# Chapter 7
# Amplification of Coherent Radiation

In the preceding chapter, we discussed the interaction of *broadband* radiation with an ensemble of two-level atomic systems. Here, we treat the interaction of *monochromatic* radiation with an ensemble of two-level atomic systems. We will show that the photon density in a disk of light traveling in an active medium increases exponentially with the traveling path length. The gain coefficient of an active medium is proportional to the Einstein coefficient of stimulated emission and to the population difference. We express the gain coefficient as the product of the gain cross section of a two-level system and the population difference.

The largest gain cross section is obtainable for an active medium with a naturally broadened $2 \rightarrow 1$ fluorescence line. Then the gain cross section at the line center is equal to the square of the wavelength of the radiation divided by $2\pi$; we assume that the medium is optically isotropic. The broadening of a transition line of an active medium operated at room temperature is always due to another mechanism (and not by natural broadening). Therefore, the gain cross section of atoms in an active medium at room temperature is smaller than the square of the wavelength of the radiation divided by $2\pi$.

In the case that an active medium is a two-band medium, it is convenient to introduce an effective gain coefficient. It is related to the difference of the density of electrons in the upper band and the transparency density.

We compare gain coefficients and gain cross sections of different active media and discuss, in particular, the gain coefficient of titanium–sapphire.

A two-dimensional active medium can interact with a light beam, which is three dimensional, in two ways: it can propagate along the two-dimensional medium or it can cross the two-dimensional medium. In the case that radiation *propagates along* an active medium, it is useful to introduce a modal gain coefficient, which is related to the average density of atomic two-level systems within a photon mode—the populations of atomic two-level systems still underly the laws governing the two-dimensional medium. In the case that radiation *crosses* an active medium, a description by use of the gain factor of radiation (rather than a gain coefficient of the active medium) is adequate.



K.F. Renk, *Basics of Laser Physics*, Graduate Texts in Physics,
DOI 10.1007/978-3-319-50651-7_7

## 7.1 Interaction of Monochromatic Radiation with an Ensemble of Two-Level Systems

In a laser, *monochromatic radiation* acts on an ensemble of two-level atomic systems. Which are, in this case, the rate equations?

We characterize monochromatic radiation (Fig. 7.1) by a spectral energy density $\rho(\nu)$ that has a constant value within a frequency interval $\nu, \nu + \mathrm{d}\nu$ and is zero outside this interval. The energy density $u(\nu)$ of the monochromatic radiation is

$$u(\nu) = \rho(\nu)\mathrm{d}\nu. \tag{7.1}$$

We assume that the spectral width of the monochromatic radiation is small compared to the linewidth $\Delta\nu_0$ of the atomic transition,

$$\mathrm{d}\nu \ll \Delta\nu_0. \tag{7.2}$$

If only natural line broadening is present, then $\mathrm{d}\nu \ll \Delta\nu_{\mathrm{nat}}$.

(We treat $\mathrm{d}\nu$ as a small but finite physical quantity; $\mathrm{d}\nu$ appears also as a differential in differential equations or integrals. The two aspects—to consider $\mathrm{d}\nu$ as a finite quantity or as a differential—are compatible with each other, *see* [20].)

We ignore, for the moment, spontaneous emission. Stimulated emission processes depopulate the upper level and absorption processes populate it. The temporal change of the population of the upper level is

$$\mathrm{d}N_2/\mathrm{d}t = -B_{21}\rho(\nu)g(\nu)\mathrm{d}\nu N_2 + B_{12}\rho(\nu)g(\nu)\mathrm{d}\nu N_1, \tag{7.3}$$

where $g(\nu)\mathrm{d}\nu$ is the portion of the transition probability in the interval $\nu$, $\nu + \mathrm{d}\nu$ and $g(\nu)$ is the lineshape function.

*Justification:* Broadband radiation with a constant spectral density in the frequency region of the spectral line leads to the temporal change of population of the upper level.

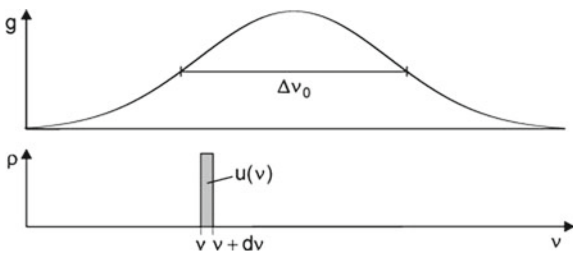


**Fig. 7.1** Monochromatic radiation

$$\frac{dN_2}{dt} = -B_{21}\rho(\nu)N_2 \int_0^\infty g(\nu')d\nu' + B_{12}\rho(\nu)N_1 \int_0^\infty g(\nu')d\nu'. \tag{7.4}$$

This is, because of $\int g d\nu = 1$, equal to the result of the preceding chapter. (We assumed that $B_{21}$ is independent of $\nu$.)

We continue the discussion of the interaction of monochromatic radiation with an ensemble of two-level atomic systems and write, with $B_{12} = B_{21}$, the decay rate in the form

$$dN_2/dt = -B_{21}\rho(\nu)g(\nu)d\nu(N_2 - N_1). \tag{7.5}$$

It follows, with $u = u(\nu) = \rho(\nu)d\nu$, that the temporal change of the population of the upper level is given by

$$dN_2/dt = -B_{21}u\; g(\nu)(N_2 - N_1). \tag{7.6}$$

The transitions $2 \to 1$ dominate ($dN_2/dt < 0$) if $N_2 - N_1 > 0$. The change $dN_2$ of the population $N_2$ is connected with a change $du$ of the energy density of the radiation,

$$du = -\,dN_2 \times h\nu. \tag{7.7}$$

Thus, we obtain

$$du/dt = h\nu g(\nu)B_{21}(N_2 - N_1)\,u. \tag{7.8}$$

We replace the energy density $u$ by the photon density, $Z = u/h\nu$, and obtain the temporal change of the population of the upper level

$$dN_2/dt = -\,h\nu\; g(\nu)B_{21}(N_2 - N_1)\;Z, \tag{7.9}$$

and the change of the photon density

$$\frac{dZ}{dt} = -\frac{d(N_2 - N_1)}{dt}. \tag{7.10}$$

It follows that

$$dZ/dt = b_{21}(N_2 - N_1)Z, \tag{7.11}$$

where

$$b_{21} = h\nu B_{21}g(\nu) \tag{7.12}$$

is the *growth rate constant*, which is a measure of the strength of stimulated emission. *The growth rate constant* $b_{21}(\nu)$ *is equal to the product of the photon energy* $h\nu$, *the Einstein coefficient of stimulated emission and the value of the lineshape function at frequency* $\nu$. We have the result: the growth rate $(\mathrm{d}Z/\mathrm{d}t)$ of the photon density is proportional to the population difference and the photon density.

According to the equation

$$\mathrm{d}N_2/\mathrm{d}t = -b_{21}Z(N_2 - N_1), \tag{7.13}$$

we also can interpret $b_{21}$ as the decay rate for stimulated decay of the population $N_2$, per unit of the photon density and per unit of population difference.

We replace the population difference by the occupation number difference, $N_2 - N_1 = (N_2 + N_1)(f_2 - f_1)$, and write

$$\mathrm{d}N_2/\mathrm{d}t = -b_{21}(N_1 + N_2)(f_2 - f_1)Z. \tag{7.14}$$

The decay rate of the population of the upper laser level is proportional to the density $N_1 + N_2$ of two-level atomic systems and to the occupation number difference $f_2 - f_1$. The net decay rate of the decay of a single two-level atomic system in an ensemble of two-level atomic systems is equal to

$$r_{21}(\nu) = -h\nu B_{21}g(\nu)(f_2 - f_1)Z. \tag{7.15}$$

Alternatively, we can write

$$r_{21}(h\nu) = r_{21}(\nu) = -h\nu \bar{B}_{21}g(h\nu)(f_2 - f_1)Z, \tag{7.16}$$

where the lineshape function $g$ is now expressed on the energy scale and where $\bar{B}_{21} = hB_{21}$.

## 7.2 Growth and Gain Coefficient

The temporal change of the photon density is equal to

$$\mathrm{d}Z/\mathrm{d}t = \gamma Z, \tag{7.17}$$

where

$$\gamma = h\nu B_{21}g(\nu)(N_2 - N_1) = b_{21}(N_2 - N_1) \tag{7.18}$$

is the *growth coefficient* of an active medium.

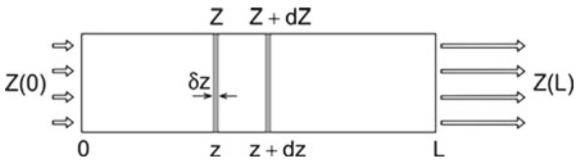


**Fig. 7.2** Monochromatic radiation in an active medium

If we suddenly turn on, at time $t = 0$, the population inversion, then the density of photons increases exponentially,

$$Z = Z_0 e^{\gamma t}. \tag{7.19}$$

$Z_0$ is the photon density at $t = 0$.

The radiation in a disk of light (thickness $\delta z$) propagating in an active medium (Fig. 7.2) is amplified. On the path from $z$ to $z + dz$, the change of the photon density within the disk of thickness $\delta z \ll dz$ is

$$dZ = b_{21}(N_2 - N_1) Z dt, \tag{7.20}$$

where

$$dt = dz/(c/n) \tag{7.21}$$

is the time the disk of light takes to travel the distance d$z$. We can write

$$dZ = \alpha(\nu)\, Z\, dz, \tag{7.22}$$

where

$$\alpha(\nu) = \frac{\gamma(\nu)}{c/n} = \frac{b_{21}(\nu)}{c/n}(N_2 - N_1) = \frac{h\nu}{c/n} B_{21} g(\nu)(N_2 - N_1) \tag{7.23}$$

is the *gain coefficient* (=small-signal gain coefficient) of an active medium. The gain coefficient is proportional to $b_{21}$ and to the population difference.

(If the energy levels are degenerate, the gain coefficient is given by

$$\alpha(\nu) = \frac{\gamma(\nu)}{c/n} = \frac{b_{21}(\nu)}{c/n}\left(N_2 - N_1 \frac{g_2}{g_1}\right) = \frac{h\nu}{c/n} B_{21} g(\nu) \left(N_2 - N_1 \frac{g_2}{g_1}\right). \tag{7.24}$$

In the following, we consider nondegenerate two-level systems.)

The photon density increases exponentially with the traveling path length,

$$Z(z) = Z(z_0)\, e^{\alpha(\nu)(z - z_0)}. \tag{7.25}$$

If $N_2 < N_1$, then $\alpha(\nu)$ is negative, and we obtain the absorption coefficient

$$\alpha_{\text{abs}} = -\alpha = (n/c)h\nu B_{21}g(\nu)(N_1 - N_2). \tag{7.26}$$

In this case, the photon density decreases exponentially according to the Lambert–Beer law

$$Z(z) = Z(z_0)\mathrm{e}^{-\alpha_{\text{abs}}(z-z_0)}. \tag{7.27}$$

We continue the discussion of gain. We can replace the population difference by the product of the occupation number difference and the density of two-level atomic systems, $N_2 - N_1 = (f_2 - f_1)(N_1 + N_2)$, and obtain

$$\alpha(\nu) = (n/c)h\nu B_{21}g(\nu)\,(N_1 + N_2)\,(f_2 - f_1). \tag{7.28}$$

*The gain coefficient is proportional to the density of two-level atomic systems and to the occupation number difference* $f_2 - f_1$. We introduce the *gain bandwidth* $\Delta\nu_{\text{g}}$ as the halfwidth of the gain curve $\alpha(\nu)$. If $B_{21}$ is independent of frequency, $\Delta\nu_{\text{g}}$ is determined by the halfwidth of the lineshape function $g(\nu)$.

If the lineshape function is given on the energy scale, then

$$\alpha(\nu) = \alpha(h\nu) = (n/c)h\nu\bar{B}_{21}g(h\nu)\,(N_1 + N_2)\,(f_2 - f_1). \tag{7.29}$$

If a line has Lorentzian shape, we can write the gain coefficient in the form

$$\alpha(\nu) = \frac{h\nu}{c/n}B_{21}\,\frac{2}{\pi\,\Delta\nu_0}\,\bar{g}_{\text{L,res}}(\nu)\,(N_2 - N_1), \tag{7.30}$$

or

$$\alpha(\nu) = (n/c)h\nu B_{21}\frac{2}{\pi\,\Delta\nu_0}\,\bar{g}_{\text{L,res}}(N_1 + N_2)\,(f_2 - f_1), \tag{7.31}$$

where $\bar{g}_{\text{L,res}}$ is the lineshape function normalized to unity at the line center.

In a light beam propagating through an active medium of length $L$, the photon density increases from the value $Z_0$ to the value

$$Z = Z_0\mathrm{e}^{\alpha(\nu)L}. \tag{7.32}$$

The *single-path gain factor* is equal to

$$G_1(\nu) = \mathrm{e}^{\alpha(\nu)L}. \tag{7.33}$$

The single-path gain is

$$\frac{Z - Z_0}{Z_0} = \frac{u - u_0}{u_0} = G_1(\nu) - 1 = \mathrm{e}^{\alpha(\nu)L} - 1. \tag{7.34}$$

If $\alpha(\nu)L \ll 1$, then

$$G_1(\nu) - 1 = \alpha(\nu)L, \tag{7.35}$$

i.e., the single-path gain $G_1 - 1$ is equal to the product of the gain coefficient and the length of the active medium.

## 7.3 Gain Cross Section

We write the gain coefficient of an active medium (containing two-level atomic systems) in the form

$$\alpha(\nu) = N_2\sigma_{21} - N_1\sigma_{12}, \tag{7.36}$$

where $\sigma_{21}$ is the *gain cross section* of a two-level atomic system and $\sigma_{12}$ the *absorption cross section*. The cross sections are equal, $\sigma_{12} = \sigma_{21}$. It follows that the gain coefficient of an active medium containing an ensemble of two-level atomic systems is given by

$$\alpha = (N_2 - N_1)\,\sigma_{21}, \tag{7.37}$$

where

$$\sigma_{21}(\nu) = \frac{b_{21}}{c/n} = \frac{h\nu}{c/n} B_{21} g(\nu). \tag{7.38}$$

The gain cross section at the frequency $\nu$ is proportional to $\nu$, to the Einstein coefficient of stimulated emission, and to the value of the lineshape function at the frequency $\nu$.

In the case that a line is naturally broadened, we can use the Einstein relations and the relation $A_{21} = \tau_{\mathrm{sp}}^{-1}$. We then find

$$\sigma_{21}(\nu) = \frac{c^2 A_{21}}{8\pi n^2 \nu^2} g_{\mathrm{nat}}(\nu) = \frac{c^2}{8\pi n^2 \nu^2} \frac{1}{\tau_{\mathrm{sp}}} g_{\mathrm{nat}}(\nu). \tag{7.39}$$

The largest gain cross section of an isotropic two-level atomic system in an active medium is obtainable if the $2 \rightarrow 1$ line is naturally broadened. Then $g(\nu_0) = 4\tau_{\mathrm{sp}}$ and

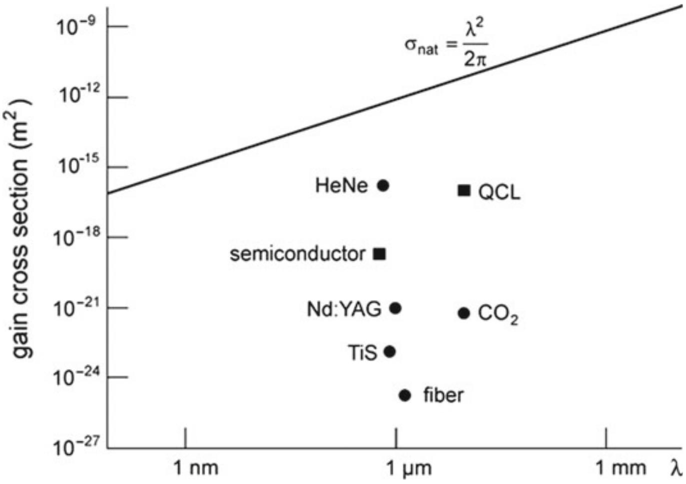


**Fig. 7.3** Gain cross sections

$$\sigma_{\text{nat}} = \sigma_{21,\text{nat}}(\nu_0) = \frac{c^2}{2\pi n^2 \nu_0^2} = \frac{(\lambda/n)^2}{2\pi}, \tag{7.40}$$

where $\lambda = c/\nu_0$ is the wavelength of the radiation in vacuum, $n$ is the refractive index of the active medium at the frequency $\nu_0$, and $\nu_0 = (E_2 - E_1)/h$. The gain cross section of a two-level system with a naturally broadened line increases with the square of the wavelength (Fig. 7.3, solid line).

We will see later, when we will discuss specific lasers (Chaps. 14–16 and chapters on semiconductor lasers), that the active media of all lasers operated at room temperature show $1 \rightarrow 2$ absorption lines and the $2 \rightarrow 1$ fluorescence lines that are not broadened by natural broadening, but that other mechanisms dominate the line broadening. Therefore, the gain cross section of radiation propagating in an active (isotropic) medium at room temperature is smaller than $(\lambda/n)^2/2\pi$.

If the gain coefficient curve is a Lorentz resonance curve, we can write

$$\sigma_{21}(\nu) = \frac{h\nu}{c/n} B_{21}\, g_{\text{L,res}}(\nu) = \frac{2}{\pi \Delta\nu_0} \frac{h\nu}{c/n} B_{21}\, \bar{g}_{\text{L,res}}(\nu). \tag{7.41}$$

The gain cross section at the center of a Lorentzian gain curve (with the halfwidth $\Delta\nu_0$) is

$$\sigma_{21,\text{L}}(\nu_0) = \frac{\Delta\nu_{\text{nat}}}{\Delta\nu_0} \sigma_{\text{nat}}(\nu_0) = \frac{\Delta\nu_{\text{nat}}}{\Delta\nu_0} \frac{(\lambda/n)^2}{2\pi}. \tag{7.42}$$

The gain cross section of radiation at the line center of a Lorentzian line is by the factor $\Delta\nu_{\text{nat}}/\Delta\nu_0$ smaller than in case of a naturally broadened line.

**Table 7.1** Gain bandwidths and gain cross sections

| Laser | $\lambda$ | $n$ | $\Delta\nu_g$ | $\Delta\nu_g/\nu_0$ | $\sigma_{21}$ ($m^2$) [$\sigma_{eff}$ ($m^2$)] |
|---|---|---|---|---|---|
| HeNe | 633 nm | 1 | 1.5 GHz | $3 \times 10^{-6}$ | $1.4 \times 10^{-16}$ |
| $CO_2$ | 10.6 µm | 1 | 69 MHz–500 GHz | $2.5 \times 10^{-6}$–$1.7 \times 10^{-2}$ | $1.2 \times 10^{-20}$ |
| Nd:YAG | 1.06 µm | 1.82 | 140 GHz | $1.4 \times 10^{-4}$ | $8.1 \times 10^{-22}$ |
| TiS ($E$ \|\| c) | 830 nm | 1.74 | 110 THz | 0.3 | $2.3 \times 10^{-23}$ |
| TiS ($E \perp$ c) | | | | | $8 \times 10^{-24}$ |
| Fiber | 1.5 µm | 1.5 | 5 THz–12 THz | $2.5$–$6 \times 10^{-2}$ | $2 \times 10^{-25}$–[$6 \times 10^{-25}$] |
| Semiconductor | 840 nm | 3.6 | 10 GHz–1 THz | $3 \times 10^{-5}$–$3 \times 10^{-3}$ | [$3 \times 10^{-19}$] |
| QCL | 5 µm | 3.4 | 10 GHz–1 THz | $2 \times 10^{-4}$–$1.6 \times 10^{-2}$ | $10^{-16}$ |

The gain cross section of radiation at the center of a Gaussian line is

$$\sigma_{21,G}(\nu_0) = \frac{\Delta\nu_{nat}}{\Delta\nu_0}\sqrt{\pi \ln 2}\,\sigma_{nat}(\nu_0) \sim 1.48\,\frac{\Delta\nu_{nat}}{\Delta\nu_0}\,\frac{(\lambda/n)^2}{2\pi}. \tag{7.43}$$

Table 7.1 shows values of gain bandwidths and gain cross sections (and of effective gain cross sections, *see* next section and Sect. 18.7). Different halfwidths, mentioned in the following, are: $\Delta\nu_g$ = gain bandwidth = halfwidth of the gain curve; $\Delta\nu_0$ = halfwidth of an absorption line; $\Delta\nu_{fluor}$ = halfwidth of the fluorescence line, measured on the frequency scale.

- *Helium–neon laser*. The gain bandwidth is equal to the halfwidth of the $2 \rightarrow 1$ fluorescence line, $\Delta\nu_g = \Delta\nu_{fluor} = \Delta\nu_0$.
- $CO_2$ *laser*. $\Delta\nu_g = \Delta\nu_{fluor} = \Delta\nu_0$.
- *Nd:YAG laser*. $\Delta\nu_g = \Delta\nu_{fluor} = \Delta\nu_0$.
- *Titanium–sapphire laser*. The gain bandwidth is very large (Sect. 7.6). In the table, the crystal anisotropy of $Ti^{3+}$:$Al_2O_3$ is taken into account. An average gain cross section is $\sigma_{21} = \frac{1}{3}(\sigma_1 + 2\sigma_2)$, where $\sigma_1$ is the gain cross section for $E$ || c and $\sigma_2$ is the gain cross section for $E \perp$ c. The experimental fluorescence curves indicate that $\sigma_2 \sim 3\sigma_1$ and that therefore $\sigma_1 \sim 1.8\,\sigma_{21}$ and $\sigma_2 \sim 0.6\,\sigma_{21}$.
- *Fiber laser*. The gain bandwidth depends on the pump strength (Chap. 18).
- *Bipolar semiconductor laser*. The gain bandwidth changes if the strength of the pumping (i.e., the current flowing through the active semiconductor medium) changes (Chaps. 21 and 22).
- *Quantum cascade laser*. The gain bandwidth varies if the strength of the pumping changes. The gain bandwidth of an active medium of a specific quantum cascade laser can be obtained by a detailed analysis of the properties of the active medium (Chap. 29).

The gain bandwidths of the different active media differ by five orders of magnitude. The titanium–sapphire laser has by far the largest gain bandwidth, corresponding to 30% of the center frequency. The gain cross section (*see* also Fig. 7.3) differs by nine orders of magnitude. The helium–neon laser and the quantum cascade laser have large values of the gain cross section. The bipolar semiconductor lasers have smaller values. $CO_2$ lasers and solid state lasers have still smaller values.

## 7.4 An Effective Gain Cross Section

We consider a two-band laser with an active medium containing $N_0$ two-level atomic systems per unit volume. Without pumping, all levels in the lower band are occupied and all levels of the upper band are empty (Fig. 7.4a); we assume that $E_{2,\mathrm{min}} - -E_{1,\mathrm{max}} \gg kT$. Pumping leads to a population in the upper band and to empty levels in the lower band. The population in the upper band and the population in the lower band are in a nonequilibrium relative to each other. At weak pumping, the population $N_2$ (=$N$) in the upper band is small and the density $N_1$ (=$N$) of empty levels in the lower band is also small. Accordingly, the relative occupation number $f_2$ (at energies near the minimum of the upper band) is small ($f_2 \ll 1$), the relative occupation number $f_1$ (at energies near the maximum of the lower band) is only slightly smaller than unity and absorption of radiation prevails. The width of the energy distributions of each of the populations is of the order of $kT$. With increasing $N$, $f_2$ increases and $f_1$ decreases until $N$ reaches the transparency density $N_{\mathrm{tr}}$, where $f_2 - f_1 = 0$ (Fig. 7.4b). The width of the energy distributions of each of the populations is still of the order of $kT$. The largest population in the upper band occurs at energies near the band minimum (energy $E_{2,\mathrm{min}}$) and the smallest population in the lower band at energies near the band maximum (energy $E_{1,\mathrm{max}}$).

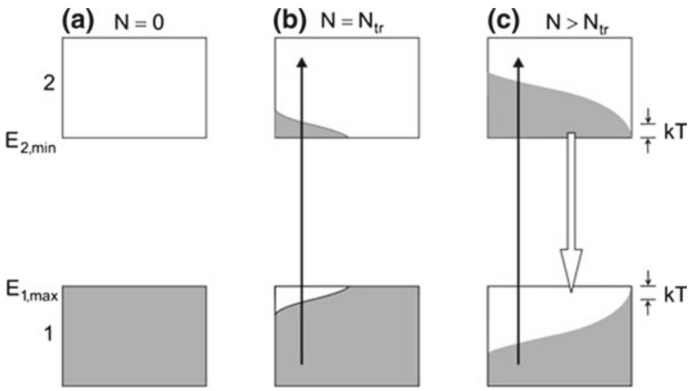


**Fig. 7.4** Two-band laser. **a** Population without pumping. **b** Quasi-thermal distributions of the populations for $N = N_{\mathrm{tr}}$. **c** Quasi-thermal distributions of the populations for $N > N_{\mathrm{tr}}$

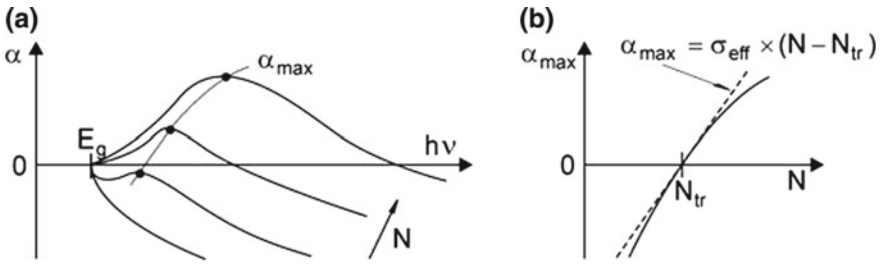


**Fig. 7.5** Gain coefficient of a two-band laser medium. **a** Frequency dependence of the gain coefficient. **b** Dependence of the maximum gain coefficient on the population $N$ in the upper band

If $N > N_{tr}$ (i.e., $f_2 > f_1$), the medium is a gain medium and the gain increases with increasing $N$. The gain bandwidth increases with increasing band filling and becomes larger than $kT$ at large filling (Fig. 7.4c); then the widths of the distributions are larger than $k\,T$.

The gain coefficient $\alpha$ depends on different parameters: temperature of the active medium; Einstein coefficient $B_{21}$; electron density $N$; energy distributions in the levels of the lower and the levels in the upper band. The $\alpha(\nu)$ curve can have a complicated shape (Fig. 7.5a). Gain occurs if the maximum $\alpha_{max}$ of the $\alpha(\nu)$ curve becomes (for $N = N_{tr}$) positive. With increasing $N$, $\alpha_{max}$ increases and the gain bandwidth increases too. Figure 7.5b (solid line) shows $\alpha_{max}$ versus $N$ for values of $N$ around the transparency density. The expansion of $\alpha_{max}$ leads to

$$\alpha_{max} = \sigma_{eff} \times (N - N_{tr}), \tag{7.44}$$

where the differential cross section

$$\sigma_{eff} = (\partial \alpha_{max} / \partial N)_{N=N_{tr}} \tag{7.45}$$

is an *effective gain cross section*. The effective gain cross section (Fig. 7.5b, dashed) corresponds to the slope of $\alpha_{max}(N)$ near $N_{tr}$. The effective gain cross section is the gain cross section related to the density of two-level systems excited in addition to the two-level systems that are, at the transparency density, already in the excited state. We will discuss later (in Chaps. 21 and 22) how we can determine the effective gain cross sections and $\alpha_{max}$ as well as gain bandwidths of bipolar semiconductor lasers; a value of an effective gain cross section is given in Table 7.1.

We will introduce two other effective gain cross sections in connection with the discussion of gain coefficient of a doped fiber (Sect. 18.7).

## 7.5 Gain Coefficients

The gain coefficients of different laser media differ markedly (Table 7.2 and Fig. 7.6—they differ by eight orders of magnitude. The gain coefficient is small for the helium–neon laser, it is large for the $CO_2$ laser and for solid-state lasers. It is very large for the (bipolar) semiconductor laser and the quantum cascade laser. The gain coefficient of a medium can be obtained by an analysis of the properties of an active medium or from the study of the laser thershold. The length $L'$ of an active medium is about 1 mm or smaller for a semiconductor laser and lies between several centimeters and about 1 m for the other lasers. We can ask whether it is possible to increase $N_2 - N_1$ in order to obtain larger gain coefficients. The answer is different for the different lasers.

**Table 7.2** Gain coefficients

| Laser | λ | $\alpha$ ($m^{-1}$) | $L'$ (m) | $G$ ($G_1$) | $\sigma_{21}$ [$\sigma_{eff}$] ($m^2$) | $N_2 - N_1$ [$N - N_{tr}$] ($m^{-3}$) |
|---|---|---|---|---|---|---|
| HeNe | 633 nm | 0.014 | 0.5 | 1.014 | $1.4 \times 10^{-16}$ | $10^{14}$ |
| $CO_2$ | 10.6 μm | 5 | 0.5 | 3 | $1.2 \times 10^{-20}$ | $1.5 \times 10^{18}$ |
| Nd:YAG | 1.06 μm | 20 | 0.1 | 50 | $8.1 \times 10^{-22}$ | $1 \times 10^{23}$ |
| TiS $E\|\|c$ | 830 nm | 20 | 0.1 | 50 | $2.3 \times 10^{-23}$ | $8 \times 10^{23}$ |
| Fiber | 1.5 μm | 0.7 | 10 | ($10^3$) | [$6 \times 10^{-25}$] | $1.2 \times 10^{24}$ |
| Semiconductor | 840 nm | 1,500 | $10^{-3}$ | (4.5) | [$3 \times 10^{-19}$] | [$6 \times 10^{21}$] |
| QCL | 5 μm | 1,000 | $10^{-3}$ | 2.7 | $10^{-16}$ | $10^{19}$ |

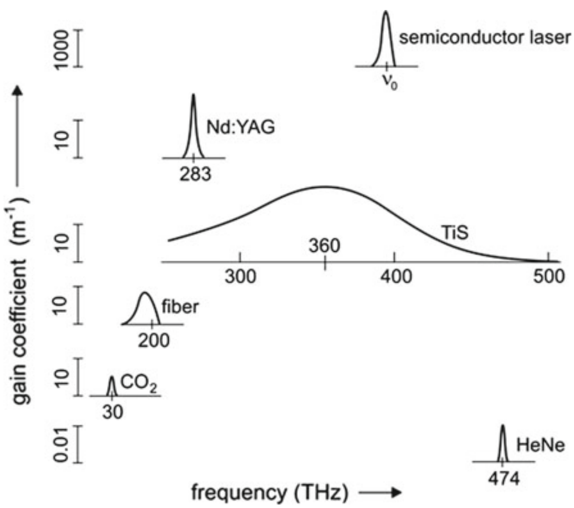


**Fig. 7.6** Gain coefficient of different laser media

- *Helium–neon laser.* The excited neon atoms have a large gain cross section. However, the population difference $N_2 - N_1$ that can be reached is very small.
- *$CO_2$ laser.* The value of the population difference given in the table is obtained at a discharge in a gas of 5 mbar pressure. Higher population differences are obtainable at higher gas pressures (e.g., at a gas pressure of 1 bar in pulsed lasers), which emit pulses of high power. The gain coefficient $\alpha$ at the line center does not change with $N_2 - N_1$ because of collision broadening (Sects. 14.2 and 14.8).
- *Nd:YAG laser.* The population difference $N_2 - N_1$ increases in case of stronger pumping. However, it is preferable to make use of stronger pumping to increase the laser output power rather than to enhance the population difference (Chap. 8).
- *Titanium–sapphire laser.* The population difference cannot increase much further, it has already a value near 10% of the density of $Ti^{3+}$ ions (at a doping level of $10^{25}$ m$^{-3}$). A further increase of $N_2 - N_1$ leads to saturation of the pump rate (Sect. 5.4).
- *Fiber laser*. An increase of the population difference (at stronger pumping) by a factor of 10 is possible; the impurity concentration, $N_0 = 7 \times 10^{25}\,\mathrm{m}^{-3}$, is by about an order of magnitude larger than for crystals. The transparency density is $N_{tr} \sim N_0/2$.
- *Bipolar semiconductor laser.* An increase of the population difference $N - N_{tr}$ and of $\alpha$ by less than an order of magnitude is possible.
- *Quantum cascade laser*. Whether the population difference can be increased depends on the specific design.

While the gain coefficient curves of gas lasers follow directly from atomic properties of gases and of solid-state laser media from atomic properties of impurity ions in solids, the situation is completely different for semiconductor and quantum cascade lasers: it is possible to choose the center frequency $\nu_0$ of a gain coefficient curve through the choice of an appropriate semiconductor material and an appropriate heterostructure. Designing bipolar semiconductor lasers is possible for almost all frequencies of radiation in the near UV, the visible and the near infrared (150–800 THz). Designing quantum cascade lasers is possible for all frequencies in the range 11–150 THz or, as cooled quantum cascade lasers operating at a temperature of 80 K, in the range 1–5 THz.

## 7.6 Gain Coefficient of Titanium–Sapphire

In the preceding section, we presented gain data of titanium–sapphire. Here, we show how we can obtain the data. The fluorescence band extends over a large wavelength range. Therefore, we have to take into account that the Einstein coefficient of spontaneous emission varies strongly with frequency. We now determine the spectral distribution $S_\nu(\nu)$ on the frequency scale from the spectral distribution $S_\lambda(\lambda)$ on the wavelength scale, represented in Fig. 5.3 (Sect. 5.3). We use the relation

$$S_\nu(\nu)|\mathrm{d}\nu| = S_\lambda(\lambda)|\mathrm{d}\lambda|, \quad (7.46)$$

where $\mathrm{d}\nu$ is a frequency range near the frequency $\nu$ and $\mathrm{d}\lambda$ the corresponding wavelength range near the wavelength $\lambda = c/\nu$. With $\nu = c/\lambda$ and $|\mathrm{d}\nu| = |\mathrm{d}\lambda|/\lambda^2$, we obtain

$$S_\nu = \frac{1}{|\mathrm{d}\nu/\mathrm{d}\lambda|}\, S_\lambda = \frac{\lambda^2}{c}\, S_\lambda. \quad (7.47)$$

$S_\nu$ is proportional to $g(\nu)$ and to $A_{21}(\nu)$, i.e.,

$$g(\nu) = K_1 \frac{S_\nu(\nu)}{A_{21}(\nu)} = K_2\, \lambda^5\, S_\lambda. \quad (7.48)$$

$K_1$ and $K_2$ are constants. Multiplying $S_\lambda$ (Fig. 5.3) by $\lambda^5$ and normalizing the maximum of the lineshape function to 1, we obtain the gain profile $\bar{g}(\nu)$ shown in Fig. 7.7 (solid line). The curve yields the center frequency $\nu_0$ ($\sim$ 360 THz) and the gain bandwidth $\Delta\nu_g$ ($\sim$110 THz). The ratio of the gain bandwidth and the center frequency is about 0.3. The gain coefficient $\alpha(\nu)$ has a Gaussian-like profile. At small frequencies ($\nu < \nu_0$), the decrease of the $\bar{g}(\nu)$ curve is less steep than for a Gaussian lineshape (dotted). A Gaussian-like gain profile, with a deviation from a Gaussian profile at small frequencies, is consistent with the vibronic character of the energy levels as we will discuss later (in Chap. 17)—the deviation is a consequence of the anharmonicity of the lattice vibrations of sapphire. We attribute the line broadening to homogeneous broadening (Sect. 17.4).

Figure 7.8 (upper part) shows the absorption coefficient of $Ti^{3+}$:$Al_2O_3$ and, furthermore, the gain coefficient of excited $Ti^{3+}$:$Al_2O_3$ in the case that 8% of the titanium ions (in a crystal containing $10^{25}$ $Ti^{3+}$ ions per $m^3$) are in the excited state. The maximum gain coefficient follows from the relation $\alpha_{max} = N\sigma_{max}$, where $N_2$ is the density of excited $Ti^{3+}$ ions. We find that the maximum cross section of stimulated emission is equal to

$$\sigma_{max} = a\sqrt{\pi \ln 2}\, \frac{\Delta\nu_{nat}}{\Delta\nu_0}\, \sigma_{nat}(\nu_0) \sim 1.48a\, \frac{\Delta\nu_{nat}}{\Delta\nu_0}\, \frac{(\lambda/n)^2}{2\pi}, \quad (7.49)$$

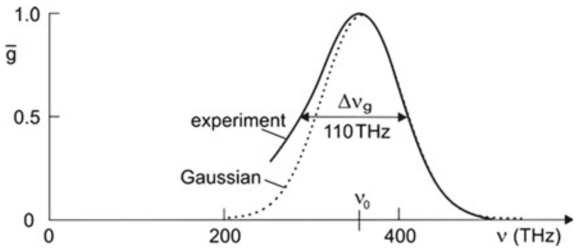


**Fig. 7.7** Gain profile of $Ti^{3+}$:$Al_2O_3$

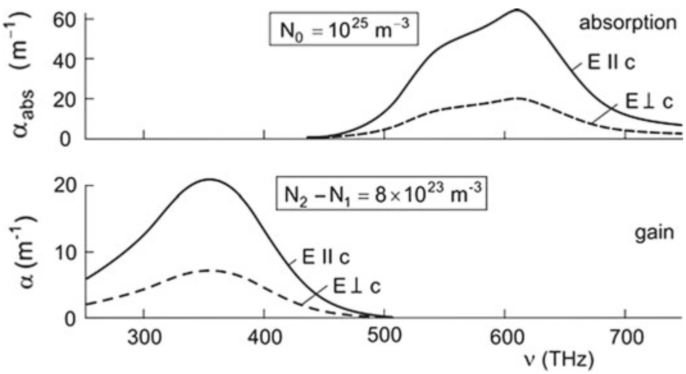


**Fig. 7.8** Absorption and gain coefficients of $Ti^{3+}$:$Al_2O_3$

where $\Delta\nu_{nat}$ is the natural linewidth, $\Delta\nu_0$ the gain bandwidth, $\sigma_{nat}(\nu_0)$ the cross section corresponding to a naturally broadened line and $n$ (=1.74) the refractive index of sapphire; the factor $\sqrt{\pi \ln 2}$ takes account of the difference of a Gaussian and a Lorentzian profile and the factor $a$ ($\sim$2) of crystal anisotropy.

## 7.7 Gain Coefficient of a Medium with an Inhomogeneously Broadened Line

We can decompose an inhomogeneously broadened line, e.g., a Gaussian line, into homogeneously broadened lines. We introduce:

- $\nu_c$ = center frequency of an inhomogeneous broadened line.
- $g_{inh}(\nu, \nu_c)$ lineshape function describing the inhomogeneous broadening.
- $\nu_0$ = resonance frequency of a specific two-level atomic system; each two-level atomic system has its own resonance frequency.
- $g_{hom}(\nu, \nu_0)$ = lineshape function describing the homogeneous broadening of a two-level atomic system that has the resonance frequency $\nu_0$.
- $dN_2^{d\nu_0} = N_2 g_{inh}(\nu_0, \nu_c)d\nu_0$ = density of two-level atomic systems in the upper laser level that have the resonance frequency in the frequency interval $\nu_0$, $\nu_0 + d\nu_0$.
- $dN_1^{d\nu_0} = N_1 g_{inh}(\nu_0, \nu_c)d\nu_0$ = density of two-level atomic systems in the lower laser level that have the resonance frequency in the frequency interval $\nu_0$, $\nu_0 + d\nu_0$.

The temporal change of the population difference is given by

$$d(N_2 - N_1)/dt = -\gamma\, Z, \tag{7.50}$$

and the temporal change of the photon density is

$$dZ/dt = \gamma\, Z, \tag{7.51}$$

where

$$\gamma(\nu) = \int_0^\infty h\nu g_{\text{hom}}(\nu - \nu_0) B_{21} (N_2 + N_1)(f_2 - f_1) g_{\text{inh}}(\nu_0) d\nu_0 \tag{7.52}$$

is the growth coefficient. We have used the relation $N_2 - N_1 = (N_2 + N_1)(f_2 - f_1)$.

The gain coefficient is $\alpha = (n/c)\gamma$. Two-level atomic systems that have different resonance frequencies $\nu_0$ contribute to the gain coefficient at frequency $\nu$. The Einstein coefficient $B_{21}$ can depend on frequency. A special case of (7.52) is the Voigt profile (Problem 14.2b).

## 7.8 Gain Characteristic of a Two-Dimensional Medium

There are two possibilities to arrange a two-dimensional active medium in a light beam. The propagation direction of the light can be parallel to the plane of the medium or perpendicular. We treat here the first case and the second case in the next section.

We consider the propagation of a parallel light beam that contains a two-dimensional active medium (Fig. 7.9). The propagation direction of the light is parallel to the plane of the medium. We introduce the *average density of two-level atomic systems in the light beam*,

$$N_{\text{av}} = \frac{N^{2\text{D}}}{a_2}, \tag{7.53}$$

where $a_2$ is the extension of the beam perpendicular to the film plane—the height of the photon mode—and $N^{2\text{D}}$ is the two-dimensional density of two-level systems in the two-dimensional medium. If $N_2^{2\text{D}} - N_1^{2\text{D}}$ is the population difference, the average population difference in the light beam is

$$(N_2 - N_1)_{\text{av}} = \frac{N_2^{2\text{D}} - N_1^{2\text{D}}}{a_2}. \tag{7.54}$$

The average density is *independent* of the thickness of the two-dimensional active medium. It follows that the temporal change of the photon density in a disk of light is equal to

$$\frac{dZ}{dt} = b_{21} \frac{N_2^{2\text{D}} - N_1^{2\text{D}}}{a_1} Z. \tag{7.55}$$

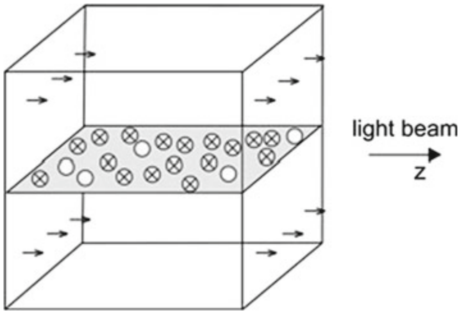


**Fig. 7.9** Two-dimensional medium in a light beam

Our procedure is justified because it does not matter at which position within the photon mode the two-level atomic systems are located. As an essential condition, we assumed that the photons in the light beam belong to a single mode.

The growth coefficient is equal to

$$\gamma = b_{21} \frac{N_2^{2D} - N_1^{2D}}{a_2}. \tag{7.56}$$

Taking into account that $b_{21} = (c/n)\sigma_{21}$, we find the gain coefficient

$$\alpha = \sigma_{21} \frac{N_2^{2D} - N_1^{2D}}{a_2}. \tag{7.57}$$

The gain coefficient $\alpha$ is inversely proportional to the extension of the photon mode perpendicular to the plane of the two-dimensional medium and is called *modal gain coefficient*. The single-path gain factor of radiation transversing a medium of length $L$ is

$$G_1 = e^{\alpha L}. \tag{7.58}$$

We introduce the two-dimensional gain characteristic

$$H^{2D}(\nu) = \sigma_{21}(\nu)(N_2^{2D} - N_1^{2D}), \tag{7.59}$$

as the product of the gain cross section and the difference of the two-dimensional populations. The modal gain coefficient is given by

$$\alpha = \frac{1}{a_2} H^{2D} \tag{7.60}$$

and the modal growth coefficient by

$$\gamma = \frac{c}{na_2} H^{2D}. \tag{7.61}$$

The two-dimensional gain characteristic completely describes the active medium while the gain coefficient and the growth coefficient depends not only on the properties of the active medium but also on the extension of the photon mode perpendicular to the plane of the two-dimensional medium. Thus, a modal gain coefficient refers to a hypothetical medium: the hypothetical medium has the height of the photon mode; it contains a homogeneous distribution of two-level systems; the density of two-level systems in the hypothetical medium is equal to the density of two-level systems in the two-dimensional medium divided by the height of the photon mode.

*Example* GaAs quantum well in a light beam in a quantum well laser (Chaps. 21 and 22).

## 7.9 Gain of Light Crossing a Two-Dimensional Medium

In the case that a light beam is crossing a two-dimensional active medium (Fig. 7.10), it is convenient to make use of the gain factor rather than the gain coefficient. The average population difference in a disk of light is

$$(N_2 - N_1)_{\mathrm{av}} = \frac{N_2^{2D} - N_1^{2D}}{\delta z}, \tag{7.62}$$

where $\delta z$ is the length of the disk. The change of the photon density within the interaction time $\delta t = n\delta z/c$ is

$$\delta Z = \frac{N_2^{2D} - N_1^{2D}}{\delta z} b_{21}\, Z\, \delta t. \tag{7.63}$$

It follows that

$$\frac{\delta Z}{Z} = \sigma_{21}(N_2^{2D} - N_1^{2D}) \tag{7.64}$$

and, with $G_1 - 1 = \delta Z/Z$, that

$$G_1 - 1 = \sigma_{21}(N_2^{2D} - N_1^{2D}) = H^{2D}(\nu). \tag{7.65}$$

The single-path gain $G_1 - 1$ of radiation crossing a two-dimensional medium is equal to the two-dimensional gain characteristic.

*Example* a light beam crossing a GaAs quantum well in a quantum well laser (Chaps. 21 and 22).

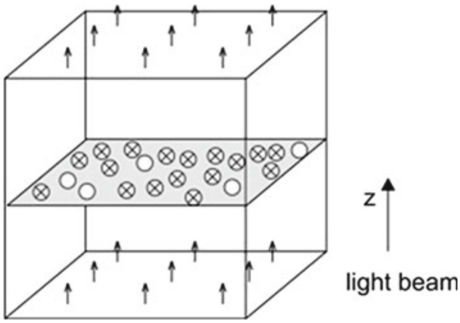


**Fig. 7.10** Light beam crossing a two-dimensional active medium

References [1–4, 6, 31, 35–37].

## Problems

**7.1 Amplification of radiation in titanium–sapphire**. Given is an active titanium–sapphire medium with a population difference $N_2 - N_1 = 10^{24}\ \mathrm{m}^{-3}$.

(a) Determine the gain coefficient at the frequency of maximum gain.
(b) Determine the single-path gain factor at the frequency of maximum gain when the crystal has a length of 10 cm.
(c) Determine the gain coefficient and the single path gain factor of radiation at a wavelength in vacuum of 1 μm.

**7.2 Gain cross section of $Ti^{3+}$ in titanium–sapphire**. Compare the gain cross section of an excited $Ti^{3+}$ ion with the gain cross section of a two-level system that has a naturally broadened line at the frequency of maximum gain coefficient of titanium–sapphire.

**7.3 Two-dimensional gain medium**. The two-level systems of a two-dimensional gain medium have a gain cross section $\sigma_{21} = 1.5 \times 10^{-19}\ \mathrm{m}^2$. The population difference is equal to $N_2^{2D} - N_1^{2D} = 10^{16}\ \mathrm{m}^{-2}$.

(a) Estimate the modal gain coefficient in the case that radiation propagates along the active medium and that the mode has a height of 800 nm.
(b) Estimate the gain for radiation traversing the medium.

**7.4 Anisotropic media**.

(a) We manipulate the two-level atomic systems of an active medium (for example, by applying a magnetic field, so that the atomic dipoles have an orientation mainly in one direction instead of a random orientation); we assume that $A_{21}$ does not change. Determine $B_{21}$ and $\sigma_{21}$ for radiation of different orientations of the electric field vector of electromagnetic radiation.

(b) We assume that we orient the two-level systems with their dipoles in a plane. Determine $B_{21}$ and $\sigma_{21}$ for radiation polarized either parallel or perpendicular to the plane.

**7.5 Oscillator strength**. The classical oscillator model of an atom provides the classical absorption cross section of an atom, $\sigma_{\rm cl}(\nu) = e^2/(4\varepsilon_0 m_0 c) g_{\rm L,res}(\nu)$, according to (9.67)

(a) Show that the classical absorption strength is

$$S_{\rm cl} \equiv \int \sigma_{\rm cl}(\nu)\mathrm{d}\nu = \frac{e^2}{4\,\epsilon_0\, m_0 c}. \tag{7.66}$$

(b) Show that the quantum mechanical absorption strength is equal to

$$S \equiv \int \sigma_{21}(\nu)\mathrm{d}\nu = \frac{n}{c} h\nu B_{21} = \frac{c^2 A_{21}}{8\pi n^2 \nu^2} = \frac{c}{8\pi n^2 \nu^2 \tau_{\rm sp}}. \tag{7.67}$$

(c) We introduce the oscillator strength $f$ via the relation

$$S = S_{\rm cl} \times f. \tag{7.68}$$

Estimate the oscillator strengths, which correspond to the absorption and to the gain cross sections of titanium–sapphire.

(d) Show that in case of a narrow line

$$S = \frac{\lambda_0^2}{8\pi n^3 \tau_{\rm sp}}, \tag{7.69}$$

where $\lambda_0 = c/\nu_0$.

**7.6 Fluorescence line and absorption cross section**.

(a) Show that we can write, in case of a narrow line caused by transitions in an ensemble of two-level atomic systems,

$$\frac{1}{\tau_{\rm sp}} \approx \frac{8\pi c n^2}{\lambda_0^4} \int \sigma_{12}(\lambda)\mathrm{d}\lambda, \tag{7.70}$$

where $\lambda_0$ is the center wavelength, $n$ the refractive index and $\sigma_{12}$ the absorption cross section.

(b) Show that this leads to the relation

$$\sigma_{12}(\lambda) = \frac{\lambda_0^4}{8\pi n^2 \tau_{\rm sp}} \frac{S(\lambda)\mathrm{d}\lambda}{\int S(\lambda)\mathrm{d}\lambda}, \tag{7.71}$$

where $S(\lambda)\mathrm{d}\lambda$ is the fluorescence intensity in the wavelength interval $\mathrm{d}\lambda$ at the wavelength $\lambda$ and $\int S(\lambda)\mathrm{d}\lambda$ is the total fluorescence intensity. This relation is sometimes called Füchtbauer-Ladenburg relation; in the 1920s, Füchtbauer studied absorption lines [48] and Ladenburg (*see* Sect. 9.10) fluorescence lines of atomic gases.

**7.7 Gain saturation**. We consider a four-level laser medium and take into account both pumping and relaxation. Instead of (7.13), we write

$$\mathrm{d}N_2/\mathrm{d}t = r - b_{21}Z(N_2 - N_1) - N_2/\tau^{\star}_{\mathrm{rel}}, \tag{7.72}$$

where $r$ is the pump rate (per unit of volume). We assume that $\tau_{\mathrm{rel}} \ll \tau^{\star}_{\mathrm{rel}}$ and therefore $N_2 \ll N_2$, and find

$$N_2 = N_{2,0}(1 + b_{21}\tau^{\star}_{\mathrm{rel}}Z). \tag{7.73}$$

We introduce the intensity $I = cZh\nu$. It follows that the large-signal gain coefficient is

$$\alpha_{\mathrm{I}} = \alpha/(1 + I/I_s), \tag{7.74}$$

where

$$I_{\mathrm{s}} = c/\left(B_{21}g(\nu)\tau^{\star}_{\mathrm{rel}}\right) \tag{7.75}$$

is the saturation intensity.

(a) Sketch gain curves for $I/I_{\mathrm{s}} = 0; 1; 10$. [*Hint*: in the case of homogeneous broadening, the whole line saturates.]
(b) Determine the saturation intensity For Nd:YAG.
(c) Determine the saturation intensity For titanium–sapphire.

**7.8 Saturation of absorption**.

(a) Consider an ensemble of two-level atomic systems and show that the large-signal absorption coefficient is

$$\alpha_{\mathrm{abs,I}} = \alpha_{\mathrm{abs}}/(1 + I/I_{\mathrm{s}}), \tag{7.76}$$

where $\alpha_{\mathrm{abs}} = -(n/c)h\nu B_{12}g(\nu)(N_2 - N_1)$ is the small-signal absorption coefficient, $I = cZh\nu$ the intensity of radiation, and

$$I_{\mathrm{s}} = c/\left(2B_{12}g(\nu)\tau^{\star}_{\mathrm{rel}}\right) \tag{7.77}$$

the saturation intensity. [*Hint*: begin with (7.26); take into account that the total population density $N_{\mathrm{tot}} = N_2 + N_1$ is constant; introduce the population difference $\Delta N$, with $\Delta N = N_2 - N_1$; then derive the differential equation for

$d(\Delta N)/dt$ and determine the steady state solution; because the lower level remains populated, the saturation intensity is smaller (by a factor two) than in case of a four-level system with a short lifetime of the lower laser level (Problem 7.7).]

(b) Determine $\Delta N$, $N_2$, and $N_1$ for $I = I_s$.
(c) Sketch absorption curves of a transition for $I/I_s = 0$; 1; 10.
(d) Why is the saturation intensity in case of saturation of absorption and in case of gain saturation independent of the populations of the two-level systems?

**7.9** Show that the transition probability for stimulated emission induced by monochromatic radiation in a frequency band $d\nu$ is given by
$w_{21\text{stim}} = B_{21}\rho(\nu)g(\nu)d\nu$.

**7.10** The photon flux in a beam of monochromatic radiation is $\Phi = cZ$, where $Z$ is the photon density.

(a) Show that the transition probability for stimulated emission for an atom in the beam is equal to $w_{21}(\nu) = \sigma(\nu)\Phi$.
(b) Determine the photon flux that is necessary to reach $w_{21} = 10^{-9}$ s for the laser materials mentioned in Table 7.1.

**7.11** Relate the transition probability for stimulated emission to the growth coefficient and to the gain coefficient.

**7.12** **Test of equations**. Compare the dimensions of left and right side of the following equations:
(7.18), (7.23), (7.38), and (7.52).
[*Hint*: for the dimension of $B_{21}$, *see* (6.11) or Table 6.1.]

**7.13** Determine $B_{12}$ for the transition that is responsible for the absorption band of titanium–sapphire for E || c (Fig. 7.7).

**7.14** **Optical thickness and self-absorption**. The optical thickness of a material is defined as the product $\alpha L$, where $\alpha$ is the absorption coefficient and $L$ the length of the material; a material is optically thick if $\alpha L \gg 1$ and optically thin if $\alpha L \ll 1$.

(a) Determine the length of a titanium-sapphire crystal for which $\alpha L = 1$ in the center of the pump band of a TiS laser. [*Hint*: make use of Fig. 7.8]
(b) Determine the thickness of a TiS crystal at which self-absorption of fluorescence radiation strongly influences the fluorescence spectrum.