Chapter 31
A Comparison of Laser Oscillators and Quasiclassical Solid State Oscillators

We present three types of quasiclassical oscillators that are able to generate microwave radiation of high frequency: Gunn oscillator (used as source of radiation up to $\sim$200 GHz); superlattice oscillator (in development, up to 200 GHz); resonant-tunnel diode oscillator (demonstrated up to 700 GHz). These oscillators are solid state oscillators, driven by active media. An active medium of a solid state oscillator makes use of the nonlinear transport in a semiconductor (Gunn oscillator) or a semiconductor heterostructure (superlattice oscillator and resonant-tunnel diode oscillator). The nonlinear transport is due to a negative mobility of conduction electrons. The origin of negative differential mobility is of quantum mechanical nature. However, the transport can be described classically.

A laser oscillator and a quasiclassical solid state oscillator have in common that gain is mediated by a high frequency polarization of an active medium and, additionally, that the active medium experiences a change during the buildup of an oscillation.

What makes the difference between a laser oscillator and a quasiclassical solid state oscillator? In a laser oscillator, polarization occurs via interaction of a high frequency field with single particles (atoms, molecules, free-electrons). In a quasiclassical solid state oscillator, polarization occurs via interaction of a high frequency field with charge density domains, i.e., with collectives of free-electrons. The formation of domains and thus of the polarization is due to nonlinear transport properties of the active medium—and not by a population inversion. A quasiclassical solid state oscillator shows an upper frequency limit that is determined by a relaxation time; this is the time it takes the electrons to establish a collective. Oscillation is only possible if the period of the high frequency field is larger than the relaxation time.

There is, beside the mechanism of interaction of radiation with a medium, a difference in the techniques used to couple radiation to a medium. The active medium of a laser fills a resonator partly or completely. A solid state diode that drives a quasiclassical solid state oscillator can have extensions that are small compared to the wavelength of the radiation. An antenna serves for coupling of the active medium to the radiation. It is possible to use an active medium of small volume because the gain of classical active media can be much larger than the gain of laser media.

K.F. Renk, *Basics of Laser Physics*, Graduate Texts in Physics,
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According to Kroemer, there are two types of domains and therefore two modes of operation of a quasiclassical solid state oscillator—the pure charge accumulation mode and the propagating dipole domain mode. Here, we treat the pure charge accumulation mode, which is rarely described in textbooks, and we present an experiment that demonstrates the occurrence of the pure charge accumulation mode in a quasiclassical solid state oscillator.

We will, furthermore, discuss a classical oscillator model—the van der Pol oscillator. The model describes an equivalent circuit containing a nonlinear resistance that drives a self-excited oscillation in the circuit. The resistance of a van der Pol oscillator does not undergo a change during the buildup of an oscillation.

The chapter provides a connection to textbooks that treat microwave oscillators.

31.1 Interaction of Radiation with an Active Medium of a Laser or a Quasiclassical Oscillator

A comparison of a laser oscillator and a quasiclassical solid state oscillator shows the following.

- A laser oscillator and a quasiclassical oscillator have in common: interaction of an active medium with a high frequency field results in a high frequency polarization, which is synchronized to the field; mutual interaction of field and polarization leads to the buildup of both field and polarization.
- An active medium of a laser is, with respect to the charge distribution ρ, homogeneous (Fig. 31.1a). The corresponding material equation has the form $\nabla \cdot \boldsymbol{D} = 0$. An active medium of a laser contains high frequency dipole moments carried by atomic excitations. Interaction of these single-particle excitations with a high frequency electromagnetic field leads to gain for the high frequency field.
- In the active medium of a solid state oscillator, the charge distribution is inhomogeneous, $\nabla \cdot \boldsymbol{D} \neq 0$ (Fig. 31.1b). The periodic buildup and destruction of charge density domains gives rise to a high frequency polarization of the active medium. A quasiclassical solid state oscillator shows an upper frequency limit that is determined by the relaxation time of the electrons, which constitute a domain. The material properties responsible for the occurrence of charge density domains are based on quantum mechanical properties of a semiconductor (or a semiconductor heterostructure).

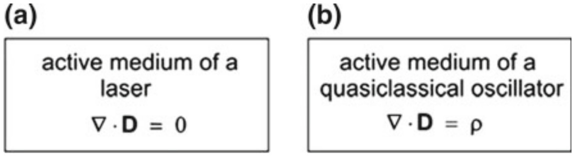

Fig. 31.1 Active media. **a** Laser medium. **b** Active medium of a solid state oscillator

31.2 Solid State Oscillators

There are various types of solid state oscillators for generation of microwave radiation. We mention three types.

- *Gunn oscillator*. The active device (active medium and electrodes together) of a Gunn oscillator is a Gunn diode. We describe a GaAs Gunn diode. The active device consists of a doped GaAs layer embedded in highly doped GaAs layers carrying metallic contacts. Nonlinearity is due to transfer of conduction electrons from a high-mobility state to a low-mobility state in GaAs. The electron transfer, which is of quantum mechanical nature, gives rise to a negative differential mobility for voltages larger than a critical voltage. The negative differential mobility causes formation of charge density domains. Gunn oscillators are available as microwave oscillators up to frequencies of $\sim$200 GHz. Gunn oscillators are described in many textbooks and survey articles; *see*, for instance, [234–239].
- *Semiconductor superlattice oscillator*. The basis of the nonlinearity of a semiconductor superlattice oscillator is the miniband transport. At voltages across a superlattice that are larger than a critical voltage, miniband electrons show a negative differential mobility. The negative differential mobility causes formation of charge density domains.
- *Resonant tunnel diode oscillator*. The active medium is a resonant-tunneling diode (Sect. 31.7).

There are two modes of operation of a Gunn oscillator or of a superlattice oscillator.

- *Pure charge accumulation mode* [239] (Fig. 31.2a). Under the action of a static field, a negative differential mobility medium extracts electrons from the cathode. The excess electrons in the medium and the positive charges at the cathode represent a dipole domain connected with a quasistatic polarization of the medium. Under the action of both a static field and a high frequency field, the number of excess electrons within the medium (and thus the density of positive charge at the cathode) increases and decreases periodically at the frequency of the high frequency field. The corresponding high frequency polarization P mediates gain. In the pure charge accumulation mode of operation of an oscillator, negative charge flows periodically from the cathode into the negative differential mobility medium and back to the cathode while the positive charge is bound to the cathode.

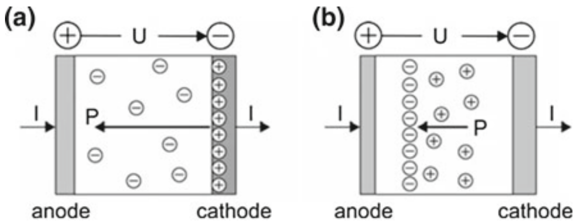

Fig. 31.2 Dipole domains in an active medium of a solid state oscillator. **a** Dipole domain caused by pure charge accumulation. **b** Propagating dipole domain

- *Propagating dipole domain mode* [239] (Fig. 31.2b). Under the action of a static field and of a high frequency field, negative and positive charges within the negative differential mobility medium separate giving rise to dipole domains. A dipole domain is formed near the cathode, travels through the medium, and disappears at the anode. The periodic formation and destruction of domains at the frequency of the high frequency field is joined with a high frequency polarization P of the active medium. The polarization mediates gain. The formation of propagating domains requires special boundary conditions for the field at the boundary between cathode and the negative differential mobility medium.

We will consider a particular solid state oscillator, namely a semiconductor superlattice oscillator, operating in a pure charge accumulation mode.

31.3 Semiconductor Superlattice Oscillator

In a semiconductor superlattice oscillator (Fig. 31.3a), a superlattice in a cavity resonator drives the oscillation. The superlattice is electromagnetically coupled to the field in the resonator via an antenna (a metal whisker). The antenna is also connected

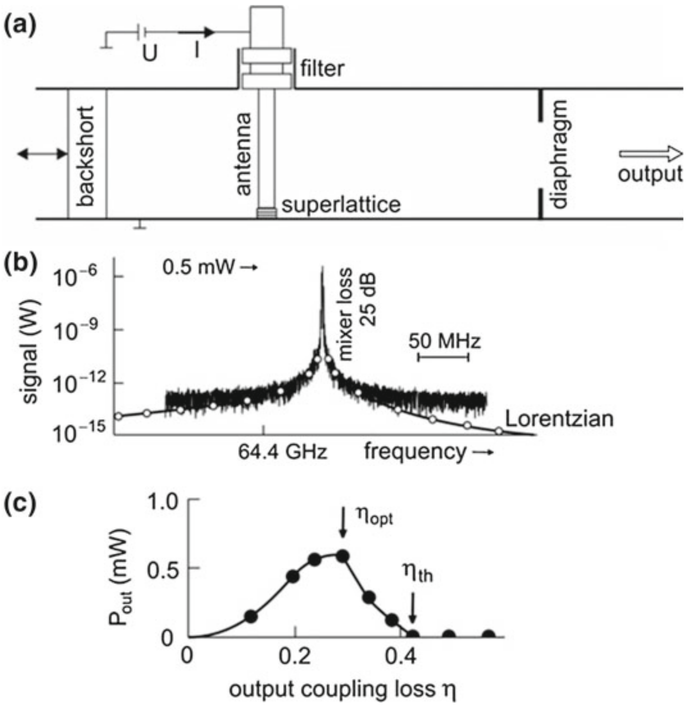

Fig. 31.3 Semiconductor superlattice oscillator. **a** Arrangement. **b** Emission spectrum. **c** Threshold behavior

to a bias circuit containing a voltage source (voltage U), which delivers a direct current I. A filter in the bias circuit avoids loss of radiation to the bias circuit. Radiation is coupled out from the resonator via the output port that contains a diaphragm. The oscillator is suited to generate microwave radiation. The emission spectrum (Fig. 31.3b), of an oscillator generating radiation near a frequency of 64 GHz, shows a bandwidth (200 kHz) that is determined by the spectrum analyzer used to register the spectrum. The emission line is, as indicated by the slope in the far wings, a Lorentzian line; a small deviation is due to background of the spectrum analyzer. For description of a superlattice oscillator, we follow [246].

Figure 31.3c (points and solid line) shows the output power P_{out} of the oscillator for different strengths η of output coupling loss; a measure of the output coupling loss η is the ratio of the aperture area and the area of the completely open output port. At small η, with radiation stored in the resonator, P_{out} is small. With increasing η, P_{out} increases, shows a maximum corresponding to optimum output coupling at η_{opt}, and then decreases to zero at the threshold loss η_{th}. A solid state oscillator shows an oscillation threshold behavior as a laser oscillator does.

To illustrate the principle of a superlattice oscillator, we consider the current-voltage (I–V) curve of a superlattice (Fig. 31.4a). With increasing voltage, the current increases linearly at small voltage, then less than linearly, reaches a peak value I_p at a critical voltage U_c, and remains constant for $U_s > U_c$. A static voltage $U_s > U_s$ causes the buildup of a high frequency current $I(t)$ and voltage $U(t)$. The active medium experiences feedback from the high frequency field stored in the resonator, which results in a reduction δI of the direct current. The current reduction is equal to the amplitude of the high frequency current (Fig. 31.4b). The current reduction occurs stepwise: at increasing U_s, the direct current shows plateau-like slopes. A current reduction, indicating oscillation, can occur already for $U < U_c$.

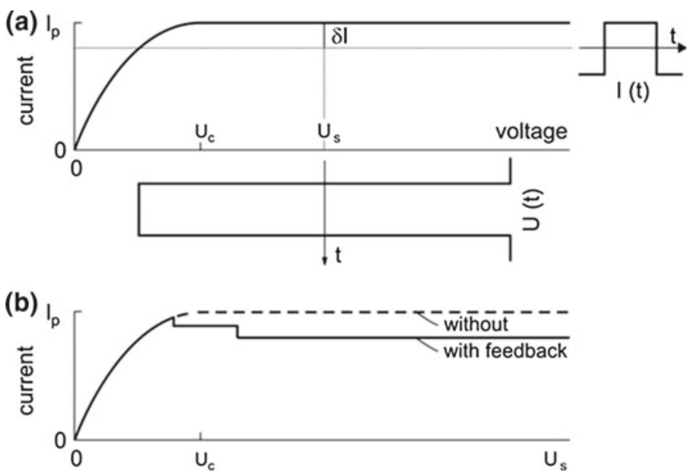

Fig. 31.4 Principle of the semiconductor superlattice oscillator. **a** I–V curve and time-dependent current and voltage. **b** I–V curve without and with feedback from radiation

We conclude from the occurrence of oscillations that the real high frequency current and high frequency voltage contain components of opposite phases, i.e., that the high frequency resistance of the superlattice is negative, which that is a condition of gain. The high frequency resistance is equal to

$$R_{\rm neg} = -\,\hat{U}/\hat{I}. \tag{31.1}$$

$\hat{U}$ is the amplitude of the high frequency voltage and $\hat{I}$ the amplitude of the high frequency current.

Example A particular GaAs superlattice (diameter 4 μm; length 0.6 μm; electron density $N_0 = 5 \times 10^{22}$ m^{-3}) has a critical voltage of 0.6 V and a peak current of 10 mA. Oscillation at 65 GHz results in a reduction of the current amplitude of $\delta I = \hat{I} = 2$ mA. The amplitude of the high frequency voltage is $\hat{U} = 0.9$ V (for $U_{\rm s} = 2\ U_{\rm c}$). Thus, the negative resistance is equal to $R_{\rm neg} = -450\,\Omega$. The experimental output power at optimum output coupling is ∼0.5 mW corresponding to an efficiency of 4% for conversion of electric power to power of microwave radiation.

31.4 Model of a Solid State Oscillator

We follow [234]. We characterize a (quasiclassical) solid state oscillator by an equivalent resonance circuit. The resonance circuit can be a parallel or series resonance circuit. We choose a parallel resonance circuit.

The equivalent circuit (Fig. 31.5a) describes a high frequency circuit containing an active device with a negative resistance $R_{\rm neg}$, a capacitance C, an inductance L and a resistance R, which accounts for loss due to emission of radiation. The active device (i.e., the active medium together with the electrodes) itself has an inductance $L_{\rm d}$ and a capacitance $C_{\rm d}$. To illustrate the principle of a negative resistance oscillator, we make use a simplified circuit (Fig. 31.5b).

- If the total resistance is negative, an initial high frequency current in the loop will grow; thus, we have the oscillation condition: the total resistance must be negative.

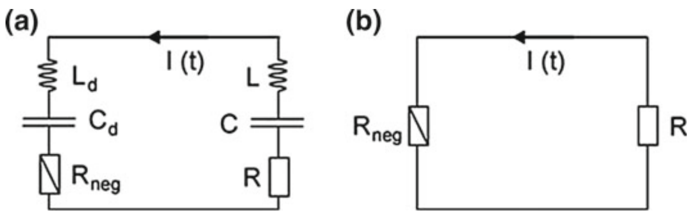

Fig. 31.5 Negative resistance oscillator. **a** Equivalent circuit and **b** simplified equivalent circuit

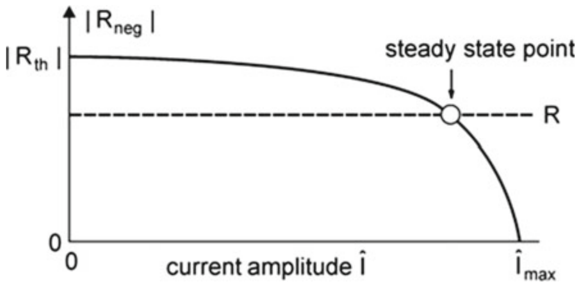

Fig. 31.6 Dependence of the magnitude of a negative resistance on the current amplitude

- At steady state oscillation, the sum of the resistances is zero; during onset of oscillation, the magnitude of the negative resistance decreases from a small-signal value to a large-signal value.
- If the total resistance is positive, an initial high frequency current will be damped and oscillation will not start.

The magnitude of the negative resistance depends on the amplitude of the high frequency current (Fig. 31.6). The absolute value of R_{neg} is largest for a small current amplitude $\hat{I}$ and is zero at maximum current amplitude $\hat{I}_{\mathrm{max}}$ obtained for $R = 0$. The resistance R determines the point of steady state oscillation.

The output power of the oscillator is

$$P_{\mathrm{out}} = (1/2) R \hat{I}^2 \tag{31.2}$$

if the condition

$$R_{\mathrm{neg}} + R = 0 \tag{31.3}$$

is satisfied. An appropriate choice of the value of R—for instance, by an appropriate choice of the output coupling aperture of the resonator—leads to optimum output coupling. In the description of an equivalent parallel circuit, the threshold condition of a solid state oscillator is given by

$$R < |R_{\mathrm{th}}|; \tag{31.4}$$

the loss resistance R must be smaller than the absolute value of the threshold resistance R_{th}.

To maintain a steady state oscillation, the high frequency voltage across the loop described by the complete equivalent circuit (*see* Fig. 31.5a) must be zero according to Kirchhoff's rules of voltages and currents in an electrical circuit,

$$I_0\,(R_{\mathrm{neg}} + \mathrm{i}\,X_{\mathrm{d}}) + I_0\,(R + \mathrm{i}\,X) = 0. \tag{31.5}$$

X_d is the reactance of the device and $X = \omega L - 1/(\omega C)$ the reactance of the resonance circuit. $R_{neg} + i\, X_d$ is the device impedance. The real part of the equation leads to (31.3) and the imaginary part to

$$X + X_d = \omega(L + L_d) - \frac{1}{\omega}\left(\frac{1}{C} + \frac{1}{C_d}\right) = 0. \tag{31.6}$$

This condition provides the oscillation frequency at steady state oscillation.

We consider an oscillator with an active element carrying a high frequency current (frequency ω) of amplitude $\hat{I}$,

$$I(t) = \hat{I} \cos \omega t. \tag{31.7}$$

The voltage across the active device is given by

$$U(t) = R_{neg}\, \hat{I}\, \cos \omega t - X_d\, \hat{I}\, \sin \omega t; \tag{31.8}$$

we neglect higher harmonics. Voltage and current have a phase shift of

$$\tan \varphi = -X_d/R_{neg}. \tag{31.9}$$

Without loss ($R = R_{neg} = 0$), the phase shift between current and voltage is $\pi/2$.

We can write the oscillator equation in the form

$$L\frac{dI}{dt} + RI + \frac{1}{C}\int I dt + U = 0. \tag{31.10}$$

Inserting (31.7) and (31.8) in (31.10) leads to the conditions of steady state oscillation, $R_{neg} + R = 0$ and $\omega L - 1/(\omega C) + X_d = 0$.

A negative resistance device based on nonlinear properties of conduction electrons in a semiconductor has internal degrees of freedom: the charge density distribution in an active device (=active medium and electrodes together) can be inhomogeneous. The degree of inhomogeneity depends nonlinearly on the voltage across the device. The value of R_{neg} depends therefore on the internal dynamics.

We consider an oscillator operated at a fixed $R = |R_{neg}|$, i.e., at a fixed static voltage. In the case that the oscillator is submitted to a small additional time dependent voltage $U_1(t)$, the oscillator equation is given by

$$L\frac{dI}{dt} + RI + \frac{1}{C}\int I dt + U = U_1(t). \tag{31.11}$$

We solve the equation by using the ansatz:

$$I(t) = \hat{I}(t) \cos[\omega t + \varphi(t)], \tag{31.12}$$

where higher harmonic currents are neglected. It follows that the high frequency voltage is equal to

$$U(t) = R_{\mathrm{neg}}\,\hat{I}\,\cos[\omega t + \varphi(t)] - X_{\mathrm{d}}\,\hat{I}\,\sin[\omega t + \varphi(t)]. \tag{31.13}$$

We assume that $\hat{I}(t)$ and $\varphi(t)$ do not vary appreciably over one cycle of the oscillation (slowly varying envelope approximation) and find the differential equations

$$\frac{\mathrm{d}I}{\mathrm{d}t} = -\hat{I}\left(\omega + \frac{\mathrm{d}\varphi}{\mathrm{d}t}\right)\sin(\omega t + \varphi) + \frac{\mathrm{d}\hat{I}}{\mathrm{d}t}\cos(\omega t + \varphi), \tag{31.14}$$

$$\int I\mathrm{d}t = \left(\frac{\hat{I}}{\omega} - \frac{\hat{I}}{\omega^2}\frac{\mathrm{d}\varphi}{\mathrm{d}t}\right)\sin(\omega t + \varphi) + \frac{1}{\omega^2}\frac{\mathrm{d}\hat{I}}{\mathrm{d}t}\cos(\omega t + \varphi). \tag{31.15}$$

Using (31.12) and (31.13), multiplying by $\cos(\omega t + \varphi)$ and $\sin(\omega t + \varphi)$ and integrating over a period $T = 2\pi/\omega$, we find from (31.14) and (31.15) *two oscillator equations*

$$\left(L + \frac{1}{\omega^2 C}\right)\frac{\mathrm{d}\hat{I}}{\mathrm{d}t} + \left(R_{\mathrm{neg}} + R\right)\hat{I} = \frac{2}{T}\int_{t-T}^{t} U_1(t)\cos(\omega t + \varphi)\mathrm{d}t \tag{31.16}$$

and

$$\left(-\omega L + \frac{1}{\omega C} - \bar{X}\right) - \left(L + \frac{1}{\omega^2 C}\right)\frac{\mathrm{d}\varphi}{\mathrm{d}t} = \frac{2}{\hat{I}T}\int_{t-T}^{t} U_1(t)\sin(\omega t + \varphi)\mathrm{d}t. \tag{31.17}$$

If an external voltage is absent, these differential equations describe self-excited oscillation of a quasiclassical solid state oscillator. The description of onset of oscillation and steady state oscillation of a specific oscillator requires knowledge about the parameters R, L, C of the passive elements and the parameters $R_{\mathrm{neg}}(\hat{I})$, $C_{\mathrm{d}}(\hat{I})$, and $L_{\mathrm{d}}(\hat{I})$ of the active device. If an external voltage is present, the equations describe phase locking of a classical oscillator to an external (weak) high frequency voltage (that is delivered, for instance, by a highly stabilized oscillator).

In comparison with a laser oscillator coupled to an external field—characterized by five differential equations of first order (Sect. 9.9)—the quasiclassical solid state oscillator coupled to an external field can be characterized by only two differential equations of first order, an equation for the amplitude of the current, and another equation for the phase between current and external field. The equations are coupled equations that have in common the parameters of the active device.

We can describe the superlattice oscillator as a regenerative amplifier with a resonator mediating feedback. Amplification of thermal radiation leads to phase and amplitude fluctuations and therefore to a noise bandwidth of the oscillator radiation. The spectral distribution of the radiation has a Lorentzian lineshape (Sect. 4.5).

The quality factor for radiation generated in a single mode oscillation is equal to (Sect. 8.9)

$$Q_{\rm rad} = Q_{\rm res} Z/Z_0. \tag{31.18}$$

$Q_{\rm res}$ is the quality factor of the resonator, Z the average occupation number of photons in the resonator mode at steady state oscillation, and Z_0 the average occupation number of thermal photons in the resonator mode without oscillation. Z follows from the relation $P_{\rm out} = Zh\nu/\tau_{\rm p}$, where $\tau_{\rm p} = Q_{\rm res}/\omega$ is an average lifetime of a photon in the resonator. The thermal occupation number is $Z_0 = kT/h\nu$; k is Boltzmann's constant and T the temperature.

Example Superlattice oscillator with a superlattice described in the preceding and the following example. Frequency $\nu = 6.5 \times 10^{10}$ Hz; $Q_{\rm res} = 30$; output power $P_{\rm out} = 0.5$ mW; $Z_0 \sim 100$; $Z = 2 \times 10^8$; $Q_{\rm rad} \sim 10^8$.

A more detailed treatment of noise in solid state oscillators can be found, for example, in [244, 245].

31.5 Dynamics of Gain Mediated by a Semiconductor Superlattice

We describe a particular superlattice (Fig. 31.7a). It consists of layers of GaAs and of AlAs in turn. The superlattice is doped and contains free-electrons. Adjacent to the superlattice are, on both ends, highly doped GaAs layers (electron concentration 2×10^{24} m^{-3}). One of these layers connects the superlattice to a highly doped GaAs substrate and the other layer is covered with a metallic contact layer.

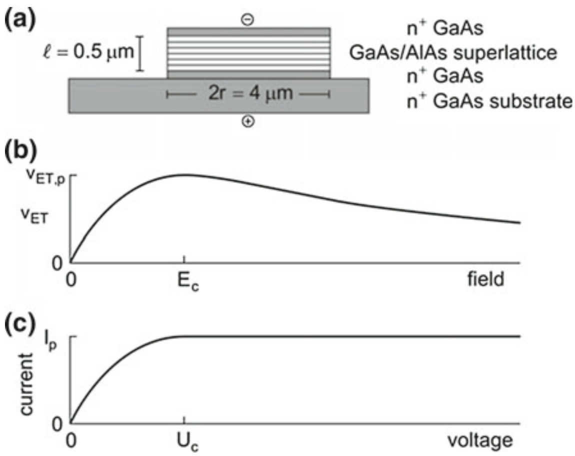

Fig. 31.7 Semiconductor superlattice. **a** Geometric structure. **b** Drift velocity-field characteristic for a homogeneous field along the superlattice axis. **c** Experimental I–V curve (*simplified*)

The drift velocity-field characteristic (Fig. 31.7b) is expected to have the form of an Esaki-Tsu characteristic:

$$\mathrm{v}_{\mathrm{ET}} = \mathrm{v}_{\mathrm{ET,m}} \frac{2\, E_{\mathrm{s}}/E_{\mathrm{c}}}{1 + (E_{\mathrm{s}}/E_{\mathrm{c}})^2} \tag{31.19}$$

E_{s} is the static field, $\mathrm{v}_{\mathrm{ET,m}}$ the maximum drift velocity reached at the critical field E_{c}. The differential mobility $\mu = d\mathrm{v}_{\mathrm{ET}}/dE_{\mathrm{s}}$ is equal to the ohmic mobility $\mu_{\mathrm{ohm}} = 2\mathrm{v}_{\mathrm{ET,m}}/E_{\mathrm{c}}$ for fields around $E_{\mathrm{s}} = 0$ and is negative for $E_{\mathrm{s}} > E_{\mathrm{c}}$. The negative differential mobility has the largest absolute value for $E = 1.7\ E_{\mathrm{c}}$, where $\mu = -\mu_{\mathrm{ohm}}/8$. We will derive the Esaki–Tsu characteristic in Sect. 32.3. We will show that the critical field is determined by the superlattice period a and a relaxation time τ according to $E_{\mathrm{c}} = \hbar/ea\tau$; the relaxation time indicates how fast an equilibrium is established in an ensemble of free-electrons in a superlattice.

If the field along the superlattice axis is homogeneous even if the field exceeds the critical field, we obtain the Esaki–Tsu I–V characteristic, which is given by

$$I_{\mathrm{ET}} = I_{\mathrm{p}} \frac{2\, U_{\mathrm{s}}/U_{\mathrm{c}}}{1 + (U_{\mathrm{s}}/U_{\mathrm{c}})^2}. \tag{31.20}$$

$U_{\mathrm{s}} = E_{\mathrm{s}}l$ is the static voltage across the superlattice, $U_{\mathrm{c}} = E_{\mathrm{c}}l$ is the critical voltage, and

$$I_{\mathrm{p}} = \pi r^2 N_0 e \mathrm{v}_{\mathrm{ET,m}} \tag{31.21}$$

is the peak current; r is the radius of the superlattice and N_0 the electron density. The ohmic resistance around $U_{\mathrm{s}} = 0$ is equal to

$$R_{\mathrm{ohm}} = \frac{U_{\mathrm{c}}}{2I_{\mathrm{p}}}. \tag{31.22}$$

Example of a superlattice (radius $r = 2\ \mu$m; $l = 0.6\ \mu$m; electron density $N_0 = 5.5 \times 10^{22}\,\mathrm{m}^{-3}$); $\mathrm{v}_{\mathrm{ET,m}} = 10^5\,\mathrm{m\,s}^{-1}$; $I_{\mathrm{p}} = 11\,$mA; $E_{\mathrm{c}} = 10^6\,\mathrm{V\,m}^{-1}$; $U_{\mathrm{c}} = 0.6\,$V; $\tau \sim 1.5 \times 10^{-13}\,$s; ohmic resistance $R_{\mathrm{ohm}} = 27\ \Omega$ (around $U_{\mathrm{s}} = 0$) and ohmic mobility $\mu_{\mathrm{ohm}} = 0.20\,\mathrm{m}^2\,\mathrm{V}^{-1}\,\mathrm{s}^{-1}$; ohmic conductivity $\sigma_{\mathrm{ohm}} = 1.8 \times 10^3\ \Omega^{-1}\,\mathrm{m}^{-1}$.

Now, the experimental I–V curve (Fig. 31.7c) shows a constant current (peak current $I_{\mathrm{p}} = N_0 e \mathrm{v}_{\mathrm{ET,m}}$) at voltages above U_{c}. The origin of the excess current $I_{\mathrm{exc}}(U) = I_{\mathrm{p}} - N_0 e \mathrm{v}_{\mathrm{ET}}(E)$, with $U = El$, are excess electrons extracted from the cathode.

A constant current ($I = I_{\mathrm{p}}$) corresponds to an excess electron density n (Fig. 31.8a) that increases with U_{s} ($>U_{\mathrm{c}}$) according to

$$n(U_{\mathrm{s}}) = N_0 \left(\frac{\mathrm{v}_{\mathrm{ET,m}}}{\mathrm{v}_{\mathrm{ET}}(U_{\mathrm{s}})} - 1 \right) = N_0\, \frac{(1 - U_{\mathrm{s}}/U_{\mathrm{c}})^2}{2U_{\mathrm{s}}/U_{\mathrm{c}}}. \tag{31.23}$$

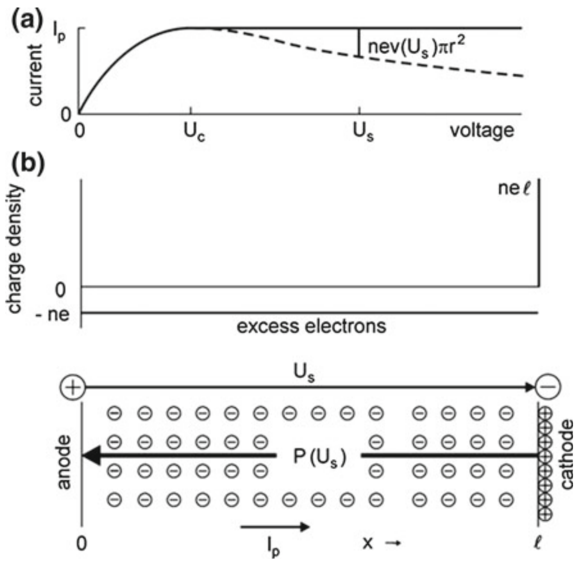

Fig. 31.8 Superlattice biased with a static voltage. **a** I–V curve and excess current at a static voltage U_s. **b** Charge density and polarization

The excess electron density is zero at $U_s = U_c$, equal to N_0 at $U_s = 3.7\ U_c$ and increases linearly with U_s at $U_s/U_c \gg 1$. For $U_s/U_c \gg 1$, the excess electron density increases linearly with the static voltage, $n = (1/2)N_0 U_s/U_c$. We suppose that the positive charges are distributed on a plane (at the cathode) adjacent to the superlattice boundary. The excess charge within the superlattice is equal to the positive charge at the cathode (Fig. 31.8b, upper part). The density of the excess charge in the superlattice is -*ne* and the area density of the charge at the cathode is nel. The excess charge in the superlattice together with the positive charge at the cathode form a dipole domain (Fig. 31.8b, lower part). It consists of a charge density domain within the superlattice and a positive area charge bound to the cathode. In the absence of current oscillations, a dipole domain is associated with a quasistatic polarization $P(U_s) = -n(U_s)el/2$. The direction of polarization is opposite to the direction of the direct current I_p.

The density $n(U_s)$ increases with increasing U_s (Fig. 31.9a). If the voltage across the superlattice suddenly changes from U_s to $U_s + U_1$, additional excess electrons flow into the superlattice until the excess charge density is equal to $n(U_s + U_1)$ in the whole superlattice. If the voltage suddenly changes from $U_s + U_1$ to $U_s - U_1$, all excess electrons escape from the superlattice. For $U_s = 2\ U_c$ and $U_1 = 1.5\ U_c$, the characteristic time of a cycle of filling of the superlattice with excess electrons and their escape is $t_c \sim 2 \times l/(0.8 v_{ET,m})$. The critical rate of generation of a full domain and its destruction is

$$\nu_c = 0.4\ v_{ET,m}/l. \tag{31.24}$$

The critical rate is ∼70 GHz at a superlattice of a length $l = 0.6\ \mu m$.

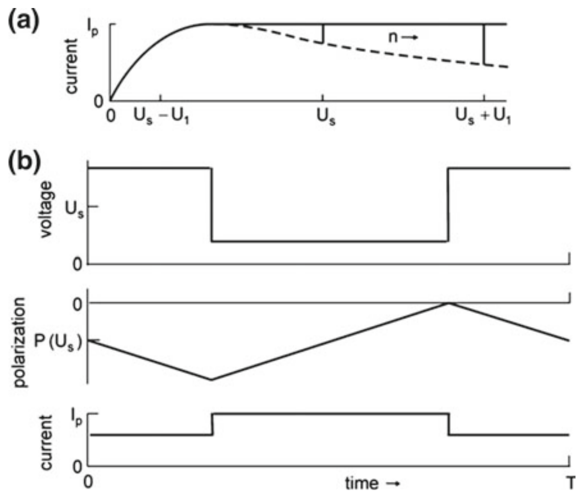

Fig. 31.9 Dynamics of gain. **a** I–V curve. **b** Voltage, polarization and current during an oscillation cycle

Under the influence of both a static and a high frequency voltage, the high frequency voltage causes a periodic change of polarization. The temporal change of polarization is equal to a polarization-current density. A high frequency voltage (frequency ν_c) of a rectangular shape (Fig. 31.9b) produces a polarization that has, in a simplified picture, a triangular shape and is phase-shifted by $\pi/2$. The current has a rectangular shape and is phase-shifted by π relative to the voltage. A Fourier transformation yields the amplitude $\hat{U} = (4/\pi)\ U_1$ of the high frequency voltage $U = \hat{U} \cos \omega t$ and the amplitude $\hat{P} = nel/2\pi$ of the high frequency polarization $P = \hat{P} \sin \omega t$. The high frequency polarization current is equal to $I = -\hat{I} \cos \omega t$, where $\hat{I} = \pi r^2 \omega \hat{P} = r^2 l \nu n e$ is the amplitude of the current; the high frequency polarization-current is continued outside the superlattice by a high frequency current (flowing through the antenna). For static voltages that are noticeably larger than U_c, the electrons are not fast enough to follow the high frequency voltage. Therefore, the effective length l_{eff} of an excess charge domain is shorter than l. We write $l_{eff} = 0.2 v_{ET,m}/\nu_c$. The product nl_{eff} $(= 1.2\ N_0 l)$ and thus $\hat{P}$ are independent of U_s. We obtain a constant current amplitude

$$\hat{I} = 1.2\, r^2 \nu l N_0 e. \tag{31.25}$$

A constant amplitude of the high frequency current results in a plateau in the $I - V$ curve for the superlattice in the oscillating state. This is in accord with the experimental result.

An analysis of the large-signal behavior of the amplitude of the high frequency voltage and the amplitude of the high frequency current leads to a negative differential resistance of the superlattice operating in the accumulation mode, $R_{acc} = -\hat{U}/\hat{I}$, which is equal to

$$R_{\rm acc} = -\frac{2\pi\,{\rm v}_{\rm ET,m}}{\nu\, l}\,\frac{\hat{U}}{U_{\rm c}}\,\frac{2\left(U_{\rm s}/U_{\rm c} + (\pi/4)\hat{U}/U_{\rm c}\right)}{\left(|1 - U_{\rm s}/U_{\rm c} - (\pi/4)\hat{U}/U_{\rm c}|\right)^2}\,R_{\rm ohm} \tag{31.26}$$

for $U_{\rm s} \sim U_{\rm c}$ and

$$R_{\rm acc} = -\frac{1.6\pi\,{\rm v}_{\rm ET,m}}{\nu\, l}\,\frac{\hat{U}}{U_{\rm c}}\,R_{\rm ohm}. \tag{31.27}$$

for $U_{\rm s}^2 \gg U_{\rm c}^2$. This analysis is oriented at the I–V curve (*see* Fig. 31.9a). It is taken into account that the flow of excess charge takes time and it was made use of (31.20)–(31.22) and (31.24).

We estimate $R_{\rm acc}$ (Fig. 31.10, solid line), using the values: $\hat{U} = 0.2\,U_{\rm c}$; $U_{\rm s} \sim U_{\rm c}$; $\hat{U} = U_{\rm s} - 0.5\,U_{\rm c}$ for $U_{\rm s} > 1.2\,U_{\rm c}$. The absolute value of $R_{\rm acc}$ has the largest value for $U_{\rm s} \sim U_{\rm c}$, has a minimum for $U_{\rm s} \sim 2\,U_{\rm c}$, and then increases with increasing $U_{\rm s}$.

A superlattice without feedback of radiation has, for $U_{\rm s} > U_{\rm c}$, a small-signal negative differential resistance. A high frequency voltage $U = \hat{U}\cos\omega t$ of small amplitude $\hat{U}$ causes the high frequency polarization $P = -(el/2)n$. This leads, with ${\rm d}n/{\rm d}t = ({\rm d}n/{\rm d}U_{\rm s}){\rm d}U_{\rm s}/{\rm d}t$, to the current amplitude

$$\hat{I} = \pi r^2 (el/2)\omega \hat{U}\,{\rm d}n/{\rm d}U_{\rm s}, \tag{31.28}$$

where

$$\frac{{\rm d}n}{{\rm d}U_{\rm s}} = \frac{N_0}{U_{\rm c}}\,\frac{U_{\rm s}^2/U_{\rm c}^2 - 1}{2U_{\rm s}^2/U_{\rm c}^2}. \tag{31.29}$$

It follows that the small-signal differential resistance for the superlattice operating in a pure charge accumulation mode, $R_{\rm acc,0} = -\hat{U}/\hat{I}$, is given by

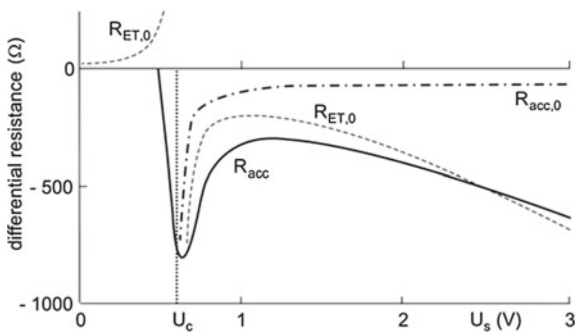

Fig. 31.10 Differential resistances of a superlattice; $R_{\rm acc,0}$, small-signal negative differential resistance of a superlattice operating in the pure charge accumulation mode; $R_{\rm acc}$, large-signal negative differential resistance of a superlattice in an oscillator operating in the pure charge accumulation mode; $R_{\rm ET,0}$, Esaki–Tsu small-signal differential resistance

$$R_{\text{acc},0} = -\frac{2\text{v}_{\text{ET,m}}}{\pi\nu\, l}\,\frac{2U_\text{s}^2/U_\text{c}^2}{U_\text{s}^2/U_\text{c}^2-1}\,R_{\text{ohm}}. \tag{31.30}$$

$R_{\text{acc},0}$ (Fig. 31.10, dashed dotted) is equal to $-\infty$ for $U_\text{s} = U_\text{c}$ and assumes the constant value $-(2\text{v}_{\text{ET,m}}/\pi\nu l)(U_\text{s}^2/U_\text{c}^2)R_{\text{ohm}}$ for $U_\text{s}^2 \gg U_\text{c}^2$.

Thermal radiation in a resonator is amplified according to the small-signal negative resistance $R_{\text{acc},0}$. Due to fluctuations of amplified thermal radiation, a superlattice can be promoted into a state of larger negative resistance. If this resistance reaches R_{acc}, stable oscillation can occur. The resistance R_{acc} corresponds to the threshold resistance R_{th} since, for $R < |R_{\text{th}}|$, feedback is strong enough to start oscillation. Then at steady state oscillation, the superlattice resistance $|R_{\text{acc}}|$ assumes the value R. Because of fluctuations of the field, i.e., because of noise, oscillation can occur also for $U_\text{s} < U_\text{c}$.

It follows from (31.19) that the small-signal Esaki-Tsu differential resistance, $R_{\text{ET},0} = 1/(\text{d}I/\text{d}U_\text{s})$ is equal to

$$R_{\text{ET},0} = \frac{(1+U_\text{s}^2/U_\text{c}^2)^2}{1-U_\text{s}^2/U_\text{c}^2}\,R_{\text{ohm}}. \tag{31.31}$$

$R_{\text{ET},0}$ (Fig. 31.10, dashed) is equal to the ohmic resistance R_{ohm} near $U_\text{s} = 0$, then increases and becomes infinitely large for $U \to U_\text{c}$. $R_{\text{ET},0}$ is negative for $U_\text{s} \geq U_\text{c}$, varies from $-\infty$ at $U_\text{s} = U_\text{c}$ to a value of $-8R_{\text{ohm}}$ for $U_\text{s} \sim 2U_\text{c}$ and is equal to $-(U_\text{s}^2/U_\text{c}^2)R_{\text{ohm}}$ for $U_\text{s}^2/U_\text{c}^2 \gg 1$.

In the voltage range of oscillation, the small-signal Esaki–Tsu resistance $R_{\text{ET},0}$ is comparable with the large-signal resistance R_{acc}. However, the absolute value of the large-signal Esaki–Tsu negative resistance R_{ET} is smaller than the absolute value of $R_{\text{ET},0}$ according to the slope of the Esaki–Tsu I–V curve. Therefore, $|R_{\text{acc}}|$ is larger than $|R_{\text{ET}}|$. This means that the interaction of the high frequency field with an electron collective of a pure charge accumulation mode is associated, with respect to the negative resistance, with a larger nonlinearity than the interaction of the high frequency field with single electrons in the case that the field in the superlattice is homogeneous.

31.6 Balance of Energy in a Superlattice Oscillator

The electric field associated with a domain has a triangular shape, with a low-field value E_1 at the anode and a high-field value E_2 at the cathode; E_1 ($<E_\text{c}$) is also the field immediately after domain destruction. From the Poisson equation

$$\nabla \cdot \boldsymbol{D}\rho, \tag{31.32}$$

we obtain the relation $ne/2 = \epsilon\epsilon_0(E_2 - E_1)/l$. A fully developed dipole domain carries the field energy $\pi r^2 l\epsilon\epsilon_0(E_2 - E_1)^2/2$. The field energy of a domain stems from the high frequency field in the resonator. During a half cycle of the field, the domain transfers its field energy to the high frequency field and during the following half cycle, energy of the high frequency field is used to build up the field of the domain.

Energy balance requires that the power delivered by the voltage source is equal to the sum of the losses (we modify a discussion in [236]):

$$\pi r^2 l \nu N_0 e U_s = \pi r^2 l \nu N_0 e E_1 l + P_{out} + P_{dom}. \quad (31.33)$$

The loss terms concern:

- $\pi r^2 l \nu N_0 e E_1 l$ = loss due to the current carried by the electrons (of density N_0).
- P_{out} = loss due to output coupling of radiation.
- P_{dom} = loss due to dissipation caused by relaxation processes during domain formation and destruction.

We find

$$U_s = E_1 l + U_{rad} + U_{dom}, \quad (31.34)$$

where $U_{rad} = P_{out}/(\pi r^2 l \nu N_0 e)$ and $U_{dom} = P_{dom}/(\pi r^2 l \nu N_0 e)$. The static voltage across the superlattice is equal to the sum of three terms: the voltage necessary to drive the normal electrons by the field E_1; the voltage U_{rad} necessary to compensate loss of radiation and the voltage U_{dom} necessary for compensation of energy of dissipation associated with domains. The normal electrons drift with the average velocity $\mathrm{v}(E_1)$ through the superlattice. The domains, with the positive charges bound to the cathode, appear and disappear at the repetition rate ν.

Example (*for the superlattice already discussed*) For $U_s = 2\ U_c$ (=1.2 V) and $P_{rad} = P_{out}$ at optimum output, the analysis yields the data: $E_1 = 0.7\ E_c$; $E_2 = 4\ E_c$; $E_1 l = 0.5$ V; $U_{rad} = 0.25$ V; $U_{dom} = 0.5$ V. Accordingly, the dissipation energy is, for $U_s = 2\ U_c$, equal to half the field energy of a fully developed domain. The direct current strength is determined by the drift velocity at the lower field and is given by the expression $I_{dc}/I_p = N_0 e \mathrm{v}(E_1)/I_p$ (~0.8).

The upper limit frequency ν_{limit} is determined by the intraminiband relaxation of the electrons in a superlattice. It follows from the intraminiband relaxation time (1.5 × 10^{-13} s) that ν_{limit} is ~1 THz. The appropriate superlattice length, according to the relation $\nu_c = 0.4\ \nu_{ET,m}/l$ has a value of ~10 nm. This means that we are no longer dealing with a superlattice but with a resonant-tunneling diode like structure (next section).

A more detailed discussion of the pure charge accumulation mode observed for superlattice oscillators can be found in [246]. The study presents a method that is suited to investigate the mechanism of gain of a solid state oscillator. Such studies may contribute to an improvement of the efficiency of microwave oscillators, particularly in the range above 100 GHz.

31.7 Resonant-Tunneling Diode Oscillator

The resonant-tunnel diode oscillator [234, 247] is a quasiclassical solid state oscillator that reaches very high oscillation frequencies. Because of small radiation power, resonant-tunneling diode oscillators are not in use.

A resonant-tunneling diode (Fig. 31.11a) consists, for instance, of two AlAs layers separated by a GaAs layer, embedded in n GaAs. The layers form a quantum well with a discrete energy level for electron motion perpendicular to the layers. Under the action of a static voltage U_s, electrons tunnel through the quantum well from one n GaAs region to the other n GaAs region, which results in a current. If the energy of the tunneling electrons coincides with the energy of the discrete energy level (Fig. 31.11b), the tunnel current has a maximum as indicated in the $I-V$ curve (Fig. 31.11c). The $I-V$ curve has, for a voltage above a critical voltage U_c, a negative slope, which corresponds to a negative differential resistance. The $I-V$ curve is a hypothetical curve: because of the negative differential resistance, the charge distribution is inhomogeneous.

The negative differential resistance gives rise to a self-excited oscillation if the active element is coupled to a resonance circuit; the oscillation frequency is determined by the resonator. Radiation generated in first order has been observed in frequency ranges from 10 GHz up to several hundred GHz. The power decreased strongly at frequencies above 100 GHz. The highest frequency of radiation emitted

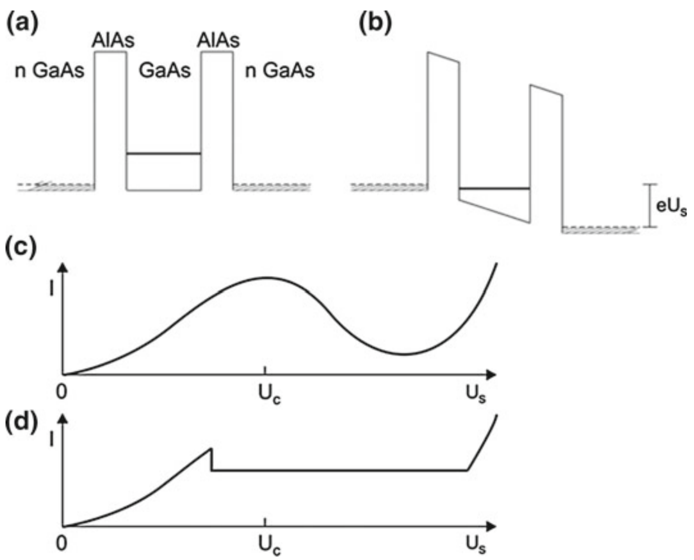

Fig. 31.11 Resonant tunnel diode. **a** Quantum well. **b** Voltage-biased quantum well. **c** Hypothetical $I-V$ curve. **d** $I-V$ curve in the case of occurrence of feedback from a high frequency field

by a GaAs/AlAs resonant-tunneling diode oscillator was near 400 GHz [247] and near 700 GHz for an InAs/AlSb resonant-tunneling diode oscillator [248].

Example InAs/AlAs resonant-tunneling diode [249]. Double barrier structure with 1.5 nm thick undoped barriers separated by a 6.4 nm thick undoped InAs quantum well; diameter 1.8 μm; current 5 mA; voltage 1.3 V; $R_{opt} \sim -50\ \Omega$; power 0.3 μW at 712 GHz.

The resonant-tunneling diode oscillators operated most likely in the pure charge accumulation mode.

31.8 Van der Pol Oscillator

We discuss a model of a classical electric oscillator, namely the van der Pol oscillator. The active device is a resistance that shows a negative differential resistance above a critical voltage U_c (Fig. 31.12a); the $I-V$ curve resembles the hypothetical $I-V$ curve of the resonant-tunneling diode. Under the action of a static voltage (bias voltage U_0), with the resistance coupled to a resonant circuit, a self-excited oscillation can occur. It is characteristic of this classical oscillator model that the current through the active device and the voltage across the device always follow the $I-V$ curve and that the curve does not change during buildup of an oscillation.

To study basic properties of a classical oscillator, we introduce an $I-V$ curve (Fig. 31.12b) that has, around the range of negative shape, a similar slope as the hypothetical $I–V$ curve of the resonant-tunneling diode and can be described by an analytical expression,

$$I(U) = I_0 - a\,(U - U_0) + b\,(U - U_0)^3 = f(U), \qquad (31.35)$$

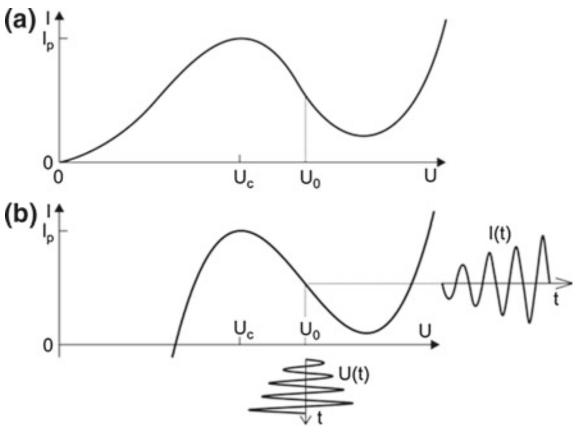

Fig. 31.12 Classical oscillator. **a** Hypothetical $I–V$ curve of a tunnel diode. **b** $I–V$ curve of a van der Pol oscillator

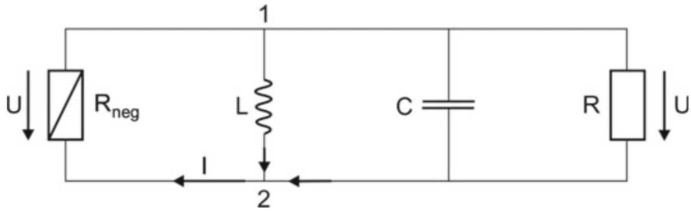

Fig. 31.13 Equivalent circuit of a negative resistance oscillator of the van der Pol type

where $a > 0$ and $b > 0$ are constants. The $I-V$ curve has the largest negative slope for $U = U_0$. The slope of the $I-V$ curve according to (31.35) is unrealistic for voltages $U \ll U_0$ and $U \gg U_0$. In the range around U_0, it shows for appropriate parameters a, b, U_0, and I_0, the hypothetical characteristic of a tunnel diode.

We consider a parallel equivalent circuit (Fig. 31.13) containing a negative resistance R_{neg}, an inductance L, a capacitance C, and a loss resistance R, which describes loss due to emission of radiation. The high frequency currents through the capacitance (I_c), through the inductance (I_L), and through the resistance (I_R) are related to the high frequency voltage U_{HF} between the points 1 and 2,

$$I_C = C\,\frac{dU_{\text{HF}}}{dt}; \quad I_L = \frac{1}{L}\int U_{\text{HF}}dt; \quad I_R = \frac{U_{\text{HF}}}{R}. \tag{31.36}$$

The sum of the total current in point 1 of the circuit must be zero,

$$I_C + I_L + I_R + I_d = 0. \tag{31.37}$$

I_d is the current through the nonlinear device. The signs follow from Kirchhoff's rules for voltages and currents in a circuit taking into account that the instantaneous voltage $U_{\text{HF}}(t)$ across the active element has a sign that is opposite to the sign of the high frequency current flowing through the resistance. The sum of all currents through a knot is zero and the sum of all voltages in a loop is zero according to Kirchhoff's rules. By differentiation, we obtain

$$CL\frac{d^2U_{\text{HF}}}{dt^2} + U_{\text{HF}} + \frac{L}{R}\,\frac{dU_{\text{HF}}}{dt} = -L\,\frac{dI_d}{dt}. \tag{31.38}$$

The current, i.e., the derivative of the current with respect to time, is the source of the high frequency voltage.

We can write

$$\frac{dI_d}{dt} = \frac{dI_d}{dU}\,\frac{dU}{dt} = \frac{df}{dU}\,\frac{dU}{dt}. \tag{31.39}$$

Then (31.36) assumes the form

$$CL\frac{\mathrm{d}^2U}{\mathrm{d}t^2} + L\left(\frac{1}{R} + f'(U_0 + U)\right)\frac{\mathrm{d}U}{\mathrm{d}t} + U = 0. \tag{31.40}$$

We omitted the subscript HF. We find, with $\omega_0^2 = 1/LC$, the differential equation

$$\frac{\mathrm{d}^2U}{\mathrm{d}t^2} + (-\gamma + \kappa)\frac{\mathrm{d}U}{\mathrm{d}t} + \omega_0^2 U = 0, \tag{31.41}$$

where

$$\gamma = \gamma(U) = -C^{-1}\left(\frac{\partial I_\mathrm{d}}{\partial U}\right)_U \tag{31.42}$$

is the growth coefficient and

$$\kappa = 1/RC \tag{31.43}$$

the damping coefficient. In the active element of a classical oscillator, the growth coefficient γ depends on the instantaneous voltage $U(t)$ at time t.

Using the analytical form (31.20) of the I–V curve, we can write

$$I_\mathrm{HF} = -aU_\mathrm{HF} + bU_\mathrm{HF}^3. \tag{31.44}$$

It follows that

$$\left(\frac{\partial I}{\partial U}\right)_U = -a + 3bU^2; \tag{31.45}$$

we again omit the subscript HF. We find the growth coefficient

$$\gamma = -a/C + 3b/C\ U^2 \tag{31.46}$$

and obtain the differential equation (*van der Pol equation*) for the high frequency voltage

$$\frac{\mathrm{d}^2U}{\mathrm{d}t^2} + \left(-\gamma_0 + \kappa + \frac{3b}{C}U^2\right)\frac{\mathrm{d}U}{\mathrm{d}t} + \omega_0^2 U = 0. \tag{31.47}$$

The differential equation describes a self-excited oscillator with the small-signal growth coefficient

$$\gamma_0 = \frac{a}{C} \tag{31.48}$$

and two damping terms. The first damping term, κ, characterizes output coupling of electromagnetic radiation and the second term intrinsic loss in the active element. This loss is zero for $U = 0$ and increases proportionally to the square of U. The van der Pol equation describes an oscillation that is strongly nonlinear, except in the case that the net gain is small,

$$\gamma_0 - \kappa \ll \omega_0. \tag{31.49}$$

In this case the van der Pol equation has a solution that corresponds to a nearly harmonic oscillation. With the ansatz

$$U = A(t) \cos \omega_0 t, \tag{31.50}$$

where $A(t)$ is a slowly varying function, $\gamma_0 - \kappa \ll \omega_0$, we obtain

$$\frac{\mathrm{d}U}{\mathrm{d}t} = \frac{\mathrm{d}A}{\mathrm{d}t} \cos \omega_0 t - \omega_0 A \sin \omega_0 t, \tag{31.51}$$

$$\frac{\mathrm{d}^2 U}{\mathrm{d}t^2} = -2\omega_0 \frac{\mathrm{d}A}{\mathrm{d}t} \sin \omega_0 t - \omega_0^2 A \cos \omega_0 t. \tag{31.52}$$

The differential equation leads, with $(\gamma_0 - \kappa) \left| \frac{\mathrm{d}A}{\mathrm{d}t} \right| \ll \omega_0 \left| \frac{\mathrm{d}A}{\mathrm{d}t} \right|$ (SVEA), to

$$-2\omega_0 \frac{\mathrm{d}A}{\mathrm{d}t} \sin \omega_0 T - (-\gamma_0 + \kappa)\omega_0 A \sin \omega_0 t - \frac{3b\omega_0}{C} A^3 \cos^2 \omega_0 t \sin \omega_0 t = 0. \tag{31.53}$$

Using the relation

$$\cos^2 \alpha \sin \alpha = \frac{1}{2}(1 + \cos 2\alpha) \sin \alpha = -\frac{1}{4} \sin \alpha + \frac{1}{4} \sin 3\alpha \tag{31.54}$$

and neglecting the higher order term $\sin 3\alpha$, we find

$$\frac{\mathrm{d}A}{\mathrm{d}t} + \frac{1}{2}(-\gamma_0 + \kappa)A + \frac{3b}{8C} A^3 = 0. \tag{31.55}$$

This differential equation has exactly the same form as the differential equation (9.144) derived for the amplitude of the field in a laser oscillator. The solution is

$$A(t) = \frac{A_\infty}{\sqrt{1 + (A_\infty / A_0)^2 \, \mathrm{e}^{-(\gamma_0 - \kappa)t}}}. \tag{31.56}$$

$A_0 = A(t = 0)$ is the initial amplitude of the voltage and

$$A_\infty = 2 \sqrt{(\gamma_0 - \kappa) C / 3b} \tag{31.57}$$

is the amplitude of the high frequency voltage at steady state oscillation. After a sudden turning on of the active element, a small high frequency voltage initiates the buildup of an oscillation. The initial high frequency voltage stems from noise in the resonance circuit.

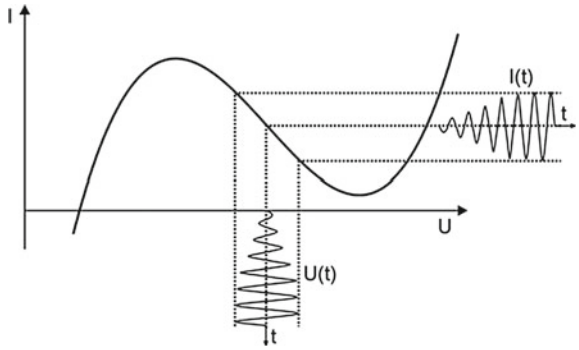

Fig. 31.14 The van der Pol oscillator at a small net gain

The van der Pol oscillator of small net gain $\gamma_0 - \kappa$ is driven in a range of the voltage amplitude that corresponds to the range of almost constant negative slope of the I–V curve (Fig. 31.14). At small amplitude of the voltage, the intrinsic damping is negligibly small. At large amplitude and steady state oscillation, the intrinsic damping becomes efficient during each cycle at instantaneous voltages in the ranges $U \approx \pm A$. This leads, as our analysis shows, to the same form of the first-order differential equation for the amplitude of the voltage in the classical oscillator as we found for the amplitude of the field in a laser oscillator, although the nonlinearities have completely different origins.

The van der Pol oscillator represents a model oscillator of a negative-resistance oscillator that is discussed in many textbooks; *see*, for instance, [250].

References [240–243].

Problems

31.1 Equivalent circuit.

(a) Replace the equivalent circuit of Fig. 31.5 by a parallel resonant circuit; the active device has the negative admittance G_d and the loss resistor the admittance G.
(b) Derive the differential equation for the high frequency voltage.
(c) Discuss the dependence of the negative admittance of the device on the voltage across the device.
(d) Show that the output power of the oscillator is $P_{out} = (1/2)G\hat{U}^2$, where $G + G_{neg} = 0$ is the condition of steady state oscillation.

31.2 Electric polarization.

(a) Determine the electric polarization of a dipole domain consisting of a positive area charge $\rho l e$ at $x = 0$ and a negative charge of density ρ in the range $0, l$.

(b) Determine the polarization of a dipole domain consisting of a negative charge of area density $\rho l e$ at $x = x_0$ and a positive charge of density ρ in the range $x_0, x_0 + l$.

31.3 Van der Pol oscillator.

(a) Evaluate for a van der Pol oscillator (with the data: $a = 10^{-2}\ \Omega^{-1}$; $b = 10^{-2}\ \Omega^{-1}\ \mathrm{V}^{-2}$; $\omega_0 = 2\pi \times 10^{10}\,\mathrm{Hz}$; $C = 1$ pF; and $G = 1\Omega^{-1}$) the small-signal net growth coefficient and show that it is small compared to ω_0.
(b) Determine the voltage amplitude for the steady state oscillation.
(c) Determine the current amplitude for the steady state oscillation. [*Hin*t: make use the relation $\cos^3\alpha = \frac{3}{4}\cos\alpha + \frac{1}{3}\cos 3\alpha$ and neglect the term with 3α.]

31.4 Van der Pol equation.

(a) Show that the van der Pol equation can be written in dimensionless units,

$$\frac{\mathrm{d}^2 y}{\mathrm{d}\tau^2} + \epsilon(-1 + y^2)\frac{\mathrm{d}y}{\mathrm{d}\tau} + y = 0,$$

where y is the voltage in dimensionless units, $\tau = \omega_0 t$ the dimensionless time and ϵ the small-signal net gain coefficient in dimensionless units.
(b) Solve the van der Pol equation for $\epsilon \ll 1$ at steady state oscillation. [*Hint*: make use of the relation $\cos^3\tau = \frac{3}{4}\cos\tau + \frac{1}{3}\cos 3\tau$ and neglect the term with $\cos 3\tau$.]

31.5 Which of the following differential equations describe a self-sustained oscillation?

(a) $\dfrac{\mathrm{d}^2 y}{\mathrm{d}t^2} + \dfrac{\mathrm{d}y}{\mathrm{d}t} + y = 0.$

(b) $\dfrac{\mathrm{d}^2 y}{\mathrm{d}t^2} - \dfrac{\mathrm{d}y}{\mathrm{d}t} + y = 0.$

(c) $\dfrac{\mathrm{d}^2 y}{\mathrm{d}\tau^2} + \varepsilon(-1 + y^2)\dfrac{\mathrm{d}y}{\mathrm{d}\tau} + y = 0.$

31.6 Compare a classical oscillator and a laser oscillator. (A classical oscillator has a stable I–V characteristic, Fig. 31.12, while a laser oscillator has a current-density-field characteristic that varies during onset of oscillation, Fig. 9.7.)

31.7 Show that the impedance $Z(\omega)$ of a resonance electrical circuit has Lorentzian lineshape.