

In this chapter, we explore the phenomenon that occurs when combining two or more waves at one point in the same medium. This phenomenon is known as **interference**. We first combine waves having the same frequencies. Then we combine waves that have slightly different frequencies. In both cases we only consider waves with small amplitudes so that we can use the **superposition principle**.

16.1 Superposition and Interference

To analyze complex combinations of traveling waves where each wave has a small amplitude, we use the superposition principle:

The Superposition Principle

If y_1 and y_2 are two traveling waves produced separately by two sources, then the resultant wave y at any point is the algebraic sum $y_1 + y_2$ when the two sources act together.

This principle is extremely important in all types of wave motion and applies not only to sound waves, but to string waves, light waves, and, in fact, to wave motion of any sort.

The general term **interference** is applied to the effect produced by two (or more) traveling waves when they are simultaneously passing through a given region. When the resultant wave has larger amplitude than that of either individual wave, we refer to their superposition as **constructive interference**. However, when the resultant

wave has smaller amplitude than that of either individual wave, we refer to their superposition as **destructive interference**.

Superposition of Sinusoidal Waves

Let us apply the principle of superposition to two sinusoidal waves traveling to the right in a homogeneous medium and having a different phase ϕ but the same frequency f , wavelength λ , and amplitude A . Accordingly, we write their individual waves as follows:

$$\begin{aligned} y_1 &= A \sin(kx - \omega t), \\ y_2 &= A \sin(kx - \omega t + \phi) \end{aligned} \quad (16.1)$$

where, as usual, $k = 2\pi/\lambda$, $\omega = 2\pi f$, and ϕ is the phase constant. The superposition of y_1 and y_2 gives the following resultant:

$$y = y_1 + y_2 = A [\sin(kx - \omega t) + \sin(kx - \omega t + \phi)] \quad (16.2)$$

To simplify the previous expression, we use the trigonometric identity:

$$\sin a + \sin b = 2 \cos \left(\frac{a-b}{2} \right) \sin \left(\frac{a+b}{2} \right) \quad (16.3)$$

If we substitute in this identity with $a = kx - \omega t$ and $b = kx - \omega t + \phi$, then $a - b = -\phi$ and $a + b = 2kx - 2\omega t + \phi$. Accordingly, we will find that the resultant wave y is reduced to:

$$y = 2A \cos \left(\frac{\phi}{2} \right) \sin \left(kx - \omega t + \frac{\phi}{2} \right) \quad (16.4)$$

The resultant wave y has the following important characteristics:

- (1) It is a sinusoidal wave and has the same frequency f and wavelength λ as any one of the contributing waves y_1 and y_2 ,
- (2) It has an amplitude of $2A \cos(\phi/2)$,
- (3) It has a phase of $\phi/2$.

Now, let us consider the following three cases:

- (a) If $\phi = 0$, then $\cos(\phi/2) = +1$ and the amplitude of y is $+2A$, i.e. twice the amplitude of either one of the individual waves. In this case, the waves are said to be **in phase** and thus *interfere constructively*, see Fig. 16.1a. In general,

constructive interference occurs if the phase ϕ is an *even* multiple of π , i.e. $\phi = 0, 2\pi, 4\pi, \dots$ rad, then $\cos(\phi/2) = \pm 1$.

- (b) If ϕ is an *odd* multiple of π , i.e. $\phi = \pi, 3\pi, 5\pi, \dots$ rad, then $\cos(\phi/2) = 0$, and the crests of one wave occur at the same positions as the troughs of the second wave to produce a resultant amplitude of zero. In this case, the waves are canceling each other out and are said to be **out of phase** and thus *interfere destructively*, see Fig. 16.1b.
- (c) If ϕ has an arbitrary value other than an odd or even multiple of π , then the resultant wave has an amplitude between 0 and $2A$, see Fig. 16.1c.

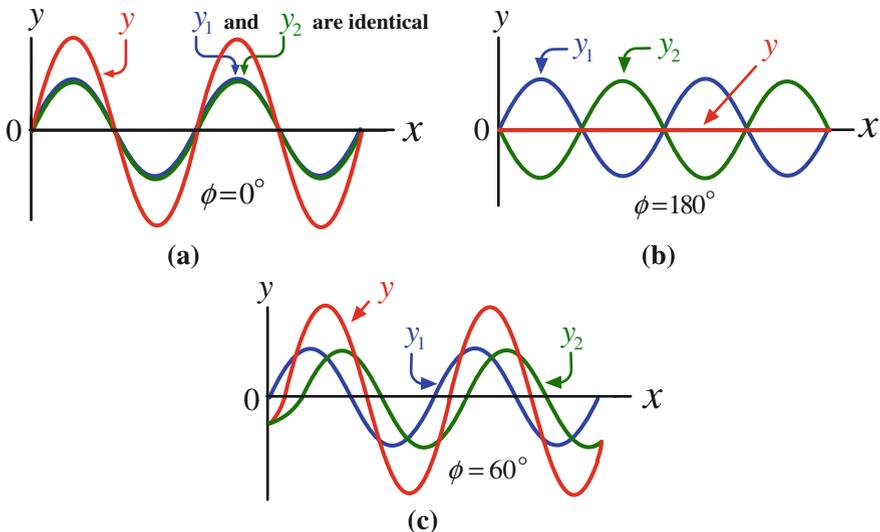


Fig. 16.1 Two identical waves, y_1 (blue) and y_2 (green), traveling in the same direction are added to each other at time $t = 0$ to give a resultant wave y (red). (a) When y_1 and y_2 are in phase ($\phi = 0$), they undergo constructive interference with a resultant wave $y = y_1 + y_2$ that has double the amplitude of either one of y_1 or y_2 . (b) When y_1 and y_2 are out of phase ($\phi = \pi$ rad = 180°), they undergo destructive interference with a resultant wave $y = y_1 + y_2 = 0$, i.e. they cancel each other out. (c) When the phase is different from 0 or π rad, the resultant wave y falls somewhere between part (a) and part (b)

16.2 Spatial Interference of Sound Waves

Figure 16.2 depicts an acoustic interferometer device used to demonstrate sound interference. Sound energy from the source S is divided into two equal parts at the

T-shaped junction of the tube. This means that the sound wave that reached the receiver R traveled along either path A or path B. The distance along any path is called the **path length** L . The upper path length L_A is adjusted by a U-shaped tube, while the lower path length L_B is kept fixed.

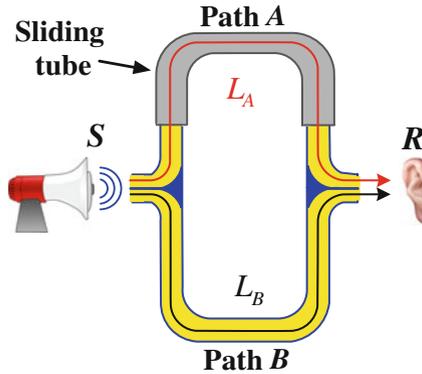


Fig. 16.2 A device used to demonstrate the interference of sound waves. Sound energy from the speaker (S) is divided into two parts at the T-shaped junction of the tube. Before reaching the receiver (R), half of the wave energy propagates through path A of length L_A , while the other half propagates through path B of length L_B . The *upper* path length L_A can be varied by sliding the U-tube *up* or *down*

Constructive interference occurs when the difference in the path length $\Delta L = |L_A - L_B|$ is given by:

$$\Delta L = |L_A - L_B| = (2n) \frac{\lambda}{2}, \quad n = 0, 1, 2, \dots \quad \left\{ \begin{array}{l} \text{Constructive} \\ \text{interference} \end{array} \right\} \quad (16.5)$$

Therefore, the two waves reaching the receiver at any time are in phase ($\phi = 0, 2\pi, 4\pi, \dots$ rad), as shown in Fig. 16.1a, and hence a maximum sound intensity is detected at the receiver R.

Destructive interference occurs when the difference in the path length $\Delta L = |L_A - L_B|$ is given by:

$$\Delta L = |L_A - L_B| = (2n + 1) \frac{\lambda}{2}, \quad n = 0, 1, 2, \dots \quad \left\{ \begin{array}{l} \text{Destructive} \\ \text{interference} \end{array} \right\} \quad (16.6)$$

Therefore, the two waves reaching the receiver at any time are completely out of phase ($\phi = \pi, 3\pi, 5\pi, \dots$ rad), as shown in Fig. 16.1b, and hence no sound is detected at the receiver R.

Because a path difference of a complete wave length λ corresponds to a phase angle of 2π rad, one can relate path difference ΔL to the phase angle ϕ by the relation:

$$\Delta L = \frac{\phi}{2\pi} \lambda \quad (16.7)$$

Example 16.1

Two identical speakers, S_1 and S_2 , are placed horizontally at a distance $d = 2$ m apart. Each emits sound waves of wavelength $\lambda = 80$ cm driven by the same oscillator, see Fig. 16.3. A listener is originally located at point O , which is midway between the two speakers. The listener walks to point P , which is a distance x from O , and reaches the *first minimum* in sound intensity. Find x .

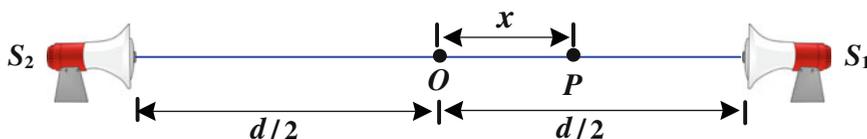


Fig. 16.3

Solution: If L_1 and L_2 are the distances from S_1 and S_2 to point P , respectively, then according to Fig. 16.3 we have:

$$L_1 = \frac{d}{2} - x, \quad L_2 = \frac{d}{2} + x$$

From these two relations and Eq. 16.6, the condition for the first destructive interference at point P leads to the following:

$$|L_2 - L_1| = \frac{\lambda}{2} \Rightarrow \left| \left[\frac{d}{2} + x \right] - \left[\frac{d}{2} - x \right] \right| = \frac{\lambda}{2} \Rightarrow x = \frac{\lambda}{4} = \frac{80 \text{ cm}}{4} = 20 \text{ cm}$$

Example 16.2

Two identical speakers, S_1 and S_2 , are placed vertically at a distance $d = 2$ m apart and emit sound waves driven by the same oscillator, see Fig. 16.4. A listener is

originally located at point O , which is a distance $R = 5$ m from the center of the line connecting the two speakers. The listener walks to point P , which is a distance $y = 0.5$ m above O , and thus reaches the first minimum in sound intensity. Find the wavelength λ of the sound wave.

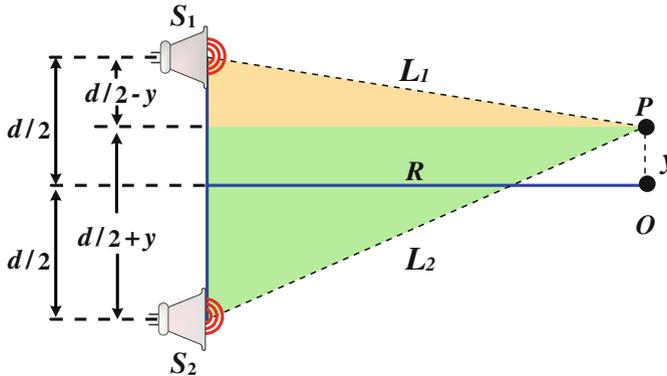


Fig. 16.4

Solution: The first minimum in sound intensity occurs when the two waves reaching the listener at point P are 180° out of phase. In other words, when their path difference equals $\lambda/2$. As per Fig. 16.4, we first calculate the path lengths L_1 and L_2 as follows:

$$L_1 = \sqrt{R^2 + (d/2 - y)^2} = \sqrt{(5 \text{ m})^2 + [(2 \text{ m})/2 - 0.5 \text{ m}]^2} = 5.0249 \text{ m}$$

and

$$L_2 = \sqrt{R^2 + (d/2 + y)^2} = \sqrt{(5 \text{ m})^2 + [(2 \text{ m})/2 + 0.5 \text{ m}]^2} = 5.2202 \text{ m}$$

Thus, from Eq. 16.6, the first destructive interference at point P leads to the following:

$$|L_2 - L_1| = \frac{\lambda}{2} \Rightarrow |5.2202 \text{ m} - 5.0249 \text{ m}| = \lambda/2 \Rightarrow 0.1953 \text{ m} = \lambda/2$$

Therefore:

$$\lambda = 0.3906 \text{ m} = 39.06 \text{ cm}$$

16.3 Standing Sound Waves

Assume we have two identical sound sources that face each other as shown in Fig. 16.5 and driven by the same oscillator. In this case, they produce two identical traveling waves each with a speed v . These waves would be moving in opposite directions in the same medium. Of course, these two waves combine according to the superposition principle.



Fig. 16.5 Two identical sound sources emitting traveling waves towards each other, each with a speed v . When the two waves overlap, they produce standing waves (not shown in the figure)

To analyze this situation, we assume that the two sound sources generate sound waves that have the same frequency f , wavelength λ , and amplitude A but differ by traveling in opposite directions. Therefore, we can write these two waves in the following form:

$$\begin{aligned} y_1 &= A \sin(kx - \omega t), \\ y_2 &= A \sin(kx + \omega t) \end{aligned} \quad (16.8)$$

where y_1 represents a wave traveling in the positive x -direction and y_2 represents a wave traveling in the negative x -direction. The superposition of y_1 and y_2 gives the following resultant:

$$y = y_1 + y_2 = A [\sin(kx - \omega t) + \sin(kx + \omega t)] \quad (16.9)$$

To simplify this expression, we use the trigonometric identity:

$$\sin(a \pm b) = \sin a \cos b \pm \cos a \sin b \quad (16.10)$$

If we substitute in this identity with $a = kx$ and $b = \omega t$, then the resultant wave y reduces to:

$$y = (2A \sin kx) \cos \omega t \quad (16.11)$$

The resultant y represented by Eq. 16.11 gives a special kind of simple harmonic motion in which every element of the medium oscillates in simple harmonic motion with the same angular frequency ω (through the factor $\cos \omega t$) and an amplitude (given by the factor $2A \sin kx$) that varies with position x . This wave is called a *standing wave* because there is no motion of the disturbance along the x -direction.

A standing wave is distinguished by stationary positions with *zero amplitudes* called *nodes* (see Fig. 16.6). This happens when x satisfies the condition $\sin kx = 0$, that is, when:

$$kx = 0, \pi, 2\pi, 3\pi, \dots$$

When using $k = 2\pi/\lambda$, these values give $x = 0, \frac{\lambda}{2}, \lambda, \frac{3\lambda}{2}, \dots$, that is:

$$x = 0, \frac{\lambda}{2}, \lambda, \frac{3\lambda}{2}, \dots = n\frac{\lambda}{2}, \quad (n = 0, 1, 2, \dots) \quad (\text{Nodes}) \quad (16.12)$$

In addition, a standing wave is distinguished by elements with *greatest possible displacements* called *antinodes* (see Fig. 16.6). This happens when x satisfies the condition $\sin kx = \pm 1$, that is, when:

$$kx = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \dots$$

Also, using $k = 2\pi/\lambda$, these values give $x = \frac{\lambda}{4}, \frac{3\lambda}{4}, \frac{5\lambda}{4}, \dots$, that is:

$$x = \frac{\lambda}{4}, \frac{3\lambda}{4}, \frac{5\lambda}{4}, \dots = (n + \frac{1}{2})\frac{\lambda}{2}, \quad (n = 0, 1, 2, \dots) \quad (\text{Antinodes}) \quad (16.13)$$

Equations 16.12 and 16.13 indicate the following general features of nodes and antinodes (see Fig. 16.6):

Soptlight

- (1) The distance between adjacent nodes is $\lambda/2$.
- (2) The distance between adjacent antinodes is $\lambda/2$.
- (3) The distance between a node and an adjacent antinode is $\lambda/4$.

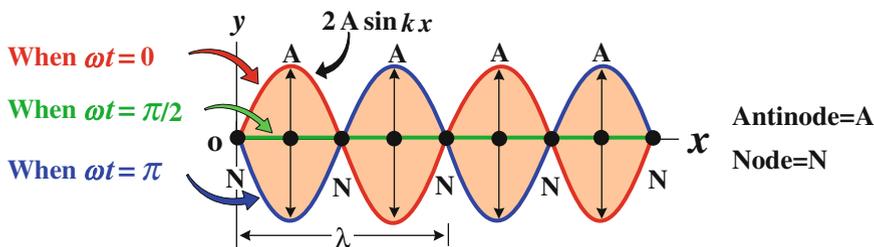


Fig. 16.6 The time dependence of the vertical displacement (from equilibrium) of any individual element in a standing wave y is governed by $\cos \omega t$. Each element vibrates within the confines of the envelope $2A \sin kx$. The *nodes* (N) are points of zero displacement, and the *antinodes* (A) are points of maximum displacement

In Fig. 16.7a, at $t = 0$ ($\omega t = 0$), the two oppositely traveling waves are in phase, producing a wave pattern in which each element of the medium is experiencing its maximum displacement from equilibrium. In Fig. 16.7b, at $t = T/4$ ($\omega t = \pi/2$), the traveling waves have moved one quarter of a wavelength (one to the right and the other to the left). At this time, each element of the medium is passing through the equilibrium position in its simple harmonic motion. The result is zero displacement for each element at all values of x . In Fig. 16.7c, at $t = T/2$ ($\omega t = \pi$), the traveling waves are again in phase, producing a wave pattern that is inverted relative to the $t = 0$ pattern. The patterns at $t = 3T/4$ and $t = T$ are similar to $t = T/4$ and $t = 0$, respectively.

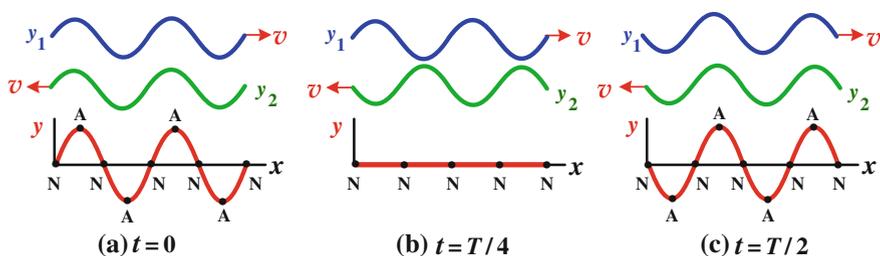


Fig. 16.7 Standing-wave patterns resulting from two oppositely traveling identical waves y_1 and y_2 at different phases. The displacement is zero for each node (N), and is maximum for each antinode (A)

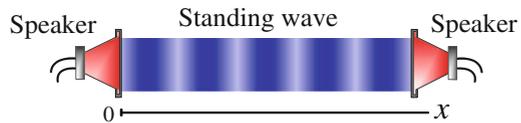
Example 16.3

Two opposing speakers are shown in Fig. 16.8. A standing wave is produced from two sound waves traveling in opposite directions; each can be described as follows:

$$y_1 = (5 \text{ cm}) \sin(4x - 2t),$$

$$y_2 = (5 \text{ cm}) \sin(4x + 2t).$$

where x and y , are in centimeters and t is in seconds. (a) What is the amplitude of the simple harmonic motion of a medium element lying between the two speakers at $x = 2.5 \text{ cm}$? (b) Find the amplitude of the nodes and antinodes. (c) What is the maximum amplitude of an element at an antinode?

Fig. 16.8

Solution: (a) Using the general form of a standing wave given by Eq. 16.11, we find $A = 5 \text{ cm}$, $k = 4 \text{ rad/cm}$, and $\omega = 2 \text{ rad/s}$. Thus:

$$y = (2A \sin kx) \cos \omega t = [(10 \text{ cm}) \sin(4x)] \cos(2t)$$

The amplitude of the simple harmonic motion of an element lying between the two speakers at $x = 2.5 \text{ cm}$ is the absolute value of the coefficient of $\cos(2t)$ evaluated at this point. Thus:

$$\begin{aligned} \text{Amplitude} &= |(10 \text{ cm}) \sin(4x)|_{x=2.5} \\ &= |(10 \text{ cm}) \sin(10 \text{ rad})| = |-5.4 \text{ cm}| = 5.4 \text{ cm} \end{aligned}$$

(b) With $k = 2\pi/\lambda = 4 \text{ rad/cm}$, we have $\lambda = \pi/2 \text{ cm}$. Then, from Eq. 16.12 we find that the nodes are located at:

$$x = n \frac{\lambda}{2} = n \frac{\pi}{4} \text{ cm}, \quad (n = 0, 1, 2, \dots)$$

From Eq. 16.13, we find that the antinodes are located at:

$$x = (n + \frac{1}{2}) \frac{\lambda}{2} = (n + \frac{1}{2}) \frac{\pi}{4}, \quad (n = 0, 1, 2, \dots)$$

(c) The maximum amplitude of antinodes will be $2A = 10 \text{ cm}$

Example 16.4

Two sinusoidal sound waves, equal in amplitude and traveling in opposite directions along the x -axis, are superimposed on each other. The resultant wave is of the form:

$$y = (2 \text{ m}) \sin\left(\frac{\pi}{L}x\right) \cos\left(\frac{\pi}{T}t\right)$$

where x is in meters and t in seconds and the arguments of the sine and cosine functions are in radians. (a) What are the mathematical formulas of the two sinusoidal sound waves that are superimposed to give this resultant? (b) Find the values of the wavelength and the frequency of the two sinusoidal waves when $L = 2 \text{ m}$ and $T = 1 \text{ s}$. (c) What are the velocities of the two sinusoidal waves?

Solution: (a) Using the general form of the standing waves given by Eq. 16.11, we find $A = 1 \text{ m}$, $k = \pi/L \text{ rad/m}$, and $\omega = \pi/T \text{ rad/s}$. Using Eq. 16.8, we find the two sinusoidal waves as follows:

$$y_1 = (1 \text{ m}) \sin\left(\frac{\pi}{L}x - \frac{\pi}{T}t\right),$$

$$y_2 = (1 \text{ m}) \sin\left(\frac{\pi}{L}x + \frac{\pi}{T}t\right)$$

(b) Using $k = 2\pi/\lambda$, and $\omega = 2\pi f$ when $L = 2 \text{ m}$ and $T = 1 \text{ s}$, we have:

$$k = \frac{2\pi}{\lambda} = \frac{\pi}{L} = \frac{\pi}{2 \text{ m}} \Rightarrow \lambda = 4 \text{ m}$$

$$\omega = 2\pi f = \frac{\pi}{T} = \frac{\pi}{1 \text{ s}} \Rightarrow f = 0.5 \text{ s}^{-1} = 0.5 \text{ Hz}$$

(c) Using $v = \omega/k$, we find the speed of each of the sinusoidal waves as follows:

$$v = \frac{\omega}{k} = \frac{2\pi f}{2\pi/\lambda} = \lambda f = (4 \text{ m})(0.5 \text{ s}^{-1}) = 2 \text{ m/s}$$

The velocity of y_1 is $v_1 = +2 \text{ m/s}$ (in the direction of increasing x) and the velocity of y_2 is $v_2 = -2 \text{ m/s}$ (in the direction of decreasing x).

16.4 Standing Sound Waves in Air Columns

In Chap. 14, we saw how a standing wave can be generated either on a stretched string with fixed ends or when one end is fixed and the other is left free to move. We learned

that this happens when the wavelengths of the waves suitably match the length of the string, in which case the superposition of the traveling and reflecting waves produce a standing wave pattern. For such a match, the wavelength corresponds to the resonant frequency of the string.

We can set up standing sound waves in air-filled pipes in a way similar to that for strings. Here is how we can compare the two:

1. The closed end of a pipe is similar to the fixed end of a string in that it must be a **displacement node**. This is because the pipe's wall at this end does not allow longitudinal motion of the air and acts like a **pressure antinode** (point of maximum pressure variation).
2. The open end of a pipe acts like the end of a string that is free to move, so there must be a **displacement antinode** there¹. This is because the pipe's open end allows longitudinal motion of the air and acts like a **pressure node** (point of no pressure variation, since the end must remain at atmospheric pressure).

It is interesting to know how sound waves reflect from the open end of a pipe. To get insight into this, we start with the fact that sound waves are in fact pressure waves. Next, we know that any compression region must be contained inside the pipe (between its two ends). Furthermore, any compression region that exists at an open end is free to expand into the atmosphere. This change in behavior of the air inside and outside the pipe is sufficient to allow some reflection.

With the boundary conditions of nodes and antinodes at the ends of air columns, we must set the normal modes of oscillations as we did in the case of stretched strings.

Air Columns of Two Open Ends

First, we consider a pipe of length L that is open at both ends. By representing the horizontal displacement of air elements on the vertical axis and applying the boundary condition that meets the case of two open ends, see Fig. 16.9, the normal modes of oscillations can be explained by considering the following first three patterns:

- (1) The first normal mode (the first harmonic, or the fundamental):

The simplest pattern is shown in Fig. 16.9a. There are two imposed antinodes

¹ The antinode of an open end of a pipe is located slightly beyond the end because sound compression reaching an open end does not reflect until it passes the end. Therefore, the effective length of the air column is little greater than the true length L of the pipe.

at the two ends and only one node in the middle of the pipe. Also, there is only half a wavelength in the length L . Thus, this standing wave pattern has:

$$\lambda_1 = 2L \quad \text{and} \quad f_1 = \frac{v}{\lambda_1} = \frac{v}{2L}.$$

(2) The second normal mode (the second harmonic):

The second pattern is shown in Fig. 16.9b. This pattern has three antinodes and two nodes. This standing wave pattern has:

$$\lambda_2 = L \quad \text{and} \quad f_2 = \frac{v}{\lambda_2} = \frac{v}{L} = 2f_1$$

(3) The third normal mode (the third harmonic):

The third pattern is shown in Fig. 16.9c. This pattern has four antinodes and three nodes. This standing wave pattern has:

$$\lambda_3 = 2L/3 \quad \text{and} \quad f_3 = \frac{v}{\lambda_3} = \frac{3v}{2L} = 3f_1$$

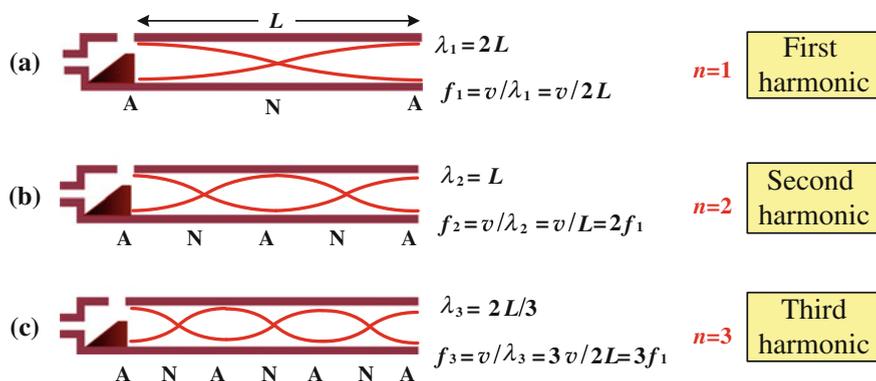


Fig. 16.9 The first three standing wave patterns (a), (b), and (c) of a longitudinal sound wave established in an organ pipe that is open to the atmosphere at both ends. The horizontal motion of air elements in the pipe is displayed vertically by using a red color. The difference between successive harmonics is the fundamental frequency f_1 , and each harmonic is an integer multiple of the fundamental frequency f_1

Generally, the relation between the wavelength λ_n of the various normal modes and the length L of a pipe of two open ends is:

$$\lambda_n = \frac{2L}{n}, \quad (n = 1, 2, 3, \dots) \quad (\text{Pipe, two open ends}) \quad (16.14)$$

Also, according to the relation $f = v/\lambda$, where the speed v of the sound wave is the same for all frequencies, the resonance frequencies f_n associated with these modes are (see Fig. 16.9):

$$f_n = \frac{v}{\lambda_n} = n \frac{v}{2L}, \quad (n = 1, 2, 3, \dots) \quad (\text{Pipe, two open ends}) \quad (16.15)$$

The expressions of λ_n and f_n are the same as for the string, except that v is the speed of waves on the strings as in Eq. 14.66, whereas v in Eq. 16.15 is the speed of sound in air. The relation between the resonance frequencies and the fundamental frequency is:

$$f_n = n f_1, \quad (n = 1, 2, 3, \dots) \quad (\text{Pipe, two open ends}) \quad (16.16)$$

Air Columns of One Closed End

Second, we consider a pipe of length L that is open at one end and closed at the other. By applying the boundary condition that meets this case, the normal modes of oscillations can be explained by considering the following first three patterns:

- (1) The first normal mode (the first harmonic, or the fundamental):

Fig. 16.10a shows the simplest pattern. The standing wave extends from an antinode at the open end to the adjacent node at the closed end. The fundamental standing wave pattern has:

$$\lambda_1 = 4L \quad \text{and} \quad f_1 = \frac{v}{\lambda_1} = \frac{v}{4L}.$$

- (2) The third normal mode (the third harmonic):

The next pattern is shown in Fig. 16.10b. This pattern has two antinodes and two nodes. Thus, this standing wave pattern has:

$$\lambda_3 = 4L/3 \quad \text{and} \quad f_3 = \frac{v}{\lambda_3} = \frac{3v}{4L} = 3f_1$$

- (3) The fifth normal mode (the fifth harmonic):

The next pattern is shown in Fig. 16.10c. This pattern has four antinodes and four nodes. Thus:

$$\lambda_5 = 4L/5 \quad \text{and} \quad f_5 = \frac{v}{\lambda_5} = \frac{5v}{4L} = 5f_1$$

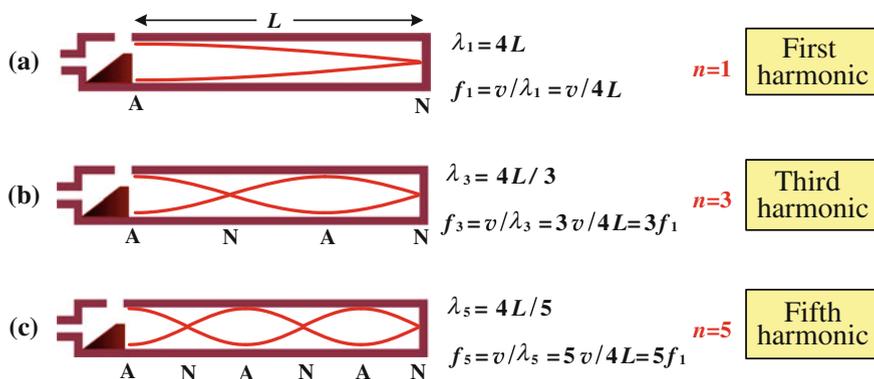


Fig. 16.10 The first three standing wave patterns (a), (b), and (c) of a longitudinal sound wave established in an organ pipe that is open to the atmosphere at only one end. The horizontal motion of air elements in the pipe is displayed vertically by using a *red* color. The harmonic frequencies are the odd-integer multiples of f_1 , and the successive difference is $2f_1$

Generally, λ_n and f_n of the various normal modes for a pipe of length L with only one end open are given as (see Fig. 16.10):

$$\lambda_n = \frac{4L}{n}, \quad (n = 1, 3, 5, \dots) \quad (\text{Pipe, one open end}) \quad (16.17)$$

$$f_n = \frac{v}{\lambda_n} = n \frac{v}{4L}, \quad (n = 1, 3, 5, \dots) \quad (\text{Pipe, one open end}) \quad (16.18)$$

$$f_n = nf_1, \quad (n = 1, 3, 5, \dots) \quad (\text{Pipe, one open end}) \quad (16.19)$$

Figure 16.11 shows a simple apparatus for demonstrating the resonance of sound waves in air columns. A tube that is open from both ends is immersed into a container filled with water, and a tuning fork of unknown frequency f and wavelength λ is placed at its top. The sound waves generated by the fork are reinforced when the length L corresponds to one of the resonance frequencies of the tube. Thus:

$$\lambda = \frac{4L_n}{n}, \quad f = \frac{v}{\lambda} = n \frac{v}{4L_n}, \quad (n = 1, 3, 5, \dots) \quad (16.20)$$

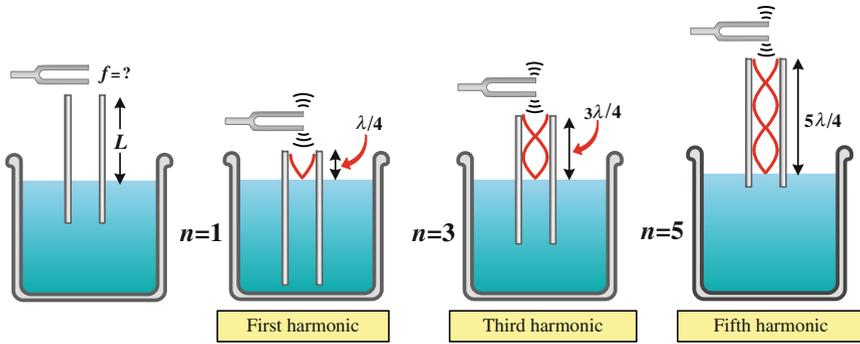


Fig. 16.11 An apparatus used to demonstrate the resonance of sound waves in a tube closed at one end. At resonance, L and λ are related

Example 16.5

When wind blows through a cylindrical drainage culvert of 2.5 m length, see Fig. 16.12, a howling noise is established. Take $v = 343$ m/s as the speed of sound in air. (a) Find the frequencies of the first three harmonics if the pipe is open at both ends. (b) How many of the harmonics fall within the normal human hearing range (from about 20 Hz \rightarrow 20,000 Hz). (c) Answer part (a) if the pipe is blocked at the other end.

Fig. 16.12



Solution: (a) When the pipe is open at both ends, we use Eq. 16.15 with $n = 1$ to find the fundamental frequency as follows:

$$f_1 = 1 \times \frac{v}{2L} = \frac{343 \text{ m/s}}{2 \times 2.5 \text{ m}} = 68.6 \text{ Hz}$$

Also, all harmonics are available for a pipe open at both ends; thus:

$$f_2 = 2f_1 = 137.2 \text{ Hz} \quad \text{and} \quad f_3 = 3f_1 = 205.8 \text{ Hz}$$

(b) We can express the frequency of the highest harmonic heard as $f_n = n f_1$, where $f_n = 20,000$ Hz and n is the number of harmonics that can be heard. Therefore:

$$n = \frac{f_n}{f_1} = \frac{20,000 \text{ Hz}}{68.6 \text{ Hz}} = 292$$

Although we get $n = 292$, practically, only the first few harmonics have amplitudes that are sufficient to be heard.

(c) Using Eq. 16.18 and substituting with $n = 1$, the fundamental frequency of a pipe closed at one end will be given by:

$$f_1 = 1 \times \frac{v}{4L} = \frac{343 \text{ m/s}}{4 \times 2.5 \text{ m}} = 34.3 \text{ Hz}$$

In this case, only the odd harmonics can exist. Thus:

$$f_3 = 3f_1 = 102.9 \text{ Hz} \quad \text{and} \quad f_5 = 5f_1 = 171.5 \text{ Hz}$$

Example 16.6

A background noise in a hall sets up a fundamental standing wave frequency in a tube of length $L = 0.7$ m. What is the value of this fundamental frequency if your ear blocks one end of the tube (see Fig. 16.13a) and when your ear is far from the tube (see Fig. 16.13b)? Take $v = 343$ m/s as the speed of sound in air.

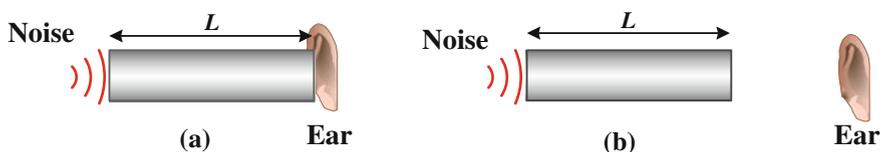


Fig. 16.13

Solution: When the tube is blocked by your ear (see Fig. 16.13a) the fundamental frequency is given by Eq. 16.18 with $n = 1$:

$$f_1 = 1 \times \frac{v}{4L} = \frac{343 \text{ m/s}}{4 \times 0.7 \text{ m}} = 122.5 \text{ Hz}$$

In addition, you can hear frequencies that are *odd integer multiples* of 122.5 Hz provided that the standing waves are formed with sufficient amplitudes.

When you move your head away enough (see Fig. 16.13b) the pipe becomes open at both ends and the fundamental frequency will be given by Eq. 16.15 with $n = 1$:

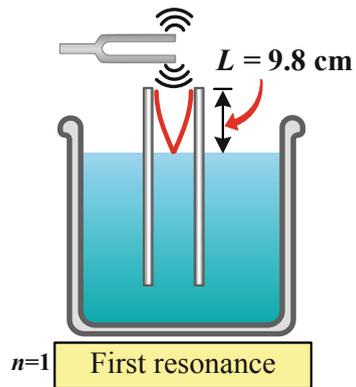
$$f_1 = 1 \times \frac{v}{2L} = \frac{343 \text{ m/s}}{2 \times 0.7 \text{ m}} = 245 \text{ Hz}$$

In addition, you can hear frequencies that are *multiples* of 245 Hz if the standing waves are formed with sufficient amplitudes.

Example 16.7

Resonance can occur in Fig. 16.14 when the smallest length of the air column is $L = 9.8 \text{ cm}$. Take $v = 343 \text{ m/s}$ as the speed of sound in air. (a) What is the frequency f of the tuning fork? (b) What is the value of L for the next two resonances?

Fig. 16.14



Solution: (a) When the tube is blocked by the water's surface, it acts as if the tube is closed at one end. Thus, for the smallest air column L_1 , the fundamental frequency is given by Eq. 16.20 with $n = 1$:

$$f = 1 \times \frac{v}{4L_1} = \frac{343 \text{ m/s}}{4 \times (0.098 \text{ m})} = 875 \text{ Hz} \quad \left\{ \begin{array}{l} \text{First resonance} \\ \text{First harmonic} \end{array} \right.$$

This frequency must be equal to the frequency f of the tuning fork.

(b) We know from Fig. 16.14 and Eq. 16.20 that the wavelength of the fundamental mode is four times the length of the air column. Thus:

$$\lambda = \frac{4L_1}{1} = 4(0.098 \text{ m}) = 0.392 \text{ m}$$

Because the frequency of the tuning fork is constant, then according to Fig. 16.11, the values of L for the next two normal modes are:

$$L_3 = \frac{3\lambda}{4} = \frac{3 \times (0.392 \text{ m})}{4} = 0.294 \text{ m} = 29.4 \text{ cm} \quad \left\{ \begin{array}{l} \text{Second resonance} \\ \text{Third harmonic} \end{array} \right\}$$

$$L_5 = \frac{5\lambda}{4} = \frac{5 \times (0.392 \text{ m})}{4} = 0.49 \text{ m} = 49 \text{ cm} \quad \left\{ \begin{array}{l} \text{Third resonance} \\ \text{Fifth harmonic} \end{array} \right\}$$

16.5 Temporal Interference of Sound Waves: Beats

Previously, we discussed the *spatial interference* of waves of same frequencies, where at fixed time the amplitude of the oscillating elements varies with the position in space. The standing waves in strings and air columns are good examples of this kind of interference.

Now, we consider another type of interference of waves having a slight difference in their frequencies, where at fixed position, the amplitude of the oscillating elements varies periodically with time. The standing wave produced by two tuning forks having a slight difference in their frequencies is a good example of this kind of interference. We refer to this *interference in time* by **temporal interference**, and this phenomenon is called **beating**:

Beating

Beating is defined as the periodic variation in amplitude at a fixed position due to the variation in the constructive and destructive interference between waves having slightly different frequencies.

Consider the time-dependent variations of the displacements of two sound waves of equal amplitude and slightly different frequencies f_1 and f_2 (angular frequencies $\omega_1 = 2\pi f_1$ and $\omega_2 = 2\pi f_2$) such that:

$$\begin{aligned} y_1 &= A \cos(k_1 x - \omega_1 t), \\ y_2 &= A \cos(k_2 x - \omega_2 t) \end{aligned} \quad (16.21)$$

At the fixed point $x = 0$ (chosen for convenience), the two wave functions become (see Fig. 16.15a):

$$\begin{aligned} y_1 &= A \cos \omega_1 t, \\ y_2 &= A \cos \omega_2 t \end{aligned} \quad (16.22)$$

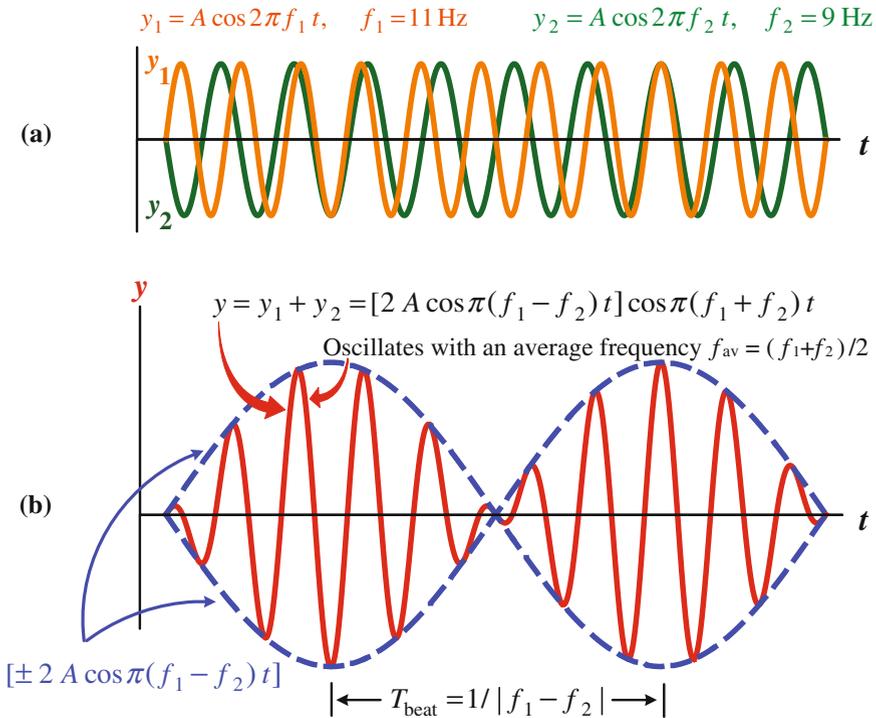


Fig. 16.15 (a) Formation of beats by combining two waves of slightly different frequencies f_1 and f_2 ($f_1 = 11 \text{ Hz}$ and $f_2 = 9 \text{ Hz}$). (b) The slowly varying amplitude envelope $\pm 2A \cos \pi(f_1 - f_2) t$ limits the amplitude of the rapid sinusoidal function $\cos \pi(f_1 + f_2) t$, which proceeds with an average frequency $f_{\text{av}} = (f_1 + f_2)/2$

The superposition of y_1 and y_2 gives the following resultant:

$$y = y_1 + y_2 = A [\cos \omega_1 t + \cos \omega_2 t] \quad (16.23)$$

To simplify this expression, we use the trigonometric identity:

$$\cos a + \cos b = 2 \cos \frac{1}{2}(a - b) \cos \frac{1}{2}(a + b) \quad (16.24)$$

If we substitute $a = \omega_1 t$ and $b = \omega_2 t$ in this identity, then the resultant wave y reduces to:

$$y = [2A \cos \frac{1}{2}(\omega_1 - \omega_2)t] \cos \frac{1}{2}(\omega_1 + \omega_2)t \quad (16.25)$$

When the difference in angular frequencies is small compared to the sum of angular frequencies, i.e.:

$$|\omega_1 - \omega_2| \ll \omega_1 + \omega_2 \quad \text{or} \quad |f_1 - f_2| \ll f_1 + f_2 \quad (16.26)$$

Then the time behavior of the factor $\cos \frac{1}{2}(\omega_1 + \omega_2)t$, is a *rapidly varying* sinusoidal oscillation, see Fig. 16.15b, with the average angular frequency $\frac{1}{2}(\omega_1 + \omega_2)$. Thus, the y equation indicates that the resultant sound wave at any given location has an effective angular frequency equal to the average angular frequency:

$$\omega_{\text{av}} = \frac{\omega_1 + \omega_2}{2} \quad \text{or} \quad f_{\text{av}} = \frac{f_1 + f_2}{2} \quad (16.27)$$

In addition, the oscillation is not precisely sinusoidal because the resultant amplitude varies with time according to the expression:

$$A_{\text{res}} = 2A \cos \frac{1}{2}(\omega_1 - \omega_2)t \quad (16.28)$$

This resultant amplitude is a *slowly varying envelope* in time, see Fig. 16.15b, that modulates the rapidly oscillating factor $\cos \frac{1}{2}(\omega_1 + \omega_2)t$. Moreover, this resultant amplitude A_{res} confirms the existence of a constructive interference when $\cos \frac{1}{2}(\omega_1 - \omega_2)t = \pm 1$. That is when:

$$\frac{1}{2}|\omega_1 - \omega_2|t = 0, \pi, 2\pi, \dots \quad (\text{Constructive interference}) \quad (16.29)$$

Also, the resultant amplitude A_{res} confirms the existence of a destructive interference when $\cos \frac{1}{2}(\omega_1 - \omega_2)t = 0$. That is when:

$$\frac{1}{2}|\omega_1 - \omega_2|t = \frac{1}{2}\pi, \frac{3}{2}\pi, \frac{5}{2}\pi, \dots \quad (\text{Destructive interference}) \quad (16.30)$$

The time between successive moments of constructive (or destructive) interferences is called the *beat period*, T_{beat} . During this time, the phase difference increases by π , i.e. $\frac{1}{2}|\omega_1 - \omega_2|T_{\text{beat}} = \pi$. Thus:

$$T_{\text{beat}} = \frac{2\pi}{|\omega_1 - \omega_2|} = \frac{1}{|f_1 - f_2|} \quad (16.31)$$

Hence, the number of beats per second, or the *beat frequency* f_{beat} , will be given by:

$$f_{\text{beat}} = |f_1 - f_2| \quad (16.32)$$

Musicians can use the beat phenomenon in tuning their instruments. If an instrument sounds different from how it is supposed to, it can be tuned by using a standard frequency until the beat disappears.

Example 16.8

Two identical violin A strings (see the left part of Fig. 16.16) of the same length and tension are tuned exactly to 440 Hz. The tension in one of them is increased by 2% (see the right part of Fig. 16.16). When both strings are struck, what will be the beat frequency between their fundamental frequencies?

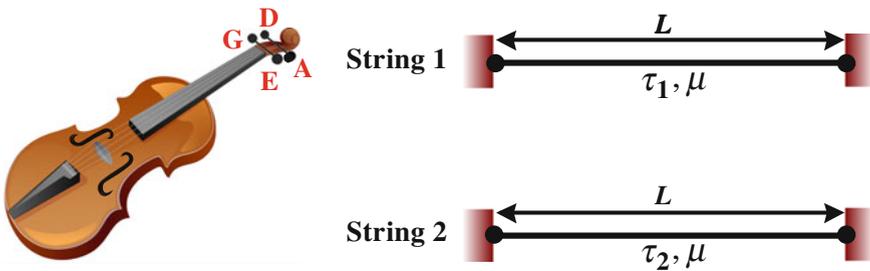


Fig. 16.16

Solution: The frequency of a string that is fixed at both ends is given by Eq. 14.68 as $f = \sqrt{\tau/\mu}/(2L)$, where L , τ , and μ are the length, tension, and mass per unit length of the string, respectively. Thus, the ratio of frequencies of the two strings after being struck is:

$$\frac{f_2}{f_1} = \frac{1}{2L} \sqrt{\frac{\tau_2}{\mu}} \bigg/ \frac{1}{2L} \sqrt{\frac{\tau_1}{\mu}} = \sqrt{\frac{\tau_2}{\tau_1}}$$

When tension τ_2 is 2% more than τ_1 , we can find the frequency f_2 of string 2 as follows:

$$\frac{f_2}{f_1} = \sqrt{\frac{1.02\tau_1}{\tau_1}} = \sqrt{1.02} = 1.01 \quad \Rightarrow \quad f_2 = 1.01f_1 = 1.01 \times (440 \text{ Hz}) = 444 \text{ Hz}$$

With the use of Eq. 16.32, the beat frequency will be:

$$f_{\text{beat}} = |f_1 - f_2| = |440 \text{ Hz} - 444 \text{ Hz}| = 4 \text{ Hz} = 4 \text{ beat/s}$$

Example 16.9

A musician wants to tune the A2 key (key No. 25) of a piano that has a *proper* fundamental frequency of 110 Hz, see Fig. 16.17. Assume he uses a fork of frequency $f_1 = 220 \text{ Hz}$ and was able to tune the A2 key after observing a beat frequency of 8 Hz. Explain the process of tuning and find the mistuned frequency.

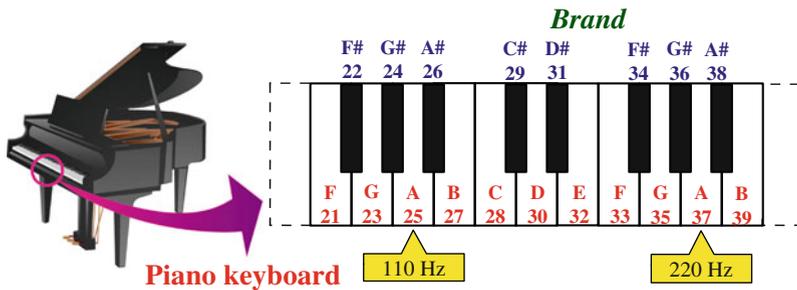


Fig. 16.17

Solution: Equation 16.25 leads to the beat phenomenon when the frequencies are close to each other, which is not the case for 110 and 220 Hz. However, based on Eq. 14.69 of fixed strings, the proper second harmonic of the string of the A2 key should be:

$$f_2 = 2 \times (110 \text{ Hz}) = 220 \text{ Hz} \quad (\text{Proper second harmonic})$$

By listening to the beats of 8 Hz between the fundamental frequency $f_1 = 220 \text{ Hz}$ of the tuning fork and the unknown mistuned second harmonic frequency f_2 of the A2 key, he can adjust the tension in the string until the beat note disappears. From the beats Eq. 16.32, he can find the mistuned frequency of the A2 key as follows:

$$\begin{aligned} f_{\text{beat}} &= |f_1 - f_2| \\ 8 \text{ Hz} &= |220 \text{ Hz} - f_2| \end{aligned}$$

Hence,

$$f_2 = \begin{cases} 228 \text{ Hz} \\ \text{or} \\ 212 \text{ Hz} \end{cases}$$

Accordingly, the musician cannot tell whether the mistuned fundamental frequency of the string was 114 Hz or 106 Hz, because both frequencies produce the same beat frequency.

16.6 Exercises

Sections 16.1 and 16.2 Superposition and Interference and Spatial Interference of Sound Waves

- (1) Two traveling waves are defined by the following relations:

$$\begin{aligned} y_1 &= (2 \text{ cm}) \sin(kx - \omega t), \\ y_2 &= (2 \text{ cm}) \sin(kx - \omega t + \phi) \end{aligned}$$

Find the amplitude of the resultant wave $y = y_1 + y_2$ when $\phi = \pi/2$ and $\phi = \pi$.

- (2) Two traveling waves are defined by the following relations:

$$\begin{aligned} y_1 &= (1.5 \text{ m}) \sin(10x - 16t), \\ y_2 &= (1.5 \text{ m}) \sin(14x - 20t) \end{aligned}$$

where x is in meters, t is in seconds, and the arguments of the sine waves are in radians. (a) What is the phase difference between the two waves when $x = 4 \text{ m}$ and $t = 2 \text{ s}$? (b) At $t = 4 \text{ s}$, apply the condition of destructive interference (phase difference $= (2n + 1)\pi$, $n = 0, 1, 2, \dots$) to find the closest positive value of x to the origin.

- (3) The two identical speakers shown in Fig. 16.18 are driven by one oscillator that has a frequency of 3,400 Hz. Take the speed of sound to be 343 m/s. (a) What are the values of x that correspond to a minimum sound intensity at point P ? (b) What are the values of x that correspond to a maximum sound intensity at point P ?
- (4) A small speaker is placed in a circular pipe of radius $r = 1.35 \text{ m}$, as shown in Fig. 16.19. Take the speed of sound to be 343 m/s and assume propagation

of one-dimensional waves for such a big radius. What are the three smallest frequencies that produce a maximum sound intensity in the tube?

Fig. 16.18 See Exercise (3)

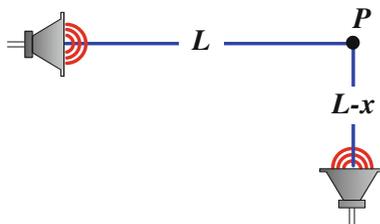
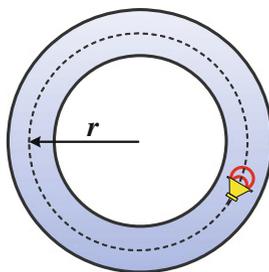


Fig. 16.19 See Exercise (4)



- (5) Two identical speakers, S_1 and S_2 , are placed vertically at a distance d apart. They emit sound waves driven by the same oscillator whose frequency is f . A listener at a distance R from the lower speaker walks straight towards it as shown in Fig. 16.20. If the speed of sound is v , show that the listener will hear a minimum sound when R satisfies the following relation:

$$R^2 = \frac{d^2 - (2n + 1)^2(v/2f)^2}{2(2n + 1)(v/2f)}, \quad (n = 0, 1, 2, \dots) \quad \left\{ \begin{array}{l} \text{Destructive} \\ \text{interference} \end{array} \right\}$$

- (6) In the previous example, assume that $d = 3$ m, $f = 350$ Hz, and $v = 343$ m/s. How many times will the listener hear a minimum in sound intensity while walking from a very far point to the nearest possible point in front of the lower speaker?

Section 16.3 Standing Sound Waves

- (7) Two waves are traveling in opposite directions and are described by the following relations:

$$y_1 = A \sin(kx - \omega t),$$

$$y_2 = \frac{1}{2}A \sin(kx + \omega t)$$

Show that the resultant of these two waves can be written as a combination of a traveling wave and a standing wave of the following form:

$$y = y_1 + y_2 = \frac{1}{2}A \sin(kx - \omega t) + (A \sin kx) \cos \omega t$$

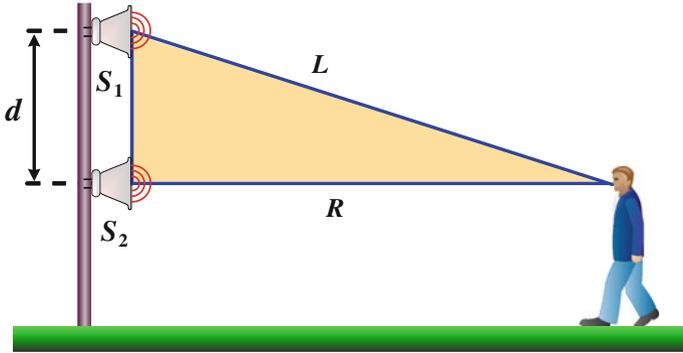


Fig. 16.20 See Exercise (5)

- (8) Two identical speakers facing each other as shown in Fig. 16.21, establish a standing wave as a result of the production of the following two oppositely traveling sound waves:

$$y_1 = (2 \text{ cm}) \sin(2.5x - 5t),$$

$$y_2 = (2 \text{ cm}) \sin(2.5x + 5t).$$

where x and y are in centimeters and t is in seconds. (a) What is the amplitude of the simple harmonic motion of an element of the medium located at $x = 4 \text{ cm}$? (b) Find the position of the nodes and antinodes. (c) What is the maximum amplitude of an element at an antinode?

- (9) The two sources shown in the evacuated vessel of Fig. 16.22, are 1.2 m apart, and send sound waves of speed $v = 2 \text{ m/s}$. Source S_1 vibrates according to the equation $(0.04 \text{ m}) \sin 10\pi t$ while source S_2 vibrates according to the equation $(0.01 \text{ m}) \sin 10\pi t$. (a) Show that S_1 sends sound in the positive x direction as:

$$y_1 = (0.04 \text{ m}) \sin(5\pi x_1 - 10\pi t)$$

where x_1 is measured from an origin located at S_1 . (b) Show that S_2 emits sound in the negative x -direction as:

$$y_2 = (0.01 \text{ m}) \sin(5\pi x_2 + 10\pi t)$$

where x_2 is measured from an origin located at S_2 . (c) Show that the equation of motion of a particle at 0.8 m from S_1 and 0.4 m from S_2 is given by:

$$y = y_1 + y_2 = (-0.03 \text{ m}) \sin 10\pi t.$$

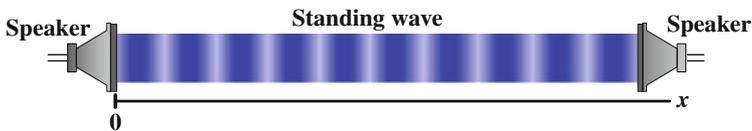


Fig. 16.21 See Exercise (8)

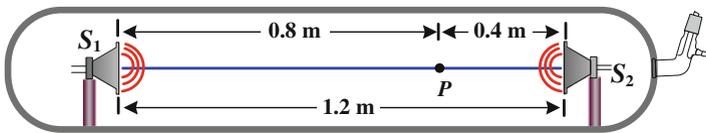


Fig. 16.22 See Exercise (9)

(10) Using direct substitution, show that the standing wave function:

$$y = 2 A \cos\left(\frac{\phi}{2}\right) \sin\left(kx - \omega t + \frac{\phi}{2}\right)$$

is a solution of the general partial linear differential equation [see Eq. 14.58]:

$$\frac{\partial^2 y}{\partial x^2} - \frac{1}{v^2} \frac{\partial^2 y}{\partial t^2} = 0$$

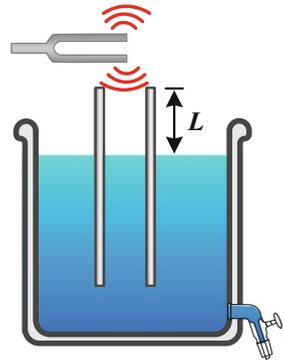
Section 16.4 Standing Sound Waves in Air columns

Note: Unless otherwise specified, use the speed of sound in this section to be 343 m/s.

- (11) An organ pipe of length 30 cm is open at both ends. What are the frequencies of the fundamental and the next two harmonics?
- (12) If the organ pipe in the previous exercise has one end closed, what are the frequencies of the fundamental and the next two harmonics?

- (13) The fundamental frequency of a pipe is found to be 110 Hz when the speed of sound is 330 m/s. (a) Find the pipe's length when it is closed at one end. (b) Find the pipe's length when it is open at both ends.
- (14) The two adjacent harmonic frequencies of an organ pipe (with both ends open) are determined to be 540 Hz and 420 Hz. (a) Find the fundamental frequency of the pipe. (b) Find the pipe's length.
- (15) Estimate the fundamental frequency that you would experience when blowing across the top of an empty cylindrical soft drink bottle that has a height of 10 cm. Assume that the bottle behaves like a tube with one end closed. Take the speed of sound to be 340 m/s. How would this frequency change if the bottle was only three quarters empty?
- (16) What would be the range of an adjustable pipe length that has two open ends if its fundamental frequency spans the human hearing range (from 20 Hz to 20 kHz)? Take the speed of sound to be 340 m/s.
- (17) A tuning fork vibrating at a frequency of 384 Hz is held over the top end of a vertical tube while the other end is partially inserted in a water tank as shown in Fig. 16.23. The water level in the tube is lowered by opening a valve in the tank so that the length L of the air column slowly increases from an initial value of 30 cm. Determine the next two values of L that correspond to resonance.

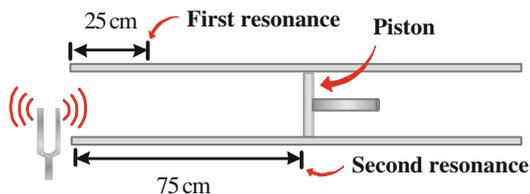
Fig. 16.23 See Exercise (17)



- (18) Assume that the speed of waves on a guitar string does not change when the string is fingered. If an unfingered string has a length $L = 0.75$ m and is tuned to play an F note (at 349 Hz). (a) How far from the end of this string must your finger be placed to play an A note (at 440 Hz). (b) What is the wavelength of the standing wave when this fingered string resonates at its fundamental frequency? (c) Find the frequency and wavelength of the sound waves that are produced by this string at that fundamental frequency.

- (19) At a temperature of 25°C , an open organ pipe produces the middle C note (262 Hz) with a fundamental standing wave. (a) What is the length of the pipe? (b) Find the frequency and wavelength of the fundamental standing wave in the pipe. (c) Find the frequency and wavelength of the sound produced in the air outside the pipe.
- (20) An open organ pipe is tuned in a room where the temperature was set to 20°C . If the temperature drops to 10°C , what would be the percentage change in frequency generated by the pipe?
- (21) In an air-filled tube closed at both ends, the distance between several nodes is 25 cm. When another gas replaces the air, the distance between that same number of nodes is 35 cm. If the speed of sound in air is 340 m/s, what is the speed in the gas?
- (22) An organ pipe can resonate at the successive harmonics of frequencies 210, 350, and 490 Hz. (a) Is this pipe open at both ends or closed at one of its ends? Explain why. (b) What is the fundamental frequency of this pipe?
- (23) A tube is open at both ends and has a length $L = 2$ m. It resonates at two successive harmonics of frequencies 355 and 440 Hz. (a) What is the fundamental frequency of this pipe? (b) What is the speed of sound in the air inside the tube?
- (24) A tube has a length $L = 2.5$ m. How many harmonics are present in this tube within the human hearing range (from 20 Hz to 20 kHz) if: (a) the tube is open at both ends, and (b) is closed at one end?
- (25) A pipe is open at one end and closed by a movable piston at the other end. A tuning fork of frequency 348 Hz is held at the open end. On a hot day a resonance occurs when the piston is at 0.25 m from the open end and again when it is at 0.75 m, see Fig. 16.24. (a) What is the speed of sound in the air inside the pipe? (b) How far from the open end will the piston be when the next resonance is experienced?

Fig. 16.24 See Exercise (25)



Section 16.5 Temporal Interference of Sound Waves: Beats

- (26) Determine the beat frequency resulting from the superposition of the two sound waves given by:

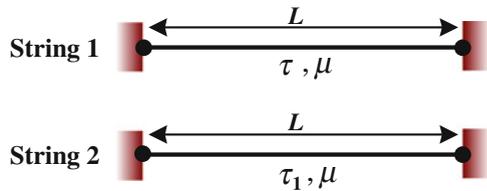
$$y_1 = (1.5 \text{ cm}) \sin(3.5x - 1376\pi t),$$

$$y_2 = (1.5 \text{ cm}) \sin(3.5x - 1364\pi t).$$

where x and y are in centimeters and t is in seconds.

- (27) Two identical violin strings have the same length L , tension τ , and exact fundamental frequency of 600 Hz. How much should we increase the tension of one of these strings to generate a sound beat of 6 Hz (see Fig. 16.25 for a new tension τ_1)?

Fig. 16.25 See Exercise (27)



- (28) A standard tuning fork of frequency 512 Hz makes a beat frequency of 4 Hz with another fork of unknown frequency. The beat frequency disappears when the prongs of the second fork are waxed. What is the frequency of the unknown fork?
- (29) A mistuned Middle C string in a piano (corresponds to key No. 40) has a *proper* fundamental frequency of 262 Hz, see Fig. 16.17. During the tuning trials, a musician hears 3 beats per second between the piano string and a standard oscillator of 262 Hz. (a) What are the possible frequencies of the string? (b) When the musician tightens the string slightly, he hears 4 beats per second. What is the frequency of the string now? (Hint: use the fact that tightening the string raises the wave speed and frequency) (c) By what percentage should the musician change the tension in the string to tune it?
- (30) At a temperature of 30°C , a source generates sound waves that propagate in the air with wavelengths $\lambda_1 = 1.62 \text{ m}$ and $\lambda_2 = 1.70 \text{ m}$. (a) What beat frequency is heard? (b) How far in space is the distance between the maximum intensities? (Hint: see Fig. 16.15b)