Chapter 8
A Laser Theory

In this chapter, we present simple laser equations describing the dynamics of laser oscillation. The equations are coupled rate equations relating the populations of the laser levels and the photon density.

The laser equations provide the threshold condition, the pump threshold, and the threshold population difference. The solutions to the equations indicate that, at steady state oscillation, clamping of the population difference occurs. Pumping with a pump power exceeding the pump threshold results in generation of laser radiation. The analysis of the laser equations allows us, furthermore, to determine the oscillation onset time and to calculate the optimum output coupling efficiency of a laser.

During the onset of laser oscillation, the interplay of the active medium with the field in a laser resonator can lead to oscillations (relaxation oscillations) of both the density of photon in the resonator and the population difference. We derive a criterion of the occurrence of relaxation oscillations. The relaxation oscillations have frequencies in the GHz range.

We perform an estimate of the laser linewidth. It is finite because of the influence of noise on laser oscillation. Amplification of radiation, which is either due to spontaneous emission by the active medium or due to thermal radiation in the laser resonator is the origin of the finite linewidth of laser radiation.

In the next chapter, we will extend the theory taking into account that laser oscillation is joined with a high frequency polarization of the active medium.

8.1 Rate Equations

To describe dynamical processes occurring in a laser, we make use of a rate equation theory; the rate equations correspond to differential equations of first order. We treat the four-level laser. The theory applies, without modification, also to the three-level laser (with the pump level coinciding with the upper laser level).

In the center of the four-level laser (Fig. 8.1) is a two-level atomic system with the upper laser level 2 and the lower laser level 1. An ensemble of two-level atomic

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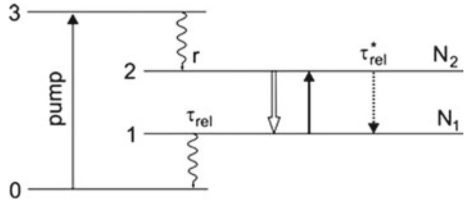

Fig. 8.1 Four-level laser

systems interacts with the laser radiation by stimulated emission and absorption of radiation. The upper laser level can relax (relaxation time $\tau^*_{\rm rel}$) by transitions to the lower laser level. Pumping into the pump level (level 3) and fast relaxation leads to a population of the upper laser level. The population rate r is a measure of the pump strength. The lower laser level is depopulated by relaxation (relaxation time $\tau_{\rm rel}$). We assume that further processes, like the relaxation $3 \rightarrow 1$ or $2 \rightarrow 0$, are negligibly week.

We describe the dynamics of the four-level laser by the *laser rate equations:*

$$\frac{\mathrm{d}N_2}{\mathrm{d}t} = r - \frac{N_2}{\tau^*_{\rm rel}} - b_{21} Z (N_2 - N_1), \tag{8.1}$$

$$\frac{\mathrm{d}N_1}{\mathrm{d}t} = -\frac{N_1}{\tau_{\rm rel}} + \frac{N_2}{\tau^*_{\rm rel}} + b_{21} Z (N_2 - N_1), \tag{8.2}$$

$$\frac{\mathrm{d}Z}{\mathrm{d}t} = b_{21} Z (N_2 - N_1) - \frac{Z}{\tau_{\rm p}}. \tag{8.3}$$

These laser equations take into account the following processes:

- The upper laser level is populated by pumping with the pump rate r. It is depopulated by relaxation with the relaxation rate $N_2/\tau^*_{\rm rel}$ and by the net effect of stimulated emission and absorption with the rate $b_{21} Z (N_2 - N_1)$.
- The lower laser level is depopulated by relaxation to the ground state with the rate $N_1/\tau_{\rm rel}$. It is populated by the relaxation of the upper laser level and the net effect of stimulated emission and absorption.
- The photon density increases according to the net effect of stimulated $2 \rightarrow 1$ transitions and absorption processes and decreases due to loss of photons in the resonator.

The three equations are nonlinear differential equations relating N_1, N_2, and Z. We list the quantities used for description of the four-level laser:

- N_2 = population of the upper laser level = density of two-level atomic systems in the upper laser level = number density (=number per m^3) of two-level atomic systems in the upper laser level.
- N_1 = population of the lower laser level = density of two-level atomic systems in the lower laser level.
- $N_2 - N_1$ = population difference.

- $N_1 + N_2$ = density of two-level atomic systems.
- τ^*_{rel} = lifetime of the upper laser level with respect to $2 \to 1$ relaxation.
- τ_{rel} = lifetime of the lower laser level with respect to $1 \to 0$ relaxation.
- r = pump rate (per unit volume) = number of two-level atomic systems in the upper laser level that are excited per m^3 and s.
- $E_{21} = E_2 - E_1$ = energy difference of the laser levels = transition energy.
- ν = frequency of the laser radiation.

Because of line broadening effects, the quantum energy $h\nu$ of a laser photon is not necessarily equal to the transition energy E_{21}. We are describing a laser that oscillates on *one* mode. We characterize the light in the laser resonator by the quantities:

- Z = photon density (=number of photons per m^3).
- τ_p = photon lifetime = average lifetime of a photon in the resonator.
- $\kappa = \kappa_i + \kappa_{out}$ $(=1/\tau_p)$ = photon loss coefficient of the resonator.
- κ_i = internal loss coefficient describing loss of photons within the resonator.
- κ_{out} = loss coefficient describing loss of photons by output coupling of radiation.
- $b_{21}(\nu) = h\nu B_{21} g(\nu)$ = growth rate constant.
- $\sigma_{21} = nb_{21}/c$ = gain cross section.
- n = refractive index of the active medium at the laser frequency.
- c = speed of light in vacuum.

8.2 Steady State Oscillation of a Laser

At steady state oscillation, the populations and the photon density are independent of time,

$$dN_2/dt = 0; \quad dN_1/dt = 0; \quad dZ/dt = 0. \tag{8.4}$$

We obtain the three laser equations

$$r - N_2/\tau^*_{rel} - b_{21} Z(N_2 - N_1) = 0, \tag{8.5}$$

$$-N_1/\tau_{rel} + N_2/\tau^*_{rel} + b_{21} Z(N_2 - N_1) = 0, \tag{8.6}$$

$$b_{21} Z(N_2 - N_1) - Z/\tau_p = 0. \tag{8.7}$$

(In the case that the laser levels are degenerate, we have to replace $N_2 - N_1$ by $N_2 - N_1 g_2/g_1$. Population inversion then corresponds to the condition $N_2 > N_1 g_2/g_1$; g_1 = degree of degeneracy of level 1 and g_2 = degree of degeneracy of level 2. In the following, we treat an ensemble of two-level atomic systems, $g_1 = g_2 = 1$.)

At steady state oscillation, the photon density is unequal to zero ($Z \neq 0$) and we can eliminate Z from the first two equations and find

$$N_{1,\infty}/\tau_{rel} = r. \tag{8.8}$$

The relaxation rate of the lower laser level is equal to the pump rate. This result is obvious (*see* Fig. 8.1): at steady state, the pumping compensates the loss of two-level atomic systems.

Equation (8.7) yields the threshold condition:

$$(N_2 - N_1)_{\text{th}} = (N_2 - N_1)_\infty = \frac{1}{b_{21}\tau_{\text{p}}} = \frac{\kappa_{\text{i}} + \kappa_{\text{out}}}{h\nu B_{21} g(\nu)}. \quad (8.9)$$

$(N_2 - N_1)_{\text{th}}$ is the *threshold population difference*. The population difference at steady state oscillation is equal to the threshold population difference,

$$(N_2 - N_1)_\infty = (N_2 - N_1)_{\text{th}}. \quad (8.10)$$

The population difference is independent of the pump rate and is "clamped" to the threshold population difference $(N_2 - N_1)_{\text{th}}$. The population difference is equal to the reciprocal of the product of the growth rate constant and the lifetime of a photon in the resonator. The threshold decreases with increasing growth rate constant and with increasing photon lifetime. The threshold condition is also discussed in the next section.

We find, with

$$(N_2 - N_1)_\infty = (N_{2,\infty} - N_{1,\infty}), \quad (8.11)$$

that

$$N_{2,\infty} = N_{1.\infty} + \frac{1}{b_{21}\tau_{\text{p}}}. \quad (8.12)$$

Both $N_{2,\infty}$ and $N_{1,\infty}$ increase linearly with the pump rate while the difference experiences clamping.

It follows from (8.5) and (8.6) that the photon density is given by

$$Z_\infty = r\left(1 - \frac{\tau_{\text{rel}}}{\tau_{\text{rel}}^*}\right)\tau_{\text{p}} - \frac{(N_2 - N_1)_\infty \tau_{\text{p}}}{\tau_{\text{rel}}^*}. \quad (8.13)$$

The photon density at steady state oscillation increases linearly with the pump rate (Fig. 8.2). The *threshold pump rate* r_{th} follows from the last equation, for $Z_\infty = 0$,

$$r_{\text{th}} = \frac{(N_2 - N_1)_\infty}{\tau_{\text{rel}}^*(1 - \tau_{\text{rel}}/\tau_{\text{rel}}^*)}. \quad (8.14)$$

The threshold pump rate (=pump rate at laser threshold) compensates the loss of N_2 population that is due to $2 \rightarrow 1$ relaxation processes.

Equation (8.13) shows that the photon density Z_∞ becomes infinitely large if $\tau_{\text{p}} = \infty$, i.e., if the lifetime of the photons is not limited by loss of photons by escape from the resonator or by loss within the resonator. A laser without any loss, $\kappa_{\text{i}} = \kappa_{\text{out}} = 0$, would contain an infinitely large number of photons.

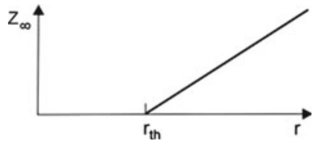

Fig. 8.2 Photon density at steady state oscillation

We now consider the case that the relaxation time of the lower level is small compared with the relaxation time of the upper laser level, $\tau_{\rm rel} \ll \tau^*_{\rm rel}$. Then the threshold pump rate is equal to

$$r_{\rm th} = \frac{(N_2 - N_1)_\infty}{\tau^*_{\rm rel}} = \frac{1}{b_{21}\tau^*_{\rm rel}\tau_{\rm p}} = \frac{\kappa_{\rm i} + \kappa_{\rm out}}{b_{21}\tau^*_{\rm rel}} \tag{8.15}$$

and the photon density at steady state oscillation is

$$Z_\infty = (r - r_{\rm th})\tau_{\rm p} = \left(\frac{r}{r_{\rm th}} - 1\right)\frac{1}{b_{21}\tau^*_{\rm rel}}. \tag{8.16}$$

Without output coupling loss ($\kappa_{\rm out} = 0$) but with internal loss, the threshold pump rate is equal to

$$r_{\rm th,i} = \frac{\kappa_{\rm i}}{b_{21}\tau^*_{\rm rel}} \tag{8.17}$$

and the photon density is

$$Z_{\infty,{\rm i}} = \frac{r}{\kappa_{\rm i}} - \frac{1}{b_{21}\tau^*_{\rm rel}}. \tag{8.18}$$

8.3 Balance Between Production and Loss of Photons

We will express the threshold condition in different ways. All formulations are equivalent.

The condition of steady state oscillation is the following: *the rate of photon production is equal to the rate of photon loss*:

$$b_{21}(N_2 - N_1)_\infty Z_\infty = \frac{Z_\infty}{\tau_{\rm p}}. \tag{8.19}$$

Dividing by Z_∞, we obtain

$$\frac{1}{b_{21}(N_2 - N_1)_\infty} = \tau_{\rm p}. \tag{8.20}$$

On the left side, we have the time it takes, in the time average, to produce one photon and on the right side, we have the average lifetime of a photon in the laser resonator. We can interpret the steady state: *during its lifetime in the resonator, a photon reproduces itself by a stimulated emission process exactly once.*

Alternatively, we can write

$$(N_2 - N_1)_\infty = \frac{1}{c\tau_p \sigma_{21}} = \frac{1}{l_p \sigma_{21}}. \quad (8.21)$$

The threshold population difference is inversely proportional to the product of the path length l_p of a photon in the resonator and of the gain cross section σ_{21}. We can also write

$$(N_2 - N_1)_\infty \sigma_{21} l_p = 1. \quad (8.22)$$

This means: On its multiple path through the resonator, a photon induces exactly one photon by a stimulated emission process. Or, on its multiple path through the resonator, a photon reproduces itself before it leaves the resonator. Finally, we write

$$(N_2 - N_1)_\infty = \frac{1}{\sigma_{21} l_p}. \quad (8.23)$$

By replacing the photon path length $l_p = 2nL/(-\ln V)$, we obtain the threshold population difference

$$(N_2 - N_1)_\infty = \frac{-\ln V}{2nL\sigma_{21}}. \quad (8.24)$$

The threshold population difference tends to zero if the V factor approaches unity.

8.4 Onset of Laser Oscillation

We assume that we suddenly, at time $t = 0$, turn on a population difference $(N_2 - N_1)_0$. The temporal change of the photon density Z in the laser resonator is, for small Z, given by the equation

$$\frac{1}{Z}\frac{dZ}{dt} = b_{21}(N_2 - N_1)_0 - \frac{1}{\tau_p}. \quad (8.25)$$

The solution is

$$Z(t) = Z_0\, e^{(\gamma_0 - \kappa)t}. \quad (8.26)$$

Z_0 is the photon density at $t = 0$,

$$\gamma_0 = b_{21}(N_2 - N_1)_0 \tag{8.27}$$

is the small-signal growth coefficient, and $\kappa = 1/\tau_p$ is the decay coefficient of the resonator with respect to the decay of a photon.

To estimate the onset time t_{on}, we now assume that the population difference remains constant during the buildup of laser oscillation and changes suddenly to the steady state value $(N_2 - N_1)_\infty$. Under this assumption, the density Z of photons in the resonator increases exponentially until it reaches the steady value Z_∞. Accordingly, we find

$$Z_\infty = Z_0\, \mathrm{e}^{(\gamma_0 - \kappa)t_{on}}. \tag{8.28}$$

It follows that the oscillation onset time is equal to

$$t_{on} = \frac{\ln(Z_\infty/Z_0)}{\gamma_0 - \kappa}. \tag{8.29}$$

This is the same result as derived earlier (in Sect. 2.9) since the gain factor is $G_0 = \mathrm{e}^{\gamma_0 T}$ and the V factor is $V = \mathrm{e}^{-\kappa T}$, where T is the round trip transit time; *see* (2.85).

According to our description of the buildup of laser oscillation, the photon density increases exponentially (Fig. 8.3) until it reaches, at the onset time t_{on}, the steady state value Z_∞. The population density decreases at the onset time t_{on} from $(N_2 - N_1)_0$ to

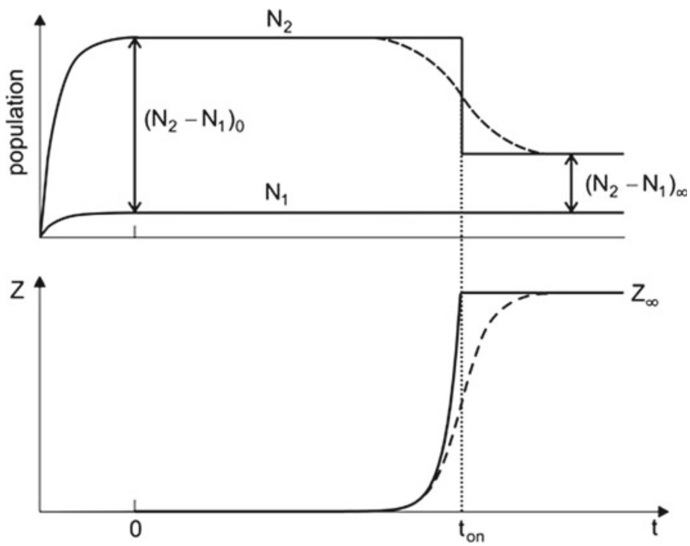

Fig. 8.3 Onset of laser oscillation

$(N_2 - N_1)_\infty$. We will later (in Sect. 9.7) show that the population difference $N_2 - N_1$ and the photon density Z are smoothly going over into their steady state values (dashed curves in Fig. 8.3).

In our discussion of the onset of laser oscillation, we assume that the relaxation time of the upper laser level is much smaller than the onset time, $\tau^*_{\mathrm{rel}} \ll t_{\mathrm{on}}$. In this case, the population N_2 reaches a constant value at $t = 0$, immediately after the start of the pumping, as indicated in Fig. 8.3. Together with the population N_2, the population N_1 reaches a constant value immediately after the start of pumping too.

The helium–neon laser belongs to the lasers that fulfill the condition of a fast relaxation in comparison with the oscillation onset time. The relaxation time τ^*_{rel} of many other laser media (for instance, of titanium–sapphire) is much larger than the oscillation onset time. Then the population and the photon density show dynamic effects, which we will discuss later (Sects. 8.8 and 9.8).

8.5 Clamping of Population Difference

The population difference at steady state oscillation is clamped to the threshold population difference. What does this mean with respect to the occupation number difference $f_2 - f_1$? It follows from the laser equations of the steady state that the density of two-level atomic systems increases with increasing pump rate according to

$$(N_1 + N_2)_\infty = \frac{1}{b_{21}\tau_{\mathrm{p}}} + 2r\tau_{\mathrm{rel}}. \tag{8.30}$$

Both $N_{1,\infty}$ and $N_{2,\infty}$ increase,

$$N_{1,\infty} = r\tau_{\mathrm{rel}}, \tag{8.31}$$

$$N_{2,\infty} = \frac{1}{b_{21}\tau_{\mathrm{p}}} + r\tau_{\mathrm{rel}}. \tag{8.32}$$

The occupation number difference is given by

$$(f_2 - f_1)_\infty = \frac{(N_2 - N_1)_\infty}{(N_2 + N_1)_\infty} = \frac{1}{(N_2 + N_1)_\infty} \times \frac{1}{b_{21}\tau_{\mathrm{p}}}. \tag{8.33}$$

The solid lines of Fig. 8.4 illustrate our result concerning a laser oscillating above threshold. With increasing pump rate, the density $N_2 + N_1$ of two-level atomic systems increases and the population difference $(N_2 - N_1)_\infty$ remains constant while the occupation number difference $f_2 - f_1$ decreases; below threshold, the populations N_2 and N_1 as well as $f_2 - f_1$ increase linearly with the pump rate (dashed lines). In a four-level laser, the occupation number difference $f_2 - f_1$ decreases with increasing pump rate. An increaasing pump rate corresponds to an increasing density of two-level systems in the active medium.

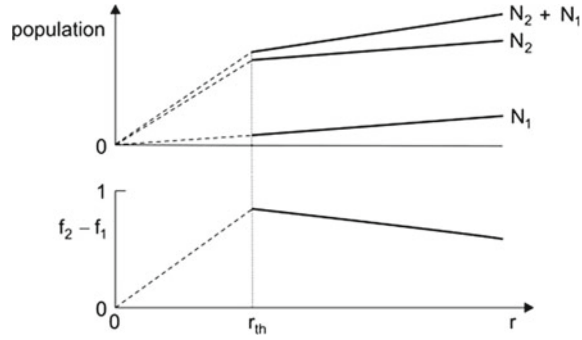

Fig. 8.4 Populations and occupation number difference of a four-level laser

If the lifetime of the lower laser level is very small, $\tau_{\rm rel} \ll \tau^*_{\rm rel}$, the two-level atomic systems are mainly in their excited states because the population of level 1 is small compared to the population of level 2 ($N_1 \ll N_2$). Then the population difference is nearly equal to the density of the two-level atomic systems, $N_2 - N_1 \approx N_2 + N_1$. Accordingly, the occupation number difference is near unity, $(f_2 - f_1)_\infty \sim 1$.

We will later find (Chaps. 21 and 22) that for a two-band laser, clamping occurs for the occupation number difference $f_2 - f_1$. The reason is that the density of twolevel systems in a two-band medium is constant and does not depend on the pump strength.

8.6 Optimum Output Coupling

How can we obtain optimum laser output? We have two limiting cases:

- If we choose an output coupling mirror of reflectivity $R = 1$, a strong laser field builds up. The laser, however, does not emit radiation.
- If we choose an output coupling mirror of a reflectivity allowing laser oscillation to occur just at threshold, then the laser field in the resonator is extremely weak. The output power of the laser is negligibly small too.

Optimum output corresponds to an intermediate case. We can choose the reflectivity of the output coupling mirror of a laser and thus the output coupling coefficient $\kappa_{\rm out}$ (Fig. 8.5a). We are now looking for the value of $\kappa_{\rm out}$ that leads to optimum output; we assume that $\tau_{\rm rel} \ll \tau^*_{\rm rel}$. We introduce the photon output coupling rate $r_{\rm out}$. At steady state, the output coupling rate (=number of photons coupled out from the resonator per m^3 and per s) is

$$r_{\rm out} = \kappa_{\rm out} Z_\infty = \frac{\kappa_{\rm out}}{\kappa_{\rm i} + \kappa_{\rm out}}(r - r_{\rm th}). \tag{8.34}$$

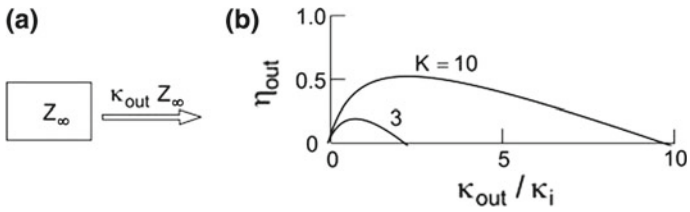

Fig. 8.5 Output coupling of radiation. **a** Output coupling coefficient. **b** Output coupling efficiency for two different pump rates

The output coupling rate is proportional to the difference of pump rate and threshold pump rate. We can write

$$r_{out} = Z_\infty \kappa_{out} = \frac{r\kappa_{out}}{\kappa_{out} + \kappa_i} - \frac{\kappa_{out}}{b_{21}\tau^*_{rel}}. \tag{8.35}$$

We define the output coupling efficiency by

$$\eta_{out} = r_{out}/r, \tag{8.36}$$

where r is the pump rate. A straightforward calculation yields

$$\eta_{out} = \frac{1}{K}\,\frac{(K-1)\kappa_{out}/\kappa_i - \kappa_{out}^2/\kappa_i^2}{1 + \kappa_{out}/\kappa_i}, \tag{8.37}$$

where the parameter

$$K = r/r_{th,i} \tag{8.38}$$

is a measure of the pump rate. By differentiating η_{out} with respect to κ_{out} and equating to zero, we find that *optimum output coupling* occurs if

$$(\kappa_{out}/\kappa_i)_{opt} = \sqrt{K} - 1. \tag{8.39}$$

The output coupling efficiency depends on the pump rate parameter K (Fig. 8.5b). At a fixed pump rate (e.g., corresponding to $K = 3$), the efficiency increases at weak output coupling ($\kappa_{out} < \kappa_i$) linearly with κ_{out}, reaches a maximum and decreases to zero at the threshold value of κ_{out}.

The maximum output coupling efficiency (Fig. 8.6) increases, for $K > 1$, with K according to

$$\eta_{out,max} = (\sqrt{K} - 1)^2/K. \tag{8.40}$$

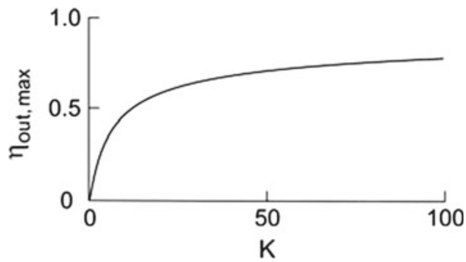

Fig. 8.6 Dependence of the maximum output coupling efficiency on the pump rate parameter K

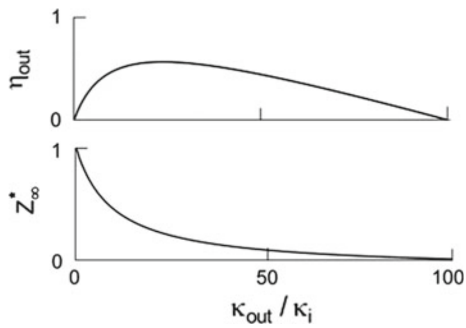

Fig. 8.7 Output coupling efficiency and density of photons in a laser resonator (for $K = 10$)

When a laser is pumped far beyond threshold, $\sqrt{K} \gg 1$, the optimum efficiency η_{out} approaches unity. In this case, the pump power is converted into energy of relaxation and energy of photons in the laser mode; almost all photons are coupled out. The intrinsic loss of photons (e.g., due to diffraction or due to absorption of radiation within the laser resonator) becomes negligibly small.

Which is the density Z_∞ of photons in the laser resonator? Using the relation

$$\kappa_{out} Z_\infty = r_{out} = \eta_{out} r, \tag{8.41}$$

we obtain, after a simple calculation,

$$Z^*_\infty = \frac{Z_\infty}{r_{th,i}/\kappa_i} = \frac{(K-1)}{1 + \kappa_{out}/\kappa_i}, \tag{8.42}$$

where the ratio $r_{th,i}/\kappa_i = (b_{21}\tau^*_{rel})^{-1}$ is a quantity that characterizes a two-level atomic system and where Z^*_∞ is the photon density in units of this quantity.

Figure 8.7 shows, for $K = 10$, the output coupling efficiency and the density of photons in the laser resonator. At optimum output coupling, the number of photons in the resonator is by far smaller than at weak output coupling. The analysis shows that optimum output coupling corresponds to a compromise between a high density of photons in the resonator and a large output coupling efficiency.

The total output coupling rate is $r_{\text{out,tot}} = r_{\text{out}} a_1 a_2 L$, where $a_1 a_2$ is the cross-sectional area of the laser mode and L the length of the active medium. The total output coupling rate corresponds to an output power $P_{\text{out}} = r_{\text{out,tot}} h\nu$.

8.7 Two Laser Rate Equations

We will replace equations (8.1) and (8.2) by one equation. By subtracting (8.2) from (8.1), we obtain, with the approximation $N_1 \ll N_2$,

$$\frac{\mathrm{d}}{\mathrm{d}t}(N_2 - N_1) = r + \frac{N_1}{\tau_{\text{rel}}} - \frac{2(N_2 - N_1)}{\tau_{\text{rel}}^*} - 2b_{21} Z (N_2 - N_1) \tag{8.43}$$

and by addition of the two equations,

$$\frac{\mathrm{d}}{\mathrm{d}t}(N_2 + N_1) = r - \frac{N_1}{\tau_{\text{rel}}} = 0. \tag{8.44}$$

Addition of (8.43) and (8.44) leads to the differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t}(N_2 - N_1) = 2r - \frac{2(N_2 - N_1)}{\tau_{\text{rel}}^*} - 2b_{21} Z (N_2 - N_1). \tag{8.45}$$

We investigate the case that the population difference is suddenly turned on. At $t = 0$, immediately after the production of the population inversion, the photon density Z is negligibly small. It follows that

$$(N_2 - N_1)_0 = r\tau_{\text{rel}}^*. \tag{8.46}$$

The population difference $(N_2 - N_1)_0$ at time $t = 0$ is equal to the pump rate multiplied by the lifetime of the upper laser level. By replacing r, we obtain (instead of originally three equations) *two laser rate equations*:

$$\frac{\mathrm{d}}{\mathrm{d}t}(N_2 - N_1) = \frac{2(N_2 - N_1)_0}{\tau_{\text{rel}}^*} - \frac{2(N_2 - N_1)}{\tau_{\text{rel}}^*} - 2b_{21} Z (N_2 - N_1), \tag{8.47}$$

$$\frac{\mathrm{d}Z}{\mathrm{d}t} = b_{21}(N_2 - N_1) Z - \frac{Z}{\tau_{\text{p}}}. \tag{8.48}$$

At steady state, $\mathrm{d}(N_2 - N_1)/\mathrm{d}t = 0$, the population difference is given by

$$(N_2 - N_1)_\infty = \frac{(N_2 - N_1)_0}{1 + b_{21}\tau_{\text{rel}}^*} Z_\infty = \frac{(N_2 - N_1)_0}{1 + Z_\infty / Z_s}, \tag{8.49}$$

where

$$Z_s = \frac{1}{b_{21}\tau^*_{\rm rel}} \tag{8.50}$$

and where Z_s is the saturation density. The decrease of the population difference during onset of laser oscillation corresponds to a decrease of gain (*gain saturation*). At the steady state, the large-signal gain coefficient is equal to

$$\alpha_\infty = \frac{b_{21}}{c/n}\frac{(N_2 - N_1)_0}{1 + Z_\infty/Z_s} = \frac{\alpha_0}{1 + Z_\infty/Z_s}, \tag{8.51}$$

where α_0 is the small-signal gain coefficient, i.e., the gain coefficient in at $Z \ll Z_s$.

We relate the initial population, the population at steady state and the photon density at steady state. We assume again that $\tau_{\rm rel} \ll \tau^*_{\rm rel}$. We find, from (8.9), (8.15) and (8.46) and with $c\sigma_{21} = b^0_{21}$, the relations

$$\frac{(N_2 - N_1)_0}{(N_2 - N_1)_\infty} = \frac{r}{r_{\rm th}} = 1 + c\tau^*_{\rm rel}\sigma_{21}Z_\infty = 1 + \tau^*_{\rm rel}b^0_{21}Z_\infty. \tag{8.52}$$

Now, the questions remain how $N_2 - N_1$ develops in the time region $t \approx t_{\rm on}$ from the initial value $(N_2 - N_1)_0$ to $(N_2 - N_1)_\infty$ and how the photon density changes from the exponential increase at $t \ll t_{\rm on}$ to the constant value Z_∞. To study the transition from the initial to the steady state, it is necessary to know more about the role of the active medium. This question is a topic of the next chapter.

We obtain a connection to the next chapter by considering the energy content of a laser. A laser contains three forms of energy: energy of excitation of two-level atomic systems $u_{\rm ex} = (N_2 - N_1)_\infty(E_2 - E_1) = h\nu/c\sigma_{21}\tau_{\rm p}$; electromagnetic field energy $u = \varepsilon_0 A^2_\infty/4$; and polarization energy of density $u_{\rm pol}$. A goal of the discussion presented in Chap. 9 will be to find out the relations between the three forms of energy during the buildup of laser oscillation as well as at steady state—we will find that, at steady state, the polarization energy density is equal to the field energy density.

8.8 Relaxation Oscillation

A *relaxation oscillation* can occur during the buildup of a laser oscillation. We search for an oscillation of the population difference and of the photon density at time $t = t_{\rm on}$. We use the ansatz

$$(N_2 - N_1) = (N_2 - N_1)_\infty + N_{\rm osc}, \tag{8.53}$$

$$Z = Z_\infty + Z_{\rm osc}. \tag{8.54}$$

$N_{osc} \equiv (N_2 - N_1)_{osc}$ is the oscillating portion of the population difference and Z_{osc} the oscillating portion of the photon density. We assume, for simplicity, that $N_{osc} \ll (N_2 - N_1)_\infty$ and $Z_{osc} \ll Z_\infty$. It follows from the laser equations that

$$\frac{\mathrm{d}N_{osc}}{\mathrm{d}t} = -\frac{2}{\tau_p} Z_{osc} - 2rb_{21}\tau_p N_{osc} \tag{8.55}$$

and

$$\frac{\mathrm{d}Z_{osc}}{\mathrm{d}t} = \left(\frac{r}{r_{th}} - 1\right) \frac{N_{osc}}{\tau^*_{rel}}. \tag{8.56}$$

We neglected the terms with the product $N_{osc}Z_{osc}$ and made use of the relations $(N_2 - N_1)_\infty = 1/(b_{21}\tau_p)$ and $Z_\infty = (r/r_{th} - 1)/(b_{21}\tau^*_{rel})$, and supposed that $\tau^*_{rel} \ll \tau_p$. By differentiating the first of the two equations and using the second equation, we find

$$\frac{\mathrm{d}^2 N_{osc}}{\mathrm{d}t} + \frac{2}{\tau^*_{rel}} \frac{r}{r_{th}} \frac{\mathrm{d}N_{osc}}{\mathrm{d}t} + \frac{2}{\tau_p \tau^*_{rel}} \left(\frac{r}{r_{th}} - 1\right) N_{osc} = 0. \tag{8.57}$$

This is the equation of a damped harmonic oscillation with the solution

$$N_{osc} = N_{osc}(0)\, \mathrm{e}^{(t-t_{on})/\tau_{damp}} \cos\left(\omega_{osc}(t - t_{on})\right). \tag{8.58}$$

$N_{osc}(0)$ is an initial value of the oscillating portion of the population difference at time $t = t_{on}$. The ansatz yields the frequency of the relaxation oscillation

$$\omega_{osc} = \frac{1}{\tau^*_{rel}} \sqrt{\frac{2\tau^*_{rel}}{\tau_p}\left(\frac{r}{r_{th}} - 1\right) - \left(\frac{r}{r_{th}}\right)^2} \tag{8.59}$$

and the damping (relaxation) time

$$\tau_{damp} = \frac{r_{th}}{r}\, \tau^*_{rel}. \tag{8.60}$$

A relaxation oscillation occurs if

$$\left(\frac{r}{r_{th}}\right)^2 < \left(\frac{r}{r_{th}} - 1\right) \frac{2\tau^*_{rel}}{\tau_p}. \tag{8.61}$$

Otherwise, the relaxation oscillation is overdamped, i.e., there is no relaxation oscillation.

At a pump rate $r = 2 \times r_{th}$, a relaxation oscillation is expected if $\tau_p \leq \tau^*_{rel}$. This is plausible. An instantaneously large population difference leads to a large

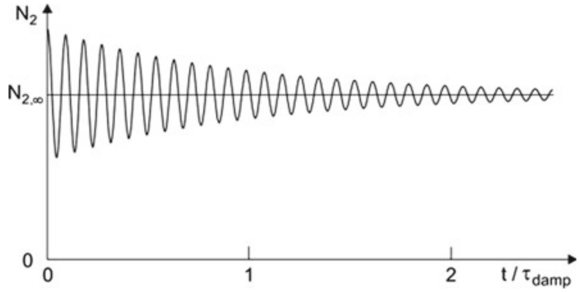

Fig. 8.8 Relaxation oscillation

photon density. This causes a strong decrease of the population difference. Since the photons leave the resonator quickly, the population can build up again and the process of decrease and enhancement of population is repeats until the damping suppresses relaxation oscillation. If the photons have a long lifetime ($\tau_p \geq \tau^*_{rel}$), an instantaneous accumulation of a population difference is not possible and there is no relaxation oscillation.

A helium–neon laser fulfills the condition $\tau_p \geq \tau^*_{rel}$; it shows no relaxation oscillation. The photon lifetime of solid-state lasers and semiconductor lasers is shorter than the relaxation time of the upper laser level, which is a condition of occurrence of relaxation oscillations.

Example Relaxation oscillation in a bipolar semiconductor laser (Fig. 8.8).

- $r/r_{th} = 2$; $\tau^*_{rel} = 4\,\text{ns}$; $\tau_p = 10^{-11}\,\text{s}$.
- $\omega_{osc} = 7 \times 10^9\,\text{Hz}$; $\nu_{osc} = 1.1\,\text{GHz}$; $\nu_{osc}^{-1} = 0.9\,\text{ns}$.
- $\tau_{damp} = 8\,\text{ns}$.

A bipolar semiconductor laser driven by a modulated current emits radiation pulses. The modulation frequency has to be smaller than the oscillation frequency. Otherwise, instabilities can occur. We will discuss in Sect. 9.9 how we can calculate the dynamics of such instabilities.

8.9 Laser Linewidth

Due to spontaneous emission, a laser line has a finite spectral width. To estimate the laser linewidth, we make use of results with respect of a Fabry–Perot resonator containing an active medium (Sect. 3.7). The power P of radiation emitted by a Fabry–Perot resonator (containing an active medium) at a frequency ν in the vicinity of a resonance frequency ν_l is given by

$$P(\nu) = \frac{K}{(1 - \sqrt{G_\infty V})^2 + 4\sqrt{G_\infty V}\,\sin^2[2\pi(\nu - \nu_l)L/c]}. \tag{8.62}$$

K is a measure of the maximum power of a particular laser. It follows that the linewidth (=*laser linewidth* [FWHM] is given by) is

$$\Delta\nu_{\rm L} = \frac{1-\sqrt{G_\infty V}}{\pi\sqrt{G_\infty V}}\,\frac{c}{2L}. \tag{8.63}$$

The linewidth is zero if $G_\infty V = 1$ but finite if $G_\infty V$ is smaller than unity. We consider the case that $G_\infty V$ is slightly smaller than unity. We can approximate, with $(1-\sqrt{a})(1+\sqrt{a}) = 1-a$ and $1+\sqrt{a} \approx 2$, the laser linewidth by

$$\Delta\nu_L = \frac{1-G_\infty V}{2\pi}\,\frac{c}{2L}. \tag{8.64}$$

We will now show that $G_\infty V$ is slightly smaller than unity because of spontaneous emission. We modify the rate equation of the photon density. The change of the density of photons during a round trip transit is the sum of the change due to stimulated and spontaneous emission,

$$\frac{{\rm d}Z}{{\rm d}t} = \frac{GV-1}{T}Z + A_{21}g(\nu)\Delta\nu\frac{1}{D(\nu)\ \Delta\nu_1 a_1 a_2 L}N_2. \tag{8.65}$$

The frequency range of spontaneous emission, covered by a mode, is $\Delta\nu_1 = c/2L = 1/T$ and the probability of spontaneous emission of radiation into one mode is the inverse of the density of states of photon modes in the resonator volume a_1a_2L within the frequency interval $\Delta\nu_1$. The condition of steady state oscillation, ${\rm d}Z/{\rm d}t = 0$, leads to the relation

$$(1-G_\infty V)\ Z_\infty = Tb_{21}N_{2,\rm th}\,\frac{1}{a_1a_2L} \tag{8.66}$$

or

$$\Delta\nu_L = \frac{b_{21}N_{2,\rm th}}{2\pi a_1 a_2 L Z_\infty}. \tag{8.67}$$

We relate the photon density and the power of the laser radiation,

$$\frac{Z_\infty a_1 a_2 L h\nu}{\tau_{\rm p}} = P_{\rm out}. \tag{8.68}$$

It follows, with the resonator linewidth $\Delta\nu_{\rm res} = (2\pi\tau_{\rm p})^{-1}$, that

$$\Delta\nu_L = \frac{h\nu}{P_{\rm out}}\ \Delta\nu_{\rm res}b_{21}N_{2,\rm th}. \tag{8.69}$$

Using the threshold condition that is still approximately valid,

$$(N_{2,\rm th}-N_{1,\rm th})\,b_{21} = N_{2,\rm th}\,\frac{N_{2,\rm th}-N_{1,\rm th}}{N_{2,\rm th}}\,b_{21} = \frac{1}{\tau_{\rm p}}, \tag{8.70}$$

we find the *Schawlow-Townes formula* [39]

$$\Delta\nu_{\mathrm{L}} = \frac{2\pi h\nu(\Delta\nu_{\mathrm{res}})^2}{P_{\mathrm{out}}} \frac{N_{2,\mathrm{th}} - N_{1,\mathrm{th}}}{N_{2,\mathrm{th}}}. \tag{8.71}$$

We define the quality factor Q_L of the laser radiation as the ratio of the laser frequency and the halfwidth of the laser line. It follows, with $P_{\mathrm{out}} = Z_{\mathrm{tot}} \times h\nu/\tau_{\mathrm{p}}$, where Z_{tot} is the total photon number in the resonator, that

$$Q_{\mathrm{L}} = \frac{\nu}{\Delta\nu_{\mathrm{L}}} = Z_{\mathrm{tot}}\, Q_{\mathrm{res}} \frac{N_{2,\mathrm{th}} - N_{1,\mathrm{th}}}{N_{2,\mathrm{th}}} = Z_{\mathrm{tot}}\, Q_{\mathrm{res}} \frac{f_2 - f_1}{f_2}. \tag{8.72}$$

$Q_{\mathrm{res}} = \nu/\Delta\nu_{\mathrm{res}} = 2\pi\nu\tau_{\mathrm{p}}$ is the Q factor of the resonator and τ_{p} the lifetime of a photon in the resonator. The quality factor of the laser radiation is proportional to: the number of photons in the resonator, the quality factor of the resonator, and the occupation number difference divided by the relative occupation number of the upper laser level. If $f_1 \ll f_2$, we have $(f_2 - f_1)/f_1 = 1$. Then the quality factor of the laser radiation is equal to the product of the number of photons in the resonator and the quality factor of the resonator,

$$Q_{\mathrm{L}} = Z_{\mathrm{tot}}\, Q_{\mathrm{res}}. \tag{8.73}$$

Since we are considering a single mode laser, Z_{∞} is equal to the occupation number n of the photons in the laser resonator mode.We obtain the simple relationship:

$$\Delta\nu_{\mathrm{L}} = \frac{\Delta\nu_{\mathrm{res}}}{n} \tag{8.74}$$

the halfwidth of the laser line is equal to the halfwidth of the resonance curve of the laser resonator divided by the occupation number of the mode that is excited in the laser resonator!

In the case that a laser is started by thermal radiation, i.e., if $h\nu \ll kT$, the quality factor of the laser radiation is smaller by the factor $h\nu/kT$,

$$Q_{\mathrm{L}} = Z_{\mathrm{tot}}\, Q_{\mathrm{res}} \frac{h\nu}{kT}. \tag{8.75}$$

Example helium–neon laser; $\nu = 5 \times 10^{14}$ Hz; $\Delta\nu_{\mathrm{res}} = 1$ MHz; $P_{\mathrm{out}} = 1$ mW; theoretical laser linewidth $\Delta\nu_L = 10^{-3}$ Hz. A laser linewidth of 0.1 Hz has been realized by thermal and mechanical stabilization. The experimental laser linewidth corresponded to a relative frequency width of the laser radiation of $\Delta\nu_L/\nu \sim 10^{-14}$.

The frequency width of laser radiation can be very narrow as a consequence of the feedback an active medium experiences from the radiation in the laser resonator. Because of thermal and mechanical fluctuations, stabilization of a continuous-wave laser, in order to use it as a frequency standard, is extremely difficult. Femtosecond lasers are more suited to develop a frequency standard (Sect. 13.7).

References [1–4, 6, 8, 31, 35–37, 39].

Problems

8.1 Threshold condition. Evaluate the threshold condition of a titanium–sapphire laser operated as cw laser. The data of the laser: Fabry–Perot resonator $L = 10\,\text{cm}$, filled with the active titanium-sapphire crystal (gain cross section $\sigma_{21} = 3 \times 10^{-23}\,\text{m}^2$; frequency $\nu = 360\,\text{THz}$); reflectivity of the output coupling mirror $R = 0.98$; cross-sectional area of the laser beam $a_1 a_2 = 0.5\,\text{mm}^2$.

8.2 Photon density, output power and efficiency. Determine the density of photons in the laser resonator and the laser output power of the laser described in Problem 8.1, for a pump power that is 10 times larger than the threshold pump power. Evaluate the efficiency of conversion of a pump photon into a laser photon.

8.3 Oscillation onset time.

(a) Show that the oscillation onset time is always large compared to the period $2\pi/\omega$ of the laser field. [*Hint*: make use of the data of Table 7.1.]
(b) Estimate the oscillation onset time of the titanium-sapphire laser (described in Problem 8.1).

8.4 Formulate the threshold condition in the case that the length L' of the active medium is smaller than the length of the laser resonator. Is the condition $GV = 1$ still valid?

8.5 Estimate the laser linewidth of a semiconductor laser of a wavelength of $0.8\,\mu\text{m}$ and an output power of 1 mW; loss factor $V_1 = 0.3$ and volume of the active medium $= 10^{-13}\,\text{m}^3$.

8.6 Coherence length. Monochromatic laser radiation consists of radiation of a line (halfwidth $\Delta\lambda$) at the laser wavelength λ.

(a) Determine the coherence length l_{coh}. [*Hint*: use as criterion that the number of wavelengths of radiation at $\lambda - \Delta\lambda/2$ and $\lambda + \Delta\lambda/2$ differs by 1.]
(b) Determine the coherence length of radiation generated by a semiconductor laser.
(c) Determine l_{coh} of radiation generated by a highly stabilized helium–neon laser.
(d) Determine the coherence length of the radiation of a hypothetical continuous wave laser at a frequency of 4×10^{14} Hz that is stabilized with a relative accuracy of 10^{-16}.