2 Vectors

When a particle moves in a straight line, we can take its motion to be positive in one specific direction and negative in the other. However, when this particle moves in three dimensions, plus or minus signs are no longer enough to specify the direction of motion. Instead, we must use a vector.

2.1 Vectors and Scalars

A **vector** has **magnitude** and **direction**, examples being displacement (change of position), velocity, acceleration, etc. Actually, not all physical quantities involve direction, examples being temperature, mass, pressure, time, etc. These physical quantities are not vectors because they do not point in any direction, and we call them **scalars**.

A vector, such as a **displacement vector**, can be represented graphically by an arrow denoting the magnitude and direction of the vector. All arrows of the same direction and magnitude denote the same vector, as in Fig. 2.1a for the case of a displacement vector.

The displacement vector in Fig. 2.1b tells us nothing about the actual path taken from point A to B. Thus, displacement vectors represent only the overall effect of the motion, not the motion itself.

Another way to specify a vector is to determine its magnitude and the angle it makes with a reference direction, as in Example 2.1.

H. A. Radi and J. O. Rasmussen, *Principles of Physics*,
Undergraduate Lecture Notes in Physics, DOI: 10.1007/978-3-642-23026-4_2,

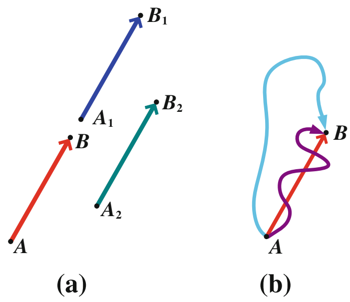


Fig. 2.1 (a) Three vectors of the same direction and magnitude represent the same displacement. (b) All three paths connecting the two points *A* and *B* correspond to the same displacement vector

Example 2.1

A person walks 3 km due east and then 2 km due north. What is his displacement vector?

Solution: We first make an overhead view of the person's movement as shown in Fig. 2.2. The magnitude of the displacement d is given by the Pythagorean theorem as follows:

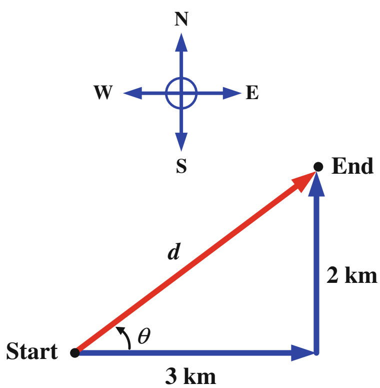
$$d = \sqrt{(3 \text{ km})^2 + (2 \text{ km})^2} = 3.61 \text{ km}$$

The angle that this displacement vector makes relative to east is given by:

$$\tan\,\theta = \frac{2 \text{ km}}{3 \text{ km}} = 0.666...$$

Then: $\theta = \tan^{-1}(0.666...) = 33.69°$
Thus, the person's displacement vector is 56.31° east of north.

Fig. 2.2



2.2 Properties of Vectors

In text books, it is common to use boldface symbols to identify vectors, such as $\boldsymbol{A}$, $\boldsymbol{B}$, etc., but in handwriting it is usual to place an arrow over the symbol, such as, $\vec{A}, \vec{B}$, etc. Throughout this text we shall use the handwriting style only and use the italic symbols A, B, etc. to indicate the magnitude of vectors.

Equality of Vectors

The two vectors $\vec{A}$ and $\vec{B}$ are said to be equal if they have the same magnitude, i.e. $A = B$, and point in the same direction; see for example the three equal vectors AB, A_1B_1, and A_2B_2 in Fig. 2.1a.

Addition of Vectors

Of course, all vectors involved in any addition process must have the same units. The rules for vector sums can be illustrated by using a graphical method. To add vector $\vec{B}$ to vector $\vec{A}$, we first draw vector $\vec{A}$ on graph paper with its magnitude represented by a convenient scale, and then draw vector $\vec{B}$ to the same scale with its tail coinciding with the arrow head of $\vec{A}$, see Fig. 2.3a. This is known as the triangle method of addition. Thus, the resultant vector $\vec{R}$ is the red vector drawn from the tail of $\vec{A}$ to the head of $\vec{B}$ and is shown in the vector addition equation:

$$\vec{R} = \vec{A} + \vec{B}\,, \tag{2.1}$$

which says that the vector $\vec{R}$ is the vector sum of vectors $\vec{A}$ and $\vec{B}$. The symbol + in Eq. 2.1 and the words "sum" and "add" have different meanings for vectors than they do in elementary algebra of scalar numbers.

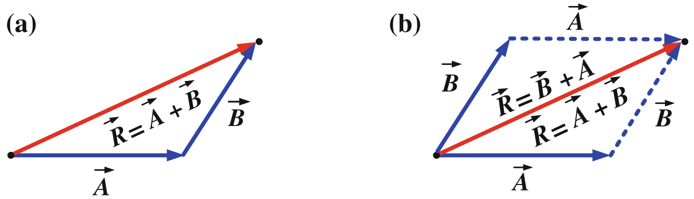


Fig. 2.3 (a) In the triangle method of addition, the resultant vector $\vec{R}$ is the red vector that runs from the tail of $\vec{A}$ to the head of $\vec{B}$. (b) In the parallelogram method of addition, the resultant vector $\vec{R}$ is the red diagonal vector that starts from the tails of both $\vec{A}$ and $\vec{B}$. This method shows that $\vec{A} + \vec{B} = \vec{B} + \vec{A}$

An alternative graphical method for adding two vectors is the parallelogram rule of addition. In this method, we superpose the tails of the two vectors $\vec{A}$ and $\vec{B}$; then the resultant $\vec{R}$ will be the diagonal of the parallelogram that starts from the tail of both $\vec{A}$ and $\vec{B}$ (which form the sides of that parallelogram), as shown in Fig. 2.3b.

Vector addition has two important properties. First, the order of addition does not matter, and this is known as the commutative law of addition, i.e.

$$\vec{A} + \vec{B} = \vec{B} + \vec{A} \qquad \text{(Commutative law)} \tag{2.2}$$

Second, if there are more than two vectors, their sum is independent of the way in which the individual vectors are grouped together. This is known as the associative law of addition, i.e.

$$\vec{A} + (\vec{B} + \vec{C}) = (\vec{A} + \vec{B}) + \vec{C} \qquad \text{(Associative law)} \tag{2.3}$$

The Negative of a Vector

The negative of a vector $\vec{B}$ is a vector with the same magnitude which points in the opposite direction, namely $-\vec{B}$, see Fig. 2.4a. Therefore, when we add a vector and its negative we will get zero, i.e.

$$\vec{B} + (-\vec{B}) = 0 \tag{2.4}$$

Adding $-\vec{B}$ to $\vec{A}$ has the same effect of subtracting $\vec{B}$ from $\vec{A}$, see Fig. 2.4b, i.e.

$$\begin{aligned}\vec{S} &= \vec{A} + (-\vec{B}) \\ &= \vec{A} - \vec{B}\end{aligned} \tag{2.5}$$

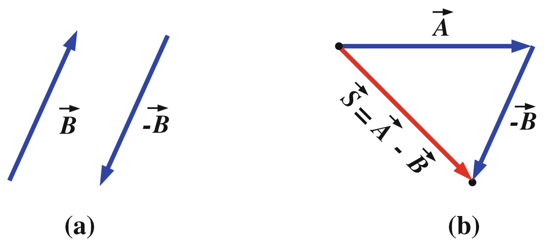


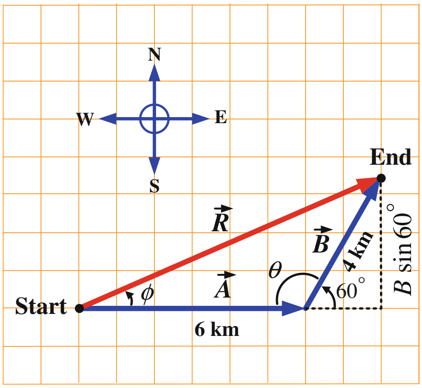
Fig. 2.4 (a) This part of the figure shows vector $\vec{B}$ and its corresponding negative vector $-\vec{B}$, both of which have the same magnitude but are opposite in direction. (b) To subtract vector $\vec{B}$ from vector $\vec{A}$, we add the vector $-\vec{B}$ to vector $\vec{A}$ to get $\vec{S} = \vec{A} - \vec{B}$

Example 2.2

A car travels 6 km due east and then 4 km in a direction 60° north of east. Find the magnitude and direction of the car's displacement vector by using: (a) the graphical method, and (b) the analytical method.

Solution: (a) Let $\vec{A}$ be a vector directed due east with magnitude $A = 6\,\text{km}$ and $\vec{B}$ be a vector directed 60° north of east with magnitude $B = 4\,\text{km}$. Using graph paper with a reasonable scale and a protractor, we draw the two vectors $\vec{A}$ and $\vec{B}$; then we measure the length of the resultant vector $\vec{R}$. The measurements shown in Fig. 2.5 indicates that $R = 8.7\,\text{km}$. Also, the angle ϕ that the resultant vector $\vec{R}$ makes with respect to the east direction can be measured and will give $\phi = 23.4°$.

Fig. 2.5



(b) The analytical solution for the magnitude of $\vec{R}$ can be obtained from geometry by using the law of cosines $R = \sqrt{A^2 + B^2 - 2AB\cos\theta}$ as applied to an obtuse triangle with angle $\theta = 180° - 60° = 120°$, see exercise (10b). Thus:

$$
\begin{aligned}
R &= \sqrt{A^2 + B^2 - 2\,AB\cos\theta} \\
&= \sqrt{(6\text{ km})^2 + (4\text{ km})^2 - 2(6\text{ km})(4\text{ km})\cos 120°} \\
&= \sqrt{(36 + 16 + 24)\ (\text{km})^2} = 8.72\text{ km}
\end{aligned}
$$

The angle that this displacement vector $\vec{R}$ makes relative to the east direction, see Fig. 2.5, is given by:

$$
\sin\phi = \frac{B\sin 60°}{R} = \frac{4\,\text{km}\sin 60°}{8.72\,\text{km}} = 0.397
$$

Then: $\phi = \sin^{-1}(0.397) = 23.41°$.

2.3 Vector Components and Unit Vectors

Vector Components

Adding vectors graphically is not recommended in situations where high precision is needed or in three-dimensional problems. A better way is to make use of the projections of a vector along the axes of a rectangular coordinate system.

Consider a vector $\vec{A}$ lying in the *xy*-plane and making an angle θ with the positive *x*-axis, see Fig. 2.6. This vector $\vec{A}$ can be expressed as the sum of two vectors $\vec{A}_x$ and $\vec{A}_y$ called the rectangular vector components of $\vec{A}$ along the *x*-axis and *y*-axis, respectively. Thus:

$$\vec{A} = \vec{A}_x + \vec{A}_y \tag{2.6}$$

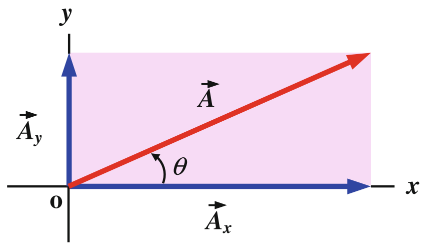


Fig. 2.6 A vector $\vec{A}$ in the *xy*-plane can be presented by its rectangular vector components $\vec{A}_x$ and $\vec{A}_y$, where $\vec{A} = \vec{A}_x + \vec{A}_y$

From the definitions of sine and cosine, the rectangular components of $\vec{A}$, namely A_x and A_y, will be given by:

$$A_x = A\cos\theta \quad \text{and} \quad A_y = A\sin\theta\,, \tag{2.7}$$

where the sign of the components A_x and A_y depends on the angle θ.

The magnitudes A_x and A_y form two sides of a right triangle that has a hypotenuse of magnitude A. Thus, from A_x and A_y we get:

$$A = \sqrt{A_x^2 + A_y^2} \quad \text{and} \quad \theta = \tan^{-1}\left(\frac{A_y}{A_x}\right) \tag{2.8}$$

The *inverse tan* obtained from your calculator is from $-90° < \theta < 90°$. This may lead to incorrect answer when $90° < \theta \le 360°$. A method used to achieve the correct answer is to calculate the angle ϕ such as:

$$\phi = \tan^{-1}\left(|A_y|/|A_x|\right) \tag{2.9}$$

Then, depending on the signs of A_x and A_y, we identify the quadrant where the vector $\vec{A}$ lies, as shown in Fig. 2.7.

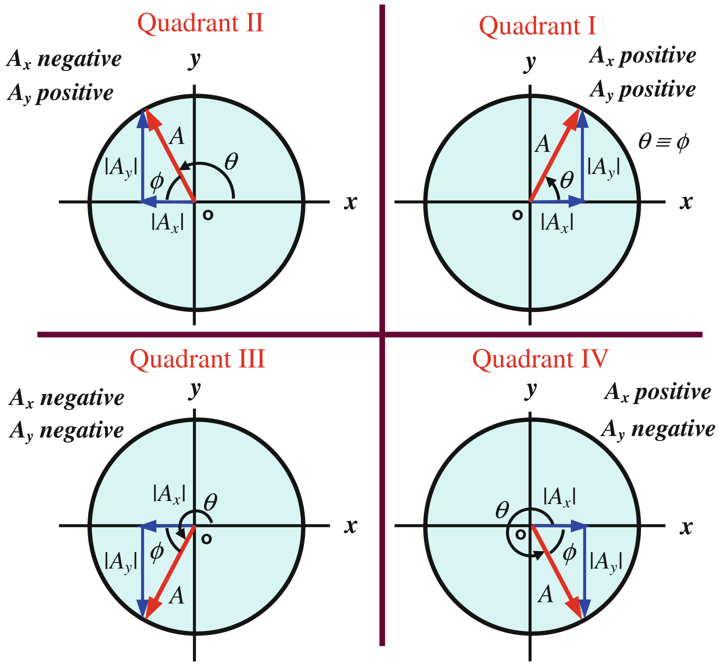


Fig. 2.7 The signs of A_x and A_y depend on the quadrant where the vector $\vec{A}$ is located

Once we determine the quadrant, we calculate θ using Table 2.1.

Table 2.1 Calculating θ from ϕ according to the signs of A_x and A_y

Sign of A_x	Sign of A_y	Quadrant	Angle θ
+	+	I	$\theta = \phi$
−	+	II	$\theta = 180° - \phi$
−	−	III	$\theta = 180° + \phi$
+	−	IV	$\theta = 360° - \phi$

Unit Vectors

A unit vector is a dimensionless vector that has a magnitude of exactly one and points in a particular direction, and has no other physical significance. The unit vectors in the positive direction of the x, y, and z axes of a **right-handed coordinate system** are often labeled $\vec{\mathrm{i}}$, $\vec{\mathrm{j}}$, and $\vec{\mathrm{k}}$, respectively; see Fig. 2.8. The magnitude of each unit vector equals unity; that is:

$$|\vec{\mathrm{i}}| = |\vec{\mathrm{j}}| = |\vec{\mathrm{k}}| = 1 \tag{2.10}$$

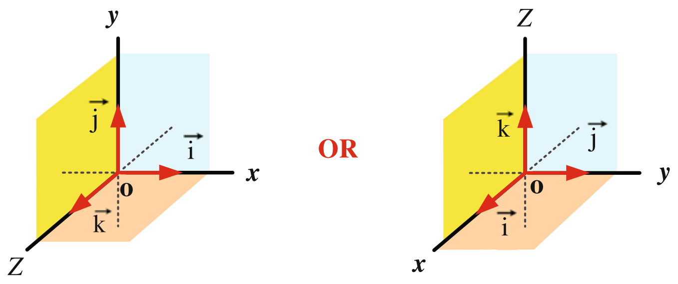
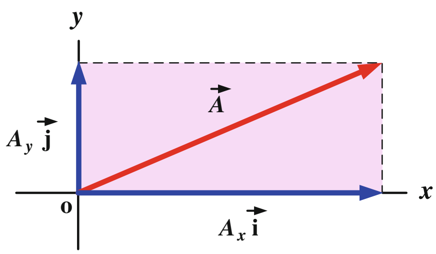


Fig. 2.8 Unit vectors $\vec{\mathrm{i}}$, $\vec{\mathrm{j}}$, and $\vec{\mathrm{k}}$ define the direction of the commonly-used right-handed coordinate system

Consider a vector $\vec{A}$ lying in the xy-plane as shown in Fig. 2.9. The product of the component A_x and the unit vector $\vec{\mathrm{i}}$ is the vector $\vec{A}_x = A_x\,\vec{\mathrm{i}}$, which is parallel to the x-axis and has a magnitude A_x. Similarly, $\vec{A}_y = A_y\,\vec{\mathrm{j}}$ is a vector parallel to the y-axis and has a magnitude A_y. Thus, in terms of unit vectors we write $\vec{A}$ as follows:

$$\vec{A} = A_x\,\vec{\mathrm{i}} + A_y\,\vec{\mathrm{j}} \tag{2.11}$$

Fig. 2.9 A vector $\vec{A}$ in the xy-plane can be represented by its rectangular components A_x and A_y and the unit vectors $\vec{\mathrm{i}}$ and $\vec{\mathrm{j}}$, and can be written as $\vec{A} = A_x\,\vec{\mathrm{i}} + A_y\,\vec{\mathrm{j}}$



This method can be generalized to three-dimensional vectors as:

$$\vec{A} = A_x\,\vec{\mathrm{i}} + A_y\,\vec{\mathrm{j}} + A_z\,\vec{\mathrm{k}} \tag{2.12}$$

We can define a unit vector $\vec{n}$ along any vector, say, $\vec{A}$, as follows:

$$\vec{n} = \frac{\vec{A}}{A}. \tag{2.13}$$

Adding Vectors by Components

Suppose we wish to add the two vectors $\vec{A} = A_x\,\vec{i} + A_y\,\vec{j}$ and $\vec{B} = B_x\,\vec{i} + B_y\,\vec{j}$ using the components method, such as:

$$\begin{aligned}\vec{R} &= \vec{A} + \vec{B}\\ &= (A_x\,\vec{i} + A_y\,\vec{j}\,) + (B_x\,\vec{i} + B_y\,\vec{j}\,)\\ &= (A_x + B_x)\,\vec{i} + (A_y + B_y)\,\vec{j}\end{aligned} \tag{2.14}$$

If the vector sum $\vec{R}$ is denoted by $\vec{R} = R_x\,\vec{i} + R_y\,\vec{j}$, then the components of the resultant vector will be given by:

$$\begin{aligned}R_x &= A_x + B_x\\ R_y &= A_y + B_y\end{aligned} \tag{2.15}$$

The magnitude of $\vec{R}$ can then be obtained from its components or the components of $\vec{A}$ and $\vec{B}$ using the following relationships:

$$R = \sqrt{R_x^2 + R_y^2} = \sqrt{(A_x + B_x)^2 + (A_y + B_y)^2} \tag{2.16}$$

and the angle that $\vec{R}$ makes with the x-axis can also be obtained by using the following relationships:

$$\theta = \tan^{-1}\left(\frac{R_y}{R_x}\right) = \tan^{-1}\left(\frac{A_y + B_y}{A_x + B_x}\right) \tag{2.17}$$

The components method can be verified using the geometrical method, as shown in Fig. 2.10.

If $\vec{A} = A_x\,\vec{i} + A_y\,\vec{j} + A_z\,\vec{k}$ and $\vec{B} = B_x\,\vec{i} + B_y\,\vec{j} + B_z\,\vec{k}$, then we can generalize the previous case to three dimensions as follows:

$$\begin{aligned}\vec{R} &= \vec{A} + \vec{B} = (A_x + B_x)\,\vec{i} + (A_y + B_y)\,\vec{j} + (A_z + B_z)\,\vec{k}\\ &= R_x\,\vec{i} + R_y\,\vec{j} + R_z\,\vec{k}\end{aligned} \tag{2.18}$$

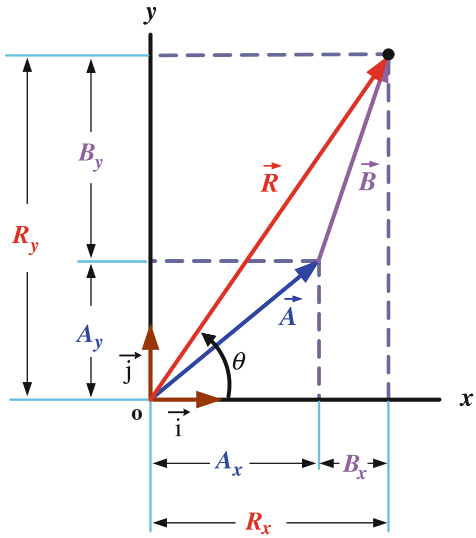


Fig. 2.10 Geometric representation of the sum of the two vectors $\vec{A}$ and $\vec{B}$, showing the relationship between the components of the resultant $\vec{R}$ and the components of $\vec{A}$ and $\vec{B}$

Example 2.3

Find the sum of the following two vectors:

$$\vec{A} = 8\,\vec{i} + 3\,\vec{j}$$

$$\vec{B} = -5\,\vec{i} - 7\,\vec{j}$$

For convenience, the units of the two vectors have been omitted, but for instance, you may take them to be kilometers.

Solution: The two vectors lie in the xy-plane, since there is no component in the z-axis. By comparing the two expressions of $\vec{A}$ and $\vec{B}$ with the general relations $\vec{A} = A_x\,\vec{i} + A_y\,\vec{j}$ and $\vec{B} = B_x\,\vec{i} + B_y\,\vec{j}$ we see that, $A_x = 8$, $A_y = 3$, $B_x = -5$, and $B_y = -7$. Therefore, the resultant vector R is obtained by using Eq. 2.14 as follows:

$$\begin{aligned}\vec{R} &= \vec{A} + \vec{B} = (A_x + B_x)\,\vec{i} + (A_y + B_y)\,\vec{j} \\ &= (8 - 5)\,\vec{i} + (3 - 7)\,\vec{j} = 3\,\vec{i} - 4\,\vec{j}\end{aligned}$$

That is: $R_x = 3$ and $R_y = -4$. The magnitude of $\vec{R}$ is given according to Eq. 2.16 as:

$$R = \sqrt{R_x^2 + R_y^2} = \sqrt{(3)^2 + (-4)^2} = \sqrt{9 + 16} = \sqrt{25} = 5$$

while the value of the angle θ that $\vec{R}$ makes with the positive x-axis is given according to Eq. 2.17 as:

$$\theta = \tan^{-1}\left(\frac{R_y}{R_x}\right) = \tan^{-1}\left(\frac{-4}{3}\right) = 360^\circ - \tan^{-1}\left(\frac{4}{3}\right) = 360^\circ - 53^\circ = 307^\circ$$

where we used Table 2.1 to calculate θ in case of negative R_y (Q IV).

2.4 Multiplying Vectors

Multiplying a Vector by a Scalar

If we multiply vector $\vec{A}$ by a scalar a we get a new vector $\vec{B}$, i.e.

$$\vec{B} = a\vec{A} \tag{2.19}$$

The vector $\vec{B}$ has the same direction as $\vec{A}$ if a is positive but has the opposite direction if a is negative. The magnitude of $\vec{B}$ is the product of the magnitude of $\vec{A}$ and the absolute value of a.

The Scalar Product (or the Dot Product)

The **scalar product** of the two vectors $\vec{A}$ and $\vec{B}$ is denoted by $\vec{A} \bullet \vec{B}$ and is defined as:

$$\vec{A} \bullet \vec{B} = AB\cos\theta \tag{2.20}$$

where A and B are the magnitudes of the two vectors $\vec{A}$ and $\vec{B}$, and θ is the angle between them, see Fig. 2.11. The two angles θ and $360^\circ - \theta$ could be used, since their cosines are the same. As we see from Eq. 2.20, the result of $\vec{A} \bullet \vec{B}$ is a scalar quantity, and is known as the **dot product** from its notation. Also, we get:

$$\vec{A} \bullet \vec{B} = AB\cos\theta = \begin{cases} +AB & \text{if } \theta = 0^\circ \\ 0 & \text{if } \theta = 90^\circ \\ -AB & \text{if } \theta = 180^\circ \end{cases} \tag{2.21}$$

According to Fig. 2.11, the dot product can be regarded as the product of the magnitude of one of the vectors with the scalar component of the second along the direction of the first. That is:

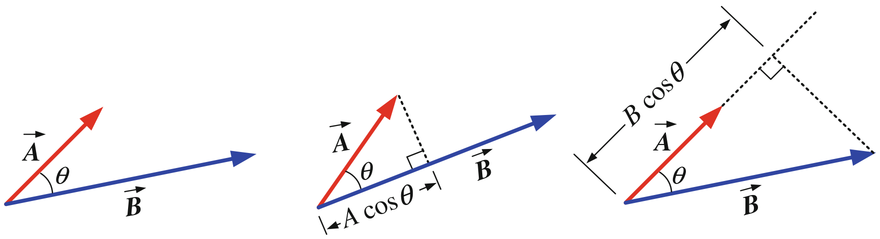


Fig. 2.11 The *left* part shows two vectors $\vec{A}$ and $\vec{B}$, with an angle θ between them. The middle and the *right* parts show the component of each vector along the other

$$\vec{A} \bullet \vec{B} = (A\cos\theta)B = A(B\cos\theta) \tag{2.22}$$

This indicates that scalar products obey the commutative and associative laws, so that:

$$\vec{A} \bullet \vec{B} = \vec{B} \bullet \vec{A} \quad \text{(Commutative law)} \tag{2.23}$$

$$\vec{A} \bullet (\vec{B} + \vec{C}) = \vec{A} \bullet \vec{B} + \vec{A} \bullet \vec{C} \quad \text{(Associative law)} \tag{2.24}$$

By applying the definition of dot product to the unit vectors $\vec{\mathrm{i}}$, $\vec{\mathrm{j}}$, and $\vec{\mathrm{k}}$, we get the following:

$$\begin{aligned} \vec{\mathrm{i}} \bullet \vec{\mathrm{i}} &= \vec{\mathrm{j}} \bullet \vec{\mathrm{j}} = \vec{\mathrm{k}} \bullet \vec{\mathrm{k}} = 1 \\ \vec{\mathrm{i}} \bullet \vec{\mathrm{j}} &= \vec{\mathrm{i}} \bullet \vec{\mathrm{k}} = \vec{\mathrm{j}} \bullet \vec{\mathrm{k}} = 0, \end{aligned} \tag{2.25}$$

where the angle between any two identical unit vectors is 0° and the angle between any two different unit vectors is 90°.

When two vectors are written in terms of the unit vectors $\vec{\mathrm{i}}$, $\vec{\mathrm{j}}$, and $\vec{\mathrm{k}}$, then to get their dot product, each component of the first vector is to be dotted into each component of the second vector. After that, we use Eq. 2.25 to get the following:

$$\begin{aligned} \vec{A} \bullet \vec{B} &= (A_x \vec{\mathrm{i}} + A_y \vec{\mathrm{j}} + A_z \vec{\mathrm{k}}) \bullet (B_x \vec{\mathrm{i}} + B_y \vec{\mathrm{j}} + B_z \vec{\mathrm{k}}) \\ &= A_x B_x + A_y B_y + A_z B_z \end{aligned} \tag{2.26}$$

Thus, from Eqs. 2.20 and 2.26, we can generally write the dot product as follows:

$$\vec{A} \bullet \vec{B} = A\,B\,\cos\,\theta = A_x B_x + A_y B_y + A_z B_z \tag{2.27}$$

Example 2.4

Find the angle between the vector $\vec{A} = 8\,\vec{i} + 3\,\vec{j}$ and the vector $B = -5\,\vec{i} - 7\,\vec{j}$.

Solution: Since $A = \sqrt{A_x^2 + A_y^2}$ and $B = \sqrt{B_x^2 + B_y^2}$, then using the dot product given by Eq. 2.20 we get:

$$\begin{aligned}\vec{A} \bullet \vec{B} &= AB\cos\theta = \sqrt{8^2 + 3^2} \times \sqrt{(-5)^2 + (-7)^2}\cos\theta \\ &= 8.544 \times 8.60\cos\theta \\ &= 73.5\cos\theta\end{aligned}$$

Keeping in mind that there is no component for $\vec{A}$ and $\vec{B}$ along the z-axis, we can find the dot product from Eq. 2.26 as follows:

$$\begin{aligned}\vec{A} \bullet \vec{B} &= A_xB_x + A_yB_y + A_zB_z \\ &= 8 \times (-5) + 3 \times (-7) + 0 \times 0 \\ &= -61\end{aligned}$$

Equating the results of the last two steps to each other, we find:

$$73.5\cos\theta = -61$$

Thus:

$$\theta = \cos^{-1}\left(\frac{-61}{73.5}\right) = 146.1^\circ.$$

The Vector Product (or the Cross Product)

The **vector product** of the two vectors $\vec{A}$ and $\vec{B}$ is denoted by $\vec{A} \times \vec{B}$ and defined as a third vector $\vec{C}$ whose magnitude is:

$$C = AB\sin\theta\ , \qquad (2.28)$$

where θ is the smaller angle between $\vec{A}$ and $\vec{B}$ (hence, $0 \leq \sin\theta \leq 1$). The direction of $\vec{C}$ is perpendicular to the plane that contains both $\vec{A}$ and $\vec{B}$, and can be determined by using the **right-hand rule**, see Fig. 2.12. To apply this rule, we allow the tail of $\vec{A}$ to coincide with the tail of $\vec{B}$, then the four fingers of the right hand are pointed along $\vec{A}$ and then "wrapped" into $\vec{B}$ through the angle θ. The direction of the erect right thumb is the direction of $\vec{C}$, i.e. the direction of $\vec{A} \times \vec{B}$. Also, the direction of $\vec{C}$ is determined by the direction of advance of a right-handed screw as shown in Fig. 2.12.

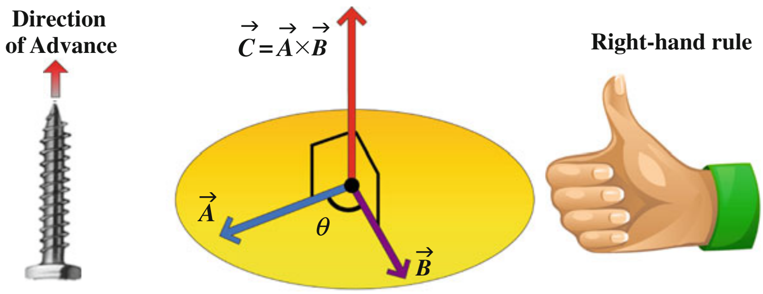


Fig. 2.12 The vector product $\vec{A} \times \vec{B}$ is a third vector $\vec{C}$ that has a magnitude of $AB \sin\theta$ and a direction perpendicular to the plane containing the vectors $\vec{A}$ and $\vec{B}$. Its sense is determined by the right-hand rule or the direction of advance of a right-handed screw

The vector product definition leads to the following properties:

1. The order of vector product multiplication is important; that is:

$$\vec{A} \times \vec{B} = -(\vec{B} \times \vec{A}) \tag{2.29}$$

   which is unlike the scalar product and can be easily verified with the right-hand rule.

2. If $\vec{A}$ is parallel to $\vec{B}$ (that is, $\theta = 0°$) or $\vec{A}$ is antiparallel to $\vec{B}$ (that is, $\theta = 180°$), then:

$$\vec{A} \times \vec{B} = 0 \quad (\text{if } \vec{A} \text{ is parallel or antiparallel to } \vec{B}) \tag{2.30}$$

3. If $\vec{A}$ is perpendicular to $\vec{B}$, then:

$$|\vec{A} \times \vec{B}| = AB \quad (\text{if } \vec{A} \perp \vec{B}) \tag{2.31}$$

4. The vector product obeys the distributive law, that is:

$$\vec{A} \times (\vec{B} + \vec{C}) = \vec{A} \times \vec{B} + \vec{A} \times \vec{C} \quad (\text{Distributive law}) \tag{2.32}$$

5. The derivative of $\vec{A} \times \vec{B}$ with respect to any variable such as t is:

$$\frac{d}{dt}(\vec{A} \times \vec{B}) = \vec{A} \times \frac{d\vec{B}}{dt} + \frac{d\vec{A}}{dt} \times \vec{B} \tag{2.33}$$

6. From the definition of the vector product and the unit vectors $\vec{i}$, $\vec{j}$, and $\vec{k}$, we get the following relationships:

$$\vec{i} \times \vec{i} = \vec{j} \times \vec{j} = \vec{k} \times \vec{k} = 0 \tag{2.34}$$

$$\vec{i} \times \vec{j} = \vec{k}, \quad \vec{j} \times \vec{k} = \vec{i}, \quad \vec{k} \times \vec{i} = \vec{j} \tag{2.35}$$

The last relations can be obtained by setting the unit vectors $\vec{i}$, $\vec{j}$, and $\vec{k}$ on a circle, see Fig. 2.13, and rotating in a clockwise direction to find the cross product of one unit vector with another. Rotating in a counterclockwise direction will involve a negative sign of the cross product of one unit vector with another, that is:

$$\vec{i} \times \vec{k} = -\vec{j}, \quad \vec{k} \times \vec{j} = -\vec{i}, \quad \vec{j} \times \vec{i} = -\vec{k} \tag{2.36}$$

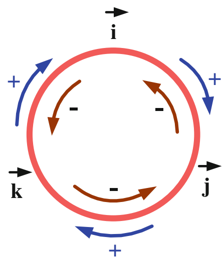


Fig. 2.13 The clockwise and counterclockwise cyclic order for finding the cross product of the unit vectors $\vec{i}$, $\vec{j}$, and $\vec{k}$

7. When two vectors $\vec{A}$ and $\vec{B}$ are written in terms of the unit vectors $\vec{i}$, $\vec{j}$, and $\vec{k}$, then the cross product will give the result:

$$\begin{aligned} \vec{A} \times \vec{B} &= (A_x \vec{i} + A_y \vec{j} + A_z \vec{k}) \times (B_x \vec{i} + B_y \vec{j} + B_z \vec{k}) \\ &= (A_y B_z - A_z B_y)\, \vec{i} + (A_z B_x - A_x B_z)\, \vec{j} \\ &\quad + (A_x B_y - A_y B_x)\, \vec{k} \end{aligned} \tag{2.37}$$

This result can be expressed in determinant form as follows:

$$\vec{A} \times \vec{B} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} = \vec{i} \begin{vmatrix} A_y & A_z \\ B_y & B_z \end{vmatrix} - \vec{j} \begin{vmatrix} A_x & A_z \\ B_x & B_z \end{vmatrix} + \vec{k} \begin{vmatrix} A_x & A_y \\ B_x & B_y \end{vmatrix} \tag{2.38}$$

Example 2.5

(a) Find the cross product of the two vectors $\vec{A} = 8\,\vec{i} + 3\,\vec{j}$ and $\vec{B} = -5\,\vec{i} - 7\,\vec{j}$.
(b) Verify explicitly that $\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$.

Solution: (a) Using Eq. 2.34 through Eq. 2.36 for the cross product of unit vectors, we will get the following for $\vec{A} \times \vec{B}$:

$$\begin{aligned}\vec{A} \times \vec{B} &= (8\,\vec{i} + 3\,\vec{j}) \times (-5\,\vec{i} - 7\,\vec{j}) \\ &= -40\,\vec{i} \times \vec{i} - 56\,\vec{i} \times \vec{j} - 15\,\vec{j} \times \vec{i} - 21\,\vec{j} \times \vec{j} \\ &= 0 - 56\,\vec{k} + 15\,\vec{k} + 0 = -41\,\vec{k}\end{aligned}$$

As an alternative method for finding $\vec{A} \times \vec{B}$, we use Eq. 2.37, with $A_x = 8$, $A_y = 3, A_z = 0, B_x = -5, B_y = -7,$ and $B_z = 0$:

$$\begin{aligned}\vec{A} \times \vec{B} &= (A_yB_z - A_zB_y)\,\vec{i} + (A_zB_x - A_xB_z)\,\vec{j} + (A_xB_y - A_yB_x)\,\vec{k} \\ &= (0)\,\vec{i} + (0)\,\vec{j} + (-56 - [-15])\,\vec{k} = -41\,\vec{k}\end{aligned}$$

(b) We can evaluate $\vec{B} \times \vec{A}$ as follows:

$$\begin{aligned}\vec{B} \times \vec{A} &= (-5\,\vec{i} - 7\,\vec{j}) \times (8\,\vec{i} + 3\,\vec{j}) \\ &= -40\,\vec{i} \times \vec{i} - 15\,\vec{i} \times \vec{j} - 56\,\vec{j} \times \vec{i} - 21\,\vec{j} \times \vec{j} \\ &= 0 - 15\,\vec{k} + 56\,\vec{k} + 0 = +41\,\vec{k}\end{aligned}$$

Therefore, $\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$.

Example 2.6

Is it possible to use the cross product to find the angle between the two vectors $\vec{A} = 8\,\vec{i} + 3\,\vec{j}$ and $\vec{B} = -5\,\vec{i} - 7\,\vec{j}$ of Example 2.5?

Solution: From Example 2.5 we found that:

$$\vec{A} \times \vec{B} = -41\,\vec{k}$$

If we let $\vec{C} = \vec{A} \times \vec{B}$, then according to Eq. 2.28 the magnitude of $\vec{C}$ is:

$$C = AB\sin\theta$$

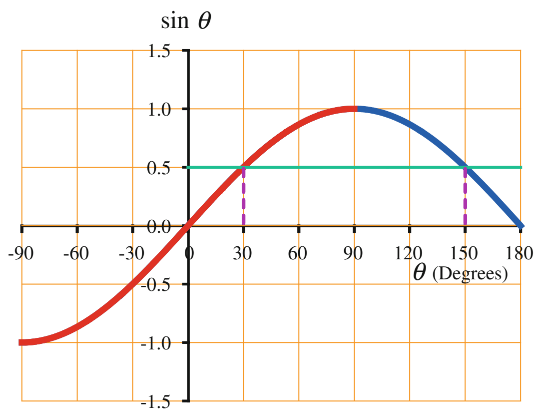
But $C = 41$. Therefore:

$$41 = \sqrt{8^2 + 3^2} \times \sqrt{(-5)^2 + (-7)^2}\,\sin\theta = 73.5\,\sin\theta$$

Thus, your calculator will give: $\theta = \sin^{-1}\left(\dfrac{41}{73.5}\right) = 33.91^\circ$

The calculator's range for $\sin^{-1}$ is only from -90° to 90°, (see the red part of the sine curve of Fig. 2.14.) So, when you calculate the inverse of a sine function, you must consider how reasonable your answer is, because there is usually another possible answer that the calculator does not display. For example, in Fig. 2.14, the horizontal line through 0.5 cuts the sine curve at 30° and 150°, i.e. the inverse sine of those two angles are equal to 0.5. But your calculator will give only the angle 30° (see the red part of the curve).

Fig. 2.14



Since $\sin\theta = \sin(180^\circ - \theta)$, then the angle between the two vectors could be either 33.91° or 146.1°. You can find the correct answer by using a graphical method or the dot product, as in Example 2.4, to prove that the correct answer is $\theta = 146.1^\circ$. Thus, the cross product is not the simplest method for determining the angle between any two vectors.

2.5 Exercises

Section 2.2 Properties of Vectors

(1) A car travels 10 km due north and then 5 km due west. Find graphically and analytically the magnitude and direction of the car's resultant displacement.

(2) A car travels 6 km due east and then 4 km in a direction 120° north of east. Use both the graphical and analytical methods to find the magnitude and direction of the car's displacement vector.

(3) Vector $\vec{A}$ has a magnitude of 10 units and makes 60° with the positive x-axis. Vector $\vec{B}$ has a magnitude of 5 units and is directed along the negative x-axis. Use geometry to find: (a) the vector sum $\vec{A} + \vec{B}$, and (b) the vector difference $\vec{A} - \vec{B}$.

(4) A car travels in a circular path of radius 10 m. (a) If the car traveled one half of the circle, find the magnitude of the displacement vector and find how far the car traveled. (b) Answer part (a) if the car makes one complete revolution.

Section 2.3 Vector Components and Unit Vectors

(5) Vector $\vec{A}$ has x and y components of 4 cm and −5 cm, respectively. Vector $\vec{B}$ has x and y components of −2 cm and 1 cm, respectively. If $\vec{A} - \vec{B} + 3\vec{C} = 0$, then what are the components of $\vec{C}$.

(6) Three vectors are oriented as shown in Fig. 2.15, where $A = 10$, $B = 20$, and $C = 15$ units. Find: (a) the x and y components of the resultant vector $\vec{D} = \vec{A} + \vec{B} + \vec{C}$, (b) the magnitude and direction of the resultant vector.

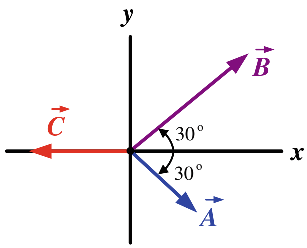


Fig. 2.15 See Exercise (6)

(7) The radar beam of a police car points at an angle of 30° away from the direction of a highway. The radar records the component of the car's speed along the beam as $v_{CR} = 120$ km/h, see Fig. 2.16. (a) What is the speed v_C of the car along the highway? (b) Can the radar beam be directed perpendicular to the direction of the highway? Why or why not?

(8) A radar device detects a rocket approaching directly from east due west. At one instant, the rocket was observed 10 km away and making an angle of 30° above the horizon. At another instant the rocket was observed at an angle of 150°

in the vertical east-west plane while the rocket was 8 km away, see Fig. 2.17. Find the displacement of the rocket during the period of observation.

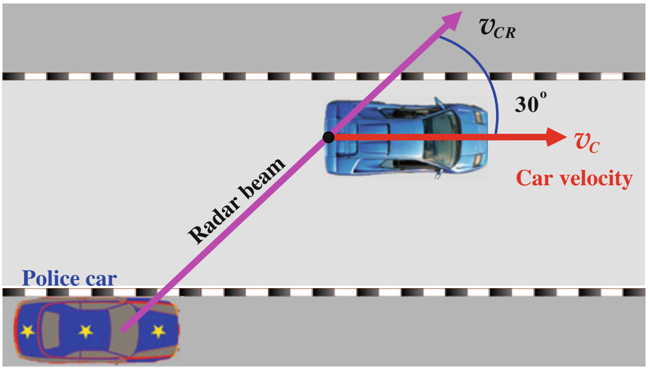


Fig. 2.16 See Exercise (7)

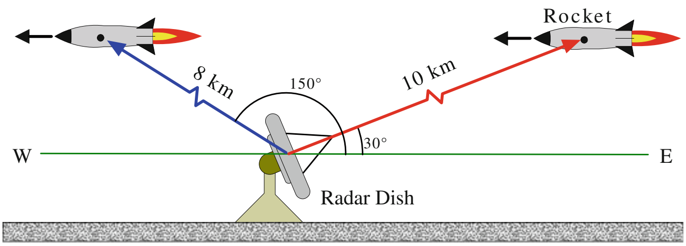


Fig. 2.17 See Exercise (8)

(9) Find the vector components of the sum $\vec{R}$ of the displacement vectors $\vec{A}$ and $\vec{B}$ whose components along three perpendicular directions are $A_x = 2$, $A_y = 1$, $A_z = 3$, $B_x = 1$, $B_y = 4$, and $B_z = 2$. Find the magnitude of $\vec{R}$.

(10) Two vectors $\vec{A}$ and $\vec{B}$ (of lengths A and B, respectively) make an angle θ with each other when they are placed tail to tail, see Fig. 2.18. (a) By taking components along two perpendicular axes, prove that the length of their vector sum $\vec{R} = \vec{A} + \vec{B}$ is:

$$R = \sqrt{A^2 + B^2 + 2AB\cos\theta}$$

(b) For the difference $\vec{C} = \vec{A} - \vec{B}$, where C is the length of the third side of a triangle formed from connecting the head of $\vec{B}$ to the head of $\vec{A}$ as in Fig. 2.19, use the same approach to prove that:

$$C = \sqrt{A^2 + B^2 - 2AB\cos\theta}$$

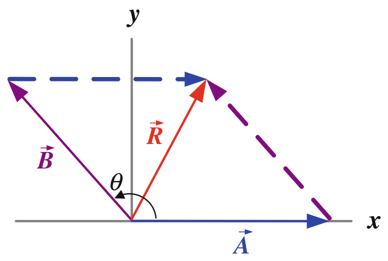


Fig. 2.18 See Exercise (10)

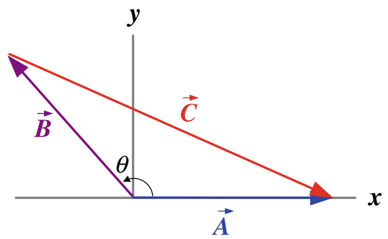


Fig. 2.19 See Exercise (10)

(11) A position vector $\vec{r} = x\,\vec{\mathrm{i}} + y\,\vec{\mathrm{j}} + z\,\vec{\mathrm{k}}$ makes angles α, β, and γ with the x, y, and z axes of a perpendicular right-handed coordinate system as in Fig. 2.20. Show that the relation between what is known as the direction cosines $\cos\alpha$, $\cos\beta$, and $\cos\gamma$ are as follows: $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$.

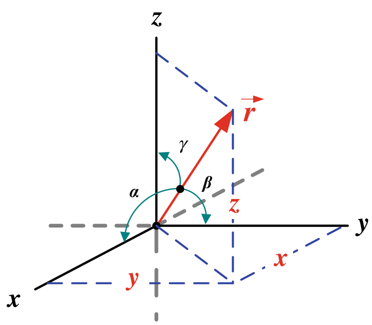


Fig. 2.20 See Exercise (11)

(12) When vector $\vec{B}$ is added to vector $\vec{A}$ we get $5\,\vec{\mathrm{i}} - \vec{\mathrm{j}}$, and when $\vec{B}$ is subtracted from $\vec{A}$ we get $\vec{\mathrm{i}} - 7\,\vec{\mathrm{j}}$. What is the magnitude and direction of $\vec{A}$?

(13) Two vectors are given by $\vec{A} = 2\vec{i} + 3\vec{j}$ and $\vec{B} = 4\vec{i} - 3\vec{j}$. Find: (a) the magnitude and direction of the vector sum $\vec{R} = \vec{A} + \vec{B}$, (b) the magnitude and direction of the vector difference $\vec{S} = \vec{A} - \vec{B}$.

(14) Two vectors are given by $\vec{A} = -\vec{i} + \vec{j} + 4\vec{k}$ and $\vec{B} = 3\vec{i} - 4\vec{j} + \vec{k}$. Find: (a) $\vec{A} + \vec{B}$, (b) $\vec{A} - \vec{B}$, and (c) a vector $\vec{C}$ such that $\vec{A} + \vec{B} + \vec{C} = 0$.

Section 2.4 Multiplying Vectors

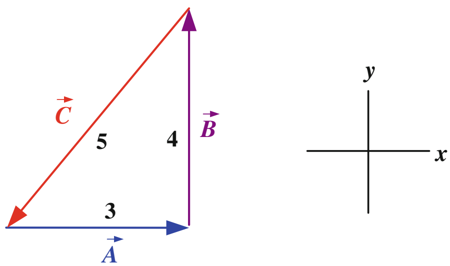
(15) Vector $\vec{A}$ has a magnitude of 3 units and lies along the negative x-axis. Vector $\vec{B}$ has a magnitude of 6 units and makes an angle 30° with the positive x-axis. (a) Find the scalar product $\vec{A} \bullet \vec{B}$ without using the concept of components. (b) Find $\vec{A} \bullet \vec{B}$ by using vector components.

(16) Show that for any vector $\vec{A}$: (a) $\vec{A} \bullet \vec{A} = A^2$ and (b) $\vec{A} \times \vec{A} = 0$.

(17) In Exercise 10, show that dotting vector $\vec{R}$ with itself and dotting vector $\vec{C}$ with itself leads directly to the results of both part (a) and part (b).

(18) For the vectors in Fig. 2.21, find the following: (a) $\vec{A} \bullet \vec{B}$, (b) $\vec{A} \bullet \vec{C}$, (c) $\vec{B} \bullet \vec{C}$, (d) $\vec{A} \times \vec{B}$, (e) $\vec{A} \times \vec{C}$, and (f) $\vec{B} \times \vec{C}$.

Fig. 2.21 See Exercise (18)



(19) (a) Show that $\vec{A} \bullet (\vec{A} \times \vec{B}) = 0$ for all vectors $\vec{A}$ and $\vec{B}$. (b) If θ is the angle between $\vec{A}$ and $\vec{B}$, then find the magnitude of $\vec{A} \times (\vec{A} \times \vec{B})$.

(20) Two vectors $\vec{A}$ and $\vec{B}$ make an acute angle θ with each other when they are placed tail to tail as shown in Fig. 2.22. (a) Prove that the area of the triangle that is contained by these two vectors is $\frac{1}{2}|\vec{A} \times \vec{B}|$. (b) Show that the area of the parallelogram formed by $\vec{A}$ and $\vec{B}$ is $|\vec{A} \times \vec{B}|$.

(21) Show that $\vec{A} \bullet (\vec{B} \times \vec{C})$ is equal in magnitude to the volume of the parallelepiped whose sides are formed from the three vectors $\vec{A}$, $\vec{B}$, and $\vec{C}$ as shown in Fig. 2.23.

Fig. 2.22 See Exercise (20)

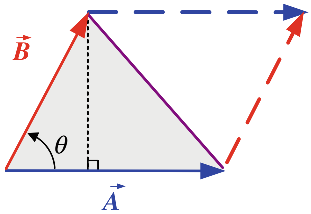
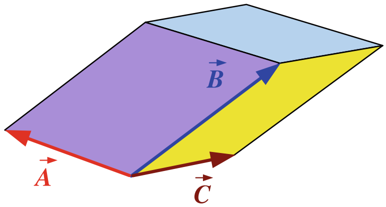
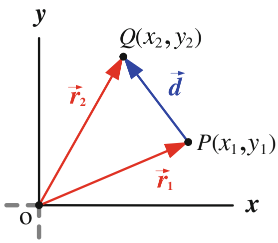


Fig. 2.23 See Exercise (21)



(22) In the xy plane, point P has coordinates (x_1, y_1) and is described by the position vector $\vec{r}_1 = x_1 \vec{i} + y_1 \vec{j}$. Similarly, point Q has coordinates (x_2, y_2) and is described by the position vector $\vec{r}_2 = x_2 \vec{i} + y_2 \vec{j}$, see Fig. 2.24. (a) Show that the displacement vector from P to Q is given by $\vec{d} = \vec{r}_2 - \vec{r}_1 = (x_2 - x_1)\vec{i} + (y_2 - y_1)\vec{j}$. (b) Find the magnitude and direction of $\vec{d}$.

Fig. 2.24 See Exercise (22)



(23) The equation $\vec{F} = q(\vec{v} \times \vec{B})$ gives the force $\vec{F}$ on an electric point charge q moving with velocity $\vec{v}$ through a uniform magnetic field $\vec{B}$. Find the force on a proton of $q = 1.6 \times 10^{-19}$ coulomb moving with velocity $\vec{v} = (2\vec{i} + 3\vec{j} + 4\vec{k}) \times 10^5$ m/s in a magnetic field of $0.5\,\vec{k}$ tesla. (The given SI units yield a force in newtons.)

(24) The electromagnetic Poynting vector $\vec{S}$ is defined by $\vec{S} = \vec{E} \times \vec{H}$, where $\vec{E}$ and $\vec{H}$ are the electric and magnetic fields. Calculate $\vec{S}$ for $\vec{E} = \vec{i} + 0.3\vec{j} + 0.5\vec{k}$ and $\vec{H} = -0.4\vec{i} + \vec{j} + 0.2\vec{k}$. You can disregard units for this calculation.