Work, Energy, and Power

6

Work, energy, and power are words that have different meanings in our everyday life. Nevertheless, physicists give them specific definitions, which we present in this chapter.

The *work-energy* power approach provides identical results to those obtained by Newtonian mechanics, but usually with simpler analysis, especially when dealing with complex situations where forces are not constant. Therefore, we will introduce two extremely important concepts: the *work-energy-theorem* and *conservation of energy*.

6.1 Work Done by a Constant Force

Consider a body that experiences a constant force $\vec{F}$ while undergoing a displacement $\vec{s}$ as it moves, see Fig. 6.1. We then define the work done by the constant force as follows:

> Work done by a constant force:
> Is defined as the product of the component of the force in the direction of the displacement and the magnitude of the displacement.

Thus:

$$W = (F\cos\theta)\, s = \vec{F} \bullet \vec{s} = \begin{cases} +Fs & \text{if } \theta = 0^\circ \\ 0 & \text{if } \theta = 90^\circ \\ -Fs & \text{if } \theta = 180^\circ \end{cases} \tag{6.1}$$

H. A. Radi and J. O. Rasmussen, *Principles of Physics*,
Undergraduate Lecture Notes in Physics, DOI: 10.1007/978-3-642-23026-4_6,

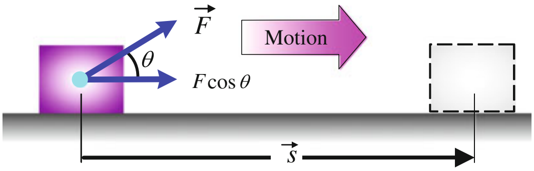


Fig. 6.1 The work done by a constant force $\vec{F}$ while undergoing a displacement $\vec{s}$ is $W = F\,s\cos\theta$

The unit of work in SI units is N.m [abbreviated by joule (J)], i.e., $1\,\text{J} = 1\,\text{N.m}$, and in cgs units is dyne.cm (abbreviated by erg), i.e. 1 erg = 1 dyne.cm. Note that $1\,\text{J} = 10^7$ erg, see Table 6.1.

Table 6.1 Units of work

System	Unit of work	Name of combined unit
SI	N.m	joule (J)
cgs	dyne.cm	erg
British	ft.lb	ft.lb

Work Done by a Weight

Consider a block of mass m to be lifted up with almost zero acceleration (i.e., $a \simeq 0$) by a constant force $\vec{F}$ applied by a person, see Fig. 6.2. While in motion, the force $\vec{F}$ and the weight $m\vec{g}$ will be oppositely directed but equal in magnitude, i.e. $\vec{F} = -m\vec{g}$. That is:

$$F = mg \tag{6.2}$$

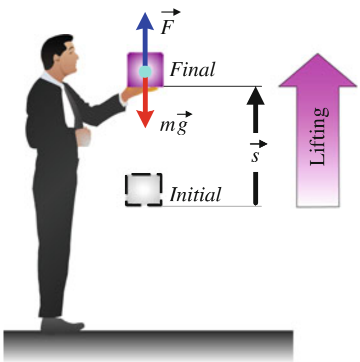


Fig. 6.2 *Lifting* a block with almost zero acceleration

If the upward displacement of the block is denoted by $\vec{s}$, as in Fig. 6.2, then we can calculate the work done by $\vec{F}$ as follows:

$$W_F = \vec{F} \bullet \vec{s} = Fs \cos 0^\circ = Fs = m\,g\,s \tag{6.3}$$

where we have used the fact that the angle between the two parallel vectors $\vec{F}$ and $\vec{s}$ is zero.

Also, we can calculate the work done by the gravitational force $m\,\vec{g}$ as follows:

$$W_g = m\,\vec{g} \bullet \vec{s} = m\,g\,s \cos 180^\circ = -m\,g\,s \tag{6.4}$$

Thus, we conclude that:

$$\boxed{W_F = m\,g\,s \quad \text{and} \quad W_g = -m\,g\,s \quad \text{(Lifting case)}} \tag{6.5}$$

where we have used the fact that the angle between the two antiparallel vectors $m\,\vec{g}$ and $\vec{s}$ is 180°. The net work $W_F + W_g$ done on the block is zero, as expected, because the net force on the block is zero. This is not, of course, to say that it takes no work to lift a block through a vertical height s. In such a context, we do not refer to the net work, but to the work done by the person.

When we lower the block vertically downward with almost zero acceleration for a displacement $\vec{s}$, see Fig. 6.3, the sign of the work done by $\vec{F}$ and $m\,\vec{g}$ will be reversed, since the sign of $\vec{s}$ has reversed.

Following similar steps, one can easily find:

$$W_F = \vec{F} \bullet \vec{s} = F\,s \cos 180^\circ = -F\,s = -m\,g\,s \tag{6.6}$$

$$W_g = m\,\vec{g} \bullet \vec{s} = m\,g\,s \cos 0^\circ = m\,g\,s \tag{6.7}$$

Thus, we conclude that:

$$\boxed{W_F = -m\,g\,s \quad \text{and} \quad W_g = m\,g\,s \quad \text{(Lowering case)}} \tag{6.8}$$

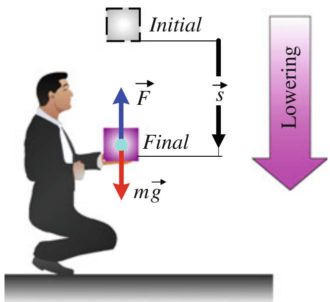


Fig. 6.3 *Lowering down* a block with almost zero acceleration

Work Done by Friction

A common example in which the work is always negative is the work done by friction. When a block slides over a rough surface due to an applied force $\vec{F}$, as shown in Fig. 6.4, the work done by the frictional force $\vec{f}_k$ while the block undergoes a displacement $\vec{s}$ is:

$$\begin{aligned} W_f &= \vec{f}_k \bullet \vec{s} = f_k\, s \cos 180^\circ \\ &= -f_k\, s \end{aligned} \tag{6.9}$$

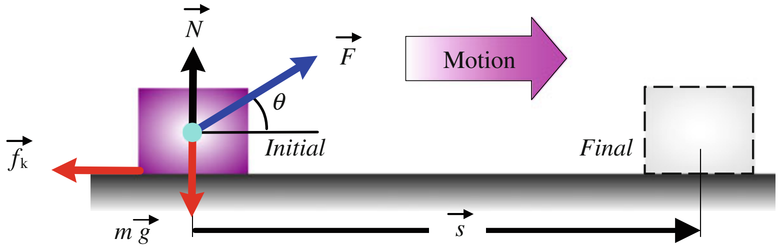


Fig. 6.4 The work done by the kinetic frictional force $\vec{f}_k$ while the block undergoes a displacement $\vec{s}$ is always negative and equals $W_f = -f_k\, s$

From Fig. 6.4, one can easily find the work done by gravity, the normal force, and the applied force as follows:

$$W_g = m\,\vec{g} \bullet \vec{s} = m\,g\,s \cos 90^\circ = 0 \tag{6.10}$$

$$W_N = \vec{N} \bullet \vec{s} = N\,s \cos 90^\circ = 0 \tag{6.11}$$

$$W_F = \vec{F} \bullet \vec{s} = F\,s \cos\theta \tag{6.12}$$

Example 6.1

A block of mass m is pushed up a rough inclined plane of angle θ by a constant force $\vec{F}$ parallel to the incline, as shown in Fig. 6.5. The displacement of the block up the incline is $\vec{d}$. (a) Find the work done by: the force $\vec{F}$, the kinetic friction $\vec{f}_k$, the force of gravity $m\,\vec{g}$, and the normal force $\vec{N}$. (b) Calculate the work done of part (a) for $m = 2$ kg, $\mu_k = 0.5$, $\theta = 30^\circ$, $F = 20$ N, and $d = 5$ m.

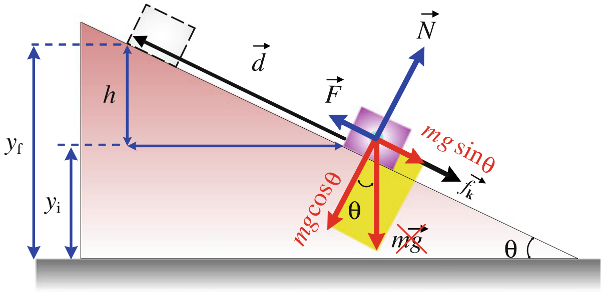


Fig. 6.5

Solution: (a) Since $\vec{F}$ is in the same direction as the displacement $\vec{d}$, we get:

$$W_F = \vec{F} \bullet \vec{d} = F\,d\cos 0° = F\,d$$

The work done by gravity is:

$$W_g = m\vec{g} \bullet \vec{d} = m\,g\,d\cos(90° + \theta) = -m\,g\,d\sin\theta = -m\,g\,h$$

where $h = y_f - y_i = d\sin\theta$ is the value of the vertical height. That is, the work done by gravity is negative and has a magnitude $m\,g$ multiplied by height h. This result and Eq. 6.4 proves that the work is independent of the path taken between any two points.

Since the force of friction $\vec{f}_k$ is opposite to the displacement $\vec{d}$, $f_k = \mu_k N$, and $N = mg\cos\theta$, the work done by friction will be:

$$W_f = \vec{f}_k \bullet \vec{d} = -f_k\,d = -\mu_k\,m\,g\,d\,\cos\theta$$

Since $\vec{N}$ is perpendicular to $\vec{d}$,we get:

$$W_N = \vec{N} \bullet \vec{d} = N\,d\,\cos 90° = 0$$

(b) Using the values given, the work done by each force will be:

$$W_F = F\,d = (20\,\text{N})(5\,\text{m}) = 100\,\text{J}$$

$$W_g = -m\,g\,d\sin\theta = -(2\,\text{kg})(9.8\,\text{m/s}^2)(5\,\text{m})(\sin 30°) = -49\,\text{J}$$

$$W_f = -\mu_k m\,g\,d\cos\theta = -0.5 \times (2\,\text{kg})(9.8\,\text{m/s}^2)(5\,\text{m})(\cos 30°) = -42.4\,\text{J}$$

Thus: $\quad W_{net} = W_F + W_g + W_f + W_N = 100 - 49 - 42.4 + 0 = 8.6\,\text{J}$

6.2 Work Done by a Variable Force

One-Dimensional Analysis

Consider an object that is being displaced along the x-axis from x_i to x_f due to the application of a varying positive force $F(x)$, as shown in Fig. 6.6a. To calculate the work done by this force, we imagine that the object undergoes a very small displacement Δx from x to $x + \Delta x$ due to the effect of an approximate constant force $F(x)$ as shown in Fig. 6.6b. For this very small displacement, we represent the amount of work done by the force by the expression:

$$\Delta W = F(x)\ \Delta x \tag{6.13}$$

which is just the area of the magnified rectangle shown in Fig. 6.6b. Then, the total work done from x_i to x_f by the variable force $F(x)$ is approximately equal to the sum of the large number of rectangles in Fig. 6.6b, i.e. the total area under the force curve. Thus:

$$W \simeq \sum_{x_i}^{x_f} F(x)\ \Delta x \tag{6.14}$$

In the limit where Δx approaches zero, the value of the sum in the last equation approaches the exact value of the area under the force curve, see Fig. 6.6c. As you probably know from calculus, the limit of that sum is called an integral and is represented by:

$$\lim_{\Delta x \to 0} \sum_{x_i}^{x_f} F(x)\ \Delta x = \int_{x_i}^{x_f} F(x)\ dx \tag{6.15}$$

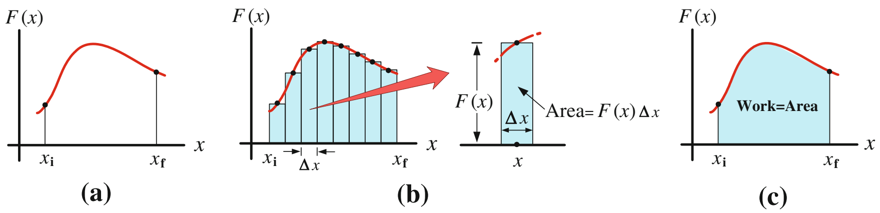


Fig. 6.6 (a) A variable force $F(x)$ displaces a body in the positive x direction from x_i to x_f. (b) The area under the curve is divided into narrow strips of thickness Δx, so that the approximate work done by the force $F(x)$ for the small displacement Δx is $\Delta W = F(x)\ \Delta x$. (c) In the limiting case, the work done by the force is the colored area under the force curve

Therefore, we can express the work done by a variable force $F(x)$ on an object that undergoes a displacement from x_i to x_f as follows:

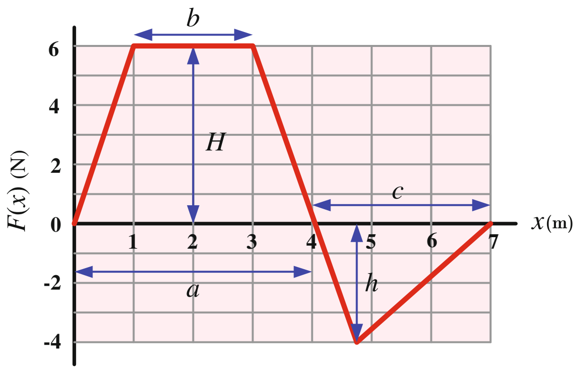
$$W = \int_{x_i}^{x_f} F(x)\, dx \tag{6.16}$$

If $F(x)$ is positive in some regions and negative in others, the last sum is called the *net signed area* and is equal to the area of the regions where $F(x) > 0$ minus the area of the regions where $F(x) > 0$.

Example 6.2

A force acting on an object varies with x as shown in Fig. 6.7. Find the work done by the force when the object undergoes a displacement from $x = 0$ to $x = 7$ m.

Fig. 6.7



Solution: The work done by the force equals the *net signed area* between the curve and the displacement from $x = 0$ to $x = 7$ m. That is, the area of the trapezoid minus the area of the triangle. Thus:

$$\begin{aligned} W &= \text{Area of the trapeziod} - \text{Area of the triangle} \\ &= \tfrac{1}{2}\,(a + b)\,H - \tfrac{1}{2}\,ch \\ &= \tfrac{1}{2} \times (4\,\text{m} + 2\,\text{m}) \times (6\,\text{N}) - \tfrac{1}{2} \times (3\,\text{m})(4\,\text{N}) \\ &= 18\,\text{J} - 6\,\text{J} = 12\,\text{J} \end{aligned}$$

Work Done by a Spring

A spring is one type of common physical system in which the force (known as the spring force) varies with position. Figure 6.8a, shows a massless block on a horizontal frictionless surface attached to the free end of a **relaxed** spring. If the spring is stretched or compressed a small distance from equilibrium, the spring will exert a force on the block. This force is given by **Hooke's law** as follows:

$$F = -k_H x \quad \text{(Hooke's law)} \tag{6.17}$$

where x is the displacement of the block from its equilibrium position ($x = 0$) and k_H is a positive constant known as the **spring constant** (or the **force constant**). The negative sign in Hooke's law indicates that the direction of the force is always opposite to the displacement. The spring force is positive (to the right) when $x < 0$, as in Fig. 6.8b, and is negative (to the left) when $x > 0$, as in Fig. 6.8c. This type of force always acts toward the equilibrium and is called a *restoring force*.

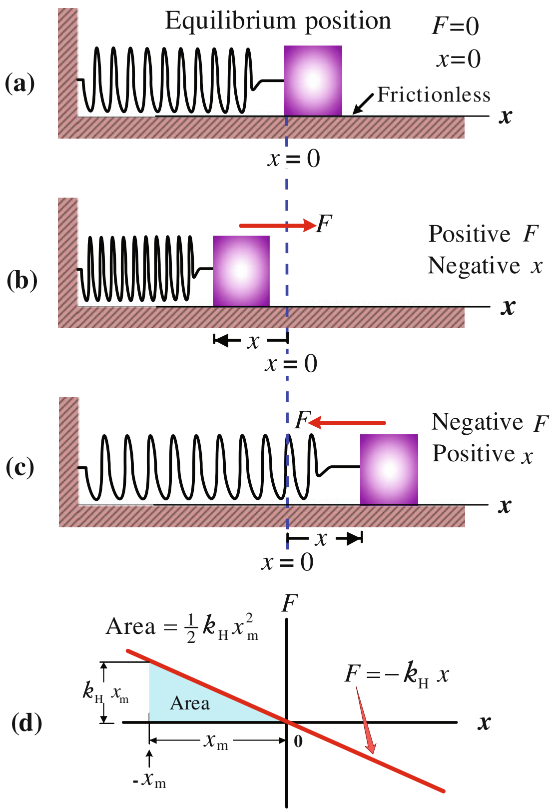


Fig. 6.8 The variation of the force of a spring on a block. (a) When $x = 0$, the force is zero (equilibrium position). (b) When x is negative, the force is positive (compressed spring). (c) When x is positive, the force is negative (stretched spring). (d) Graph of F versus x. The work done by the spring force as the block moves from $-x_m$ to 0 is the colored triangular area which equals $\frac{1}{2}k_H x_m^2$

If we allow the block to compress the spring a distance of x_m from its equilibrium position and then release the block, it will move from $-x_m$ through the equilibrium position $x = 0$ to x_m. In the absence of friction, the block will oscillate indefinitely between $-x_m$ and x_m. In this case x_m is called the *amplitude* of the oscillations.

To calculate the work done by the spring force on the body as it moves from $x_i = -x_m$ to $x_f = 0$, we use Hooke's law in Eq. 6.16 as follows:

$$W_s = \int_{x_i}^{x_f} F(x)\,dx = \int_{-x_m}^{0} (-k_H\,x)\,dx = \left[-\tfrac{1}{2}k_H\,x^2\right]_{-x_m}^{0} = \tfrac{1}{2}k_H\,x_m^2 \tag{6.18}$$

Note that the work done by the spring force is positive because the spring force is in the same direction as the displacement. We can reach the same result of Eq. 6.18 if we plot F versus x, as shown in Fig. 6.8d, and then calculate the area of the colored triangle that has a base x_m and height $k_H\,x_m$. On the other hand, when $x_i = 0$ and $x_f = x_m$, we can find that $W_s = -\frac{1}{2}k_H\,x_m^2$. In this part of the motion, the spring force is to the left and the displacement is to the right, resulting in a negative work. Generally, if the block undergoes an arbitrary displacement from x_i to x_f, the work done by the spring force will be given by:

$$W_s = \int_{x_i}^{x_f} F(x)\,dx = \int_{x_i}^{x_f} (-k_H\,x)\,dx = \tfrac{1}{2}k_H\,x_i^2 - \tfrac{1}{2}k_H\,x_f^2 \tag{6.19}$$

This shows that the work done is zero for any motion that has $x_i = x_f$.

Let us calculate the work done by the *applied force* $\vec{F}_{app}$ when the block moves *very slowly* from x_i to x_f, see Fig. 6.9a and b. To find this work we notice that $\vec{F}_{app}$ is equal and opposite to the spring force $\vec{F}$ at any displacement, i.e. $F_{app} = -F = -(-k_H\,x) = k_H\,x$. Thus:

$$W_{F_{app}} = \int_{x_i}^{x_f} F_{app}\,dx = \int_{x_i}^{x_f} k_H\,x\,dx = \left[\tfrac{1}{2}k_H\,x^2\right]_{x_i}^{x_f} = \tfrac{1}{2}k_H\,x_f^2 - \tfrac{1}{2}k_H\,x_i^2 \tag{6.20}$$

Comparing Eq. 6.19 and Eq. 6.20 we find that $W_{F_{app}} = -W_s$, as expected. If we plot F_{app} versus x, as shown in Fig. 6.9c, then the work done by F in compressing the spring *very slowly* from $x_i = 0$ to $x_f = -x_m$ equals the area of the colored triangle that has a base x_m and height $k_H\,x_m$, i.e. $W_{app} = \frac{1}{2}k_H\,x_m^2$ (F_{app} *and the displacement are negative*).

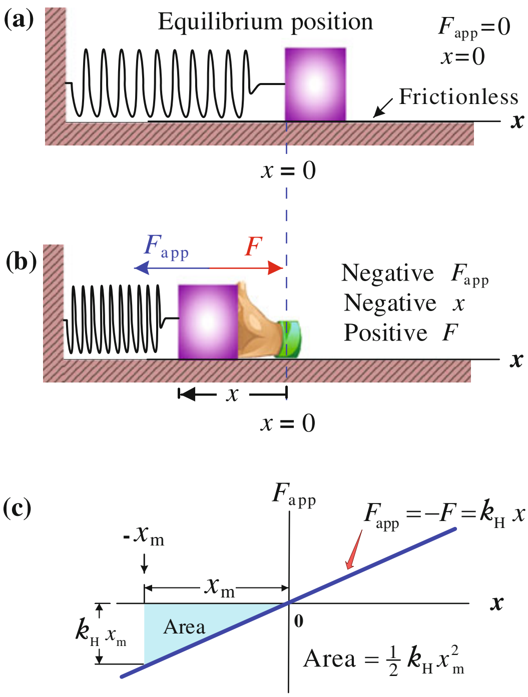


Fig. 6.9 (a) When $x = 0$, the applied force is zero (equilibrium position). (b) When x is negative, the applied force is negative (compressed spring). (c) Graph of the applied force versus x. The work done by the applied force as the block moves very slowly from $x = 0$ to $x = -x_m$ is the colored triangular area which equals $\frac{1}{2}k_H x_m^2$

Example 6.3

An applied force $F_{app} = -5$ N is exerted on the block that is attached to the free end of the spring of Fig. 6.9b. As a result of this force, the spring is compressed by 1 cm from its relaxed length. (a) What is the spring constant of the spring? (b) What force does the spring exert on the block if the spring is compressed by 2.5 cm? (c) How much work does the spring force do on the block as the spring is compressed from the relaxed state by 2.5 cm? (d) How much work does the spring force do on the block during a total displacement starting from a compression of 2.5 cm, passing through the equilibrium, and then to a stretch of 2.0 cm?

Solution: (a) The compressed spring pushes the block with a force $F = -F_{app} = +5$ N. From $F = -k_H x$, with $x = -1$ cm, we have:

$$k_H = -\frac{F}{x} = -\frac{5\,\text{N}}{(-1 \times 10^{-2}\,\text{m})} = 500\,\text{N/m}$$

(b) Using Hooke's law, with $x = -2.5\,\text{cm} = -2.5 \times 10^{-2}$ m, we have:

$$F = -k_H x = -(500\,\text{N/m})(-2.5 \times 10^{-2}\,\text{m}) = 12.5\,\text{N}$$

(c) Since the spring is initially at its relaxed state, the work done by the spring force on the block from $x_i = 0$ to $x_f = -2.5 \times 10^{-2}$ m will be:

$$W_s = \tfrac{1}{2}k_H x_i^2 - \tfrac{1}{2}k_H x_f^2 = 0 - \tfrac{1}{2}(500\,\text{N/m})(-2.5 \times 10^{-2}\,\text{m})^2 = -0.156\,\text{J}$$

The work is negative because the spring force and the displacement are in opposite directions. Note that the amount of work done by the spring on the block would be the same when stretching by 2.5 cm.

(d) For this case, we have $x_i = -2.5 \times 10^{-2}$ m (the spring is initially compressed) and $x_f = +2.0 \times 10^{-2}$ m (the spring is finally stretched). Then Eq. 6.19 becomes:

$$\begin{aligned} W_s &= \tfrac{1}{2}k_H x_i^2 - \tfrac{1}{2}k_H x_f^2 = \tfrac{1}{2}k_H (x_i^2 - x_f^2) \\ &= \tfrac{1}{2}(500\,\text{N/m})[(-2.5 \times 10^{-2}\,\text{m})^2 - (+2.0 \times 10^{-2}\,\text{m})^2] = 0.06\,\text{J} \end{aligned}$$

Three-Dimensional Analysis

Consider a particle that is acted upon by a three-dimensional force of the following form:

$$\vec{F} = F_x\,\vec{i} + F_y\,\vec{j} + F_z\,\vec{k} \tag{6.21}$$

where the components F_x, F_y, and F_z are generally a function of the position vector $\vec{r}$ of the particle. Furthermore, let the particle move through an incremental displacement $d\vec{r}$, i.e.

$$d\vec{r} = dx\,\vec{i} + dy\,\vec{j} + dz\,\vec{k} \tag{6.22}$$

In this case, the increment of work dW done on the particle by the force $\vec{F}$ during the incremental displacement $d\vec{r}$ is giving by:

$$\begin{aligned} dW &= \vec{F} \bullet d\vec{r} = (F_x\,\vec{i} + F_y\,\vec{j} + F_z\,\vec{k}) \bullet (dx\,\vec{i} + dy\,\vec{j} + dz\,\vec{k}) \\ &= F_x\,dx + F_y\,dy + F_z\,dz \end{aligned} \tag{6.23}$$

The work W done by the force $\vec{F}$ on the particle when it moves from an initial position r_i of coordinates (x_i, y_i, z_i) to a final position r_f of coordinates (x_f, y_f, z_f) can be represented by:

$$W = \int_{r_i}^{r_f} dW = \int_{x_i}^{x_f} F_x \, dx + \int_{y_i}^{y_f} F_y \, dy + \int_{z_i}^{z_f} F_z \, dz \tag{6.24}$$

When $\vec{F}$ has only an x component, this equation reduces to Eq. 6.16.

6.3 Work-Energy Theorem

Consider a particle of mass m, moving with acceleration $a = a(x)$ along the x-axis under the effect of a net force $F(x)$ that points along this axis. Thus, according to Newton's second law of motion we have $F(x) = ma$. The work done by this net force on the particle as it moves from an initial position x_i to a final position x_f can be found using Eq. 6.16 as follows:

$$W = \int_{x_i}^{x_f} F(x) \, dx = \int_{x_i}^{x_f} m \, a \, dx \tag{6.25}$$

We can write the quantity $m \, a \, dx$ in the last equation as:

$$m \, a \, dx = m \frac{dv}{dt} \, dx \tag{6.26}$$

Since v is a function of time, then we can use the "chain rule" to have:

$$\frac{dv}{dt} = \frac{dv}{dx} \frac{dx}{dt} = \frac{dv}{dx} \, v \tag{6.27}$$

Then Eq. 6.26 becomes:

$$m \, a \, dx = m \frac{dv}{dx} \, v \, dx = m \, v \, dv \tag{6.28}$$

Substituting this result back into Eq. 6.25 yields:

$$W = \int_{v_i}^{v_f} m \, v \, dv = m \int_{v_i}^{v_f} v \, dv = m \left[\tfrac{1}{2} v^2 \right]_{v_i}^{v_f} = \tfrac{1}{2} m \, v_f^2 - \tfrac{1}{2} m \, v_i^2 \tag{6.29}$$

Note that when we change the integration variable from x to v we are required to change the limits of the integration to the new variable.

For a particle of mass m that has a speed v well below the speed of light, we define its kinetic energy as:

Kinetic Energy

The kinetic energy K of a particle is defined as the product of one half of its mass and the square of its speed, i.e.

$$K = \frac{1}{2} m\, v^2 \tag{6.30}$$

Kinetic energy is a scalar quantity and has the same units as work. In SI units we have:

$$1\,\text{J} = 1\,\text{kg.m}^2/\text{s}^2 = 1\,\text{N.m} \tag{6.31}$$

We can view kinetic energy as the energy associated with the motion of an object. It is more convenient to express Eq. 6.29 as:

$$W = \Delta K = K_f - K_i \tag{6.32}$$

where K_i is the particle's initial kinetic energy and K_f is its final kinetic energy after the work is done. That is, the work done by the *net force* in displacing a particle equals the change in its kinetic energy. If there are many forces, such as an applied force $\vec{F}$, a gravitational force $m\vec{g}$, a spring force $\vec{F}_s$, a frictional force $\vec{f}$, etc, the work done by the net force in displacing a particle will be equal to the sum of the work done by all the forces acting on the particle. That is:

$$W_{net} = W_F + W_g + W_s + W_f + \ldots = K_f - K_i = \Delta K \tag{6.33}$$

Equation 6.32 is known as the **work-energy theorem**. This theorem is valid even when the force varies in direction and magnitude while the particle (or the object) moves along an arbitrary curved path in three dimensions.

Example 6.4

A box of mass $m = 10\,\text{kg}$ is initially at rest on a rough horizontal surface, where the coefficient of kinetic friction between the box and the surface is $\mu_k = 0.2$. The box is then pulled horizontally by a force $F = 50\,\text{N}$ that makes an angle $\theta = 60°$ with the horizontal, see Fig. 6.10. (a) Use the work-energy theorem to find the speed v_f of the box after it moves a distance s of 4 m. *(b) Repeat part (a) using Newtonian mechanics.

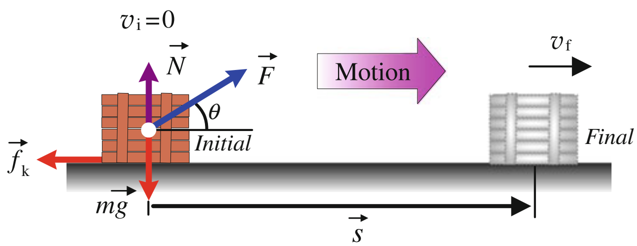


Fig. 6.10

Solution: (a) Both the weight-gravitational force $m\vec{g}$ and the normal force $\vec{N}$ do no work, since the displacement is horizontal, i.e. $W_g = W_N = 0$. The work done by the applied force is:

$$W_F = \vec{F} \bullet \vec{s} = F\, s \cos\theta = (50\,\text{N})(4\,\text{m})(\cos 60°) = 100\,\text{J}$$

The magnitude of the frictional force is $f_k = \mu_k N$, where in this case $N = mg - F \sin\theta$. Therefore, the work done by friction is:

$$\begin{aligned} W_f &= \vec{f}_k \bullet \vec{s} = -f_k\, s = -\mu_k(mg - F \sin\theta)\, s \\ &= -0.2 \times [(10\,\text{kg})(9.8\,\text{m/s}^2) - (50\,\text{N})(0.866)](4\,\text{m}) \\ &= -43.76\,\text{J} \end{aligned}$$

Thus, the net work done on the box is:

$$W_{\text{net}} = W_F + W_g + W_N + W_f = 100\,\text{J} + 0 + 0 + (-43.76\,\text{J}) = 56.24\,\text{J}$$

Applying the work-energy theorem with $v_i = 0$ gives:

$$W_{\text{net}} = K_f - K_i = \tfrac{1}{2} m\, v_f^2 \quad \Rightarrow \quad v_f = \sqrt{\frac{2\, W_{\text{net}}}{m}} = \sqrt{\frac{2 \times 56.24\,\text{J}}{10\,\text{kg}}} = 3.35\,\text{m/s}$$

*(b) Applying Newton's second law in the component form, then for the horizontal component, we find that:

$$\Sigma F_x = F \cos\theta - f_k = m\, a$$

Thus, the acceleration of the box will be given by:

$$a = \frac{F\cos\theta - \mu_k(mg - F\sin\theta)}{m}$$
$$= \frac{(50\,\text{N})(\cos 60°) - 0.2 \times [(10\,\text{kg})(9.8\,\text{m/s}^2) - (50\,\text{N})(0.866)]}{10\,\text{kg}} = 1.406\,\text{m/s}^2$$

To find the final speed, we use the kinematic equation $v_f^2 = v_i^2 + 2as$ when $v_i = 0$ to get:

$$v_f = \sqrt{2as} = \sqrt{2 \times (1.406\,\text{m/s}^2)(4\,\text{m})} = 3.35\,\text{m/s}$$

Because the forces are constants in this example, the analysis used by Newtonian mechanics is easier than that of the work-energy theorem.

6.4 Conservative Forces and Potential Energy

In the previous section we introduced the concept of kinetic energy and found that it can change only if work is done on the object. In this section we introduce another form of energy, called *potential energy*, associated with the position or configuration of an object, and can be thought of as a stored energy that can be converted to kinetic energy or to work. We begin by defining the following:

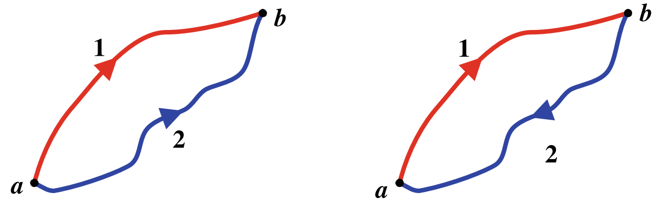
(a) Conservative and Non-conservative Forces

Conservative Forces

In Example 6.1 we were able to see that the work done by gravity depends only on the initial and final vertical coordinates and hence is independent of the path taken between any two points. Also, we found the same holds true in the case of a spring. In addition, we can easily see from Sect. 6.2 that the net work done on the object by the gravitational force during a round trip is zero. When a force exhibits these properties, it is called a **conservative force**.

With reference to the arbitrary paths of Fig. 6.11a, we can write the first condition for a conservative force as:

$$W_{ab}(\text{path 1}) = W_{ab}(\text{path 2}) \tag{6.34}$$



(a) (b)

Fig. 6.11 (a) A conservative force acts on a particle moving from point a to point b by following either path 1 or path 2. (b) A conservative force acts on a particle moving in a round trip from point a to point b along path 1 and then back to point a along path 2

i.e., the work done by a conservative force on a particle moving from a to b along path 1 is the same as from a to b along path 2. In words:

> Spotlight
> The net work done by a conservative force on a particle moving between any two points does not depend on the path taken.

Also, with reference to the arbitrary paths of Fig. 6.11b, we can write the second condition for a conservative force as:

$$\left.\begin{array}{c} W_{ab}(\text{path 1}) = -W_{ba}(\text{path 2}) \\ \text{or} \\ W_{ab}(\text{path 1}) + W_{ba}(\text{path 2}) = 0 \end{array}\right\} \tag{6.35}$$

That is, the work done by a conservative force on a particle that moves in a round trip from a to b along path 1 and then from b to a along path 2 is zero. In other words:

> Soptlight
> The net work done by a conservative force on a particle that is moving around any closed path is zero.

From the work-energy theorem, $W = 0$ for a round trip, which means that the particle will return to its starting point with the same kinetic energy it had when it started its motion.

We recall from Example 6.1 that the work done by the gravitational force as a particle of mass m moves between two points of elevations y_{i} and y_{f} can be written as:

$$W_g = -mgh = -mg(y_{\mathrm{f}} - y_{\mathrm{i}}) \tag{6.36}$$

which satisfies the two conditions of a conservative force.

Non-conservative Forces

Not all forces are conservative. For example, let us allow a book to slide across a table that is not frictionless, see Fig. 6.12a. During the sliding, the kinetic frictional force does negative work on the book, slowing it by transferring energy from its kinetic energy to thermal energy of the book-table system. This energy transfer cannot be reversed. So, this force is not conservative. Therefore, all types of frictional forces are **non-conservative** forces. That is:

> Soptlight
> The work done by a non-conservative force on a particle that is moving between any two points depends on the path taken by the particle.

With reference to the arbitrary paths of Fig. 6.12a, we can write the first condition for a non-conservative force as:

$$W_{AB}(\text{path 1}) \neq W_{AB}(\text{path 2}) \quad \text{(Non-conservative forces)} \tag{6.37}$$

i.e., the work done by a non-conservative force on a particle moving from A to B along path 1 is always not the same along path 2.

Also, with reference to the arbitrary paths of Fig. 6.12b, we can write the second condition for a non-conservative force as:

$$\left.\begin{array}{c} W_{AB}(\text{path 1}) \neq -W_{BA}(\text{path 2}) \\ \text{or} \\ W_{AB}(\text{path 1}) + W_{BA}(\text{path 2}) \neq 0 \end{array}\right\} \tag{6.38}$$

That is, the work done by a non-conservative force on a particle that moves in a round trip from A to B along path 1 and then from B to A along path 2 is not zero.

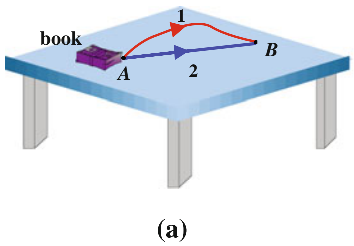


Fig. 6.12 (a) The work done by the force of friction depends on the path taken as the book is moved from A to B. (b) The work done by the force of friction in a round trip from point A to point B along path 1 and then back to point A along path 2 is not zero

(b) Potential Energy

We found that the work done by a conservative force is a function of the particle's initial and final coordinates and neither depends on the path taken nor depends on its velocity. Therefore, we can define a function U called the "*potential energy*" such that the work done by a conservative force equals the decrease of potential energy. That is:

$$W_c = -\Delta U = U_\mathrm{i} - U_\mathrm{f} \tag{6.39}$$

where the subscript "c" refers to a conservative force and the change in potential energy is defined as $\Delta U = U_\mathrm{f} - U_\mathrm{i}$. For a particle moving along the x axis under the effect of a conservative force $\vec{F}$ that has an x component F_x, we can express Eq. 6.39 as follows:

$$\Delta U = U_\mathrm{f} - U_\mathrm{i} = -\int_\mathrm{i}^\mathrm{f} F_x\,dx \quad \text{or} \quad F_x = -\frac{dU}{dx} \tag{6.40}$$

It is often convenient to choose some selected initial configuration that has a potential U_i as a **reference point** and measure all potential energy differences with respect to this point. Usually, we set $U_\mathrm{i} = 0$ at this point because it does not really matter what value we assign to U_i.

Gravitational Potential Energy

Consider a particle with mass m moving vertically along the y axis from point y_i to point y_f. Of course, the displacement will be an upward vector while the weight

$m\vec{g}$ will be a downward vector. To find the corresponding change in gravitational potential energy of the particle-Earth system, we change the integration in Eq. 6.40 to be along the y axis and substitute $-mg$ for the force F_x. Thus:

$$\Delta U = U_{\mathrm{f}} - U_{\mathrm{i}} = -\int_{y_{\mathrm{i}}}^{y_{\mathrm{f}}} (-mg)\, dy = mg \int_{y_{\mathrm{i}}}^{y_{\mathrm{f}}} dy = mg\left[y\right]_{y_{\mathrm{i}}}^{y_{\mathrm{f}}} = mg(y_{\mathrm{f}} - y_{\mathrm{i}})$$

That is:

$$\boxed{\Delta U = U_{\mathrm{f}} - U_{\mathrm{i}} = mg(y_{\mathrm{f}} - y_{\mathrm{i}}) = mg\Delta y} \tag{6.41}$$

Only the change in gravitational potential energy ΔU is physically important. So, according to the previous result, we can set $U_{\mathrm{i}} = 0$ when $y_{\mathrm{i}} = 0$. This gives:

$$U_{\mathrm{f}} - 0 = mg(y_{\mathrm{f}} - 0)$$

which is generally written as follows:

$$\boxed{U = mgy} \quad \text{(Gravitational potential energy)} \tag{6.42}$$

That is, the gravitational potential energy associated with the particle-Earth system depends on the vertical position y (or the height) of the particle relative to the reference position $y = 0$, and does not depend on the horizontal position. We can think of $U = mgy$ as the configuration energy stored in the particle-Earth system.

Elastic Potential Energy

Now Consider a block attached to a spring with a spring constant k_{H} as in Fig. 6.8. As the block moves from position x_{i} to position x_{f} the spring force $F = -k_{\mathrm{H}}\, x$ does work on the block. To find the corresponding change in elastic potential energy of the block-spring system, we substitute $-k_{\mathrm{H}}\, x$ for the force F_x in Eq. 6.40 to get:

$$\Delta U = U_{\mathrm{f}} - U_{\mathrm{i}} = -\int_{x_{\mathrm{i}}}^{x_{\mathrm{f}}} (-k_{\mathrm{H}}\, x)\, dx = k_{\mathrm{H}} \int_{x_{\mathrm{i}}}^{x_{\mathrm{f}}} x\, dx = k_{\mathrm{H}} \left[\tfrac{1}{2}x^2\right]_{x_{\mathrm{i}}}^{x_{\mathrm{f}}} = \tfrac{1}{2}k_{\mathrm{H}}\, x_{\mathrm{f}}^2 - \tfrac{1}{2}k_{\mathrm{H}}\, x_{\mathrm{i}}^2$$

That is:

$$\boxed{\Delta U = U_{\mathrm{f}} - U_{\mathrm{i}} = \tfrac{1}{2}k_{\mathrm{H}}\, x_{\mathrm{f}}^2 - \tfrac{1}{2}k_{\mathrm{H}}\, x_{\mathrm{i}}^2} \tag{6.43}$$

We set $U_{\mathrm{i}} = 0$ at the equilibrium position of the block, i.e. when $x_{\mathrm{i}} = 0$. This gives:

$$U_{\mathrm{f}} - 0 = \tfrac{1}{2}k_{\mathrm{H}}\, x_{\mathrm{f}}^2 - 0$$

which is generally written as follows:

$$U = \tfrac{1}{2} k_{\mathrm{H}}\, x^2 \quad \text{(Elastic potential energy)} \tag{6.44}$$

Example 6.5

A ball of mass $m = 0.2$ kg is at the level of a second balcony which is 10 m above the ground, see Fig. 6.13. (a) What is the gravitational potential energy of the ball if we take the reference point $y = 0$ to be: (1) at the ground, (2) at the first balcony, (3) at the second balcony, and (4) at the top of the building? (b) If the ball drops to the ground, for each of the reference points of part (a), what is the change of potential energy of the ball due to the fall?

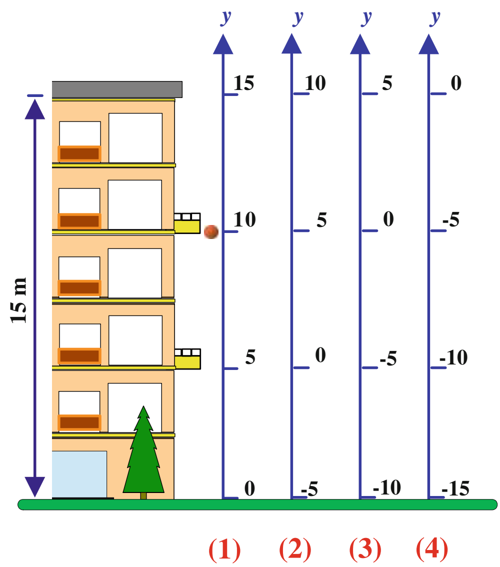


Fig. 6.13 Example 6.5

Solution: (a) Using Eq. 6.42, we can calculate the potential energy U of the ball for each choice of $y = 0$ as follows:

coordinate choice (1): $U = mgy = (0.2\,\text{kg})(9.8\,\text{m/s}^2)(10\,\text{m}) = 19.6\,\text{J}$

coordinate choice (2): $U = mgy = (0.2\,\text{kg})(9.8\,\text{m/s}^2)(5\,\text{m}) = 9.8\,\text{J}$

coordinate choice (3): $U = mgy = (0.2\,\text{kg})(9.8\,\text{m/s}^2)(0\,\text{m}) = 0\,\text{J}$

coordinate choice (4): $U = mgy = (0.2\,\text{kg})(9.8\,\text{m/s}^2)(-5\,\text{m}) = -9.8\,\text{J}$

(b) For all the coordinate choices, we have $\Delta y = -10$ m. So Eq. 6.41 will give the same change in potential energy as follows:

$$\Delta U = mg\Delta y = (0.2\,\text{kg})(9.8\,\text{m/s}^2)(-10\,\text{m}) = -19.6\,\text{J}$$

Thus, although the value of U depends on the choice of where we let $y = 0$, the change in potential energy does not. In fact, only the change ΔU, not the value of U, in potential energy is physically important.

6.5 Conservation of Mechanical Energy

When a conservative force does work W_c on a particle, the work-energy theorem tells us that there will be a change in its kinetic energy given by Eq. 6.32, which can be rewritten as:

$$W_c = \Delta K \tag{6.45}$$

and a change in potential energy given by Eq. 6.39, rewritten:

$$W_c = -\Delta U \tag{6.46}$$

By equating the last two equations we get:

$$\Delta K = -\Delta U \tag{6.47}$$

or:

$$\Delta K + \Delta U = \Delta(K + U) = 0 \tag{6.48}$$

If we define the total mechanical energy E as the sum of the kinetic energy K and potential energy U, i.e.

$$\boxed{E = K + U}\,, \tag{6.49}$$

then Eq. 6.48 gives:

$$\boxed{\Delta E = 0} \tag{6.50}$$

which is called the **principle of conservation of mechanical energy**.

Conservation of Mechanical Energy:
When only a conservative force acts on a system, the kinetic energy and the potential energy can change. However, their sum, the mechanical energy E of the system, does not change. That is:

$$E_i = E_f \tag{6.51}$$

or:

$$K_i + U_i = K_f + U_f \tag{6.52}$$

If more than one conservative force acts on the system, where each one is associated with a potential energy, then the conservation of mechanical energy will take the form:

$$K_i + \sum U_i = K_f + \sum U_f \tag{6.53}$$

Example 6.6

A frictionless roller-coaster is given a maximum possible initial speed $v_\circ = 6\,\text{m/s}$ when it is at height $y_\circ = 6\,\text{m}$ above the ground and moves freely afterwards, see Fig. 6.14 and take $g = 10\,\text{m/s}^2$. (a) What will be the roller-coaster's speed when it reaches the lowest point at $y_1 = 4\,\text{m}$? (b) What will be its maximum height y_2?

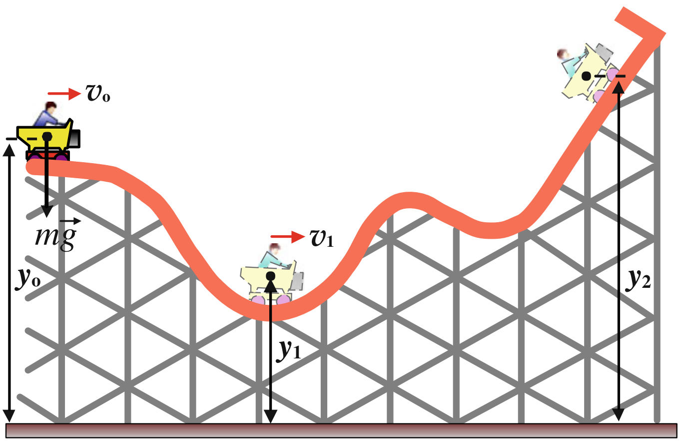


Fig. 6.14

Solution: (a) The only force that contributes to the work is the force of gravity. Therefore, we can use the law of conservation of mechanical energy. Initially, we have $K_i = \frac{1}{2}m\,v_\circ^2$ and $U_i = m\,g\,y_\circ$. Finally, at the lowest point we have $K_f = \frac{1}{2}m\,v_1^2$ and $U_f = m\,g\,y_1$. Thus, according to Eqs. 6.51 and 6.52, we get:

$$E_i = E_f \quad \Rightarrow \quad K_i + U_i = K_f + U_f \quad \Rightarrow \quad \frac{1}{2}m\,v_\circ^2 + m\,g\,y_\circ = \frac{1}{2}m\,v_1^2 + m\,g\,y_1$$

i.e. $$v_1^2 = v_\circ^2 + 2g(y_\circ - y_1)$$

Then: $$v_1 = \sqrt{(6\,\text{m/s})^2 + 2(10\,\text{m/s}^2)(6\,\text{m} - 4\,\text{m})} = 8.7\,\text{m/s}$$

(b) The roller-coaster will stop momentarily when it reaches the maximum height y_2, i.e. $v_2 = 0$. Accordingly, Eq. 6.52 gives:

$$E_i = E_f \quad \Rightarrow \quad K_i + U_i = K_f + U_f \quad \Rightarrow \quad \tfrac{1}{2}m\,v_\circ^2 + m\,g\,y_\circ = \tfrac{1}{2}m\,v_2^2 + m\,g\,y_2$$

Then: $$y_2 = y_\circ + \tfrac{1}{2}v_\circ^2/g \quad \Rightarrow \quad y_2 = 6\,\text{m} + \tfrac{1}{2}(6\,\text{m/s})^2/(10\,\text{m/s}^2) = 7.8\,\text{m}$$

6.6 Work Done by Non-conservative Forces

In real-life systems, the total mechanical energy is not constant due to the presence of non-conservative forces, such as friction or any applied forces. When the work done by all the non-conservative forces on a particle is W_{nc} and the work done by the conservative force is W_c, then the work-energy theorem tells us that there will be a change in the particle's kinetic energy given by Eq. 6.32 as:

$$W_{nc} + W_c = \Delta K \tag{6.54}$$

Since Eq. 6.39 gives $W_c = -\Delta U$, then this equation becomes:

$$W_{nc} = \Delta K + \Delta U = (K_f - K_i) + (U_f - U_i) \tag{6.55}$$

Since $E = K + U$ as we saw in Eq. 6.49, this equation becomes:

$$W_{nc} = \Delta E = E_f - E_i \tag{6.56}$$

which generally can be stated as:

Spotlight

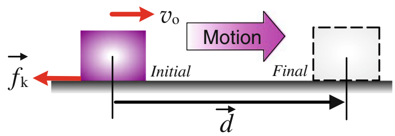
The work done by all non-conservative forces W (or W_{nc}) equals the change in the total mechanical energy of the system.

When there are no non-conservative forces present, $W_{nc} = 0$ and hence $E_f = E_i$; that is, the total mechanical energy is conserved.

Example 6.7

A block of initial speed v_o slides across a floor, see Fig. 6.15. A kinetic frictional force of magnitude $f_k = 50\,\text{N}$ does work on the block, stopping it over a displacement of magnitude $d = 2\,\text{m}$. Find the dissipated mechanical energy.

Fig. 6.15



Solution: From Eq. 6.9, the work done by friction is given by:

$$W_{nc} \equiv W_f = -f_k\, d = -(50\,\text{N})(2\,\text{m}) = -100\,\text{J}$$

From Eq. 6.56 and the above result, the dissipated mechanical energy is:

$$\Delta E = W_{nc} = -100\,\text{J}$$

Example 6.8

A boy of mass $m = 30\,\text{kg}$ slides down a curved track of height $h = 3\,\text{m}$, see Fig. 6.16. The boy starts at point i with a speed $v_i = 0$ and reaches the bottom of the track at point f with a speed v_f. (a) If the track is frictionless, i.e. $f_k = 0$, then find the speed v_f. (b) If the track is rough and $v_f = 5\,\text{m/s}$, then find the work done by friction.

Solution: (a) The normal force $\vec{N}$ does no work on the boy since it is always perpendicular to each displacement element on the curved track. The only force that has a change in potential energy is $m\vec{g}$. Therefore, we can use the law of conservation of mechanical energy. Initially, we have $K_i = 0$ and $U_i = m\,g\,h$. At the end we have $K_f = \frac{1}{2}m\,v_f^2$ and $U_f = 0$. Thus, according to Eq. 6.51, we get:

$$E_i = E_f \quad \Rightarrow \quad K_i + U_i = K_f + U_f \quad \Rightarrow \quad 0 + m\,g\,h = \tfrac{1}{2}m\,v_f^2 + 0$$

$$\text{i.e.,} \quad v_f = \sqrt{2\,g\,h} = \sqrt{2(9.8\,\text{m/s}^2)(3\,\text{m})} = 7.67\,\text{m/s}$$

(b) In the presence of a non-conservative frictional force, i.e. $W_{nc} \neq 0$, then mechanical energy is not conserved and we can use Eq. 6.56 to find the work done by friction on the boy as follows:

$$W_{nc} \equiv W_f = E_f - E_i = (K_f + U_f) - (K_i + U_i) = (\tfrac{1}{2}m\,v_f^2 + 0) - (0 + m\,g\,h)$$
$$= \tfrac{1}{2}(30\,\text{kg})(5\,\text{m/s})^2 - (30\,\text{kg})(9.8\,\text{m/s}^2)(3\,\text{m}) = -507\,\text{J}$$

Note that W_f is negative, since the work done by friction is negative.

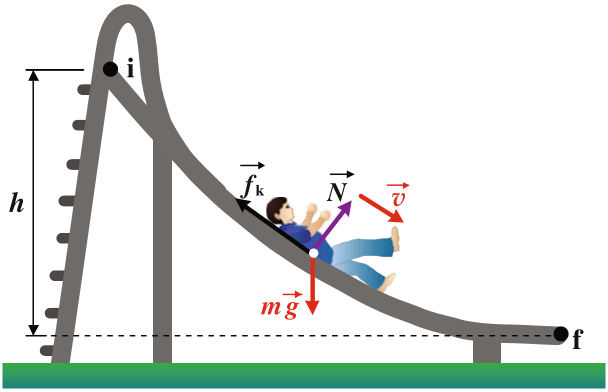


Fig. 6.16

Example 6.9

A block of mass $m = 2$ kg is placed on a rough horizontal surface against a compressed spring with a spring constant of $k_H = 2{,}000$ N/m. The spring is compressed a distance of $x = 20$ cm, see Fig. 6.17. The block is released, and then it moves to the right until it stops completely after rising onto a rough track of height $h = 0.5$ m. Find the work done by friction.

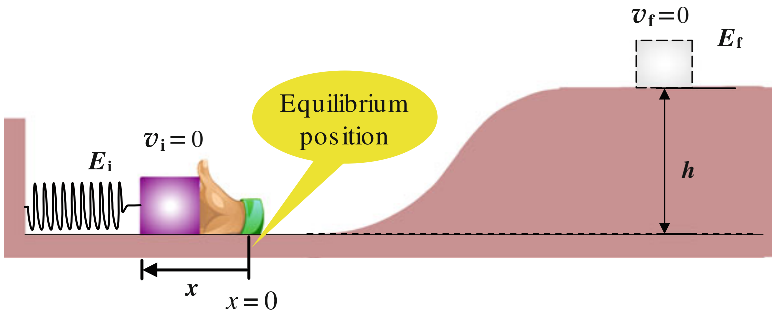


Fig. 6.17

Solution: As in Example 6.8, the normal force $\vec{N}$ does no work on the block since it is always perpendicular to each displacement element on the horizontal and curved parts of the track. The only force that has a change in potential energy is the force of gravity. Initially, we have $K_i = 0$ and only an elastic potential energy $\frac{1}{2}k x^2$, i.e. $\Sigma U_i = \frac{1}{2}k x^2$. At the end we have $K_f = 0$ and only a gravitational energy $m g h$, i.e. $\Sigma U_f = m g h$. In the presence of a non-conservative frictional force, $W_{nc} \neq 0$, the mechanical energy is not conserved. We then use Eq. 6.56 to find the work done by friction as follows:

$$\begin{aligned} W_{nc} &= E_f - E_i \\ &= (K_f + \Sigma U_f) - (K_i + \Sigma U_i) \\ &= (0 + m g h + 0) - (0 + 0 + \tfrac{1}{2}k_H x^2) \\ &= m g h - \tfrac{1}{2}k_H x^2 \\ &= (2\,\text{kg})(9.8\,\text{m/s}^2)(0.5\,\text{m}) - \tfrac{1}{2}(2{,}000\,\text{N/m})(0.2\,\text{m})^2 = -30.2\,\text{J} \end{aligned}$$

W_{nc} is negative since the work done by friction is always negative.

6.7 Conservation of Energy

When a block slides across a rough floor through a displacement $\vec{d}$, the frictional force $\vec{f}_k$ (which is an external force) does work on the block, and we found from Example 6.7 that $W_{nc} = W_f = -f_k\, d$. This allows us to write the dissipated mechanical energy given by Eq. 6.56 as follows:

$$\Delta E = \Delta K + \Delta U = -f_k\, d \tag{6.57}$$

In fact, this dissipated energy is transferred as thermal energy to the block and the floor. So, the energy of the block, which is considered to be our system, is not conserved.

When we expand our system to include both the block and the floor, the frictional force is no longer an external force, and the energy transfer will be within the system. So, again we have an *isolated system* within which energy is conserved.

To find this conservation principle, we look at the decrease ΔE in Eq. 6.57 as the total amount of energy transferred as thermal energy to the block and floor. If ΔE_{int} represents the change in the thermal energy (which is an *internal energy*) of the system consisting of the block and floor, then we get:

$$\Delta E_{\text{int}} = -\Delta E \tag{6.58}$$

which gives:

$$\Delta E + \Delta E_{\text{int}} = \Delta K + \Delta U + \Delta E_{\text{int}} = 0 \tag{6.59}$$

This means that, although the mechanical energy of the block is not conserved, the sum of the mechanical energy of the block and the thermal energy of the block and floor is conserved. This sum is called the total energy E_{tot} of the block-floor system. This conservation principle is called **the law of conservation of energy** and written as:

$$\Delta E_{\text{tot}} = \Delta K + \Delta U + \Delta E_{\text{int}} = 0 \quad \left\{ \begin{array}{l} \text{conservation of energy} \\ \text{for an isolated system} \end{array} \right\} \tag{6.60}$$

This law of conservation is not derived, but instead based on countless experiments done by scientists and engineers.

If the system is not isolated and applied external forces transfer energy to or from the system, then the work done *on* the system by external forces will be:

$$W = \Delta E_{\text{tot}} = \Delta K + \Delta U + \Delta E_{\text{int}} \quad \text{(non isolated system)} \tag{6.61}$$

For example, in Fig. 6.18, if we consider the rope to be external to the system, then the frictional force exerted by the rope on the metal rings of the system does an amount of work W on the system, transferring energy from the system to thermal energy in the rope while the values of K, U, and E_{int} change.

Fig. 6.18 A firewoman wrapping a rope around metal rings so that the rope rubs against the rings while she is descending from a helicopter. Doing so, she will transfer energy from the gravitational potential energy of a system consisting of her, her gear, and the Earth to thermal energy gained by the rope and the rings. While descending slowly, this allows most of the transferred energy to go to the rope and the rings rather than to her kinetic energy

Example 6.10

A steel ball of mass $m = 5$ g is projected vertically downward from a height $h = 14.8$ m with an initial speed $v_\circ = 10$ m/s, see part a of Fig. 6.19. The ball penetrates itself in sand to a depth $d = 20$ cm, see part c of the figure. Neglect air resistance and take g to be $10\,\text{m/s}^2$. (a) What is the change in the mechanical energy of the ball? (b) What is the change in the internal energy of the ball-Earth-sand system? (c) What is the magnitude of the average force exerted by the sand on the ball in part b of the figure?

Solution: (a) Let us take the reference point $y = 0$ to be at the point where the ball stops completely, as shown in part c of the figure. Therefore, at the stopping depth d, the kinetic energy and the potential energy are zero. Thus:

$$\begin{aligned}\Delta E &= E_f - E_i \\ &= \Delta K + \Delta U \\ &= (K_f - K_i) + (U_f - U_i) \\ &= (0 - \tfrac{1}{2} m v_\circ^2) + (0 - mg[h + d]) \\ &= -\tfrac{1}{2} m v_\circ^2 - mg(h + d)\end{aligned}$$

Inserting the given data into the final expression, we find:

$$\begin{aligned}\Delta E = &- \tfrac{1}{2}(5\times10^{-3}\,\text{kg})(10\,\text{m/s})^2 \\ &- (5\times10^{-3}\,\text{kg})(10\,\text{m/s}^2)(14.8\,\text{m} + 20\times10^{-2}\,\text{m}) \\ = &-0.25 - 0.75 = -1\,\text{J}\end{aligned}$$

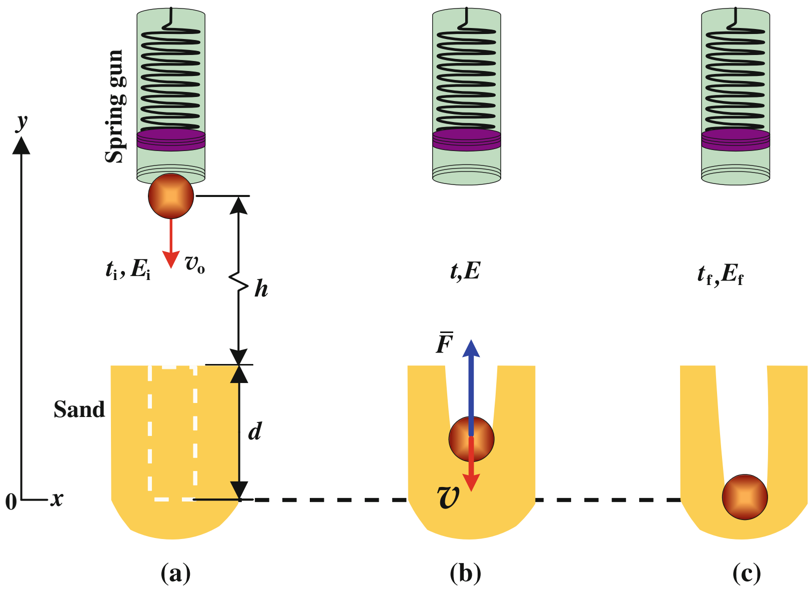


Fig. 6.19

(b) This system is isolated, and we can apply Eq. 6.59 as follows:

$$\Delta E + \Delta E_{\text{int}} = 0$$

or

$$\Delta E_{\text{int}} = -\Delta E = -(-1\,\text{J}) = 1\,\text{J}$$

That is to say, as the ball moves through the sand, the sand exerts an upward force on the ball and thus dissipates all the mechanical energy of the ball, transforming it to thermal energy of the sand and ball.

(c) When the ball reaches the surface of the sand, its mechanical energy will be the same as the initial mechanical energy E_i, since air resistance is neglected.

Then, as the ball moves through the sand, an average upward force $\overline{F}$ dissipates all its mechanical energy by the time the ball moves a distance d. Thus, the change in mechanical energy $\Delta E = E_f - E_i$ will be transferred to thermal energy of the sand and the ball. So, Eq. 6.57 can be written as:

$$\Delta E = E_f - E_i = -\overline{F}\, d$$

Solving this for $\overline{F}$, we find the following:

$$\overline{F} = -\frac{\Delta E}{d} = -\frac{(-1\,\text{J})}{20 \times 10^{-2}\,\text{m}} = 5\,\text{N}$$

We can arrive at this answer by using the techniques of Chap. 3 by finding the ball's speed at the surface of the sand and then its average deceleration within the sand. Then, using Newton's second law, we can find $\overline{F}$. Obviously, more algebraic steps would be required.

6.8 Power

It is more interesting to know not only the work done on an object, but also the time rate at which work is being done. This rate is defined as the **power**.

If ΔW is the work done by an applied force on an object during a time interval Δt, then the average power $\overline{P}$ during this time interval is defined as:

$$\overline{P} = \frac{\Delta W}{\Delta t} \tag{6.62}$$

The instantaneous power P is the limiting value of this average power as Δt approaches zero, i.e.

$$P = \lim_{\Delta t \to 0} \frac{\Delta W}{\Delta t} = \frac{dW}{dt} \tag{6.63}$$

The SI unit of power is joule per second (J/s), called a **watt** (W). In the British system, the unit of power is foot-pound per second (ft.lb/s). Often the term horsepower (hp) is used. These units relate as follows:

$$\left.\begin{aligned} &1\text{ watt} = 1\,\text{W} = 1\,\text{J/s} = 0.738\,\text{ft.lb/s} \\ &1\text{ horsepower} = 1\,\text{hp} = 550\,\text{ft.lb/s} = 746\,\text{W} \end{aligned}\right\} \tag{6.64}$$

From Eq. 6.62, we see that the work can be expressed as power multiplied by time, as in the common unit, the kilowatt-hour, Thus:

$$1\ \text{kilowatt-hour} = 1\ \text{kW.h} = (10^3\ \text{W})(3{,}600\ \text{s}) \\ = 3.6 \times 10^6\ \text{J} = 3.6\ \text{MJ} \tag{6.65}$$

It is important to realize that a kW.h is a unit of energy, not power. For example, our electric bills are usually in kW.h, and this gives the consumed amount of energy, whereas an electric bulb rated at a power of 100 W means it would consume 3.6×10^5 J of energy in 1 h.

We can express the rate at which a force $\vec{F}$ does work on a particle (or a particle-like object) in terms of that force and the body's velocity $\vec{v}$. In Eq. 6.23, we were able to express the work done dW on the particle by a force $\vec{F}$ during a displacement $d\vec{r}$ as $dW = \vec{F} \bullet d\vec{r}$. Therefore, the instantaneous power can be written as:

$$P = \frac{dW}{dt} = \frac{\vec{F} \bullet d\vec{r}}{dt} = \vec{F} \bullet \frac{d\vec{r}}{dt}$$

Recognizing $d\vec{r}/dt$ as the instantaneous velocity $\vec{v}$, we get:

$$P = \vec{F} \bullet \vec{v} = F\,v\,\cos\theta = \begin{cases} +F\,v & \text{if } \theta = 0° \\ 0 & \text{if } \theta = 90° \\ -F\,v & \text{if } \theta = 180° \end{cases} \tag{6.66}$$

Positive power means that energy is transferred to the particle, while negative power means that energy is transferred from the particle.

Example 6.11

An elevator loaded fully with passengers has a mass $M = 2{,}000$ kg. When the elevator ascends, an almost constant frictional force $f = 5{,}000$ N acts against its motion, see Fig. 6.20. What power must be delivered by the motor (the tension T) to lift the elevator at: (I) a constant speed v of 4 m/s, see part (a) of the figure? (II) a constant acceleration a of 1.5 m/s^2 that produces a speed $v = at$, see part (b) of the figure?

Solution: (I) Let T be the force supplied by the elevator's motor to pull the elevator upward. From Newton's second law and from the fact that $a = 0$ (since v is a constant in part a of Fig. 6.20), we get:

$$T - f - M\,g = 0$$

Using M as the total mass of the elevator and the passengers and inserting the given data into this expression, we find:

$$T = f + M\,g$$
$$= 5{,}000\,\text{N} + (2{,}000\,\text{kg})(9.8\,\text{m/s}^2)$$
$$= 24{,}600\,\text{N}$$

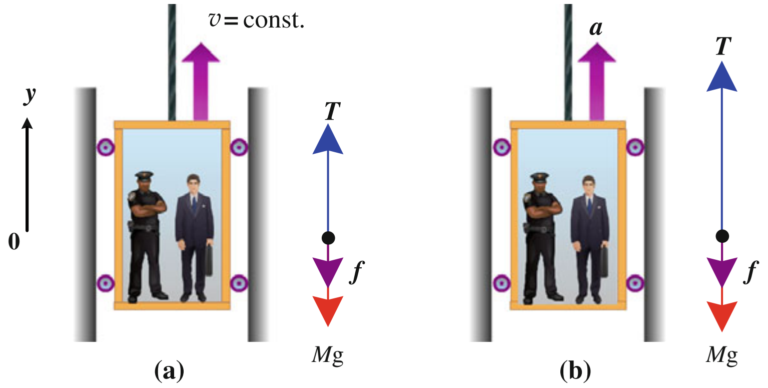


Fig. 6.20

Then, using Eq. 6.66 and the fact that $\vec{T}$ is in the same direction as $\vec{v}$ gives:

$$P = \vec{T} \bullet \vec{v} = T v \cos 0° = (24{,}600\,\text{N})(4\,\text{m/s})$$
$$= 98{,}400\ \text{W} = 98.4\,\text{kW} \simeq 132\,\text{hp}$$

This means that to maintain a constant speed of 4 m/s, a force of magnitude 24,600 N is required to transfer energy *to* the elevator at a rate of 98,400 J/s.

(II) Applying Newton's second law to part (b) of the figure gives:

$$T - f - Mg = Ma$$

Inserting the given data into this expression, we find:

$$T = f + M(g + a)$$
$$= 5{,}000\,\text{N} + (2{,}000\,\text{kg})[9.8\,\text{m/s}^2 + 1.5\,\text{m/s}^2]$$
$$= 27{,}600\,\text{N}$$

Then, using Eq. 6.66 we get:

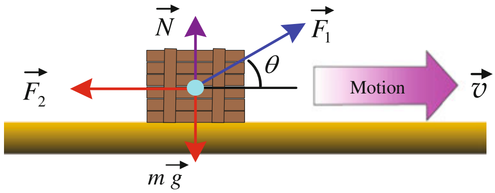
$$P = \vec{T} \bullet \vec{v} = T v \cos 0° = T\ at$$
$$= (41{,}400\ t)\ \text{W}$$

This indicates that the required power increases linearly with time t.

Example 6.12

Two forces $\vec{F}_1$ and $\vec{F}_2$ are acting on a box that slides horizontally to the right across a frictionless surface, see Fig. 6.21. Force $\vec{F}_1$ has a magnitude of 5 N and makes an angle $\theta = 60°$ with the horizontal. Force $\vec{F}_2$ is against the motion and has a magnitude of 2 N. The speed v of the box at a certain instant is 4 m/s. What is the power due to each force that acts on the box at that instant, and what is the net power? Is the net power changing with time?

Fig. 6.21



Solution: The weight $m\vec{g}$ and the normal force $\vec{N}$ are perpendicular to the velocity $\vec{v}$. Thus, their work done is zero, and hence the power due to each of them on the block is zero. We use Eq. 6.66 to find the power due to $\vec{F}_1$ and $\vec{F}_2$. First, for the force $\vec{F}_1$ that is applied at an angle $\theta = 60°$ to the velocity $\vec{v}$, we have:

$$P_1 = \vec{F}_1 \bullet \vec{v} = F_1 v \cos 60° = (5\,\text{N})(4\,\text{m/s})(0.5) = 10\,\text{W}$$

which indicates that the force $\vec{F}_1$ is transferring energy *to* the box at a rate of 10 J/s. Similarly, for $\vec{F}_2$ we have:

$$P_2 = \vec{F}_2 \bullet \vec{v} = F_2 v \cos 180° = (2\,\text{N})(4\,\text{m/s})(-1) = -8\,\text{W}$$

which indicates that the force $\vec{F}_2$ is transferring energy *from* the box at a rate of 8 J/s.

The net power is the sum of the individual powers. Thus:

$$P_{\text{net}} = P_1 + P_2 = 10\,\text{W} + (-8\,\text{W}) = 2\,\text{W}$$

This indicates that the net rate of energy transfer *to* the box is positive. So, the kinetic energy of the box will increase, and hence its speed. Consequently, the net power will increase with time.

6.9 Exercises

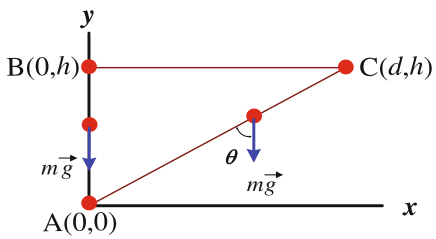
Section 6.1 Work Done by a Constant Force

(1) A 100 kg object moves in a straight line with a speed of 20 m/s. The object is to be stopped by a deceleration of $2\,\text{m/s}^2$. (a) What is the magnitude of the force required? (b) What distance does the object travel? (c) What work is done by the decelerating force? (d) Answer parts (a) to (c) for a deceleration of $4\,\text{m/s}^2$?

(2) How much work is done in moving a body of mass 2 kg vertically upward from an elevation of 1 m to an elevation of 3 m, (a) by gravity? (b) by an external agent that is slowly moving the body? (c) Answer parts (a) and (b) for a downward motion from an elevation of 3 m to an elevation of 2 m.

(3) Using Fig. 6.22, find the work done by the weight $m\vec{g}$ of a particle of mass m, as the particle is moved (by application of any other constant forces) from: (a) A to B, (b) B to A, (c) A to B to C, (d) A to C directly, and (e) A to B to C to A.

Fig. 6.22 See Exercise (3)



(4) A coin of mass $m = 0.5$ g slides a distance $d = 0.5$ m along a tabletop. If the coefficient of kinetic friction between the coin and the table is $\mu_k = 0.7$, find the work done on the coin by friction.

(5) A block of mass m is pushed along a rough horizontal surface by a constant horizontal force $\vec{F}$. The displacement of the block along the surface is $\vec{d}$. (a) Find the mathematical expression that represents the work done by: the force $\vec{F}$, the kinetic friction $\vec{f}_k$, the gravitational force $m\vec{g}$, and the normal force $\vec{N}$. (b) Calculate the work done when $m = 2$ kg, $\mu_k = 0.5$, $F = 20$ N, and $d = 5$ m.

(6) A block moves up an incline of angle $\theta = 30°$ under the action of the three forces shown in Fig. 6.23. Force $\vec{F}_1$ has a magnitude of 30 N and is parallel to the plane. Force $\vec{F}_2$ has a magnitude of 20 N and is normal to the plane. Force

$\vec{F}_3$ of 40 N is horizontal. Find the work done by each force as the block moves a distance $d = 2$ m up the incline.

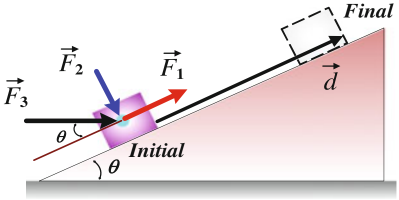


Fig. 6.23 See Exercise (6)

Section 6.2 Work Done by a Variable Force

(7) A force acting in the x direction on an object varies with x as shown in Fig. 6.24. Find the work done by the force in the intervals: (a) $0 \leq x \leq 1$ m, (b) $1\text{ m} \leq x \leq 3$ m, (c) $3\text{ m} \leq x \leq 4$ m, (d) $4\text{ m} \leq x \leq 7$ m, and (e) $0 \leq x \leq 7$ m.

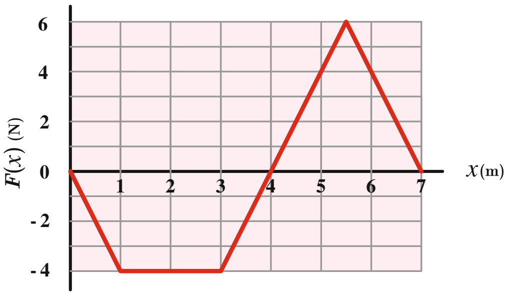


Fig. 6.24 See Exercise (7)

(8) A particle is subject to a force $f(x) = (2 + 0.5\,x)$ N. As the particle moves from $x = 0$ to $x = 8$ m, find the work done by the force using: (a) Equation 6.16, and (b) a graphical method.

(9) A smooth track in the form of a quarter of a circle of radius $r = 40$ cm lies in a vertical plane as shown in Fig. 6.25. A bead of mass 4 g moves from P_1 to P_2 under the effect of a force $\vec{F}(s)$ that is always acting tangentially to the track and of magnitude $F(s) = (10 - 2\,s)$ N, where the arc length s is measured in meters. (a) Find the work done by the applied force $\vec{F}$. (b) Find the work done by weight $m\vec{g}$.

(10) A force is used to compress a spring with a spring constant $k_H = 300$ N/m, see Fig. 6.26. (a) How much work does the applied force do when compressing

the spring a distance of 6 cm? (b) When the block is released, how much work does the spring force do on the block during a total displacement starting from a compression of 6 cm to a stretch of 4 cm?

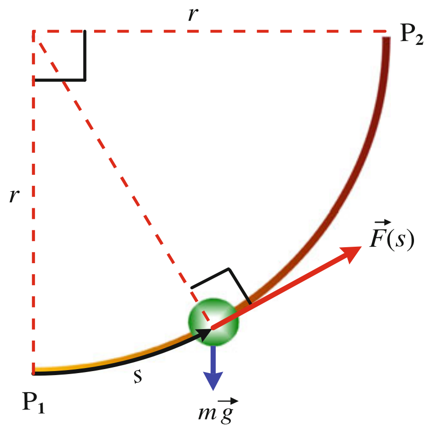


Fig. 6.25 See Exercise (9)

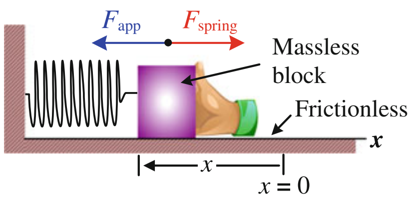


Fig. 6.26 See Exercise (10)

(11) A small sphere of weight $m\,g$ hangs from a string of length L, as shown in Fig. 6.27. A variable horizontal force $\vec{F}$, which starts from zero and gradually increases, is used to pull the sphere slowly (i.e., equilibrium exists at all the times) until the string makes an angle θ with the vertical. (a) Use Eq. 6.16 to show that the work done by the force $\vec{F}$ is $W_F = m\,g\,L(1 - \cos\theta)$. (b) Use the concept of equilibrium to reach the same answer without performing integration.

(12) The average resistive force against a nail penetrating a hard material is given by $\vec{F} = -kx^4\vec{i}$, where k is a constant and x is the penetration depth. Find the work done by this force when penetrating this material for a distance d.

(13) A bead is moving along the circumference of a circular hoop of radius R under a constant force of magnitude F. The force always makes an angle θ with

respect to the tangent to the circle. Find the work done by this force during one revolution.

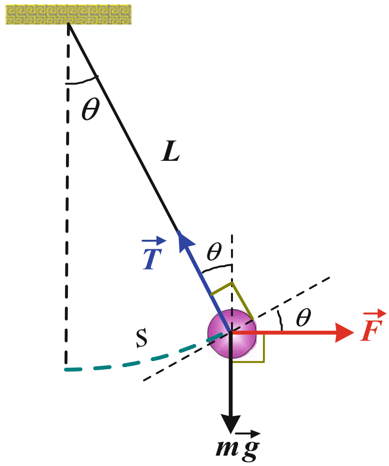


Fig. 6.27 See Exercise (11)

Section 6.3 Work-Energy Theorem

(14) A car is moving at 100 km/h. If its mass is 1,000 kg, what is its kinetic energy?

(15) A 120 g mass has a velocity $\vec{v} = (3\vec{i} + 4\vec{j})$m/s at a certain instant. What is its kinetic energy?

(16) Use the work-energy theorem to find the magnitude of the force required to accelerate a car of mass 1,300 kg from rest to 25 m/s in a distance of 100 m?

(17) The speed of a 10 kg object changes from 4 to 10 m/s. What is its change in kinetic energy?

(18) The velocity of a 0.4 kg object changes from $\vec{v}_i = (4\vec{i} + 3\vec{j})$ to $\vec{v}_f = (12\vec{i} - 9\vec{j})$ m/s. What is its change in kinetic energy?

(19) A force acting on a body that moves along the x-axis produces a velocity-time graph as shown in Fig. 6.28. If the body has a mass $m = 2$ kg, then find the change in kinetic energy in the intervals: (a) $0 \leq t \leq 1$ s, (b) $1\text{ s} \leq t \leq 3$ s, (c) $3\text{ s} \leq t \leq 5$ s, (d) $5\text{ s} \leq t \leq 7$ s, and (e) $0 \leq t \leq 7$ s.

(20) A force acts on a body of mass $m = 2$ kg that moves along the x-axis. The force varies with x as shown in Fig. 6.29. If the body was initially at rest, then find the change in kinetic energy in the intervals: (a) $0 \leq x \leq 1$ m, (b) $0 \leq t \leq 2$ m, and (c) $0 \leq x \leq 3$ m.

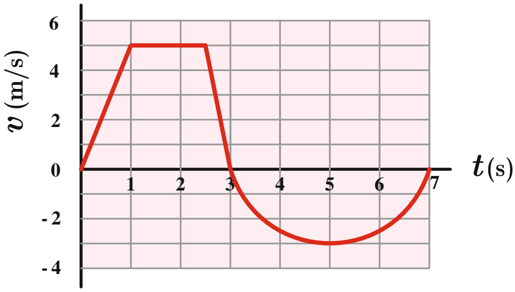


Fig. 6.28 See Exercise (19)

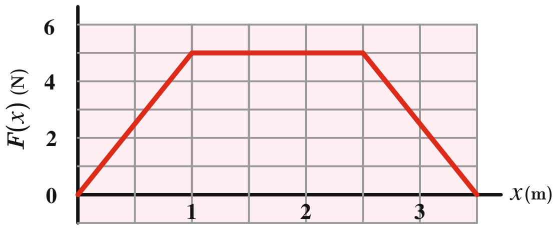


Fig. 6.29 See Exercise (20)

(21) A block of mass $m = 15$ kg slides from rest down a frictionless incline of inclination angle $\theta = 30°$ and is stopped by a spring that has a spring constant $k_H = 5{,}000$ N/m, see Fig. 6.30. The block moves a total distance $d = 1.5$ m from the point of release to the point where it stops momentarily as the spring reaches its maximum compression. Use the work-energy theorem to find the maximum compression of the spring.

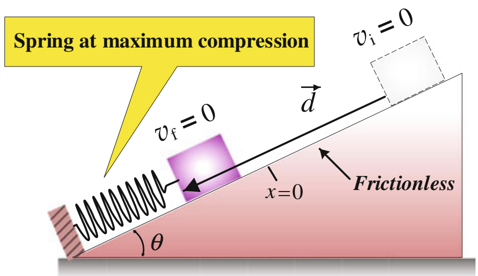


Fig. 6.30 See Exercise (21)

(22) A force acts on a particle of mass $m = 5$ kg and changes its velocity from $\vec{v_i} = (3\,\vec{i} + 4\,\vec{j})$ to $\vec{v_f} = (6\,\vec{i} + 8\,\vec{j})$ m/s. How much work is applied to this particle by this force?

Section 6.5 Conservation of Mechanical Energy

(23) A body of mass $m = 5\,\text{kg}$ is released from rest from a height of 2 m above the ground. (a) What is the kinetic energy of the body just before hitting the ground? (b) At that point, what is its speed?

(24) A freely falling ball of mass $m = 0.5\,\text{kg}$ passes a window 1.5 m high. (a) How much did the kinetic energy of the ball increase as it fell past the window? (b) If its speed at the top of the window was 2 m/s, what will its speed be at the bottom of the window?

(25) A pendulum bob has a mass $m = 0.5\,\text{kg}$. It is suspended by a cord of length $L = 2\,\text{m}$ which is pulled back through an angle of 90° and released, see Fig. 6.31. (a) What is its maximum potential energy relative to its lowest position? (b) What is its maximum speed at point B? (c) What is its speed at point C when the cord makes an angle $\theta = 60°$ with the vertical?

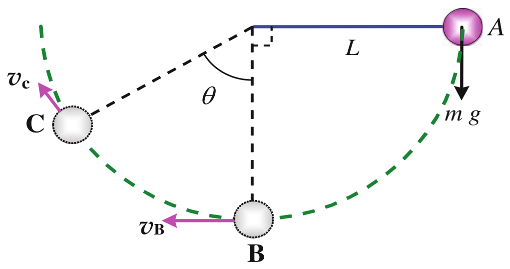


Fig. 6.31 See Exercise (25)

(26) In the track shown in Fig. 6.32, section AB is a quadrant of a circle of radius $r = 1\,\text{m}$. A block is released at A and slides without friction until it reaches point B, then moves a distance $d = 4\,\text{m}$ on a horizontal rough plane before stopping at point C. (a) Haw fast is the block moving at point B? (b) What is the coefficient of kinetic friction between the block and the plane?

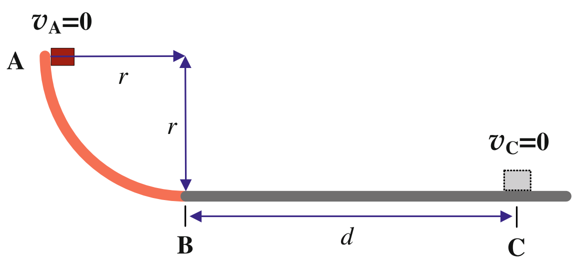
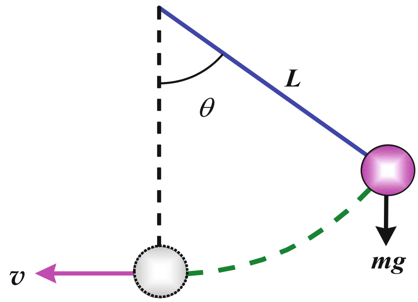


Fig. 6.32 See Exercise (26)

(27) A pendulum bob is pulled aside from its equilibrium position through an angle θ and then released, see Fig. 6.33. Show that the pendulum bob will pass through the equilibrium position with a speed $v = \sqrt{2gL(1 - \cos\theta)}$, where L is the length of the pendulum. When $\theta = 90°$, show that the relation of v will give an identical result to the result obtained in part (b) of exercise 25.

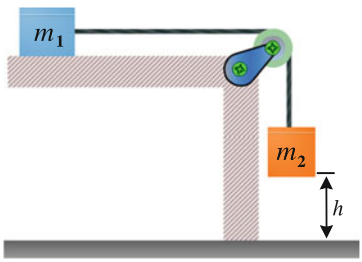
Fig. 6.33 See Exercise (27)



(28) A spring has one of its ends fixed and the other attached to a block of mass m that rests on a frictionless horizontal surface. The application of a horizontal force F on the block causes the spring to stretch a distance d from its equilibrium. The spring is held at this position momentarily and then the block is released. Find the speed of the block when the spring returns: (a) to half its original extension ($d/2$), and (b) to its natural length.

(29) Two blocks of masses $m_1 = 4$ kg and $m_2 = 5$ kg are connected by a massless string that passes over a massless frictionless pulley as shown in Fig. 6.34. Block m_1 is initially at rest on a smooth horizontal plane while block m_2 is at a height $h = 0.75$ m above the ground. Use conservation of mechanical energy to find the speed of the masses just before m_2 hits the ground.

Fig. 6.34 See Exercise (29)



(30) Figure 6.35 shows a proposed roller-coaster track. Each car starts from rest at point A, where $y_A = 21$ m and it will roll freely without friction along the track. It is important that there be at least some small normal force exerted by the track on the car at all points; otherwise, the car will leave the track. What is the minimum safe value for the radius of the curvature at point B?

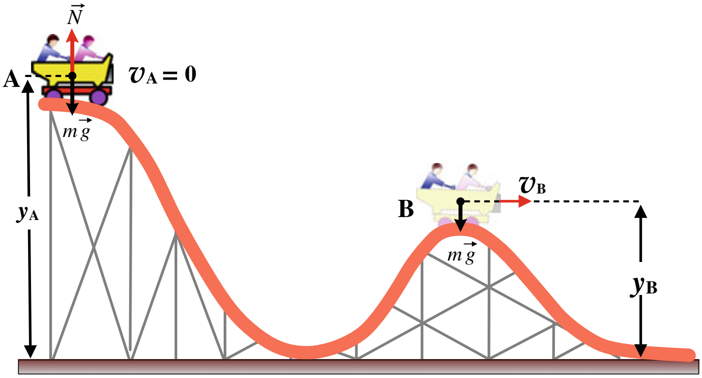
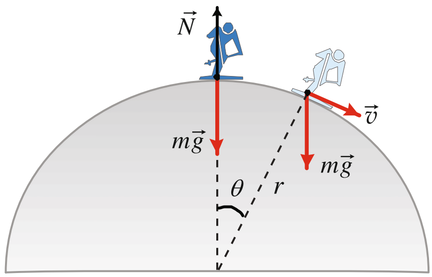


Fig. 6.35 See Exercise (30)

(31) A skier of mass m starts sliding from rest at the top of a solid frictionless hemisphere of radius r, see Fig. 6.36. At what angle θ will the skier leave the sphere?

Fig. 6.36 See Exercise (31)



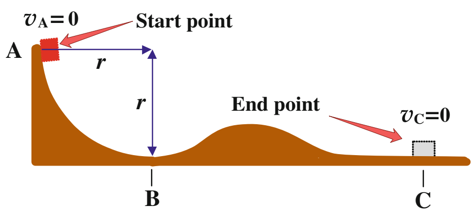
Sections 6.6 and 6.7 Work Done by Non-conservative Forces—Conservation of Energy

(32) If the mass of the block in Example 6.7 is 0.5 kg, then find the value of the speed $v_\circ$.

(33) If the mass of the boy in Example 6.8 is 50 kg, then redo parts (a) and (b) of the example and comment on the obtained results.

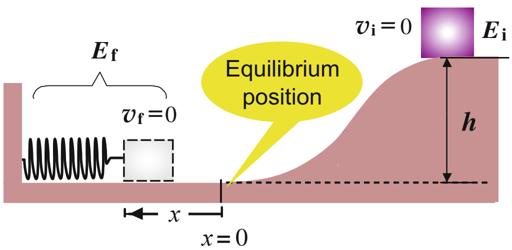
(34) In the rough track shown in Fig. 6.37, section AB is a quadrant of a circle of radius $r = 2$ m. A block of mass $m = 5$ kg is released at A and slides until it stops completely at point C. (a) Find the work done by friction. (b) What is the effect of having a more/less rough track on the block?

Fig. 6.37 See Exercise (34)



(35) A block of mass $m = 5$ kg is placed on the edge of a rough surface of height $h =$ 0.5 m, see Fig. 6.38. The block is released and moves until it stops momentarily after compressing a horizontal spring (with a spring constant $k_H = 2{,}000$ N/m) by a compression distance $x = 10$ cm. Find the work done by friction. Will the block ever be able to go back to its original location and why?

Fig. 6.38 See Exercise (35)



(36) (a) If the block in Exercise 35 traveled a total distance of 50 cm before coming to a momentary stop, estimate the average force of friction (assume it is roughly constant) on the block. (b) After the maximum compression of the spring is reached, the block starts its journey back on the surface. If the block reaches a

second momentary stop after moving a distance of 20 cm on the surface, what is the maximum height that the block can reach?

(37) A roller-coaster car of mass $m = 750$ kg starts from rest at the top of a hill 30 m high, see Fig. 6.39. The roller-coaster travels a total distance of 250 m without leaving the track and reaches a vertical height of only 25 m on the second hill before coming to a momentary stop. Find the thermal energy produced during the free motion and estimate the average frictional force on the car.

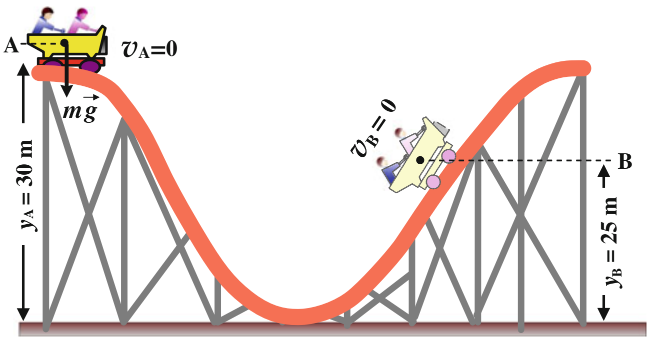


Fig. 6.39 See Exercise (37)

(38) A steel ball of mass $m = 0.5$ kg is projected horizontally with an initial speed $v_\circ = 10$ m/s, see Fig. 6.40a. The ball penetrates into a wall of clay until it stops at a depth $d = 20$ cm, see Fig. 6.40b–c. (a) What is the change in the mechanical energy of the ball ? (b) What is the change in the internal energy of the ball-Earth-wall system? (c) What is the magnitude of the average force exerted by the wall on the ball during the penetration process?

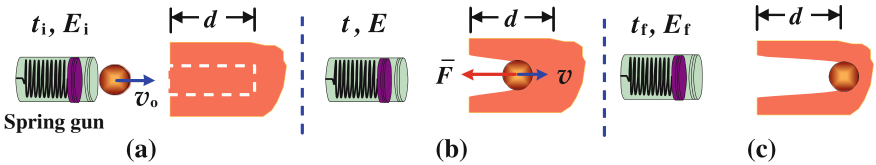


Fig. 6.40 See Exercise (38)

Section 6.8 Power

(39) How much average power in kilowatts and horsepower is required to lift a block of 100 kg to a height of 10 m in 30 s?

(40) At 30 piasters (Egyptian pound = 100 Piaster) per kilowatt-hour of electricity, what is the cost of operating a 5-hp motor for 2 h?

(41) An elevator fully loaded with passengers has a mass $M = 2{,}000$ kg. As the elevator descends, an almost constant frictional force $f = 4{,}000$ N acts against its motion. What power must be delivered by the motor to descend the elevator at: (a) a constant speed v of 4 m/s, and (b) a constant acceleration a of 1.5 m/s^2 that produces a speed $v = at$?

(42) A constant horizontal force $F = 20$ N acts on a block of mass $m = 4$ kg resting on a horizontal plane. The block starts from rest at $t = 0$. Show that the instantaneous power delivered by the force at any time t is given by $P = F^2t/m$, and find its value at $t = 5$ s.

(43) A car generates 20 hp when traveling at a constant speed of 100 km/h. What is the total resistive force that acts on the car?

(44) A car of mass $m = 1{,}500$ kg accelerates from rest to 100 km/h in 8 s. What is the average power delivered by its engine?

(45) A car of mass m accelerates with acceleration a up an inclined plane of angle θ as in Fig. 6.41. The drag force f_D consists of rolling friction α (N) and air drag $\beta\, v^2$(N), i.e. $f_D = \alpha + \beta\, v^2$, where α and β are constants and v is the speed of the car. (a) Find the force F that propels the car. (b) Show that $P = mva + mvg \sin\theta + \alpha\, v + \beta\, v^3$ is the power delivered to the wheels by the engine, where mva is the power delivered to accelerate the car, $mvg \sin\theta$ is the power to overcome gravity, $\alpha\, v$ is the power to overcome rolling friction, and $\beta\, v^3$ is the power to overcome air drag. (c) Calculate the various components of P and hence the total P if we take $m = 1{,}000$ kg, $a = 2$ m/s^2, $v = 20$ m/s, $\alpha = 200$ N, $\beta = 0.5$ kg/m, and $\theta = 15°$.

Fig. 6.41 See Exercise (45)