Sources of Magnetic Field 26

In this chapter we complete the description of magnetic interactions by briefly exploring the origins of magnetic fields.

26.1 The Biot-Savart Law

Based on quantitative experiments, Biot and Savart were able to arrive at a mathematical expression that describes the magnetic field at any point in terms of the current or the charge that produces the field.

Consider a point P at a distance r from: (a) an element $d\vec{s}$ chosen in the direction of a *steady* current I, (b) a point charge q moving with velocity $\vec{v}$, see Fig. 26.1. Biot and Savart proposed that the magnetic field produced by the element, or by the charge, would be:

$$d\vec{B} = \frac{\mu_\circ}{4\pi}\frac{I\,d\vec{s} \times \hat{\vec{r}}}{r^2} \quad \text{and} \quad \vec{B} = \frac{\mu_\circ}{4\pi}\frac{q\,\vec{v} \times \hat{\vec{r}}}{r^2} \qquad \text{(Biot-Savart law)} \qquad (26.1)$$

where $\hat{\vec{r}}$ is a unit vector directed from $d\vec{s}$ or q toward point P. The product $I\,d\vec{s}$ is called the *differential current element*, and $\mu_\circ$ is a constant called the **permeability of free space**' which has the exact value:

$$\mu_\circ = 4\pi \times 10^{-7}\ \text{T.m/A} \qquad (26.2)$$

H. A. Radi and J. O. Rasmussen, *Principles of Physics*,
Undergraduate Lecture Notes in Physics, DOI: 10.1007/978-3-642-23026-4_26,

Fig. 26.1 (a) The differential magnetic field vector $d\vec{B}$ at point P, which is located by a position vector $\vec{r}$ drawn from a differential current element $I\,d\vec{s}$ to P. (b) In case of a point charge q moving with a velocity $\vec{v}$, the magnetic field $\vec{B}$ is related to the product $q\vec{v}$

To find the magnetic field $\vec{B}$ created at some point by a current of an extended circuit, we integrate Eq. 26.1 over all current elements as follows:

$$\vec{B} = \frac{\mu_\circ I}{4\pi}\int \frac{d\vec{s} \times \hat{\vec{r}}}{r^2} \tag{26.3}$$

It is useful to compare the Biot-Savart law with Coulomb's Law as follows:

Biot-Savart law ($d\vec{B}$)	Coulomb's law ($d\vec{E}$)
$d\vec{B}$ is due to differential current element $I\,d\vec{s}$, a vector	$d\vec{E}$ is due to differential charge dq, a scalar
$1/r^2$ distance dependence	$1/r^2$ distance dependence
Proportional to electric current I	Proportional to electric charge dq
Lateral, perpendicular to the $\vec{r}$ direction	Radial, in the $\vec{r}$ direction

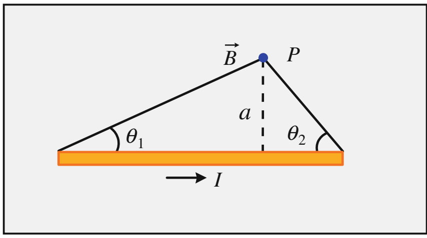
Some Applications of the Biot-Savart Law

In some situations, the integrand of Eq. 26.3 needs lengthy mathematical steps. For those interested, several mathematical and integration techniques are given at the end of this book. In this section we avoid the complexity arising from integrating Eq. 26.3 and only present the results for some cases.

1. Magnetic Field on the Extension of a Straight Wire

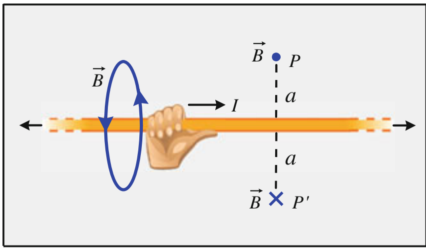
$$\vec{B} = 0 \tag{26.4}$$

2. Magnetic Field Surrounding a Thin Straight Wire



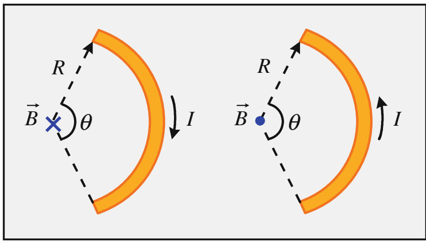
$$B = \frac{\mu_\circ I}{4\pi\, a}(\cos\theta_1 + \cos\theta_2) \quad (26.5)$$

3. Magnetic Field Surrounding a Very Long Straight Wire



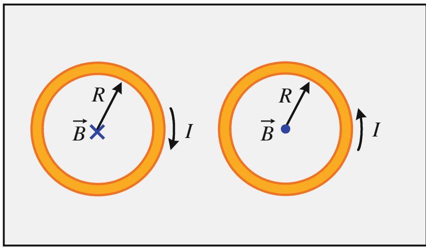
$$B = \frac{\mu_\circ I}{2\pi\, a} \quad (26.6)$$

4. Magnetic Field Due to a Curved Wire Segment



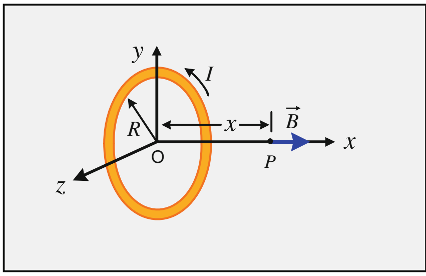
$$B = \frac{\mu_\circ I}{4\pi\, R}\theta \quad (26.7)$$

5. Magnetic Field at the Center of a Circular Wire Loop



$$B = \frac{\mu_\circ I}{2R} \quad (26.8)$$

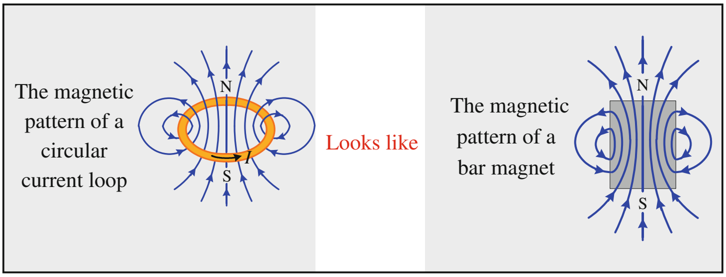
6. Magnetic Field on the Axis of a Circular Wire Loop



$$B = \frac{\mu_\circ I}{2} \frac{R^2}{(R^2 + x^2)^{3/2}}$$

$$B = \frac{\mu_\circ I R^2}{2 x^3} \quad (\text{for } x \gg R) \tag{26.9}$$

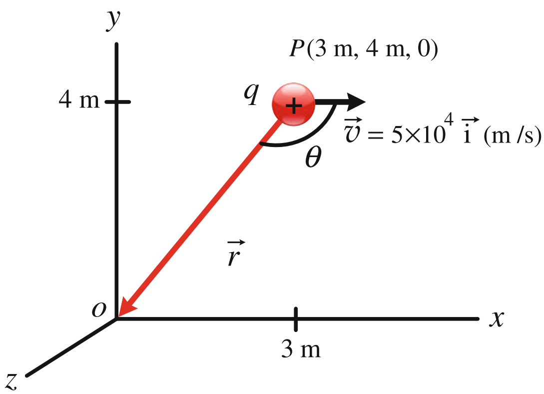
7. Sketch of $\vec{B}$ Along the Axis of a Loop and a bar Magnet



Example 26.1

A point charge $q = 6\,\mu\text{C}$ is moving in a straight line with a velocity $\vec{v} = 5 \times 10^4\,\vec{\text{i}}$ (m/s). When the charge is at the location $P(3\text{ m}, 4\text{ m}, 0)$, find the magnetic field produced by this point charge at the origin o, see Fig. 26.2.

Fig. 26.2



Solution: For a point charge q moving with a velocity $\vec{v}$, Eq. 26.1 leads to:

$$\vec{B} = \frac{\mu_\circ}{4\pi}\,\frac{q\,\vec{v} \times \hat{\vec{r}}}{r^2}$$

From the figure, we can find r and $\hat{\vec{r}}$ (from the point charge) as follows:

$$\vec{r} = [-3\,\vec{\mathrm{i}} - 4\,\vec{\mathrm{j}}\,](\mathrm{m})$$

$$r = \sqrt{(-3\,\mathrm{m})^2 + (-4\,\mathrm{m})^2} = 5\ \mathrm{m}$$

Thus: $$\hat{\vec{r}} = \frac{\vec{r}}{r} = \frac{[-3\,\vec{\mathrm{i}} - 4\,\vec{\mathrm{j}}\,](\mathrm{m})}{5\ \mathrm{m}} = -0.6\,\vec{\mathrm{i}} - 0.8\,\vec{\mathrm{j}}$$

Substituting the above results into the equation for $\vec{B}$ we obtain:

$$\vec{B} = \frac{\mu_\circ}{4\pi}\,\frac{q\,\vec{v} \times \hat{\vec{r}}}{r^2} = \frac{\mu_\circ}{4\pi}\,\frac{q\,(v\,\vec{\mathrm{i}}\,) \times [-0.6\,\vec{\mathrm{i}} - 0.8\,\vec{\mathrm{j}}\,]}{r^2} = -\frac{\mu_\circ}{4\pi}\,\frac{q\,v\,(0.8\,\vec{\mathrm{k}}\,)}{r^2}$$

$$= -(10^{-7}\ \mathrm{T.m/A})\frac{(6\times 10^{-6}\ \mathrm{C})(5\times 10^{4}\ \mathrm{m/s})(0.8)}{(5\ \mathrm{m})^2}\vec{\mathrm{k}}$$

$$= -9.6 \times 10^{-10}\ \vec{\mathrm{k}}\ (\mathrm{T})$$

Indeed this is a very small value for the magnetic field produced by this charge, which is equivalent to the charge of about 4×10^{13} protons.

Example 26.2

Two very long parallel straight wires carry currents that are perpendicular to the page. Wire ① carries a current $I_1 = 3$ A out of the page and passes through the origin o of the x-axis, while wire ② carries a current $I_2 = 2$ A into the page and passes through the x-axis at a distance $d = 0.6$ m from the origin. (a) On the x-axis, show the directions of the magnetic fields, to right of wire ②, between the two wires, and to the left of wire ①. (b) To the right of wire ②, find a distance a at which the resultant magnetic field is zero.

Solution: (a) Using the right hand rule presented in the figure of Eq. 26.6, we can draw the direction of $\vec{B}_1$ of wire ① and $\vec{B}_2$ of wire ② on the three regions of the x-axis as shown in Fig. 26.3:

(b) From Eq. 26.6 the magnitudes of the magnetic-field vectors $\vec{B}_1$ and $\vec{B}_2$ at point P are:

$$B_1 = \frac{\mu_\circ I_1}{2\pi(d+a)}, \quad \text{and} \quad B_2 = \frac{\mu_\circ I_2}{2\pi a}$$

For $I_1 > I_2$

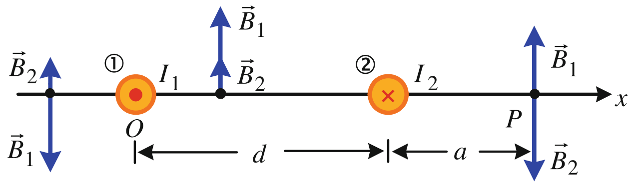


Fig. 26.3

When the magnitudes of the opposite two vectors $\vec{B}_1$ and $\vec{B}_2$ are equal, the resultant magnetic field becomes zero. Therefore, we have:

$$\frac{\mu_\circ I_1}{2\pi(d+a)} = \frac{\mu_\circ I_2}{2\pi a} \quad \Rightarrow \quad \frac{I_1}{d+a} = \frac{I_2}{a} \quad \Rightarrow \quad aI_1 = I_2(d+a)$$

$$\Rightarrow \quad a\left(\frac{I_1}{I_2} - 1\right) = d$$

Thus:

$$a = \frac{d}{\left(\frac{I_1}{I_2} - 1\right)} = \frac{0.6 \text{ m}}{\left(\frac{3 \text{ A}}{2 \text{ A}} - 1\right)} = 1.2 \text{ m}$$

Since $I_1 > I_2$, P is the only point at which $B_{\text{net}} = 0$ on the x-axis.

Example 26.3

Two straight wires ① and ③, each of length $L = 4$ cm, are connected by a quarter circular arc wire ② of radius $R = 3$ cm, as shown in Fig. 26.4. Determine the magnitude and direction of the magnetic field at the center P of the arc, when the current I is 2 A.

Solution: There is no contribution to the field at point P from the lower wire ①, since P is on the extension of the wire, i.e. $B_1 = 0$.

From Eq. 26.7, the quarter circular arc wire ② has a magnetic field:

$$B_2 = \frac{\mu_\circ I}{8R} \quad \text{(Directed out of the page)}$$

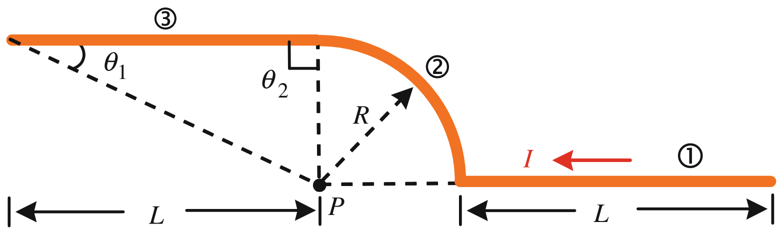


Fig. 26.4

According to Eq. 26.5, point P is at a distance $R = 3$ cm from the straight wire ③ and subtends two angles with the wire, θ_1 and θ_2. From the figure, we get:

$$\cos\theta_1 = L/\sqrt{L^2 + R^2} = 4/5 \quad \text{and} \quad \cos\theta_2 = \cos 90^\circ = 0$$

Thus: $$B_3 = \frac{\mu_\circ I}{4\pi R}(\cos\theta_1 + \cos\theta_2) = \frac{\mu_\circ I}{5\pi R} \quad \text{(Directed out of the page)}$$

The total magnetic field is the superposition of the fields from the three wires. Thus, the resultant magnetic field is:

$$\begin{aligned} B &= B_1 + B_2 + B_3 = 0 + \frac{\mu_\circ I}{8R} + \frac{\mu_\circ I}{5\pi R} \\ &= \frac{\mu_\circ I}{R}\left(\frac{1}{8} + \frac{1}{5\pi}\right) = \frac{(4\pi \times 10^{-7}\,\text{T.m/A})(2\text{A})}{3 \times 10^{-2}\,\text{m}}\left(\frac{1}{8} + \frac{1}{5\pi}\right) \\ &= 1.58 \times 10^{-5}\,\text{T} = 15.8\,\mu\text{T} \quad \text{(Directed out of the page)} \end{aligned}$$

26.2 The Magnetic Force Between Two Parallel Currents

Figure 26.5 shows a portion of length ℓ of two *long* straight parallel wires separated by a distance a and carrying currents I_1 and I_2 in the same direction. Since each wire lies in the magnetic field established by the other, each will experience a magnetic force.

Wire ② sets up a magnetic field $\vec{B}_2$ perpendicular to wire ①. According to Eq. 25.19, the magnetic force on a length ℓ of wire ① is $\vec{F}_1 = I_1\,\vec{\ell} \times \vec{B}_2$, where the direction of $\vec{F}_1$ is toward wire ②. Since $\vec{\ell} \perp \vec{B}_2$, the magnitude of $\vec{F}_1$ is $F_1 = I_1\,\ell\,B_2$. When we substitute with the magnitude of B_2 given by Eq. 26.6, we get:

$$F_1 = I_1 \, \ell \, B_2 = I_1 \, \ell \left(\frac{\mu_\circ \, I_2}{2\pi \, a} \right) = \frac{\mu_\circ \, I_1 \, I_2}{2\pi \, a} \ell \tag{26.10}$$

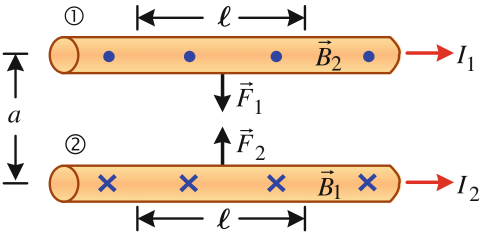


Fig. 26.5 Two parallel wires carrying currents in the same direction attract each other. Wire ② sets up a magnetic field $\vec{B}_2$ at wire ① and wire ① sets up a magnetic field $\vec{B}_1$ at wire ②

We can show that the magnetic force $\vec{F}_2$ on wire ② has the same magnitude as $\vec{F}_1$ but is opposite in direction, i.e. the two wires attract each other. We denote the magnitude of the force between the two wires by the symbol F_B and write this magnitude per unit length as:

$$\frac{F_B}{\ell} = \frac{\mu_\circ}{2\pi} \frac{I_1 I_2}{a} \tag{26.11}$$

If the two currents were antiparallel (i.e. the wires were parallel but the currents were opposite in direction), then the wires would repel.

Spotlight
Parallel currents attract and antiparallel currents repel.

Example 26.4

A battery of 12 V is connected to a resistor of resistance $R = 3\ \Omega$ by two parallel wires each of length $L = 50$ cm and separated by a distance $a = 2$ cm, see Fig. 26.6. All the connecting wires have negligible resistance. Find the magnitude of the magnetic force between the two wires. Will the wires repel or attract each other?

Solution: According to the figure, the battery sets a clockwise current I in the circuit, and the current in the parallel two wires have the same value but opposite direction. The value of this current is:

$$I = \frac{\Delta V}{R} = \frac{12\ \text{V}}{3\ \Omega} = 4\ \text{A}$$

From Eq. 26.11, the magnetic force between the two wires is:

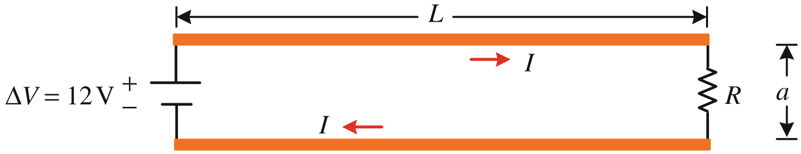


Fig. 26.6

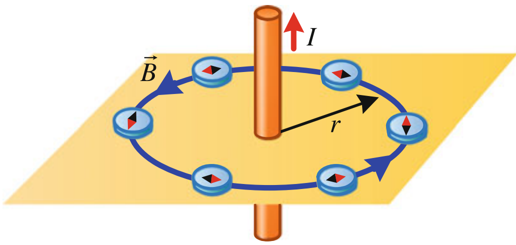
$$F_B = \frac{\mu_\circ}{2\pi}\frac{I^2}{a}L = \frac{4\pi \times 10^{-7}\,\mathrm{T.m/A}}{2\pi}\frac{(4\,\mathrm{A})^2}{2 \times 10^{-2}\,\mathrm{m}} \times 50 \times 10^{-2}\,\mathrm{m}$$
$$= 8 \times 10^{-5}\,\mathrm{N}$$

Since the currents in the two wires are antiparallel, the wires will repel each other, but with a very small force due the smallness of $\mu_\circ$.

26.3 Ampere's Law

When Oersted traced the magnetic field near a long vertical wire carrying a current I by a compass, he found that its needle deflects in a direction tangent to any circular path concentric with the wire, i.e. the needle points in the direction of $\vec{B}$, see Fig. 26.7.

Fig. 26.7 The compass needle deflects in a direction tangent to a circle of radius r, which is the direction of $\vec{B}$ created by I



The same results can be obtained when we use the Biot-Savart Law to calculate the magnetic field around a long straight wire carrying a current. The magnitude of $\vec{B}$ was given by Eq. 26.6.

The work of Oersted and Biot-Savart was continued by Ampere. Ampere's work lead to what is now known as Ampere's law, a law used in the cases of steady currents, which can be stated as follows:

Ampere's law

The *line integral* of the *tangential* magnetic field around a closed path is proportional to the net *conduction steady* current I enclosed by the path. That is:

$$\oint \vec{B} \bullet d\vec{s} = \mu_\circ I \quad \text{(Ampere's law)} \tag{26.12}$$

As a check for the long wire of Fig. 26.7, let us consider an element $d\vec{s}$ on the circular path and integrate the product $\vec{B} \bullet d\vec{s}$ over this closed path. Since $\vec{B}$ is parallel to $d\vec{s}$, then $\vec{B} \bullet d\vec{s} = B\,ds$. Thus:

$$\oint \vec{B} \bullet d\vec{s} = \oint B\,ds = B \oint ds = B(2\pi r) \tag{26.13}$$

By Ampere's law, this result should be equal to $\mu_\circ I$. Therefore:

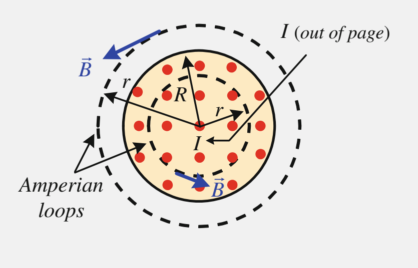
$$B = \frac{\mu_\circ I}{2\pi r} \tag{26.14}$$

This result is in complete agreement with Eq. 26.6 obtained by using the Biot-Savart law; however, Ampere's law saves considerable effort when we deal with problems that have some ***symmetry***.

Some Applications of Ampere's Law

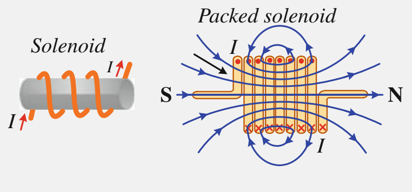
In these applications, we avoid solving the integrand of Eq. 26.12 and only present the results of some well-known cases.

1. Magnetic Field Inside and Outside a Long Straight Wire



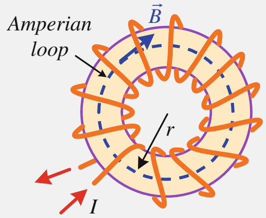
$$B = \begin{cases} \dfrac{\mu_\circ I}{2\pi R^2} r & (\text{for } r \le R) \\ \dfrac{\mu_\circ I}{2\pi r} & (\text{for } r \ge R) \end{cases} \tag{26.15}$$

2. Magnetic Field of a Solenoid of *n* Turns per Unit Length



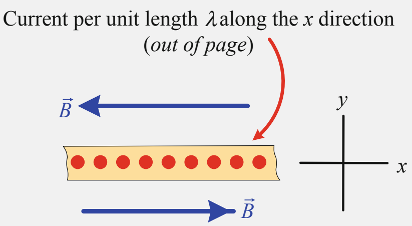
$$B = \mu_{\circ}\, n\, I \left\{ \begin{array}{l} \text{inside the} \\ \text{solenoid and} \\ \text{if } n \text{ is large} \end{array} \right\} \tag{26.16}$$

3. Magnetic Field of a Toroid of *N* Total Turns (or *n* turns/m)



$$B = \frac{\mu_{\circ} N I}{2\pi\, r} = \mu_{\circ}\, n\, I \tag{26.17}$$

4. Magnetic Field Produced by an Infinite Current Sheet



$$B = \frac{\mu_{\circ} \lambda}{2} \tag{26.18}$$

Example 26.5

A long wire of radius $R = 10$ mm carries a current $I = 3$ A. What are the magnitudes of the magnetic field at a point 5 mm and a point 50 mm from the axis of the wire?

Solution: For a point inside the wire we use Eq. 26.15 for $r \leq R$:

$$B = \frac{\mu_\circ I}{2\pi R^2}\, r = \frac{(4\pi \times 10^{-7}\,\text{T.m/A})(3\,\text{A})}{(2\pi)(10 \times 10^{-3}\,\text{m})^2} \times (5 \times 10^{-3}\,\text{m}) = 3 \times 10^{-5}\,\text{T}$$

For a point outside the wire we use Eq. 26.15 for $r \geq R$:

$$B = \frac{\mu_\circ I}{2\pi r} = \frac{(4\pi \times 10^{-7}\,\text{T.m/A})(3\,\text{A})}{(2\pi)(50 \times 10^{-3}\,\text{m})} = 1.2 \times 10^{-5}\,\text{T}$$

Example 26.6

A solenoid of length $L = 0.5$ m carries a current $I = 2$ A. The solenoid consists of six closely-packed layers, each of 800 turns. What is the magnitude of the magnetic field inside the solenoid?

Solution: The diameter of winding does not enter into the solenoid Eq. 26.16. The number of turns per unit length is:

$$n = \frac{(\text{No. of layers})(\text{No. of turns per layer})}{L} = \frac{6 \times 800\,\text{turns}}{0.5\,\text{m}} = 9{,}600\,\text{turns/m}$$

Since n is large, then from Eq. 26.16 we have:

$$B = \mu_\circ n I = (4\pi \times 10^{-7}\,\text{T.m/A})(9{,}600\,\text{turns/m})(2\,\text{A}) = 2.41 \times 10^{-2}\,\text{T}$$

Example 26.7

In a fusion reactor, a toroid has inner and outer radii $a = 0.5$ m and $b = 1.5$ m, respectively. The toroid has 900 turns and carries a current of 12 kA. What is the magnitude of the magnetic field at a point located on a circle having the average radius of the toroid?

Solution: With $R = (a + b)/2 = (0.5 + 1.5)/2 = 1$ m, Eq. 26.17 gives:

$$B = \frac{\mu_\circ N\,I}{2\pi R} = \frac{(4\pi \times 10^{-7}\ \text{T.m/A})(900\ \text{turns})(12 \times 10^3\ \text{A})}{(2\pi)(1\ \text{m})} = 2.16\ \text{T}$$

26.4 Displacement Current and the Ampere-Maxwell Law

Ampere's law is incomplete when the conduction current is not steady. We can show this by considering the region near a parallel-plate capacitor while the capacitor is charging, see Fig. 26.8a. A variable conduction current $i = dq/dt$ reaches one plate and the same conduction current i leaves the other plate. There is no current flow across the space between the plates. Experiments show the establishment of a magnetic field between the two plates as well as on both sides of the plates. In addition, experiments show that the value of $\oint \vec{B} \bullet d\vec{s}$ is the same for the three circular loops labeled ①, ②, and ③ in Fig. 26.8a. But according to Ampere's law, $\oint \vec{B} \bullet d\vec{s}$ must be zero for loop ②, because the conduction current is zero.

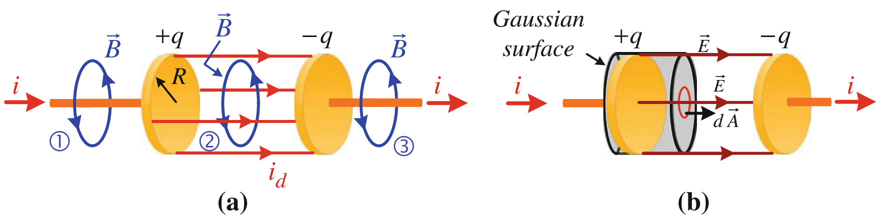


Fig. 26.8 (a) The displacement current i_d between the plates of a capacitor. (b) The Gaussian surface that encloses the varying charge q

Maxwell solved this problem by postulating an additional term to the right side of Ampere's law that is related to the changing electric field between the plates of the capacitor. This term is referred to as the *displacement current* i_d between the plates. This current is defined as:

$$i_d = \epsilon_\circ \frac{d\Phi_E}{dt} \tag{26.19}$$

The displacement current i_d between the plates is equivalent to the conduction current i in the wires, i.e. $i_d = i$, and hence produces the same magnetic effects observed experimentally, see Fig. 26.8a.

Maxwell added the displacement current i_d to the varying conduction current i and expressed Ampere's law as follows:

$$\oint \vec{B} \bullet d\vec{s} = \mu_\circ (i + i_d) = \mu_\circ \left(i + \epsilon_\circ \frac{d\Phi_E}{dt} \right) \quad \text{(Ampere–Maxwell law)} \tag{26.20}$$

When there is a conduction current but no change in electric flux (only like loops ① and ③), the second term is zero. When there is a change in electric flux but no conduction current (only like loop ②), the first term is zero.

Spotlight

Magnetic fields are produced both by conduction currents i and by displacement currents i_d, created by a time varying electric flux.

To establish the relation Eq. 26.20, we apply Gauss's law for the Gaussian surface shown in Fig. 26.8b. According to Gauss's law, see Eq. 21.7, this surface encloses a net charge q, and we have:

$$\Phi_E = \oint \vec{E} \bullet d\vec{A} = \frac{q}{\epsilon_\circ} \tag{26.21}$$

As q changes, $\vec{E}$ changes too, and the rate at which q changes gives the displacement current postulated by Maxwell. Thus:

$$i_d = \frac{dq}{dt} = \epsilon_\circ \frac{d\Phi_E}{dt} \tag{26.22}$$

Example 26.8

The circular capacitor of Fig. 26.8 a has a radius $R = 10\,\text{cm}$ and a charge $q = (4 \times 10^{-4}\,\text{C}) \sin(2 \times 10^4\, t)$ that varies with time t. In the region between the plates, find the displacement current and the maximum value of the magnetic field at radius $r = 15\,\text{cm}$.

Solution: From Eq. 26.22, we find the displacement current as:

$$i_d = \frac{dq}{dt} = \frac{d}{dt}[(4 \times 10^{-4}\,\text{C}) \sin(2 \times 10^4\, t)] = (8\,\text{A}) \cos(2 \times 10^4\, t)$$

For a maximum displacement current $(i_d)_{\text{max}}$ of 8 A at a point between the plates, we use Eq. 26.15 for $r \geq R$ to find B_{max}:

$$B_{\text{max}} = \frac{\mu_\circ (i_d)_{\text{max}}}{2\pi r} = \frac{(4\pi \times 10^{-7}\,\text{T.m/A})(8\,\text{A})}{(2\pi)(15 \times 10^{-2}\,\text{m})} = 1.07 \times 10^{-5}\,\text{T}$$

26.5 Gauss's Law for Magnetism

As in the case of an electric flux, we calculate the magnetic flux throughout a particular surface S, see Fig. 26.9, as follows:

$$\Phi_B = \int \vec{B} \bullet d\vec{A} \tag{26.23}$$

The SI unit for the magnetic flux is tesla-square meter, which is called weber (abbreviated Wb). Thus, 1 weber = 1 Wb = 1 Tm^2.

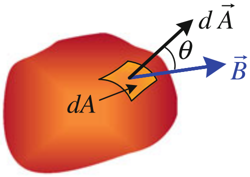


Fig. 26.9 The differential surface vector area $d\vec{A}$ is perpendicular to the differential area dA and pointing outwards. When the magnetic field $\vec{B}$ makes an angle θ with $d\vec{A}$, the differential flux $d\Phi_B$ is $\vec{B} \bullet d\vec{A}$

Since magnetic fields form closed loops, i.e. the magnetic field lines do not begin or end at any point, and for a closed surface the number of lines entering that surface equals the number of lines leaving it. Thus, the net magnetic flux over a closed surface is zero. This is known as Gauss's law for magnetism and can be stated as:

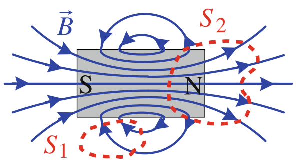
Gauss's Law for Magnetism
The *net magnetic flux* throughout any closed surface is always zero:

$$\oint \vec{B} \bullet d\vec{A} = 0 \quad \text{(Gauss's law for magnetism)} \tag{26.24}$$

Example 26.9

Find the net magnetic flux through the closed surfaces S_1 and S_2 of Fig. 26.10, which are represented by dashed lines intersecting the page.

Fig. 26.10



Solution: According to Gauss's law for magnetism, we must have:

$$\oint_{S_1} \vec{B} \bullet d\vec{A} = 0, \quad \text{and} \quad \oint_{S_2} \vec{B} \bullet d\vec{A} = 0$$

Notice that surface S_2 encloses only the north pole of the magnet, and that the south pole is associated with the left boundary of S_2.

26.6 The Origin of Magnetism

We have seen how to generate a magnetic field by allowing an electric current to pass through a wire. Moreover, we found that the magnetic pattern of a circular current loop has a North Pole and a South Pole with a magnetic dipole moment $\vec{\mu}$ producing a magnetic pattern that looks like the magnetic pattern produced by a bar magnet. (Searches for magnetic monopoles in cosmic rays or elsewhere have been negative.)

In addition, there are two subatomic ways that produce a magnetic field in space, each one involving a magnetic dipole moment. These require an understanding of *quantum physics*, which is beyond the scope of this study. Therefore, we shall only begin our study by presenting the results of the classical model of atoms and electrons.

Orbital Magnetic Dipole Moments of Atoms

In the *classical Bohr model of hydrogen atoms*, we assume that an electron of mass m_e and charge $-e$ moves around a fixed nucleus with a constant speed v in a circular orbit of radius r, see Fig. 26.11.

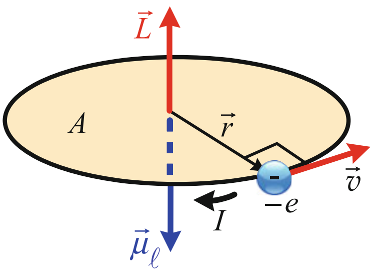


Fig. 26.11 The classical model of a hydrogen atom, where an electron moves with a constant speed in a circular orbit about a nucleus. The direction of the associated current is opposite to the direction of the electron's motion

Because the electron travels a circumference $2\pi r$ in an interval of time $T = 2\pi r/v$, the current I associated with this motion is:

$$I = \frac{e}{T} = \frac{ev}{2\pi r} \tag{26.25}$$

The magnitude of the orbital magnetic dipole moment associated with this orbiting current is $\mu_\ell = IA = \pi r^2 I$, where A is the circular area enclosed by the electron's orbit. Thus, using Eq. 26.25, we get:

$$\mu_\ell = \pi r^2 I = \tfrac{1}{2} e\, v\, r \tag{26.26}$$

From the definition of the orbital angular momentum $\vec{L} = \vec{r} \times \vec{p}$, where $\vec{p} = m_e \vec{v}$ is the momentum of the electron, we see that the angle between $\vec{r}$ and $\vec{p}$ is 90°. Then $L = m_e v r$ and μ_ℓ and L are given by:

$$\mu_\ell = \frac{e}{2m_e} L \tag{26.27}$$

Because the electron is a negatively charged particle, the vectors $\vec{\mu}_\ell$ and $\vec{L}$ are opposite to each other, see Fig. 26.11. Thus:

$$\vec{\mu}_\ell = -\frac{e}{2m_e} \vec{L} \tag{26.28}$$

The orbital angular momentum $\vec{L}$ cannot be measured. Instead, only its components along an axis can be measured. A fundamental outcome of quantum physics is that the orbital angular momentum and its components are quantized (which means having discrete restricted values). The quantization rules of $\vec{L}$ and its component along the z axis, L_z, have only the values given by:

$$\begin{aligned} L &= \sqrt{\ell(\ell+1)}\,\hbar, && (\ell = 0, 1, 2, \ldots) \\ L_z &= m_\ell\,\hbar, && (m_\ell = -\ell, \ldots, -1, 0, +1, \ldots, +\ell) \end{aligned} \tag{26.29}$$

where ℓ is the *orbital quantum number*, m_ℓ is the *orbital magnetic quantum number*, $\hbar = h/2\pi$, and h is an ever-present constant in quantum physics known as Planck's constant, which has the value:

$$h = 6.63 \times 10^{-34}\ \text{J.s} \quad \text{and} \quad \hbar = 1.05 \times 10^{-34}\ \text{J.s} \tag{26.30}$$

Figure 26.12 displays a vector model for the orbital angular momentum in case of $\ell = 1$.

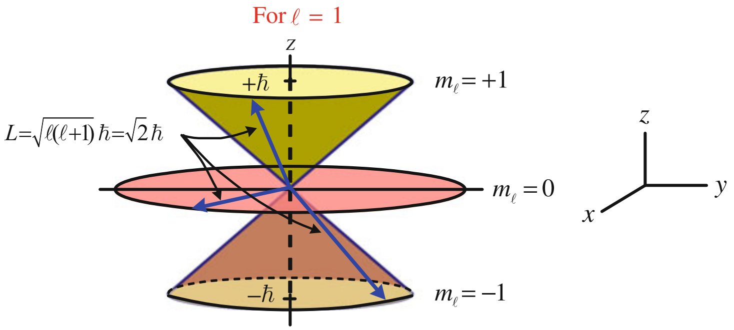


Fig. 26.12 For every value of $L_z = m_\ell\, \hbar$, there is an equal probability of finding $\vec{L}$ anywhere on the surface of a symmetrical cone about the z axis. The vector $\vec{L}$ rotates randomly about this axis, such that it has a constant value $\sqrt{\ell(\ell+1)}\,\hbar$ and a constant component $L_z = m_\ell\, \hbar$, but L_x and L_y are unknown and satisfy the average values $\overline{L}_x = \overline{L}_y = 0$

We can relate the component $\mu_{\ell,z}$ to Lz by rewriting Eq. 26.28 in component form as follows:

$$\mu_{\ell,z} = -m_\ell \frac{e\hbar}{2m_e} = -m_\ell\, \mu_B \tag{26.31}$$

where the quantity μ_B is called the Bohr magneton and is given by:

$$\mu_B = \frac{e\hbar}{2m_e} = 9.27 \times 10^{-24}\ \text{J/T}(\equiv \text{A.m}^2) = 5.79 \times 10^{-5}\ \text{eV/T} \tag{26.32}$$

When an electron is placed in an external magnetic field $\vec{B}$, a torque $\vec{\tau} = \vec{\mu}_\ell \times \vec{B}$ is exerted on its orbital magnetic dipole moment. This reminds us of the corresponding equation for the torque exerted by an electric field $\vec{E}$ on an electric dipole moment $\vec{p}$, $\tau = \vec{p} \times \vec{E}$; see Eq. 22.39. In each case, the torque exerted by the field (either $\vec{B}$ or $\vec{E}$) is equal to the vector product of the dipole moment and the field. In strict analogy to the $U = -\vec{p} \bullet \vec{E}$, see Eq. 22.42, a potential energy U_ℓ can be associated with the orientation of the orbital magnetic dipole moment $\vec{\mu}_\ell$, and it is given by:

$$U_\ell = -\vec{\mu}_\ell \bullet \vec{B} \tag{26.33}$$

If the direction of the magnetic field is taken to be along the z-axis, then the orientation potential energy can be written as:

$$U_\ell = -\mu_{\ell,z} B \tag{26.34}$$

Quantization of the component of the orbital magnetic moment gives:

$$U_\ell = +m_\ell \frac{e\hbar}{2m_e} B \quad \text{or} \quad U_\ell = +m_\ell\, \mu_B B \tag{26.35}$$

We used the word "orbital" in both classical and quantum studies, but in quantum physics we must make it clear that all electrons do not orbit the atomic nucleus like planets orbiting the sun.

Although all materials contain electrons, most of them do not exhibit magnetic properties. The main reason is due to the *cancelation of the randomly oriented orbital magnetic dipole moments of atoms*. Then, for most materials the magnetic effect produced by the electronic orbital motion is either zero or very small.

Spin Magnetic Dipole Moments of Electrons

In addition to the orbital angular momentum $\vec{L}$, an electron has an intrinsic angular momentum called the **spin angular momentum** (or just **spin**) $\vec{S}$. The vector $\vec{S}$ is a purely quantum-mechanical physical quantity that has no classical analog. Associated with this spin is an **intrinsic-spin magnetic dipole moment** $\vec{\mu}_s$. Experiments indicate that the $\vec{S}$ and S_z are quantized and related to $\vec{\mu}_s$ and $\mu_{s,z}$ as follows:

$$\begin{array}{llll} S = \sqrt{s(s+1)}\,\hbar \quad (s = \frac{1}{2}), & S_z = m_s\,\hbar & (m_s = -\frac{1}{2}, +\frac{1}{2}) \\ \vec{\mu}_s = -\frac{e}{m_e}\vec{S}\,, & \mu_{s,z} = -2m_s\mu_B & (m_s = -\frac{1}{2}, +\frac{1}{2}) \end{array} \tag{26.36}$$

where s is the *spin quantum number* and m_s is the *spin-projection magnetic quantum number*. There are two possibilities of finding the atomic electron, either in a state with $m_s = -\frac{1}{2}$ or in a state with $m_s = +\frac{1}{2}$.

When the electron is placed in an external magnetic field $\vec{B}$, the potential energy U_s associated with orientation of the spin magnetic dipole moment $\vec{\mu}_s$ is similarly given by:

$$U_s = -\vec{\mu}_s \bullet \vec{B} \tag{26.37}$$

When $\vec{B}$ is along the z-axis, $\mu_{s,z}$ can take only two possible values (up or down), and hence, the potential energy U_s takes the two values:

$$U_s = -\mu_{s,z}\,B = \pm\mu_B\,B = \begin{cases} +\mu_B\,B & \text{if } m_s = +\frac{1}{2} \quad (\text{then } \mu_{s,z} = -\mu_B) \\ -\mu_B\,B & \text{if } m_s = -\frac{1}{2} \quad (\text{then } \mu_{s,z} = +\mu_B) \end{cases} \tag{26.38}$$

In both cases, $\vec{S}$ will rotate about $\vec{B}$ with angular frequency given by:

$$\vec{\omega} = \frac{\mu_B}{\hbar}\vec{B} \tag{26.39}$$

In addition, the lowest energy $(-\mu_B B)$ occurs when $\mu_{s,z}$ is lined up with $\vec{B}$ and the highest energy $(+\mu_B B)$ occurs when $\mu_{s,z}$ is in the opposite direction of $\vec{B}$, see Fig. 26.13. The difference in energy between these two orientation levels is $\Delta U_s = 2\,\mu_B\,B$.

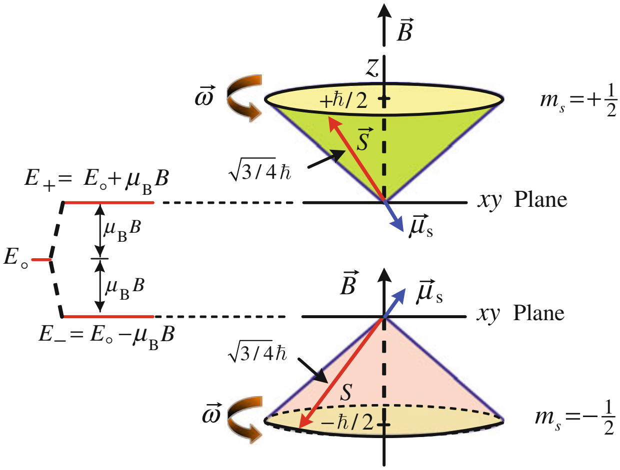


Fig. 26.13 In the presence of a magnetic field $\vec{B}$, the energy $E_\circ$ of the electron splits into two levels with a difference of $2\,\mu_B\,B$. In each level, $\vec{S}$ (or $\vec{\mu}_s$) will rotate about $\vec{B}$ with angular frequency $\vec{\omega} = \mu_B\,\vec{B}/\hbar$

Protons and neutrons have intrinsic magnetic dipole moments given by similar formulas, but are an order of 10^3 smaller than that of the electron. This is because the mass of proton m_p and the mass of neutron m_n are much greater than the mass of the electron m_e.

26.7 Magnetic Materials

Some materials exhibit weak magnetic properties, and others exhibit strong magnetic properties due to the alignment of the magnetic moments of their atoms. We consider a small volume V of one of these materials and assume that the magnetic moment of a typical atom/molecule is $\vec{\mu}_{\text{atomic}}$. Then the total magnetic moment within V is the vector sum $\sum \vec{\mu}_{\text{atomic}}$. The magnetic state of this material is described by a quantity called the **magnetization vector** $\vec{M}$ and is defined as:

$$\vec{M} = \frac{\sum \vec{\mu}_{\text{atomic}}}{V} \tag{26.40}$$

Spotlight

The magnetization of a material is defined as the magnetic moment per unit volume.

The unit of magnetization is A/m. If the atomic magnetic dipole moments of a magnetic material are randomly oriented, or there are none, then $\sum \vec{\mu}_{\text{atomic}} = 0$ and $\vec{M} = 0$.

Consider a region in which a current-carrying conductor produces a magnetic field $\vec{B}_\circ$. If this region is filled with a magnetic material that produces a magnetic field $\vec{B}_M$, then the total magnetic field in this region will be:

$$\vec{B} = \vec{B}_\circ + \vec{B}_M \tag{26.41}$$

To find the relation between $\vec{B}_M$ and $\vec{M}$, we consider a solenoid of length L having N turns and carrying a current I. In vacuum the magnetic field inside the solenoid is given by Eq. 26.16 as $B_\circ = \mu_\circ n I = \mu_\circ N I/L$. Multiplying and dividing the right hand side of this equation by the cross-sectional area A of the solenoid allows us to write this equation in terms of the total magnetic moment of all the solenoid loops $\sum \mu_{\text{coil}} = N\ I\ A$ and the solenoid volume $V = LA$ as:

$$B_\circ = \mu_\circ n\, I = \mu_\circ \frac{N\ I\ A}{LA} = \mu_\circ \frac{\sum \mu_{\text{coil}}}{V} \tag{26.42}$$

This relation can be written in vector form as:

$$\vec{B}_\circ = \mu_\circ \frac{\sum \vec{\mu}_{\text{coil}}}{V} \tag{26.43}$$

When a magnetic material fills the solenoid, the contribution resulting from the alignment of the atomic-induced magnetic dipole moments $\sum \vec{\mu}_{\text{atomic}}$ produces a magnetic field $\vec{B}_M$ that can be written in a form similar to Eq. 26.43 as:

$$\vec{B}_M = \mu_\circ \frac{\sum \vec{\mu}_{\text{atomic}}}{V} \tag{26.44}$$

The ratio $\sum \vec{\mu}_{\text{atomic}}/V$ was defined in Eq. 26.40 as the magnetization vector $\vec{M}$ of the magnetic material. Thus:

$$\vec{B}_M = \mu_\circ \vec{M} \tag{26.45}$$

Therefore, the total magnetic field inside the solenoid will be:

$$\vec{B} = \vec{B}_\circ + \mu_\circ \vec{M} \tag{26.46}$$

In Eq. 26.43, it is convenient to introduce the **magnetic field strength** $\vec{H} = \sum \vec{\mu}_{\text{coil}}/V$. This field is a quantity related to the magnetic field resulting from the conduction current. Therefore:

$$\vec{B}_\circ = \mu_\circ \vec{H} \tag{26.47}$$

Thus, Eq. 26.46 can be written as:

$$\vec{B} = \mu_\circ (\vec{H} + \vec{M}) \tag{26.48}$$

Note that $\vec{B}$ is composed of $\mu_\circ \vec{H}$ (associated with the conduction current) and $\mu_\circ \vec{M}$ (resulting from the magnetization of the material that fills the solenoid). Since $B_\circ = \mu_\circ n I$ and $B_\circ = \mu_\circ H$, then:

$$H = n I \quad \text{(Solenoid or a toroid)} \tag{26.49}$$

Magnetic materials are classified into three categories:

Diamagnetic	*where atoms have **no** permanent magnetic moments*
Paramagnetic } Ferromagnetic }	*where atoms have permanent magnetic moments*

26.8 Diamagnetism and Paramagnetism

When a diamagnetic or paramagnetic material is placed in an external magnetic field, the magnetization vector $\vec{M}$ is proportional to the magnetic field strength $\vec{H}$, and we can write:

$$\vec{M} = \chi \vec{H} \tag{26.50}$$

where χ is a dimensionless factor called the **magnetic susceptibility**, which measures the responsiveness of a material to being magnetized.

Substituting Eq. 26.50 for $\vec{M}$ into Eq. 26.48 gives:

$$\vec{B} = \mu_\circ(\vec{H} + \vec{M}) = \mu_\circ(\vec{H} + \chi\vec{H}) = \mu_\circ(1+\chi)\vec{H} \tag{26.51}$$

or:

$$\vec{B} = \mu_m \vec{H} \tag{26.52}$$

where μ_m is called the **magnetic permeability** of the material and is related to its magnetic susceptibility χ by the relation:

$$\mu_m = \mu_\circ(1+\chi) \begin{cases} <\mu_\circ & \text{For diamagnetic materials} \\ >\mu_\circ & \text{For paramagnetic materials} \end{cases} \tag{26.53}$$

The factor $K_m = \mu_m/\mu_\circ$ is called the relative permeability of the material.

Diamagnetic Materials

A material is considered diamagnetic if its atoms have zero net angular momentum and hence no permanent magnetic moment. Diamagnetic materials interact weakly with the applied magnetic field, in which case χ is very small negative value and $\vec{M}$ is opposite to $\vec{H}$. This causes diamagnetic materials to be weakly repelled by a magnet. Diamagnetism is present in all materials, but its effects are much smaller than those in paramagnetic or ferromagnetic materials.

To understand this interaction we consider the motion of two electrons orbiting a nucleus with the same speed but in opposite directions, see Fig. 26.14a. The magnetic moments of the two electrons in this figure are in opposite directions and therefore cancel.

In the presence of a uniform magnetic field $\vec{B}$ directed out of the page, as shown in Fig. 26.14b, both of the electrons experience an extra magnetic force $(-e)\,\vec{v} \times \vec{B}$. Thus:

- For the electron in the left of Fig. 26.14b, the extra magnetic force is radially inward, increasing the centripetal force. If this electron is to remain in the same circular path, it must speed up to $\vec{v}\,'$, so that mv'^2/r equals the total newly increased centripetal force. Therefore, its *inward magnetic moment thus increases*.

- For the electron in the right of Fig. 26.14b, the extra magnetic force is radially outward, decreasing the centripetal force. If this electron is to remain in the same circular path, it must slow down to $\vec{v}''$, so that mv''^2/r equals the total newly decreased centripetal force. Therefore, its *outward magnetic moment thus decreases*.

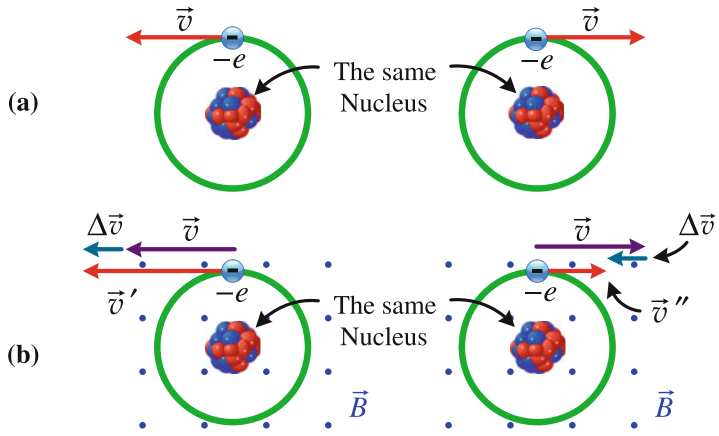


Fig. 26.14 (a) Two atomic electrons orbiting a fixed nucleus with the same speed but in opposite directions (separated for clarity). (b) When a magnetic field is applied out of the page, the magnetic force increases the speed of the left electron and decreases the speed of the right one

As a result, the change in the magnetic moment of the two electrons is into the page, opposite to the external applied magnetic field. Because the permanent magnetic moments of the two electrons cancel each other, only an induced magnetic moment opposite to the applied magnetic field will remain. The induced magnetic moments that cause diamagnetism are of the order of $10^{-5}\,\mu_B$. This value is much smaller than that of the permanent magnetic moments of the atoms of paramagnetic and ferromagnetic materials. However, the alignments produced in the diamagnetism decrease with temperature. Therefore, diamagnetism disappears in all materials at sufficiently high temperatures.

Certain types of superconductors (a substance of zero electric resistance) exhibit diamagnetism below some critical temperature. As a result, the superconductor can repel a permanent magnet.

Paramagnetic Materials

Atoms of paramagnetic materials have permanent magnetic moments that interact with each other very weakly, resulting in a very small positive magnetic susceptibility χ. Therefore, $\vec{M}$ is in the same direction as $\vec{H}$. However, the thermal motion of the molecules reduces the alignments, and this tends to randomize the magnetic dipole moments' orientations. The degree to which the magnetic moments line up with an external magnetic field depends on the strength of the field and on the temperature.

Even in a very strong magnetic field B of 1 T and a typical atomic magnetic moment μ of 1 μ_B, the difference in potential energy ΔU when the magnetic moment is parallel the field (lower energy) and when the moment antiparallel the field (higher energy) is:

$$\Delta U = 2\mu_B B = 2 \times (5.79 \times 10^{-5}\ \text{eV/T})(1\ \text{T}) = 1.2 \times 10^{-4}\ \text{eV}$$

At a normal temperature $T = 300$ K, the typical thermal energy $k_B T$ is:

$$k_B T = (8.62 \times 10^{-5}\ \text{eV/T})(300\ \text{K}) = 2.6 \times 10^{-2}\ \text{eV}$$

Therefore, $k_B T \gtrsim 200\ \ \Delta U$. Thus, at room temperature and even in a very strong magnetic field, most of the magnetic moments will be randomly oriented unless the temperature is very low.

In 1895, Pierre Curie discovered that M is directly proportional to the external magnetic field $B_\circ$ and inversely proportional to the kelvin temperature, when $B_\circ/T$ is very small; that is:

$$M = C\frac{B_\circ}{T} \quad \text{(Curie's law)} \tag{26.54}$$

where the constant C is a known as Curie's constant. This law shows that $M = 0$ when $B_\circ = 0$. Even if $B_\circ$ is very large ($\sim$2 T), deviation from Curie's law can be observed at extremely low temperatures (i.e. at a few kelvins). In addition, as $B_\circ$ increases (or T decreases), Eq. 26.54 will no longer be valid, and quantum physics indicates that the magnetization M approaches some maximum value $M_{\max}$, which corresponds to a complete alignment of all permanent magnetic dipole moments.

Table 26.1 gives the magnetic susceptibility of some materials.

Table 26.1 Magnetic susceptibility of some diamagnetic and paramagnetic materials at 300 K

Diamagnetic material	χ	Paramagnetic material	χ
Bismuth	-1.7×10^{-5}	Aluminum	2.3×10^{-5}
Carbon (graphite)	-1.4×10^{-5}	Calcium	1.9×10^{-5}
Copper	-9.8×10^{-6}	Chromium	2.7×10^{-4}
Carbon (Diamond)	-2.2×10^{-5}	Lithium	2.1×10^{-5}
Gold	-3.6×10^{-5}	Magnesium	1.2×10^{-5}
Lead	-1.7×10^{-5}	Niobium	2.6×10^{-4}
Mercury	-2.9×10^{-5}	Oxygen	2.1×10^{-6}
Nitrogen	-5.0×10^{-9}	Platinum	2.9×10^{-4}
Silver	-2.6×10^{-5}	Potassium	5.8×10^{-6}
Silicon	-4.2×10^{-6}	Tungsten	6.8×10^{-5}

26.9 Ferromagnetism

Materials such as iron, cobalt, nickel, gadolinium, dysprosium, and alloys containing these materials usually exhibit strong magnetic properties and are called **ferromagnetic materials**. These materials contain permanent atomic magnetic moments that tend to align even in the presence of a weak external magnetic field and remain magnetized after the magnetic field is removed. These alignments can only be understood in quantum-mechanical terms.

Consider a specimen of ferromagnetic material, such as iron in its crystalline form. Such a crystal would be made of microscopic regions called **magnetic domains**. Each domain would be less than 1 mm wide and would have all its atomic magnetic moments aligned. The boundaries between domains that have different magnetic-moment orientations are called **domain walls**. Depending on the structure and type of the material, the volume of each magnetic domain would vary from about 10^{-12} to 10^{-8} m^3 and contain about 10^{18} to 10^{22} molecules.

If magnetic domains of a particular ferromagnetic material specimen are randomly oriented as shown in Fig. 26.15a, then the entire specimen would not display a net magnetic dipole moment.

As the unmagnetized ferromagnetic specimen is placed in an external magnetic field $\vec{B}_\circ$ that increases gradually, then the specimen would experience the following two types of domain interactions:

- Reversible magnetization by domain growth:
  When the applied magnetic field $\overrightarrow{B}_\circ$ is weak, a growth in volume of the domains that are oriented along $\overrightarrow{B}_\circ$ occurs at the expense of those that are not, see Fig. 26.15b. In this case, the specimen is magnetized, and this magnetization is reversible. That is, we have *reversible domains* when $\overrightarrow{B}_\circ$ is removed.
- Irreversible magnetization by domain alignments and rotations:
  As the applied magnetic field $\overrightarrow{B}_\circ$ strengthens, the domains align even more, and after a particular threshold the material manifests *irreversible domains* if $\overrightarrow{B}_\circ$ is removed. But if the magnetic field $\overrightarrow{B}_\circ$ becomes even stronger, the *irreversible domains rotate* and start to align more and more in the direction of $\overrightarrow{B}_\circ$, see Fig. 26.15c. In both cases, the specimen remains magnetized at ordinary temperatures even after $\overrightarrow{B}_\circ$ is removed.

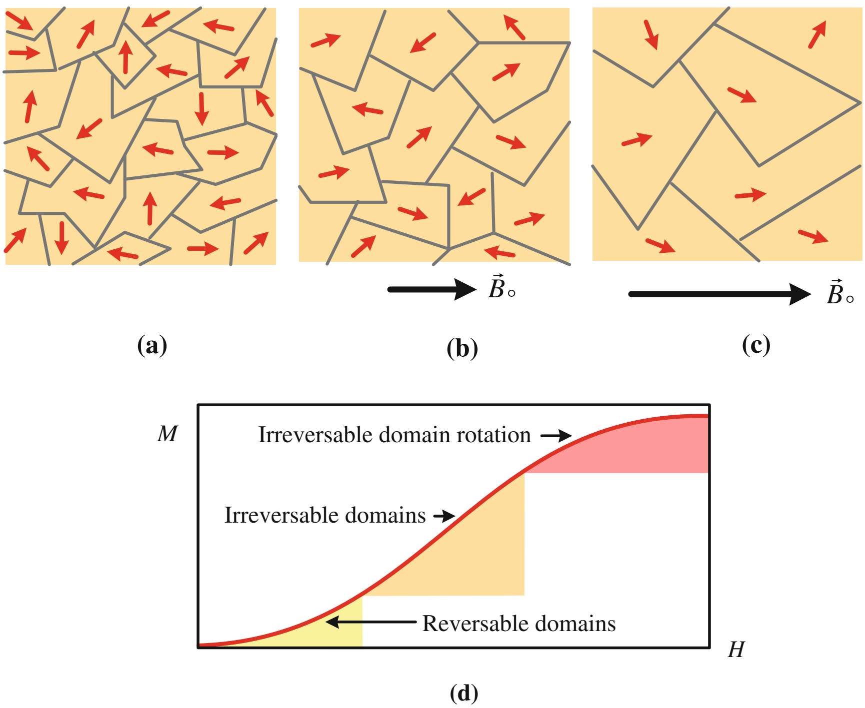


Fig. 26.15 (a) An unmagnetized specimen having magnetic domains with random magnetic dipole orientations. (b) A growth in volume of domains that are oriented along $\overrightarrow{B}_\circ$. (c) When the magnetic field becomes much stronger, the *domains rotate* and align more in the direction of $\overrightarrow{B}_\circ$. (d) Variation of the magnetization M as a function of H (or $B_\circ = \mu_\circ H$). As H increases, the domains become more and more aligned until saturation is reached

For a ferromagnetic material, χ and hence μ_M are very large, but the relation between $\vec{M}$ and $\vec{H}$ is not linear. This is because μ_M is not only a characteristic of ferromagnetic material, but also depends on $\vec{B}_\circ$ and on the previous state of the material, as we will see shortly.

Hysteresis

Measurements of the magnetic properties are usually done using a toroid (or a solenoid) of N turns with an initially unmagnetized ferromagnetic core, see Fig. 26.16.

Suppose that when the switch S in Fig. 26.16 is open (i.e. the current I in the windings is zero and $B_\circ = 0$), the ferromagnetic core is unmagnetized ($B = 0$). Then, we perform the following:

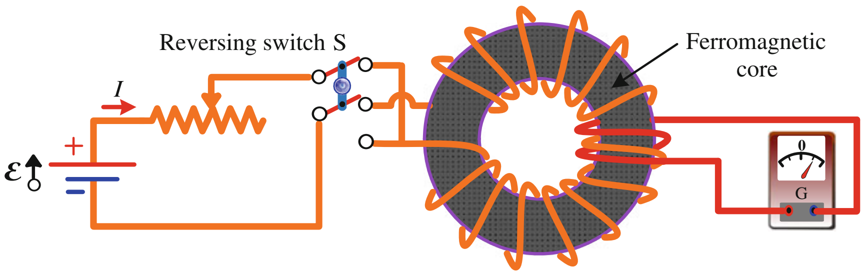


Fig. 26.16 A circuit used to study the properties of a ferromagnetic material that fills the core of a toroid, where the magnetic flux is measured by a galvanometer

1. When we close the switch and slowly increase the current in the circuit, the toroid magnetic field $B_\circ = \mu_\circ H$ increases linearly with I, but the total magnetic field $B = \mu_m H$ ($B \gg B_\circ$) follows the curve shown in the **magnetization curve** of Fig. 26.17. Initially, at point O, the domains of the core are randomly oriented. As $B_\circ$ increases gradually, the domains become more and more aligned until we reach the *saturation* point a where nearly all domains are aligned. Increasing $B_\circ$ further has a small effect on increasing B.
2. Next, we reduce the external magnetic field by decreasing the current in the coil until I becomes zero, We notice that the curve follows the path ab, where $B_\circ = 0$ at point b. This point indicates that $B \neq 0$ even though the external field $B_\circ$ is zero (that is $B = B_M$). In other words, some permanent magnetism remains, and the domains do not become completely random as they were initially.

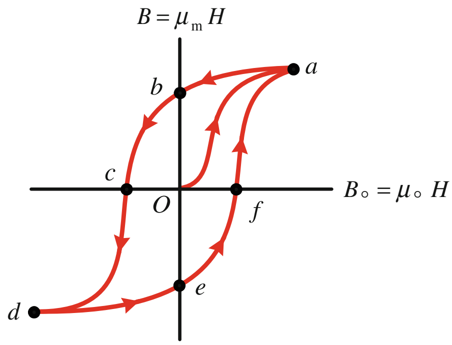


Fig. 26.17 Hysteresis curve for a ferromagnetic material

3. When the direction of the current is reversed and increased gradually (i.e. the direction of the external magnetic field $B_\circ$ is reversed), enough domains reorient their magnetic moments until the material is again unmagnetized at point *c*, where $B = 0$.
4. An increase in the reverse current causes the ferromagnetic material to be magnetized in the opposite direction, until we reach the *saturation* at point *d*.
5. Finally, if the current is again reduced to zero and then increased in the original positive direction, the total magnetic field follows the path *defa*.

We notice that the magnetic field did not pass through the origin (point *O*) in the loop *abcdefa*. This effect is called **magnetic Hysteresis**, while this loop is called the **Hysteresis loop**. Points *b* and *e* on the hysteresis loop indicate that the ferromagnetic material has a 'memory' because it remains magnetized even when the external field is removed. The area of this cycle is proportional to the thermal energy used to align the domains.

The area of the hysteresis loop depends on the properties of the ferromagnetic material under investigation. Two classifications arise as follows, depending on how big or small the loop area is:

1. **Hard ferromagnetic material** (Hard in a magnetic sense): If the hysteresis loop is wide as shown in Fig. 26.18a, the material can turn into a strong permanent magnet that cannot be easily demagnetized by an external magnetic field.
2. **Soft ferromagnetic material** (Soft in a magnetic sense): If the hysteresis loop is narrow, as shown in Fig. 26.18b, the material can be easily magnetized and demagnetized (such as iron, which is perfect for making electromagnets and transformers). An ideal soft ferromagnetic material would exhibit no hysteresis and would therefore have no residual magnetization at all.

A ferromagnetic material can be demagnetized by hitting it hard, heating it, or reversing the magnetizing current repeatedly while decreasing its magnitude, see Fig. 26.18c. As an example, the heads of a tape recorder can demagnetize tapes this way.

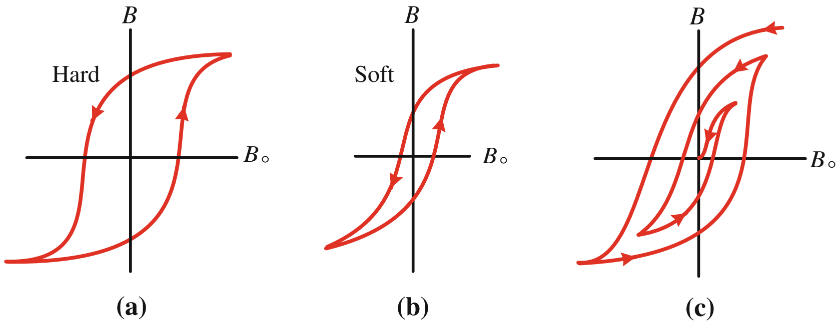


Fig. 26.18 Hysteresis curve for: (a) a hard ferromagnetic, (b) a soft ferromagnetic. (c) Demagnetizing a ferromagnetic material can be done by successive hysteresis loops

Ferromagnetic materials are no longer ferromagnetic above a critical temperature called the Curie temperature, T_{Curie}. Above this temperature, they are generally paramagnetic (for iron this temperature is about 1,040 K = 770°C).

Example 26.10

A toroid has 100 turns/m of wire carrying a current of 3 A. The core of the toroid is filled with powdered steel whose magnetic permeability μ_m is 100 $\mu_\circ$ (i.e. with relative permeability $K_m = \mu_m/\mu_\circ = 100$). Find the magnitude of the magnetic field strength H, the magnitude of the magnetic field $B_\circ$ produced by the toroid, and the magnitude of the magnetic field B inside the steel.

Solution: Using Eq. 26.49, we find H as follows:

$$H = n\,I = (100\text{ turns/m})(3\text{ A}) = 300\text{ A/m}$$

Using Eq. 26.17, we find the $B_\circ$ as follows:

$$B_\circ = \mu_\circ H = (4\pi \times 10^{-7}\text{ T.m/A})(300\text{ A/m}) = 3.77 \times 10^{-4}\text{ T}$$

Then using Eq. 26.52, we find B in the steel core as follows:

$$B = \mu_m H = 100 \times (4\pi \times 10^{-7}\,\text{T.m/A})(300\,\text{A/m}) = 0.038\,\text{T}$$

The value of B inside the steel is about 100 times the value $B_\circ$ in the absence of a steel core.

Example 26.11

(a) A substance has a magnetization of magnitude $M = 10^6$ A/m and a magnetic field of magnitude $B = 4$ T. Find the magnitude of the magnetic field strength H that produces this field. (b) A solenoid of $n = 590$ turns/m carries a current $I = 0.3$ A. If the solenoid's core is iron of magnetic permeability $\mu_m = 4{,}500\,\mu_\circ$, find the magnitude of the magnetic field in its core.

Solution: (a) Using Eq. 26.48, we find B as follows:

$$B = \mu_\circ(H + M) \quad \Rightarrow \quad H = \frac{B}{\mu_\circ} - M = \frac{4\,T}{4\pi \times 10^{-7}\,\text{T.m/A}} - 10^6\,\text{A/m}$$
$$= 2.2 \times 10^6\,\text{A/m}$$

(b) Using Eqs. 26.52 and 26.49, we find B as follows:

$$B = \mu_m H = 4{,}500\,\mu_\circ n\,I$$
$$= (4{,}500)(4\pi \times 10^{-7}\,\text{T.m/A})(590\,\text{turns/m})(0.3\,\text{A}) = 1\,\text{T}$$

26.10 Some Applications of Magnetism

Electromagnets

If a soft iron rod is placed inside a solenoid carrying a current, the magnetic field increases greatly due to the domain alignments. This setup is referred to as an **electromagnet**. The alloys of iron used in an electromagnet gain and lose magnetism quite quickly when the current in the solenoid is turned on or off. Electromagnets are used in many applications, such as in motors, generators, etc.

One simple use of electromagnets is in doorbells, where a rod of soft iron is attached to a spring and partially fitted inside a coil, see Fig. 26.19a. Pushing the doorbell button closes the circuit and the coil becomes a magnet and hence exerts a force on the rod. The rod is then pulled into the coil and strikes the bell, see Fig. 26.19b. If the circuit is then opened, the rod quickly loses its magnetization and the spring pulls the rod back to its initial position.

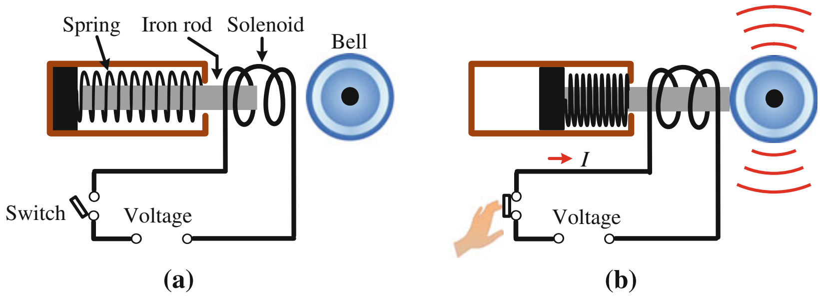


Fig. 26.19 Using the property of soft iron in doorbells. (a) The initial state when the circuit is open. (b) The circuit is closed

Magnetic Circuit Breakers

If the current in a circuit is larger than it should be, the circuit wires might become very hot and may burn. Circuit breakers are installed to prevent overloading by the current in a circuit. These ensure that the current never exceeds a particular value. Modern circuit breakers contain a magnetic sensing coil as shown in Fig. 26.20a. Inside the coil of this figure is a non-magnetic tube containing a spring-based moving iron rod.

When the contacts are closed by a switch and the operating current I is less than or equal to the maximum current I_{max} rated for this circuit breaker, the current flowing through the sensing coil establishes a magnetic field around it. In this case, the field is not strong enough to pull the armature, so the contacts are kept closed, as shown in Fig. 26.20a. However, when the current exceeds I_{max}, the strength of the magnetic field increases enough for the rod to compress the spring and move toward the pole piece. Once it reaches it, the pole piece gets magnetized and attracts the armature, pulling the contacts open. This unlatching of the trip mechanism happens very quickly (<10 ms) and thereby opens the contacts, see Fig. 26.20b.

In the case of a short circuit, the increase in magnetic field is so rapid that the armature is attracted to the pole instantaneously without any rod movement, allowing the circuit breaker to trip much faster.

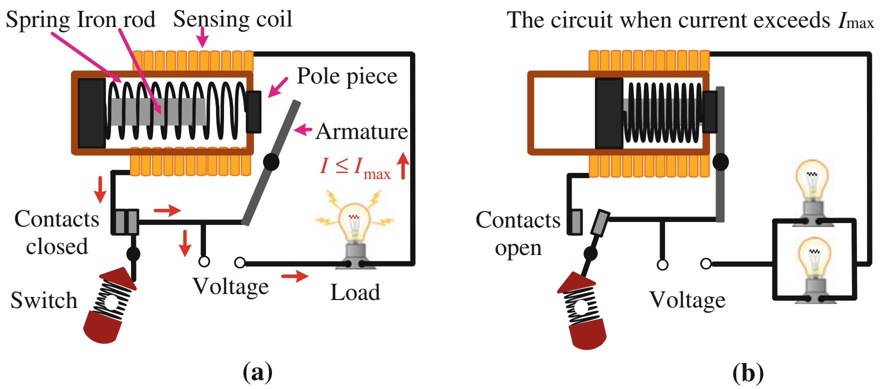


Fig. 26.20 (a) The circuit breaker is closed when the current I is maximum, $I = I_{max}$, or even when $I \le I_{max}$. (b) When I exceeds I_{max}, the armature unlatches a trip mechanism, and the circuit is opened

26.11 Exercises

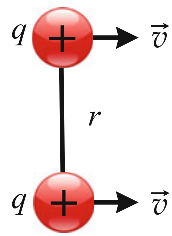
Section 26.1 The Biot-Savart Law

(1) A point charge $q = 25\,\mu\text{C}$ is moving in a straight line with a velocity $\vec{v} = 5 \times 10^4\,\vec{i}$ (m/s). If the charge is at location $P(0, 4\,\text{m}, 0)$ at time t, find the magnetic field produced by this point charge at: (a) the origin $O(0, 0, 0)$, (b) the point $Q(3\,\text{m}, 0, 0)$.

(2) An electron in the hydrogen atom orbits a fixed proton at a radius $r = 5.29 \times 10^{-11}$ m with a speed $v = 2.4 \times 10^6$ m/s. What is the magnitude of the magnetic field at the proton?

(3) Two point-particles of equal charge q are at a distance r apart. The particles are moving with the same velocity $\vec{v}$, see Fig. 26.21. What is the ratio of the magnitudes of the magnetic and electrostatic forces that each particle exerts on the other? [Hint: use $c^2 = 1/\mu_\circ\epsilon_\circ$, where c is the speed of light]

Fig. 26.21 See Exercise (3)



(4) Two very long straight conducting wires lie in the xy plane and are parallel to the y-axis. One wire is at $x = +L$ and the other is at $x = -L$, where $L = 8$ cm. Points P and Q are on the x-axis at $x = +a$ and $x = -a$, where $a = 4$ cm, see Fig. 26.22. The current in each wire is $I = 10$ A. If the currents in the wires are in the positive y direction, find the magnitude and direction of the magnetic field at point P and point Q.

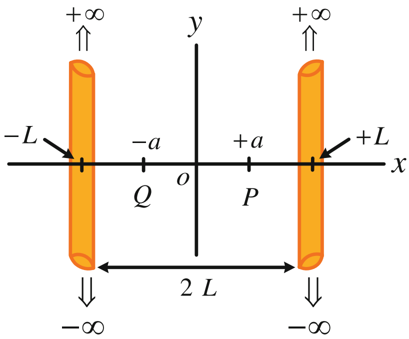


Fig. 26.22 See Exercise (4)

(5) Redo Exercise 4 when the current in the wire at $x = +L$ is in the positive y direction while the current in the wire at $x = -L$ is in the negative y direction. Then calculate the magnitude and direction of the magnetic field at point o.

(6) Two long parallel wires are at two corners of an equilateral triangle of side $a = 5$ cm, as shown in the cross-sectional view of Fig. 26.23. The current in each wire is 10 A. Find the magnitude and direction of the magnetic field at the unoccupied corner P.

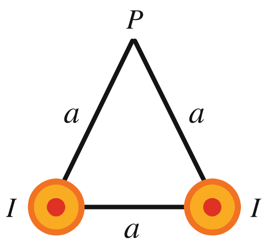


Fig. 26.23 See Exercise (6)

(7) Four long parallel wires are at the four corners of a square which has a diagonal of length $2a$, where $a = 10$ cm, see Fig. 26.24. The magnitudes of the currents in the four wires are the same, i.e. $I_1 = I_2 = I_3 = I_4 = 2$ A. Point P is at the

center of the square. Find $\vec{B}$ at P when: (a) all currents are out of the page, (b) I_1 and I_2 are out of the page while I_3 and I_4 are into th page, (c) I_1 and I_3 are out of the page while I_2 and I_4 are into the page.

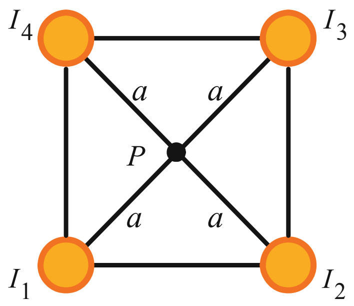


Fig. 26.24 See Exercise (7)

(8) The wire shown in Fig. 26.25 carries a current $I = \sqrt{2}$ A. Find the magnetic field $\vec{B}$ at point P due to each wire segment and then find the resultant magnetic field.

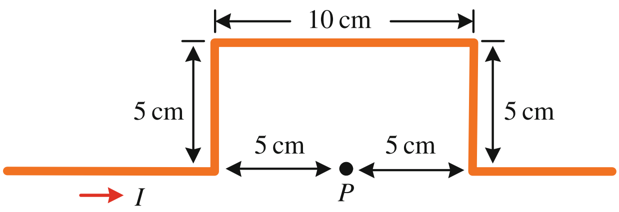


Fig. 26.25 See Exercise (8)

(9) A circular loop of radius $R = 4$ cm carries a current $I = 2$ A. What is the magnitude of the magnetic field on the axis of the loop at its center? What about 2 cm from its center, 4 cm from its center, and 10 cm from its center?

(10) For the circular loop of Exercise 9, how far from the center of the loop and along its axis are the points where the magnetic field is 10% of the field at the center? What about with 1%, and 0.1%?

(11) Two straight wires ① and ③, each of length $L = 4$ cm, are connected by a quarter circular arc wire ② of radius $R = 3$ cm, as shown in Fig. 26.26. Determine the magnitude and direction of the magnetic field at the center P of the arc, when the current I is 2 A.

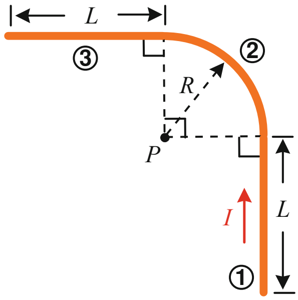


Fig. 26.26 See Exercise (11)

(12) Use the same values of Exercise 11 to calculate the magnitude and direction of the magnetic field at the center P of the arc, shown in Fig. 26.27, when the arc subtended an angle $\theta = \pi/3$.

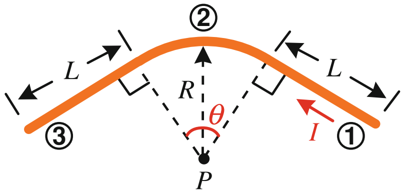


Fig. 26.27 See Exercise (12)

(13) Figure 26.28 shows two identical coaxial coils, called *Helmholtz coils*, each having a radius R, N coil turns, and are separated by a distance R. The coils carry equal currents I such that their axial magnetic fields add. (a) When $R = 30\,\text{cm}$, $N = 350$ turns, and $I = 20\,\text{A}$, find the magnitude of the magnetic field B_P at point P, which is a point that exists midway between the coils centers. (b) Show that B_P can be written as $B_P = 8\,\mu_\circ\, N\, I/(5^{3/2}\, R)$.

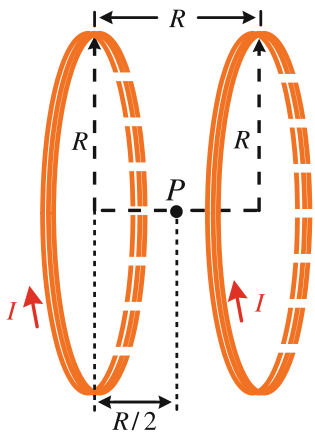


Fig. 26.28 See Exercise (13)

Section 26.2 The Magnetic Force Between Two Parallel Currents

(14) Two long parallel wires are separated by a distance $a = 5$ cm and carry antiparallel currents of the same magnitude, $I_1 = I_2 = 4$ A. (a) What is the magnitude of the magnetic field created by each wire at the location of the other? (b) What is the magnitude of the force per unit length that each wire exerts on the other? Is this force attractive or repulsive?

(15) Two long parallel wires in the xy plane are separated by a distance $2a$ and carry equal currents I in opposite directions. The origin of the x-axis is taken to be midway between the wires and x is the position of an arbitrary point P from that origin, see the cross-sectional view of Fig. 26.29. (a) Derive an expression for the magnitude of the resultant magnetic field $B(x)$ as a function of the position x. (b) Plot $B(x)$ for $-60\,\text{mm} < x < 60\,\text{mm}$ for $I = 20$ A and $a = 30$ mm.

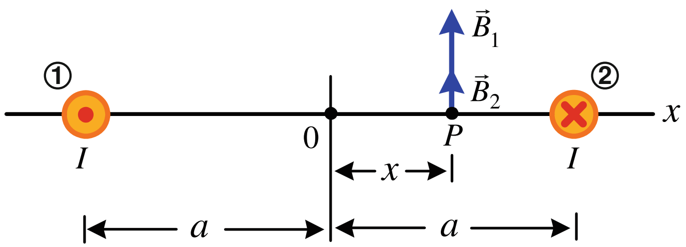


Fig. 26.29 See Exercise (15)

(16) Figure 26.30 shows a very long wire which carries a current $I_1 = 10$ A and a rectangular loop which carries a current $I_2 = 15$ A. Both the wire and the loop lie in one plane. Take $a = 0.1$ m, $b = 0.2$ m, and $c = 0.3$ m. (a) Find the magnitude and direction of the force exerted by the long wire on the wires ② and ④. (b) Find the direction of the force exerted by the long wire on the wires ① and ③. (c) Find the total force exerted by the long wire on the loop.

Section 26.3 Ampere's Law

(17) A long thin-walled conducting cylindrical shell of radius R carries a current I, see Fig. 26.31. Use Ampere's law to find the magnitude of the magnetic field inside and outside the shell.

Fig. 26.30 See Exercise (16)

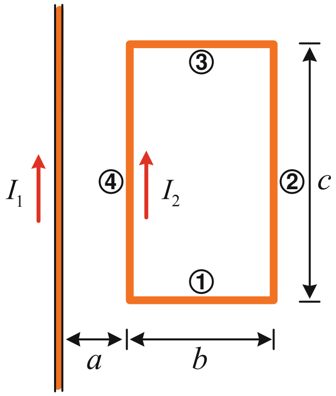
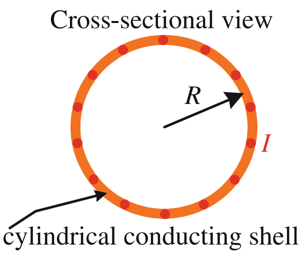
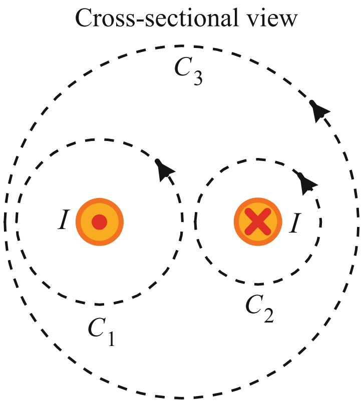


Fig. 26.31 See Exercise (17)



(18) Figure 26.32 shows two antiparallel currents of the same magnitude, $I = 10$ A. Evaluate the line integral $\oint \vec{B} \bullet d\vec{s}$ around the closed paths C_1, C_2, and C_3, where each line integral is taken with $d\vec{s}$ in a counterclockwise direction. Which path can be used to find the magnetic field at some point?

Fig. 26.32 See Exercise (18)



(19) A very long coaxial cable consists of a central wire, surrounded by a rubber layer, which is surrounded by a concentric conducting shell of radius $R = 3$ mm, which is surrounded by another rubber layer, see Fig. 26.33. The current I_1 in the inner wire is 1 A out of the page and the current I_2 in the outer conducting shell is 2 A into the page. Find the magnitude and direction of the magnetic field at $r_a = 2$ mm and $r_b = 4$ mm.

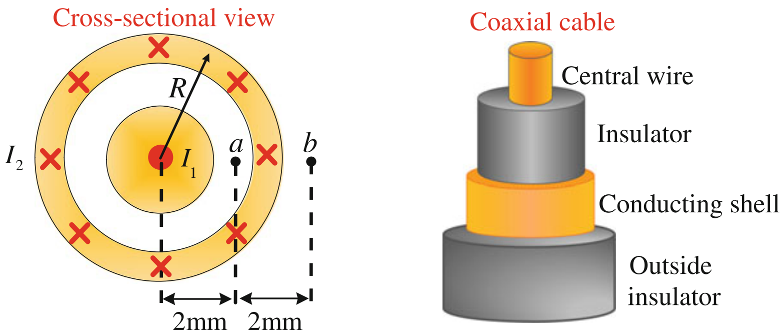


Fig. 26.33 See Exercise (19)

(20) A long wire of radius $R = 2$ cm carries a steady current $I = 50$ A. What are the magnitudes of the magnetic fields from the axis of the wire: (a) at a point 1 cm from the center, (b) on the surface, and (c) at a point 4 cm from the center?

(21) A long conducting cylindrical shell of inner radius a and outer radius b carries a current I uniformly distributed across the cross-sectional area of the shell. Find the magnitude of the magnetic field at points of radii: $r < a$, $a < r < b$, and $r > b$.

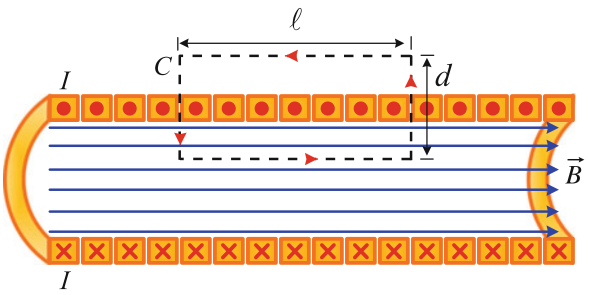
(22) A solenoid with n turns per unit length carries a current I, see Fig. 26.34. Apply Ampere's law to the rectangular path shown in the figure and derive an expression for the magnetic field B. For a packed solenoid such as this, assume that B is uniform inside and $B = 0$ outside.

(23) A solenoid of length $\ell = 0.25$ m carries a current $I = 10$ A. The solenoid consists of twenty closely packed layers, each of 500 turns. What is the magnitude of the magnetic field inside the solenoid?

(24) A solenoid has a length $\ell = 10$ cm. A superconducting fine wire (with almost zero resistance at low temperature) is wound in 10 layers such that $n = 4 \times 10^4$ turns per meter. (a) What is the number of turns per layer? (b) What is the

magnitude of the magnetic field B produced inside the solenoid when the current I in the wire is 60 A?

Fig. 26.34 See Exercise (22)



(25) Assume the wire of the solenoid of exercise 24 has a resistance of $10^4\ \Omega$ at room temperature. A 12 V battery is applied to the solenoid terminals. Find B under these conditions.

(26) An insulating cylindrical shell has a radius $R = 0.5$ cm and length $\ell = 10$ cm. A fine wire of diameter $d = 0.4$ mm is wound in many layers to establish a magnetic field of magnitude $\pi \times 10^{-2}$ T inside the cylindrical solenoid when the current is 2 A. (a) Determine the number of layers of wire needed. (b) Determine the length of the wire.

(27) Figure 26.35 shows a toroid with N turns that carries a current I. The toroid has an inner radius a and an outer radius b. Apply Ampere's law to the circular path C of radius r shown in the figure to derive an expression for the magnetic field B that is only confined to the space enclosed by the windings, and hence $B = 0$ anywhere else. Show that B is approximately uniform when $(b - a)/2 \ll R$, where $R = (a + b)/2$ is the mean radius of the toroid.

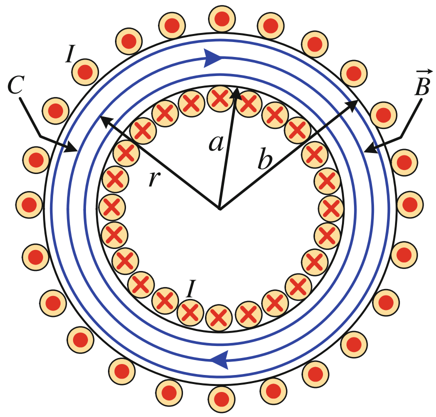
(28) A plastic ring of mean radius 6 cm is wound with $N = 1{,}000$ turns of wire. If the current in this toroid is $I = 0.6$ A, find the magnitude of the magnetic field on the mean circumference.

(29) A tightly wound toroid of inner radius $a = 5$ cm and outer radius $b = 7$ cm has $N = 3{,}300$ turns of wire and carries a current $I = 2$ A. Find the magnitude of the magnetic field: (a) at any point on the circumference of a circle of radius $r = 5.5$ cm. (b) on the mean circumference, which has a radius $r = 6$ cm.

(30) An infinite conducting sheet lying in the xz plane carries a current in the positive z direction, see Fig. 26.36. The current per unit length (or the linear current density) along the x-axis is λ. (a) Use the Biot-Savart law and the symmetry of

the problem to show that for every point P (such that $y > 0$) and every point P' (such that $y < 0$), the magnetic field $\vec{B}$ is parallel to the sheet and directed as shown in the figure. (b) Apply Ampere's law to the rectangular path shown in the figure to derive an expression for the magnitude of the magnetic field B.

Fig. 26.35 See Exercise (27)



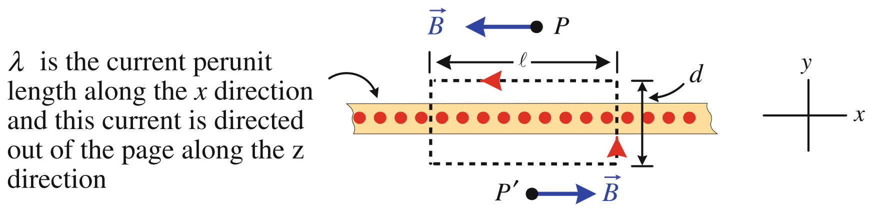
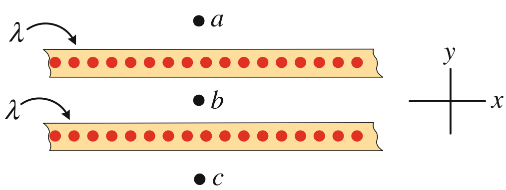


Fig. 26.36 See Exercise (30)

(31) Figure 26.37 shows two parallel infinite conducting sheets, each carries λ amperes of current per unit length out of the page. Find the magnetic field at points a, b, and c.

Fig. 26.37 See Exercise (31)



Section 26.4 Displacement Current and the Ampere–Maxwell Law

(32) A capacitor has circular plates, each of radius $R = 5$ cm. At a particular instant, the capacitor is charging by a current of 0.2 A. (a) What is the displacement current between the plates? (b) What is the rate of change of electric flux between the plates? (c) What is the magnitude of the magnetic field at $r = 8$ cm from the capacitor's axis in the region between the plates?

(33) A capacitor has circular plates, each of radius $R = 10$ cm. At a particular instant, the capacitor is charging by a current of 0.3 A. (a) What is the rate of change of electric field between the plates? (b) Apply Ampere-Maxwell Law to find the magnitude of the magnetic field at $r = 5$ cm from the capacitor's axis in the region between the plates, see Fig. 26.38.

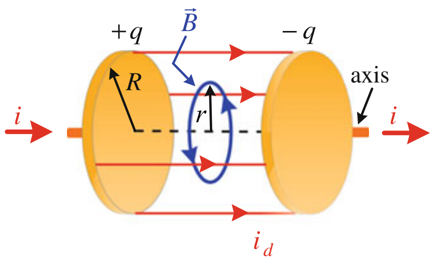


Fig. 26.38 See Exercise (33)

Section 26.5 The Origin of Magnetism

(34) What is the value of the orbital angular momentum of an electron having orbital quantum number $\ell = 2$? For this electronic state, what is the measured component of the orbital magnetic dipole moment $\mu_{\ell,z}$ when its orbital magnetic quantum numbers are $m_\ell = 0$, $m_\ell = 1$, and $m_\ell = -2$?

(35) An atomic electron has an orbital angular momentum with $m_\ell = 0$. (a) What are the measured components L_z and $\mu_{\ell,z}$? (b) When an external magnetic field of magnitude $B = 40$ mT is applied along the z-axis, find the potential energy U_ℓ associated with the orientation of the orbital magnetic dipole moment $\vec{\mu}_\ell$. (c) Repeat parts (a) and (b) for orbital angular momentum with $m_\ell = -2$.

(36) What is the potential energy U_s associated with the orientation of the spin magnetic dipole moment $\vec{\mu}_s$ of an atomic electron when an external magnetic field of magnitude $B = 0.5$ T is applied along the z-axis. Then what is the energy difference between parallel and antiparallel alignment of $\mu_{s,z}$?

(37) An external magnetic field $\vec{B}$ of magnitude 35 T is produced along the z direction in a short period by a pulsed coil. An electron whose $\mu_{s,z}$ was parallel to $\vec{B}$ experiences a "spin flip" so that the final orientation of $\mu_{s,z}$ is antiparallel to $\vec{B}$. Find the change in the orientation potential energy of the electron.

Section 26.7 Magnetic Materials

Section 26.8 Diamagnetism and Paramagnetism

(38) A small magnetic disk has a radius of 1.2 cm and thickness of 0.2 cm. The disk has a uniform magnetization throughout its volume and along its axis. The magnetic moment of the disk is 10^{-2} A.m^2. (a) What is the magnetization $\vec{M}$ of the disk? (b) If the magnetization is due to the alignment of N atoms along the disk axis, each with magnetic moment of 1 μ_B (1 Bohr Magneton), what is the value of N?

(39) Figure 26.39 shows a diamagnetic loop before and after applying an external magnetic field $\vec{B}$. (a) What is the direction of the loop's net magnetic dipole moment $\vec{\mu}$ before and after the application of $\vec{B}$? (b) Is the conventional current counterclockwise or clockwise? (c) What is the direction of the magnetic force on the loop?

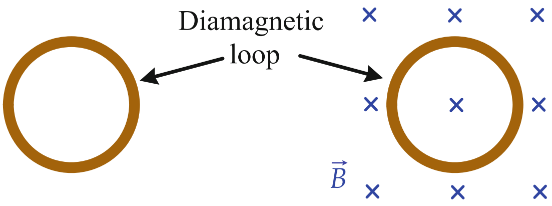


Fig. 26.39 See Exercise (39)

(40) The magnetic field inside a solenoid carrying a current is decreased by 5×10^{-3}% when sample of liquid is inserted into its core. What is the magnetic susceptibility of the liquid?

(41) The number of turns for a toroid is $N = 1{,}200$ and the current it carries is $I = 1.5$ A. The core has an average circumference of 100 cm and a cross-sectional area of 2 cm^2. (a) If the core is air, find the magnetic field strength H, the magnetic field $B_\circ$, and the total flux Φ_B in the toroid. (b) If the core is filled with bismuth of magnetic susceptibility $\chi = -2 \times 10^{-6}$, find the mag-

netization M of bismuth, the magnetic field B in bismuth, and the total flux Φ_B.

(42) A solenoid 0.4 m long is tightly wound with 800 turns of copper wire. The current in the winding is $I = 2$ A. (a) If the core is air, find at the center of the core the magnetic field strength H and the magnetic field $B_\circ$. (b) If the solenoid has an aluminum core of magnetic susceptibility $\chi = 2.3 \times 10^{-5}$, find the magnetization M of aluminum and the magnetic field B.

(43) Repeat part (b) of Exercise 42 for a tungsten core of magnetic susceptibility $\chi = 6.8 \times 10^{-5}$.

(44) If all atoms in a material have their magnetic moments aligned, then the maximum magnetization is given by $M_{\text{max}} = n\mu_{\text{atomic}}$, where n is the number of atoms per unit volume. Aluminum has a density of 2.7×10^3 kg/m^3, molecular mass of 27 kg/kmol, and atomic magnetic moment $\mu_{\text{atomic}} = 9.27 \times 10^{-24}$ J/T $= 1\,\mu_{\text{B}}$, where the quantity μ_{B} is called the *Bohr magneton*. Find M_{max} and $\mu_\circ M_{\text{max}}$.

Section 26.9 Ferromagnetism

(45) Assume that all iron atoms in an iron rod are completely aligned and each atom has an approximate magnetic dipole moment $\mu_{\text{Iron}} = 1.9 \times 10^{-23}$ J/T. Iron has density of 7.8×10^3 kg/m^3 and molecular mass of 55.85 kg/kmol. (a) Find the maximum magnetization M_{max}. (b) Find the dipole moment of the rod if it is 10 cm long, 2 cm wide, and 0.5 cm thick. (c) When a magnetic field of 0.5 T is applied perpendicular to the rod, find the torque exerted by the field.

(46) A 4-A current flows through the wire of a toroid that has 250 turns per meter. The toroid's core is iron of magnetic permeability $\mu_{\text{m}} = 2{,}100\,\mu_\circ$, find the magnitude of the magnetic field in its core.

(47) A solenoid of 50 cm long, 1.5 cm in diameter, and 500 turns is filled with iron core. When a current of 10 A flows through the wire of the solenoid, the magnetic field inside it reaches 2 T. What is the permeability of the iron?