel moves in a complex path called a cycloid while its center of mass moves in a straight line.

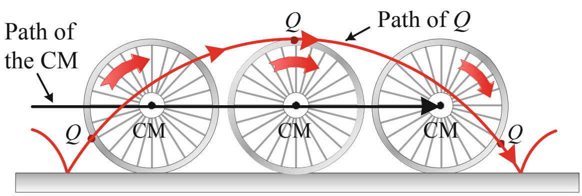
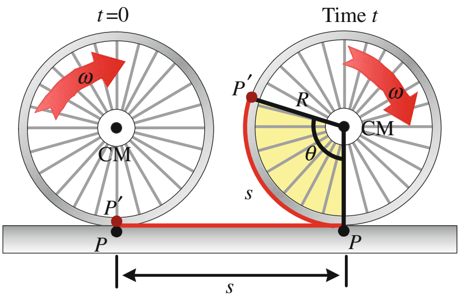


Fig. 8.22 When a wheel rolls without slipping on a flat surface, each point on the circumference (such as point Q) traces out a cycloid, while the center of mass (CM) traces out a straight line

Now, consider a wheel of a bicycle of radius R rolling without slipping on a horizontal surface at as shown in Fig. 8.23. Initially, the two points P and P' coincide, where P is the point of contact and $P'$is a point on the rim of the wheel.

Fig. 8.23 When a wheel rolls through an angle θ due to a rotation about the center of mass CM, its CM moves a linear distance $s = R\theta$



During a time interval t, both the point of contact P and the center of mass CM move a linear distance s; while the point on the rim P' moves an arc length s that subtends an angle θ at CM. Thus:

$$s = r\,\theta$$

Consequently, the linear speed of the center of mass will be given by:

$$v_{\text{CM}} = \frac{ds}{dt} = R\,\frac{d\theta}{dt} = R\,\omega \tag{8.42}$$

where ω is the angular speed of the wheel about its center of mass.

Rolling as Pure Rotation

To compare rolling-without-slipping motion with pure rotational motion, we consider Fig. 8.24. In this figure, the point of contact P is instantaneously at rest and the wheel rotates about an axis passing through this point. Since the point CM is at a distant R from P, and we proved that the CM has linear velocity $v_{\text{CM}} = R\,\omega$, then, in order to preserve Eq. 8.42, the instantaneous angular velocity about P must be the same as the instantaneous angular velocity ω about CM. In addition, the linear speed of point Q must be $2v_{\text{CM}}$.

As a result, rolling on a flat surface without slipping is equivalent to experiencing pure rotation about an axis through the point of contact P. Therefore, we can express the rolling kinetic energy of the wheel as:

$$K_{\text{Roll}} = \tfrac{1}{2} I_P\,\omega^2 \tag{8.43}$$

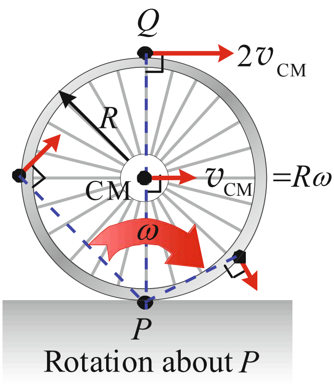


Fig. 8.24 Rotation about an axis through P with an angular velocity ω is equivalent to rotation about the CM with the same angular velocity

where I_P is the moment of inertia of the wheel about an instantaneous axis of rotation through P. By applying the parallel-axis theorem, we substitute $I_P = I_{CM} + M R^2$ into Eq. 8.43 to obtain:

$$K_{Roll} = \tfrac{1}{2} I_{CM}\, \omega^2 + \tfrac{1}{2} M\, R^2 \omega^2$$

By using $v_{CM} = R\,\omega$, the relation leads to:

$$K_{Roll} = \tfrac{1}{2} I_{CM}\, \omega^2 + \tfrac{1}{2} M v_{CM}^2 \tag{8.44}$$

Based on this relation, it seems natural to consider this type of rolling as a combination of rotational and translational motions. This consideration is explained graphically in Fig. 8.25.

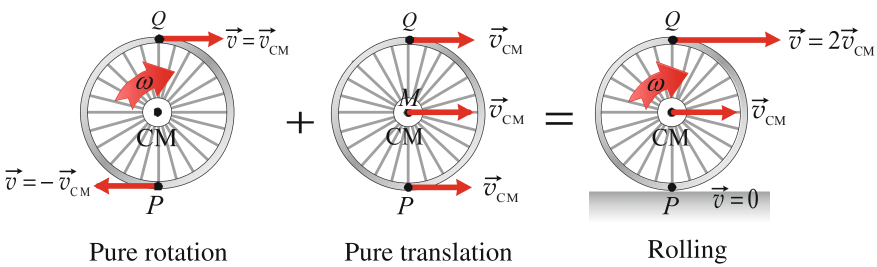


Fig. 8.25 Rolling without slipping can be considered as a combination of pure rotation and pure translation

Rolling with Friction

When the linear speed v_{CM} or the angular speed ω of a wheel changes, then a frictional force *tends to slide* the wheel at the point of contact P. Before sliding occurs, this frictional force is a static force f_s. Right on the verge of sliding, this frictional force is a maximum static force $f_{s,\max}$. When sliding occurs, this frictional force is a kinetic force f_k.

Figure 8.26a shows a wheel being rotated faster and faster (ω increases). The increase in ω tends to slide the point of contact P to the left. In Fig. 8.26b, the wheel tends to rotate more slowly, and the decrease in ω tends to slide the point of contact P to the right.

Figure 8.26c shows a wheel rolling down an incline without sliding. The weight $M\vec{g}$ at its center cannot cause rotation about the CM. Since $M\vec{g}$ tends to slide the wheel down the incline, a frictional force $\vec{f}_s$ must act at the point of contact P to oppose the sliding tendency; and this force has a moment arm about the center of mass.

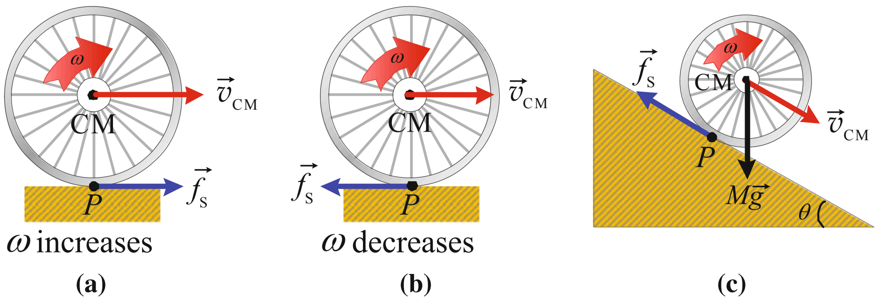


Fig. 8.26 (a) A wheel rolls horizontally without sliding while increasing its angular speed. The frictional force $\vec{f}_s$ acts at P to the right in order to oppose the sliding tendency. (b) Just like in (a) but with a decreasing angular speed. (c) A wheel rolls without sliding on an incline. The frictional force $\vec{f}_s$ acts at P to oppose the sliding tendency due to the wheel's weight $M\vec{g}$

Example 8.13

A disk of mass $M = 1.5$ kg and radius $R = 8$ cm rolls horizontally without sliding with a center-of-mass speed $v_{CM} = 4$ m/s. (a) What is the angular speed of the disk? (b) What is the kinetic energy of the rolling disk?

Solution: (a) Using Eq. 8.42, we have:

$$\omega = \frac{v_{CM}}{R} = \frac{4\,\text{m/s}}{8 \times 10^{-2}\,\text{m}} = 50\,\text{rad/s} \simeq 8\,\text{rev/s}$$

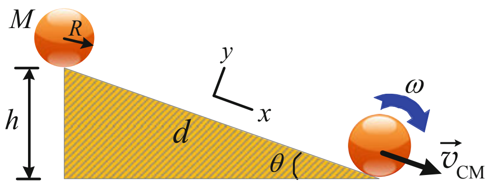
(b) The rolling kinetic energy of the disk is:

$$\begin{aligned} K_{Roll} &= K_R + K = \tfrac{1}{2} I_{CM}\,\omega^2 + \tfrac{1}{2} M v_{CM}^2 \\ &= \tfrac{1}{2}\left(\tfrac{1}{2} M R^2\right)\omega^2 + \tfrac{1}{2} M v_{CM}^2 \\ &= \tfrac{1}{4}(1.5\,\text{kg}) \times (0.08\,\text{m})^2 (50\,\text{rad/s})^2 + \tfrac{1}{2}(1.5\,\text{kg})(4\,\text{m/s})^2 = 18\ \text{J} \end{aligned}$$

Example 8.14

A solid sphere of mass M and radius R rolls without sliding when released from rest at the top of a frictional plane having a height h and inclination angle θ, see Fig. 8.27. The sphere starts at the top of the inclined plane and rolls to the bottom of the incline. Find the speed and acceleration of the sphere's center of mass when it reaches the bottom of the incline.

Fig. 8.27



Solution: Generally, the rolling kinetic energy of the sphere is:

$$K_{Roll} = K_R + K = \tfrac{1}{2} I_{CM}\,\omega^2 + \tfrac{1}{2} M v_{CM}^2$$

Using $v_{CM} = R\omega$ and $I_{CM} = \tfrac{2}{5} M R^2$ for a solid sphere, we can express K_{Roll} as a function of v_{CM} throughout the relation:

$$K_{Roll} = \tfrac{7}{10} M v_{CM}^2$$

We define the bottom of the incline to have zero gravitational potential energy. When rolling without sliding, the center of the sphere falls a vertical distance h, and the conservation of mechanical energy gives:

$$K_f + U_f = K_i + U_i$$

where $K_f = K_{Roll}$, $U_f = 0$, $K_i = 0$, and $U_i = M\,g\,h$. Thus:

$$\tfrac{7}{10} M v_{CM}^2 + 0 = 0 + M\,g\,h$$

Hence, we can express the dependence of v_{CM} on h as follows:

$$v_{CM} = \sqrt{\tfrac{10}{7} gh}$$

Notice that this is less than the speed $\sqrt{2gh}$ when an object slides on a frictionless incline without rolling (see Examples 5.5 and 6.8).

Using the kinematic equation $v^2 = v_\circ^2 + 2a(x - x_\circ)$ for the translational motion of the sphere along the incline, with $v \equiv v_{CM}$, $v_\circ = 0$, $a = a_{CM}$, and $(x - x_\circ) = d = h/\sin\theta$, we have:

$$\frac{10gh}{7} = 0 + 2\,a_{CM}\frac{h}{\sin\theta}$$

$$a_{CM} = \tfrac{5}{7} g \sin\theta$$

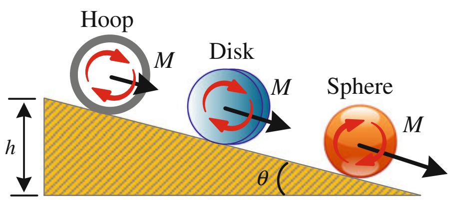
Notice also that this is less than the acceleration $g\sin\theta$ when an object slides down a frictionless incline without rolling (see Example 5.5).

The independence of v_{CM} and a_{CM} on R and M indicates that all homogeneous solid spheres experience the same speed and acceleration on a given incline.

Example 8.15

Three objects (a solid sphere, a disk, and a thin hoop) each having a mass M are at rest at the same height h. At the exact same instant, these objects start to roll without sliding down the incline of Fig. 8.28. In what order do they arrive at the bottom?

Fig. 8.28



Solution: For the given list of objects, we set $I_{CM} = \beta M R^2$, where $\beta = 0.4$ for the sphere, $\beta = 0.5$ for the disk, and $\beta = 1$ for the thin hoop. Therefore, using $K_{Roll} = \frac{1}{2} I_{CM}\,\omega^2 = M\,g\,h$ and $v_{CM} = R\omega$, the speed of the center of mass of any one of these objects at the bottom of the incline will be:

$$v_{CM} = \sqrt{\frac{2gh}{\beta + 1}}, \quad \beta = \begin{cases} 0.4 & \text{(for a sphere)} \\ 0.5 & \text{(for a disk)} \\ 1 & \text{(for a hoop)} \end{cases}$$

Note that v_{CM} does not depend on the object's mass M or radius R, but only depends on the shape (through the parameter β) and the height h. Moreover, according to the value of β, the sphere will attain the largest value of v_{CM}, followed by the disk, and finally the hoop will attain lowest value of v_{CM}, see Fig. 8.28.

In all cases, the acceleration of the center of mass is given by:

$$a_{CM} = \frac{g \sin\theta}{(1 + \beta)}$$

This is less than $g \sin\theta$ for the case of a box that slides down a frictionless incline of the same angle.

Table 8.2 summarizes the angular quantities and their linear analogs.

Table 8.2 Analogy between some linear and angular quantities and their connecting relations

Linear	Angular	Connecting relation
x	θ	$x = r\theta$
v	ω	$v = r\,\omega$
a	α	$a_t = r\,\alpha$
m	I	$I = \sum m\,r^2$
F	τ	$\tau = r\,F \sin\theta$
$K = \frac{1}{2}mv^2$	$K_R = \frac{1}{2}I\omega^2$	
$W = Fd$	$W = \tau\theta$	
$P = Fv$	$P = \tau\omega$	
$\sum F = m\,a$	$\sum \tau = I\alpha$	

8.10 Exercises

Section 8.1 Radian Measures

(1) As fractions of π and as numerical values, express the following angles in radians: 30°, 45°, 60°, 90°, 180°, 270°, 360°.

(2) The Moon, which is 3.8×10^5 km away from the Earth, subtends an angle of about 0.4° to us. Estimate the radius of the Moon.

(3) A circle has a radius of 2 m. (a) What angle in radians and degrees is subtended by an arc that is 1.5 m in length? (b) What arc length is subtended by an angle of 1.2 rad between two radii in this circle?

(4) Through how many revolutions must a car wheel turn if the wheel has a radius of 0.5 m and the car travels 2 km?

Section 8.2 Rotational Kinematics; Angular Quantities

(5) A drill starts from rest and after 4.5 s reaches a rate of 4×10^4 rev/min. What is the drill's average angular acceleration?

(6) A motor rotates at a rate of 9×10^3 rpm. When the motor is turned off, it takes 5 s to stop rotating. What is the average angular acceleration during this period?

(7) A player throws a baseball in a straight line towards a target at a speed of 90 km/h. While traveling, the ball spins at a rate of 1,800 rev/min. If the target is 10 m away, how many revolutions does the ball make on its way to the target?

(8) A reference line in a rotating fan has an angular position given by $\theta = 4t^2 - 14t + 6$, where θ is in radians and t is in seconds. (a) Find ω and α as a function of time. (b) Find the times when the angular position θ and the angular velocity ω become zero.

(9) A wheel rotates with an angular acceleration $\alpha = 6at - 2b$. At $t = 0$, the wheel has an angular speed $\omega_\circ$ and angular position $\theta_\circ$. Write down the equations for the angular speed and angular position as a function of time t.

(10) A wheel with six spokes is rotating at an angular speed $\omega = 240$ rev/min about an axle passing through its central axis at O, see Fig. 8.29. A dart of length $L = 10$ cm is shot parallel to the wheel's axle towards the wheel. Assume the dart and the spokes are very thin. (a) What is the minimum speed that the dart must have in order to miss any one of the spokes? (b) Does it matter where between the axle and the rim of the wheel you must aim the dart?

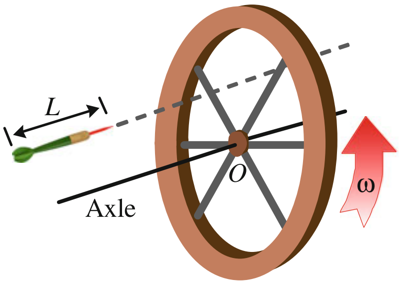


Fig. 8.29 See Exercise (10)

Section 8.3 Constant Angular Acceleration

(11) If the angular accelerations in Exercises 5 and 6 are constant, then find the change in angle through the corresponding rotational period. Provide your answer in radians, fractions of π, revolutions, and degrees.

(12) A wheel turning at an angular speed of 20 rev/s is brought to rest after 40 rev under a constant angular deceleration. (a) What is its angular deceleration? (b) How long does it take to stop?

(13) A car motor slows down from 5×10^3 rpm to 2×10^3 rpm in 2 s under a constant angular deceleration. (a) What is its angular deceleration? (b) Find the total number of revolutions of the motor in this period.

(14) A fan originally turning at 15 rev/s decelerates with $\alpha = -4\,\text{rad/s}^2$. (a) How long does the fan take to stop? (b) How many revolutions does it turn during this time period?

(15) A centrifuge rotates at an angular speed of 3.6×10^3 rev/min. When the centrifuge is turned off, it rotates 60 rev before coming to test. What is its angular deceleration? Assume it to be constant.

Section 8.4 Angular Vectors

(16) A wheel is mounted on fixed supports that are on a turntable that rotates about its axle with an angular speed $\omega_1 = 3$ rad/s, see Fig. 8.30. The turntable is rotating horizontally at an angular speed $\omega_2 = 4$ rad/s. Take the unit vectors along x, y, and z as $\vec{i}$, $\vec{j}$, and $\vec{k}$, respectively. (a) What are the directions of $\vec{\omega}_1$ and $\vec{\omega}_2$ at the instant shown in the figure? (b) Find the magnitude and direction of the resultant angular velocity $\vec{\omega}$ at the instant shown in the figure. (c) Find the magnitude and direction of the angular acceleration of the wheel $\vec{\alpha}_1$ at any time and then at the instant shown in the figure.

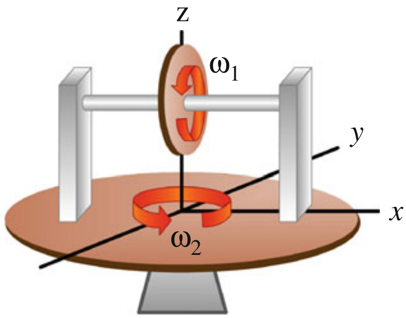


Fig. 8.30 See Exercise (16)

Section 8.5 Relating Angular and Linear Quantities

(17) A wheel 0.4 m in diameter rotates uniformly at an angular speed of 3.6×10^2 rev/min. (a) What is its angular speed in rad/s? (b) Find the linear speed and acceleration of a point on its rim.

(18) Figure 8.31 shows a synchronized analog 12-hour clock. Find the angular velocity of: (a) the second hand, (b) the minute hand, and (c) the hour hand. (d) Find the angular acceleration of each hand.

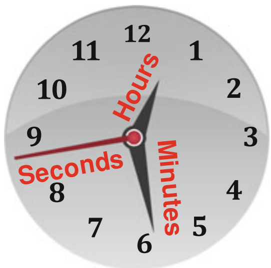


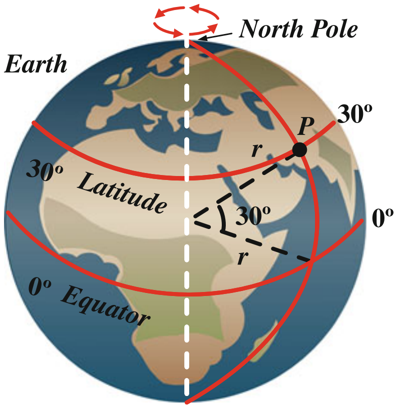
Fig. 8.31 See Exercise (18)

(19) In exercise 18, assume that the radii of the second hand, minute hand, and hour hand are 20, 15, and 10 cm, respectively. Find the linear speed of the tip of each hand.

(20) A merry-go-round completes one revolution in 1.5 s. (a) What is the linear speed of a child seated 3 m from the center? (b) Find the magnitude of the child's tangential and radial accelerations.

(21) Assume a point P is located at a latitude of exactly 30° N and is at a distance $r = 6.4 \times 10^6$ m away from Earth's center, see Fig. 8.32. As the Earth revolves about its axis, calculate: (a) the angular speed of the Earth, (b) the linear speed and magnitude of the acceleration of the point P, (c) the linear speed of a point on the equator.

Fig. 8.32 See Exercise (21)



(22) If the lower string in Example 8.5 is removed, then find the proper angular speed ω that allows the ball to rotate with the same radius $r = 0.2$ m and angle $\theta = 30°$.

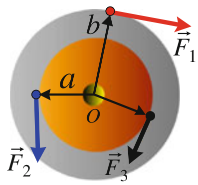
(23) A car accelerates uniformly from rest to a speed of 20 m/s during a 20 s time interval. The radius of the wheels of the car is 0.4 m. What is: (a) the angular acceleration of each wheel, and (b) the number of revolutions turned by each wheel during this time?

Section 8.6 Rotational Dynamics; Torque

(24) The pedals of a bike have a circular radius $r = 15$ cm. Find the maximum torque that can be exerted by the weight of a 70-kg person riding this bike?

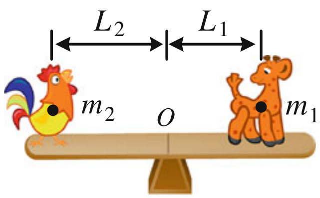
(25) The wheels in Fig. 8.33 have radii $a = 10$ cm and $b = 15$ cm. A frictional torque of 1.5 m.N opposes the motion when it rotates about an axle O perpendicular to the page. Find the net torque on the wheel when three forces of magnitudes $F_1 = 19$ N, $F_2 = 38$ N, and $F_3 = 45$ N are applied.

Fig. 8.33 See Exercise (25)



(26) A child wants to horizontally balance two toys of masses $m_1 = 0.1$ kg and $m_2 = 0.2$ kg by placing them at distances L_1 and L_2, respectively, from the central pivot of a seesaw of a massless board, see Fig. 8.34. (a) What is the ratio L_1/L_2 required to accomplish this balance? (b) If the child sets the toys 8 cm from the pivot, what is the magnitude and direction of the net torque?

Fig. 8.34 See Exercise (26)



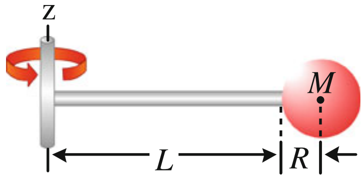
Section 8.7 Newton's Second Law for Rotation

(27) A rod of length 2 L is composed of an aluminum part with uniform length L and mass m_A and a brass part with uniform length L and mass m_B. Find the moment of inertia of the rod about an axis perpendicular to it yet passing through its center.

(28) A sphere of mass M and radius R is attached to one end of a massless rod of length L. The system rotates about the z-axis as shown in Fig. 8.35. (a) Use the parallel-axis-theorem to find the moment of inertia of the system about the z-axis. (b) Consider the sphere as a point particle and calculate its moment of inertia about the z-axis. (c) Find the percentage error introduced by the point approximation if $L = 0.5$ m and $R = 0.1$ m.

(29) A 1-kg wheel has a moment of inertia $I = 0.02\,\text{kg.m}^2$. The angular speed of the wheel reduces uniformly from 30 rev/s to zero after 150 rev. Find the torque used to slow down the wheel's rotation.

Fig. 8.35 See Exercise (28)

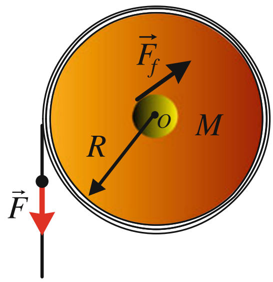


(30) Redo Example 8.8, this time assuming that a frictional torque $\vec{\tau}_f$ of magnitude 1.2 m.N exists at the axle.

(31) Redo Example 8.9, this time assuming that a frictional torque τ_f of magnitude 0.4 m.N exists at the pivot.

(32) A cord is wrapped around a pulley of mass $M = 2.5$ kg and radius $R = 0.2$ m. A constant force $\vec{F}$ of magnitude 30 N is applied to the cord as shown in Fig. 8.36. With the presence of a frictional torque $\vec{\tau}_f$ at the axle of magnitude 1.5 m.N, the pulley accelerates uniformly from rest to 21 rev/s in 2.8 s. (a) Find the moment of inertia of the pulley. (b) Does this moment of inertia equal the one obtained from the formula presented in Fig. 8.15? Explain.

Fig. 8.36 See Exercise (32)



(33) A pendulum of mass m with a string of length L is pulled aside to make an angle θ with the vertical. At the instant when the pendulum is released, find the torque on the mass m about the suspension point and its angular acceleration.

(34) A disk of mass M and radius R is attached to one end of a uniform rod of mass m and length L, as shown in Fig. 8.37. The other end of the rod is pivoted at P and the system is allowed to rotate freely. The system is released when the rod makes an angle θ with the vertical. Find the angular acceleration just after the system is released.

(35) An Atwood's machine consists of two boxes of masses $m_2 = 6$ kg and $m_1 = 4$ kg, which are connected by a massless cord that passes over a pulley, see

Fig. 8.38. The pulley has a moment of inertia $I = 5 \times 10^{-3}$ kg.m^2 and radius $R = 5$ cm. The cord does not slip over the pulley. Find the acceleration of the system and the tension in each cord. Take $g = 10$ m/s^2.

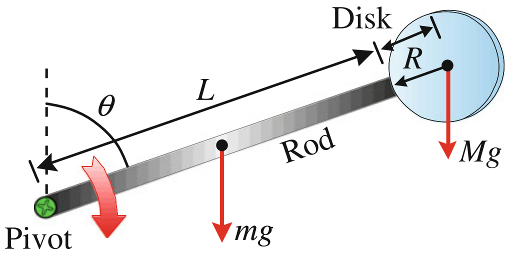


Fig. 8.37 See Exercise (34)

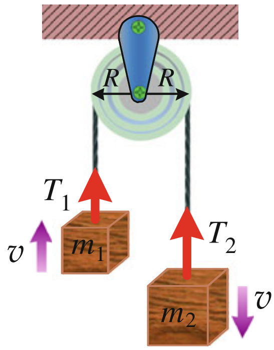


Fig. 8.38 See Exercise (35)

Section 8.8 Kinetic Energy, Work, and Power in Rotation

(36) What is the energy of an engine that has a moment of inertia $I = 5 \times 10^{-2}$ kg.m^2 and is rotating at 1,500 rpm?

(37) Two small balls of masses $M = 4$ kg and $m = 2$ kg are connected by a horizontal massless rod of length $L = 3$ m. The system is rotating with an angular speed $\omega = 8$ rad/s about an axle at a distance x from the mass M, see Fig. 8.39. Find the kinetic energy of the system and the net force on each mass: (a) when $x = L/2$, (b) when $x_{CM} = mL/(M + m)$; which is the position of the center of mass of the system.

(38) A horizontal massless rod is pivoted at one end. Three equal point masses are attached to this rod and are equidistant from each other and the pivot, see

Fig. 8.40. The system is released from its horizontal position. How fast will the bottom mass be moving when the rod becomes vertical?

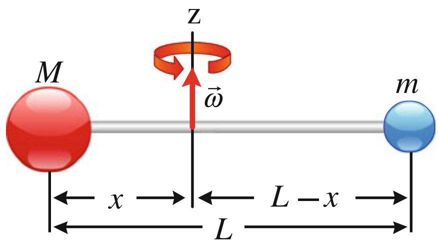


Fig. 8.39 See Exercise (37)

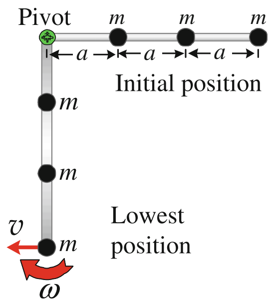


Fig. 8.40 See Exercise (38)

(39) The angular speed of a wheel increases from 60 to 180 rev/min when 125 J of work is added. What is its moment of inertia?

(40) Assume that the disk of Fig. 8.21 has a mass $M = 12$ kg and radius $R = 30$ cm. As in the figure, the disk is attached coaxially to the massless shaft of an electric motor. When the driving motor is disconnected, the motor slows down from 580 rpm to rest in 5 s. (a) What is the required power output of the motor to maintain a steady angular speed of 580 rpm? (b) How much torque does the motor deliver to maintain this steady angular speed?

Section 8.9 Rolling Motion

(41) A cylinder of mass $M = 2$ kg and radius $R = 5$ cm rolls horizontally over the floor without sliding with a center of mass speed $v_{CM} = 0.8$ m/s. (a) What is the angular speed of the cylinder about its axis? (b) What are the magnitude

and direction of the speed and acceleration of a point on the top of the cylinder? (c) What is the kinetic energy of the rolling cylinder?

(42) A thread of negligible mass is wound around a cylinder of mass $M = 4$ kg and radius $R = 2$ cm. If the thread is unwound under a constant force of magnitude $F = 30$ N and the cylinder rolls without sliding, as shown in Fig. 8.41. (a) What is the direction of the frictional force? (b) What is acceleration of the center of mass? (c) What is the value of the frictional force?

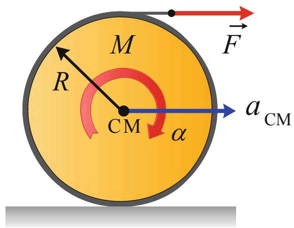


Fig. 8.41 See Exercise (42)

(43) A hoop slides when projected horizontally at time $t = 0$ with an initial speed $v_\circ$. The frictional force causes the hoop to slow down and acquire an angular speed. Show that the hoop stops sliding and starts rolling when it has a speed $v_\circ/2$ at time $t = Mv_\circ/2f$, see Fig. 8.42.

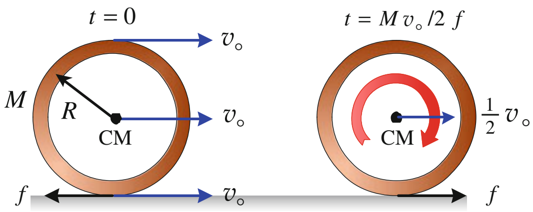


Fig. 8.42 See Exercise (43)

(44) A ball of radius r and mass m starts from rest and rolls without slipping on a track in the shape of a quarter circle of radius R, as shown in Fig. 8.43. Use conservation of mechanical energy to show that the ball's speed at the lowest point b is $v_b = \sqrt{10g\,(R - r)\,/7}$.

(45) A yo-yo of mass M and moment of inertia I has an axle of radius R. One end of a light string, assumed with negligible thickness, is tied to the axle and then wound several times around it. For idealized yo-yo, the thickness of wounded string can be neglected. While holding the other end of the string, the yo-yo

is released from rest, dropping as the string unwinds. Show that the linear acceleration, angular acceleration, and tension in the string of the yo-yo are given by:

$$a = \frac{g}{1 + I/MR^2}, \quad \alpha = \frac{g}{R + I/MR}, \quad T = \frac{Mg}{1 + MR^2/I}$$

Fig. 8.43 See Exercise (44)