Chapter 9
Driving a Laser Oscillation

We investigate the role of electric polarization of a laser medium in order to obtain further insight into dynamical processes occurring in a laser. The reader, who does not wish to interrupt the description of a laser and its operation, may skip over this chapter.

We study interaction of a medium with a high frequency field by use of Maxwell's equations. We derive five coupled differential equations of second order. Applying the slowly varying amplitude approximation, we can reduce the equations to five nonlinear differential equations of first order. The equations relate: population difference; amplitude of the field; phase of the field; amplitude of the polarization; and phase of the polarization.

We can reduce the five differential equations to three in the case that transverse relaxation of the polarization is absent and that the three relevant frequencies—laser frequency, atomic transition frequency, and resonance frequency of the laser resonator—coincide with each other. The three equations relate population difference, amplitude of the field, and amplitude of the polarization. The solutions yield the temporal development of population difference, amplitude of the field, and amplitude of the polarization during onset of laser oscillation.

We finally derive, in the slowly varying amplitude approximation, the laser equations in the case that transverse relaxation is present and that the laser frequency is equal to the resonance frequency of the resonator but differs from the atomic transition frequency. We obtain five nonlinear coupled differential equations of first order (Lorenz–Haken equations).

This chapter begins with an introduction of the electric polarization of a medium. We make use of the classical oscillator model of an atom. We derive a classical expression of the dielectric susceptibility, which relates the field in a medium and the polarization of the medium. We determine the classical absorption coefficient of a medium. Comparing the classical absorption coefficient with the absorption coefficient derived earlier by quantum mechanical arguments, we obtain a quantum mechanical expression of the dielectric susceptibility. The susceptibility of an active medium depends

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linearly on the population difference. We mention the Kramers–Kronig relations, which relate real and imaginary part of a physical response function—the polarization of a medium is the response to an external field and the dielectric susceptibility is the corresponding response function.

According to Maxwell's equations, a laser oscillation can be driven either by a high frequency electric polarization or by a high frequency current.

9.1 Maxwell's Equations

To describe the response of a medium to an electromagnetic field, we make use of Maxwell's equations,

$$\nabla \times \boldsymbol{H} = \boldsymbol{j} + \frac{\partial \boldsymbol{D}}{\partial t}, \tag{9.1}$$

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}, \tag{9.2}$$

$$\nabla \cdot \boldsymbol{E} = 0, \tag{9.3}$$

$$\nabla \cdot \boldsymbol{B} = 0. \tag{9.4}$$

$\nabla = (\partial/\partial x, \partial/\partial y, \partial/\partial z)$ is the *del operator* (=*Nabla operator*), $\boldsymbol{E}$ the electric field, $\boldsymbol{D}$ the displacement field, $\boldsymbol{j}$ the electric current density of a high frequency electric current carried by free-electrons, $\boldsymbol{H}$ the magnetic field and $\boldsymbol{B}$ the magnetic induction. We exclude, with $\nabla \cdot \boldsymbol{E} = 0$, local charge accumulations. Material equations provide further relations:

$$\boldsymbol{D}(\boldsymbol{E}) = \varepsilon_0 \boldsymbol{E} + \boldsymbol{P}(\boldsymbol{E}), \tag{9.5}$$

$$\boldsymbol{j} = \boldsymbol{j}(\boldsymbol{E}), \tag{9.6}$$

$$\boldsymbol{B} = \mu_0 \boldsymbol{H}, \tag{9.7}$$

where $\varepsilon_0 = 8.86 \times 10^{-12}\,\mathrm{A\,s\,V^{-1}\,m^{-1}}$ is the electric field constant, $\mu_0 = 4\pi \times 10^{-7}\,\mathrm{V\,s\,A^{-1}\,m^{-1}}$ the magnetic field constant and $\varepsilon_0 \mu_0 = c^{-2}$. We ignore, using the relation $\boldsymbol{B} = \mu_0 \boldsymbol{H}$, magnetic effects. The displacement field is the sum of the field (times ε_0) and the high frequency polarization $\boldsymbol{P}$, which itself depends on the field. The current density depends on the field. The quantities $\boldsymbol{E}$, $\boldsymbol{P}$ and $\boldsymbol{j}$ can depend on time and location. We can write the first Maxwell equation in the form

$$\nabla \times \boldsymbol{H} = \boldsymbol{j} + \varepsilon_0 \frac{\partial \boldsymbol{E}}{\partial t} + \frac{\partial \boldsymbol{P}}{\partial t}. \tag{9.8}$$

We now assume that the response is linear and introduce the *dielectric susceptibility* $\tilde{\chi}$, the *dielectric constant* $\tilde{\varepsilon}$ and the *conductivity* $\tilde{\sigma}$ by the relations:

$$\boldsymbol{P} = \varepsilon_0\, \tilde{\chi}\, \boldsymbol{E}, \tag{9.9}$$

$$\boldsymbol{D} = \varepsilon_0\, \tilde{\varepsilon}\, \boldsymbol{E}, \tag{9.10}$$

$$\tilde{\varepsilon} = 1 + \tilde{\chi}, \tag{9.11}$$

$$\boldsymbol{j} = \tilde{\sigma}\, \boldsymbol{E}. \tag{9.12}$$

The function $\tilde{\chi}$ is a *linear response function.* It characterizes the linear response of the polarization of a dielectric medium to a high frequency electric field. Correspondingly, $\tilde{\varepsilon}$ is the linear response function for the dielectric displacement while $\tilde{\sigma}$ is the linear response function for the current density.

The first Maxwell equation describing linear response is given by

$$\nabla \times \boldsymbol{H} = \boldsymbol{j} + \varepsilon_0\, \frac{\partial(\tilde{\varepsilon}\, \boldsymbol{E})}{\partial t} = \tilde{\sigma}\, \boldsymbol{E} + \varepsilon_0\, \frac{\partial \tilde{\varepsilon}}{\partial t}\, \boldsymbol{E} + \varepsilon_0\, \tilde{\varepsilon}\, \frac{\partial \boldsymbol{E}}{\partial t}. \tag{9.13}$$

When a medium is in thermal equilibrium, then, $\partial\tilde{\varepsilon}/\partial t = 0$. But, the dielectric constant can be time-dependent, $\partial\tilde{\varepsilon}/\partial t \neq 0$, for an active medium—which is always in a nonequilibrium state.

The first Maxwell equation contains the displacement current density

$$\boldsymbol{j}_{\mathrm{d}} = \frac{\partial \boldsymbol{D}}{\partial t} = \varepsilon_0 \frac{\partial \boldsymbol{E}}{\partial t} + \frac{\partial \boldsymbol{P}}{\partial t}, \tag{9.14}$$

which is the sum of the displacement current density $\varepsilon_0 \partial\boldsymbol{E}/\partial t$ and the polarization current density $\partial\boldsymbol{P}/\partial t$. The polarization current density corresponds to the portion of the displacement current density that is due to polarization of a medium.

We assume that high frequency field, polarization and current are oriented along x. A field

$$\tilde{E} = A\, \mathrm{e}^{\mathrm{i}\omega t}, \tag{9.15}$$

gives rise to a polarization

$$\tilde{P} = \varepsilon_0 \tilde{\chi}\, \tilde{E} = P_1 - \mathrm{i}\, P_2 = \varepsilon_0 \chi_1 A \cos\omega t - \mathrm{i}\, \varepsilon_0 \chi_2 A \sin\omega t, \tag{9.16}$$

with

$$\tilde{\chi} = \chi_1 - \mathrm{i}\, \chi_2. \tag{9.17}$$

The real part of the polarization,

$$P_1 = \varepsilon_0 \chi_1\, A\, \cos\omega t, \tag{9.18}$$

has the same phase as the field. The imaginary part $-\mathrm{i}P_2$, where

$$P_2 = \varepsilon_0 \chi_2 \, A \, \sin \omega t \tag{9.19}$$

has a phase of 90° relative to the field. We will see that gain occurs if $\chi_2(\omega)$ is negative. *Note*: throughout the book, we discuss, for convenience, the negative imaginary part (e.g., χ_2) of a complex quantity, which characterizes a material property, rather than the imaginary part ($-\chi_2$) itself.

The polarization is phase shifted relative to the field,

$$P = \varepsilon_0 \chi \, A \, \cos\,[\omega t + \varphi(\omega)], \tag{9.20}$$

where

$$\chi = \sqrt{\chi_1^2 + \chi_2^2} \tag{9.21}$$

is the absolute value of the susceptibility and where φ is the phase between polarization and field. The phase is given by the relation

$$\tan \varphi = \chi_2/\chi_1. \tag{9.22}$$

We now describe the linear response of a conductive gain medium. A high frequency field

$$\tilde{E}(\omega) = A \, \mathrm{e}^{\mathrm{i}\omega t} \tag{9.23}$$

gives rise to a high frequency current of current density

$$\tilde{j} = \tilde{\sigma} \, \tilde{E}, \tag{9.24}$$

where $\tilde{\sigma}$ is the complex high frequency conductivity,

$$\tilde{\sigma} = \sigma_1 - \mathrm{i}\,\sigma_2. \tag{9.25}$$

The current density,

$$\tilde{j} = j_1 - \mathrm{i}\, j_2 = \sigma_1 A \cos \omega t - \mathrm{i}\sigma_2 A \sin \omega t, \tag{9.26}$$

has a real part

$$j_1 = \sigma_1 \, A \, \cos \omega t \tag{9.27}$$

that has the same phase as the field. We will see that gain occurs if $\sigma_1(\omega)$ is negative. The (negative) imaginary part

$$j_2 = \sigma_2 \, A \, \sin \omega t \tag{9.28}$$

has a phase of 90° relative to the field and corresponds to a lossless current. The current is phase shifted relative to the field,

$$j = \sigma\, A\, \cos\left[\omega t + \varphi(\omega)\right], \tag{9.29}$$

where

$$\sigma = \sqrt{\sigma_1^2 + \sigma_2^2} \tag{9.30}$$

is the absolute value of the conductivity and where φ is the phase between current density and field. The phase is given by the relation

$$\tan\varphi = \sigma_2/\sigma_1. \tag{9.31}$$

We can introduce a *generalized dielectric constant* $\tilde{\varepsilon}_{\text{gen}}$ and a *generalized conductivity* $\tilde{\sigma}_{\text{gen}}$. We write the first Maxwell equation in different ways:

$$\nabla \times \boldsymbol{H} = \boldsymbol{j} + \frac{\partial \boldsymbol{D}}{\partial t} = \tilde{\sigma}\tilde{\boldsymbol{E}} + \varepsilon_0\tilde{\varepsilon}\,\frac{\partial \boldsymbol{E}}{\partial t} = (\tilde{\sigma} + \mathrm{i}\omega\varepsilon_0\tilde{\varepsilon})\,\boldsymbol{E} = \tilde{\sigma}_{\text{gen}}\,\boldsymbol{E} = \mathrm{i}\omega\varepsilon_0\tilde{\varepsilon}_{\text{gen}}\,\boldsymbol{E}. \tag{9.32}$$

We obtain the relation

$$\tilde{\sigma}_{\text{gen}} = \sigma_{\text{gen},1} - \mathrm{i}\sigma_{\text{gen},2} = \mathrm{i}\;\omega\varepsilon_0\tilde{\varepsilon}_{\text{gen}} \tag{9.33}$$

or, alternatively,

$$\tilde{\varepsilon}_{\text{gen}} = \varepsilon_{\text{gen},1} - \mathrm{i}\varepsilon_{\text{gen},2} = \frac{\tilde{\sigma}_{\text{gen}}}{\mathrm{i}\omega\varepsilon_0}. \tag{9.34}$$

It follows that

$$\varepsilon_{\text{gen},1} = \varepsilon_1 - \frac{\sigma_2}{\omega\varepsilon_0}, \tag{9.35}$$

$$\varepsilon_{\text{gen},2} = \varepsilon_2 + \frac{\sigma_1}{\omega\varepsilon_0}. \tag{9.36}$$

This formulation is useful for determination of optical constants and other optical properties of a medium in thermal equilibrium or of an active medium too. The complex refractive index $n_1 - \mathrm{i}n_2$ follows from the relation

$$(n_1 - \mathrm{i}n_2)^2 = \varepsilon_{\text{gen},1} - \mathrm{i}\varepsilon_{\text{gen},2}. \tag{9.37}$$

We obtain

$$n_{1,2} = \sqrt{\frac{1}{2}\left(\varepsilon_{\text{gen},1} \pm \sqrt{\varepsilon_{\text{gen},1}^2 + \varepsilon_{\text{gen},2}^2}\right)}. \tag{9.38}$$

9.2 Possibilities of Driving a Laser Oscillation

It follows from Maxwell's equations, with

$$\nabla \times (\nabla \times \boldsymbol{E}) = \nabla \cdot \nabla \cdot \boldsymbol{E} - \nabla^2 \boldsymbol{E}, \tag{9.39}$$

that

$$\frac{\partial^2 \boldsymbol{E}}{\partial t^2} - \frac{1}{\mu_0 \varepsilon_0} \nabla^2 \boldsymbol{E} = -\frac{1}{\varepsilon_0} \frac{\partial^2 \boldsymbol{P}}{\partial t^2} - \frac{1}{\varepsilon_0} \frac{\partial \boldsymbol{j}}{\partial t}. \tag{9.40}$$

On the right side, we have two terms, the second derivative of the polarization and the derivative of the electric current density with respect to time. There are two possibilities to obtain gain:

- A high frequency polarization can be the origin of gain or
- A high frequency electric current can be the origin of gain.

In the following sections (Sects. 9.3–9.6), we will present a model of a dielectric medium. The model is suited to study basic properties of lasers (Sects. 9.7–9.10). The model describes a dielectric medium that shows a homogeneously broadened narrow line.

We will begin with a derivation of the classical susceptibility and the classical absorption coefficient α_{cl} of an ensemble of classical oscillators. A comparison of the classical absorption coefficient with the quantum mechanical expression of the absorption coefficient α_{abs} (Sect. 7.2) will lead to a procedure that allows us to change from classical expressions of the susceptibility to quantum mechanical expressions. The model yields the complex susceptibility of a dielectric medium consisting of an ensemble of two-level atomic systems.

We will show later (in Chaps. 19 and 32) how a high frequency electric current carried by free-electrons can give rise to gain.

9.3 Polarization of an Atomic Medium

We make use of the classical oscillator model to describe the interaction of an atom with an electromagnetic field. An electric field

$$E = \mathrm{Re}\,[\tilde{E}] = \frac{1}{2}\,(A\mathrm{e}^{\mathrm{i}\omega t} + c.c.) = A\cos\omega t, \tag{9.41}$$

excites an oscillator to a forced oscillation described by the equation of motion

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + \beta\frac{\mathrm{d}x}{\mathrm{d}t} + \omega_0^2 x = \frac{q}{m_0}\,E, \tag{9.42}$$

where $q = -e$ is the electron charge (e = elementary charge), m_0 the electron mass, and β the damping constant with respect to the energy; the decay constant with respect to the amplitude is $\beta/2$. We write the displacement as a complex quantity

$$x = \mathrm{Re}[\tilde{x}] = \frac{1}{2}(\tilde{x}\mathrm{e}^{\mathrm{i}\omega t} + c.c.), \tag{9.43}$$

where $\tilde{x}(\omega)$ is a frequency-dependent complex amplitude of the oscillation. We assume that the amplitude (envelope) is slowly varying, $|\mathrm{d}\tilde{x}/\mathrm{d}t| \ll \omega|\tilde{x}|$ (slowly varying envelope approximation, SVEA) and find, with $\beta = \Delta\omega_0$, the solution

$$\tilde{x} = -\frac{e}{m_0}\,\frac{1}{\omega_0^2 - \omega^2 + \mathrm{i}\omega\Delta\omega_0}\,A. \tag{9.44}$$

(We can write

$$\tilde{x} = -\frac{e}{m_0}\,\tilde{G}_{\mathrm{L}}(\omega)\,A, \tag{9.45}$$

where

$$\tilde{G}_{\mathrm{L}}(\omega) = \frac{1}{\omega_0^2 - \omega^2 + \mathrm{i}\omega\Delta\omega_0} \tag{9.46}$$

is the complex Lorentz response function in general form; *see* Sect. 9.11.

In the following, we assume that $\beta \ll \omega_0$, i.e., that $\Delta\omega_0 \ll \omega_0$ and we restrict the frequency ω to a range around ω_0 so that $|\omega - \omega_0| \ll \omega_0$. We obtain from (9.44), with $\omega_0^2 - \omega^2 = 2\omega_0(\omega_0 - \omega)$, the solution

$$\tilde{x} = -\frac{e}{2m_0\omega_0}\,\frac{1}{\omega_0 - \omega + \mathrm{i}\Delta\omega_0/2}\,A. \tag{9.47}$$

An oscillating electron is connected with an oscillating electric dipole moment

$$\tilde{p} = -e\tilde{x}. \tag{9.48}$$

It follows that

$$\frac{d^2 p}{dt^2} + \beta \frac{dp}{dt} + \omega_0^2 p = \frac{e^2}{m_0} E. \tag{9.49}$$

The ansatz

$$p = \frac{1}{2} (\tilde{p}\, e^{i\omega t} + c.c.), \tag{9.50}$$

with the complex amplitude $\tilde{p}$ of the dipole moment, leads to

$$\tilde{p} = \frac{e^2}{2 m_0 \omega_0} \frac{1}{\omega_0 - \omega + i\Delta\omega_0/2} A. \tag{9.51}$$

The dipole moment shows, as the displacement, a resonance at the frequency ω_0.

A medium consisting of an ensemble of two-level atomic systems of density N experiences, under the action of an electric field, the electric polarization

$$\boldsymbol{P} = \sum_{i=1}^{N} \boldsymbol{p}_i, \tag{9.52}$$

where $\boldsymbol{p}_i$ is the electric dipole moment of the ith two-level atomic system and N the number of two-level systems per unit volume. Without a high frequency electric field, the dipole moments are zero, $\boldsymbol{p}_i = 0$, and there is no polarization. Under the action of a high frequency field, atomic dipole moments and polarization can oscillate synchronously to the field.

The differential equation

$$\frac{d^2 P}{dt^2} + \beta \frac{dP}{dt} + \omega_0^2 P = \frac{N e^2}{m_0} E, \tag{9.53}$$

that follows from the equation of motion of a single dipole relates a high frequency polarization P and a high frequency electric field. The ansatz

$$P = \mathrm{Re}\,[\tilde{P}] = \frac{1}{2}(\tilde{P}\, e^{i\omega t} + c.c.), \tag{9.54}$$

where $\tilde{P}$ is the complex amplitude of the polarization, yields

$$\tilde{P} = \frac{N e^2}{2 m_0 \omega_0} \frac{1}{\omega_0 - \omega + i\Delta\omega_0/2} A. \tag{9.55}$$

The polarization has the same frequency as the electric field. The amplitude is proportional to the amplitude of the field that produces the polarization. The polarization has a resonance at ω_0. We find the electric susceptibility

$$\tilde{\chi} = \frac{Ne^2}{2\varepsilon_0 m_0 \omega_0} \frac{1}{\omega_0 - \omega + \mathrm{i}\Delta\omega_0/2}. \tag{9.56}$$

The real part and the (negative) imaginary part of the susceptibility are

$$\chi_1(\omega) = \frac{Ne^2}{2\varepsilon_0 m_0 \omega_0} \frac{\omega_0 - \omega}{(\omega - \omega_0)^2 + (\Delta\omega_0)^2/4} = \frac{\omega_0 - \omega}{\Delta\omega_0/2}\, \chi_2(\omega), \tag{9.57}$$

$$\chi_2(\omega) = \frac{Ne^2}{2\varepsilon_0 m_0 \omega_0} \frac{\Delta\omega_0/2}{(\omega - \omega_0)^2 + (\Delta\omega_0)^2/4} = \frac{N\pi e^2}{2\varepsilon_0 m_0 \omega_0}\, g_{\mathrm{L,res}}(\omega), \tag{9.58}$$

where $g_{\mathrm{L,res}}(\omega)$ is the Lorentz resonance function. To obtain the susceptibility on the frequency scale, we replace $g_{\mathrm{L,res}}(\omega)$ by $(1/2\pi)g_{\mathrm{L,res}}(\nu)$ and ω_0 by $2\pi\nu_0$. We find

$$\chi_1(\nu) = \frac{\nu_0 - \nu}{\Delta\nu_0/2}\, \chi_2(\nu), \tag{9.59}$$

$$\chi_2(\nu) = N \frac{e^2}{8\pi\varepsilon_0 m_0 \nu_0}\, g_{\mathrm{L,res}}(\nu). \tag{9.60}$$

We assumed in our derivation of the susceptibility that all atomic dipoles have the same resonance frequency and the same damping constant. This corresponds to homogeneous line broadening. The damping can be due to emission of radiation (resulting in natural line broadening) or due to other energy relaxation processes that lead to a Lorentzian line.

9.4 Quantum Mechanical Expression of the Susceptibility of an Atomic Medium

We characterize a dielectric medium, which we assume to be optically isotropic, by the complex displacement field

$$\tilde{D} = \varepsilon_0 \tilde{E} + \tilde{P} = \varepsilon_0 \tilde{E} + \varepsilon_0 \tilde{\chi} \tilde{E} = \tilde{\varepsilon} \tilde{E}, \tag{9.61}$$

where $\varepsilon = \varepsilon_1 - \mathrm{i}\varepsilon_2$ is the complex dielectric constant. The real part is $\varepsilon_1 = 1 + \chi_1$ and the imaginary part $\varepsilon_2 = \chi_2$. We obtain the complex refractive index $n_1 - \mathrm{i}n_2$ from the relation

$$(n_1 - \mathrm{i}n_2)^2 = \varepsilon_1 - \mathrm{i}\varepsilon_2. \tag{9.62}$$

The wave vector of a plane wave travelling in z direction is

$$k = (n_1 - \mathrm{i}n_2)\,\frac{\omega}{c}, \tag{9.63}$$

where $n_1 \equiv n$ is the *refractive index*. The complex field of the plane wave is

$$\tilde{E} = A\,\mathrm{e}^{\mathrm{i}(\omega t - kz)} = A\,\mathrm{e}^{-(n_2\omega/c)z}\,\mathrm{e}^{\mathrm{i}[\omega t-(n_1\omega/c)z]}. \tag{9.64}$$

In the vicinity of the center of a Lorentzian line, the imaginary part of the refractive index is $n_2 \approx \frac{1}{2}\chi_2$. At frequencies around the resonance frequency ω_0, the energy density in the wave is

$$u = u_0\mathrm{e}^{-\alpha_{\mathrm{cl}}z}, \tag{9.65}$$

where $u_0 = \frac{1}{2}\varepsilon_0 A^2$ is the energy density at $z = 0$ and

$$\alpha_{\mathrm{cl}}(\omega) = \frac{\pi e^2 N}{2\varepsilon_0 m_0 c}\,g_{\mathrm{L,res}}(\omega) \tag{9.66}$$

is the classical absorption coefficient. The absorption coefficient on the frequency scale is given by

$$\alpha_{\mathrm{cl}}(\nu) = N\frac{e^2}{4\varepsilon_0 m_0 c}\,g_{\mathrm{L,res}}(\nu). \tag{9.67}$$

We now compare this formula with the quantum mechanical expression for the absorption coefficient, which follows from (7.23),

$$\alpha_{\mathrm{abs}}(\nu) = \frac{1}{c}h\nu B_{12}g_{\mathrm{L,res}}(\nu)(N_1 - N_2). \tag{9.68}$$

The two expressions of $\alpha_{\mathrm{cl}}(\nu)$ and $\alpha_{\mathrm{abs}}(\nu)$ are in accord with each other if we replace

$$\frac{Ne^2}{4\varepsilon_0 m_0} \rightarrow (N_1 - N_2)h\nu B_{12},\ \text{or, respectively,}\ (N_1 - N_2)h\omega B_{12}^{\omega} \tag{9.69}$$

on the ω scale. The replacement in the expressions of the susceptibility leads to

$$\chi_1(\omega) = \frac{\omega_0 - \omega}{\Delta\omega_0/2}\,\chi_2(\omega), \tag{9.70}$$

$$\chi_2(\omega) = (N_1 - N_2)\frac{2\pi}{\omega_0}\,b_{21}(\omega), \tag{9.71}$$

where $\Delta\omega_0$ is the halfwidth of the atomic transition and where

$$b_{21}(\omega) = \hbar\omega_0 B_{21}^{\omega} g_{\mathrm{L,res}}(\omega). \tag{9.72}$$

The complex susceptibility $\tilde{\chi} = \tilde{\chi}_1 - \mathrm{i}\tilde{\chi}_2$ is equal to

$$\tilde{\chi}(\omega) = 2\hbar B_{21}^{\omega}(N_1 - N_2)\,\frac{1}{\omega_0 - \omega + \mathrm{i}\Delta\omega_0/2}. \tag{9.73}$$

The susceptibilities on the frequency scale are given by

$$\chi_1(\nu) = \frac{\nu_0 - \nu}{\Delta\nu_0/2}\,\chi_2(\nu) = hB_{21}(N_1 - N_2)\,g_{\mathrm{L,disp}}(\nu), \tag{9.74}$$

$$\chi_2(\nu) = (N_1 - N_2)\frac{1}{\nu_0}\,b_{21}(\nu) = hB_{21}(N_1 - N_2)\,g_{\mathrm{L,res}}(\nu), \tag{9.75}$$

where $b_{21}(\nu) = h\nu B_{21} g_{\mathrm{L,res}}(\nu)$.

The susceptibilities of an inactive medium (Fig. 9.1a) indicate that the interaction of an electric field with a medium is strongest at resonance, where χ_2 has a maximum and χ_1 is zero. Absorption of radiation is strongest at the transition frequency ω_0. In comparison with a nonactive medium, the susceptibilities of an active medium (Fig. 9.1b) have opposite signs. The gain coefficient of an active medium will be largest if the frequency of the radiation is equal to the transition frequency.

We have the important result: the imaginary part of the susceptibility of a medium that shows a homogeneously broadened transition caused by energy relaxation has the shape of a Lorentz resonance function while the real part has the shape of a Lorentz dispersion function.

The replacement of the classical expressions of the susceptibilities by the quantum mechanical expressions in (9.53) leads to the differential equation

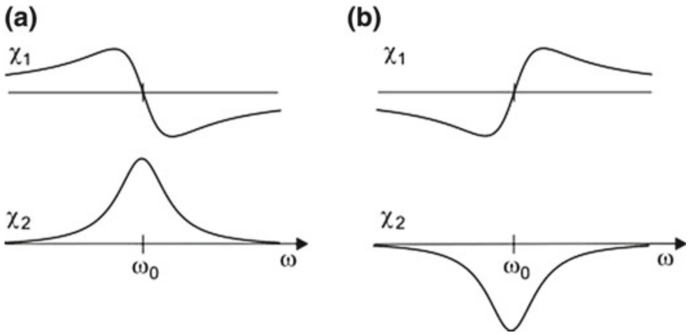


Fig. 9.1 Dielectric susceptibilities **a** of an inactive medium and **b** of an active medium

$$\frac{\mathrm{d}^2P}{\mathrm{d}t^2} + \Delta\omega_0 \frac{\mathrm{d}P}{\mathrm{d}t} + \omega_0^2 P = \frac{2}{\pi}\varepsilon_0 \hbar\omega B_{21}^{\omega}(N_1 - N_2)E. \tag{9.76}$$

P and E are real quantities.

At the center of a resonance line, at the frequency ω_0, the real part of the susceptibility is zero. In the vicinity of ω_0, the real part of the susceptibility is proportional to the frequency difference,

$$\chi_1 = \frac{8}{\Delta\omega_0}\hbar B_{21}^{\omega}(N_1 - N_2)(\omega_0 - \omega). \tag{9.77}$$

It follows that the refractive index varies with frequency—i.e., the medium shows *dispersion*—and that the refractive index is approximately given by

$$n(\omega) = n(\omega_0) + (\mathrm{d}n/\mathrm{d}\omega)_{\omega_0}(\omega_0 - \omega). \tag{9.78}$$

Differentiation of $n_1^2 - n_2^2 = 1 + \chi_1$ leads to $2n_1\mathrm{d}n_1/\mathrm{d}\omega = \varepsilon_0\mathrm{d}\chi_1/\mathrm{d}\omega$ and where we wrote n instead of n_1. Differentiation of $n^2 = 1 + \chi_1$ leads to $2n\ \mathrm{d}n/\mathrm{d}\omega = d\chi_1/d\omega$ and to

$$\left(\frac{\mathrm{d}n}{\mathrm{d}\omega}\right)_0 = -\frac{4}{n\Delta\omega_0^2}\,\hbar B_{21}^{\omega}(N_1 - N_2), \tag{9.79}$$

where $\Delta\omega_0$ is the atomic linewidth.

On the frequency scale, the change of the refractive index is

$$\left(\frac{\mathrm{d}n}{\mathrm{d}\nu}\right)_0 = \frac{-2}{n\pi\Delta\nu_0^2}\,hB_{21}(N_1 - N_2). \tag{9.80}$$

Example Dispersion of an active medium. We estimate the dispersion of optically pumped titanium–sapphire ($N_2 - N_1 = 10^{24}$ m^3; $\mathrm{B}_{21}^{\omega} = 2\pi \times 1.7 \times 10^{18}$ m^3 J^{-1} s^{-2}). In comparison with a Lorentzian line caused by natural broadening, $\mathrm{d}n/\mathrm{d}\omega$ is reduced by a factor $\Delta\omega_0/(1.48\Delta\omega_{\mathrm{nat}}) = 4 \times 10^{10}$; $\Delta\omega_{\mathrm{nat}} = 2.6 \times 10^5\,\mathrm{s}^{-1}$ and $\Delta\omega_0 = 2\pi \times 100\,\mathrm{THz}$. It follows that $\mathrm{d}n/\mathrm{d}\omega = 4 \times 10^{-11}$ s.

We summarize here the results with respect to the complex susceptibility that characterizes a resonance line whose shape is determined by energy relaxation. The complex susceptibility is given by

$$\tilde{\chi} = a\omega\tilde{G}_{\mathrm{L}}, \tag{9.81}$$

where $a = 4(N_1 - N_2)\hbar B_{12}$ is a measure of the strength of the transition and where

$$\tilde{G}_{\mathrm{L}} = G_{\mathrm{L,disp}} - \mathrm{i}G_{\mathrm{L,res}} = \frac{1}{\omega_0^2 - \omega^2 + \mathrm{i}\omega\Delta\omega_0} \tag{9.82}$$

is the (general) complex Lorentz function. At frequencies around a narrow Lorentzian resonance, the susceptibility is $\tilde{\chi} = a'\tilde{g}_L$, where $a' = a/2$ and

$$\tilde{g}_\mathrm{L} = g_\mathrm{L,disp} - \mathrm{i} g_\mathrm{L,res} = \frac{1}{\omega_0 - \omega + \mathrm{i}\Delta\omega_0/2}. \tag{9.83}$$

Many textbooks treat the theory of classical dispersion (*see* preceding section) or present quantum mechanical derivations of the susceptibility; *see*, for instance, [5, 6].

9.5 Polarization of an Active Medium

A field $E = A\cos\omega t$ in a medium produces a polarization. In the special case that the frequency of the field is equal to the resonance frequency of the atomic transition ($\omega = \omega_0$), we obtain

$$P = \varepsilon_0 \chi_2 \sin(\omega_0 t) A = \frac{2\pi\,\varepsilon_0}{\omega_0}(N_1 - N_2)\, b_{21}(\omega_0)\sin(\omega_0 t) A. \tag{9.84}$$

The polarization of an inactive medium (Fig. 9.2a) is delayed (by $\pi/2$ for $\omega = \omega_0$) with respect to the field that creates the polarization while the polarization of an active medium (Fig. 9.2b) is advanced (by $\pi/2$) with respect to the field.

An external electric field in a nonactive medium delivers energy to an ensemble of two-level atomic systems. The power transferred to a single two-level system is equal to

$$\frac{\text{force times path}}{\text{time}} = qE\dot{x} = E\dot{p}. \tag{9.85}$$

The power, averaged over time, a field (frequency $\omega = \omega_0$) delivers to a medium of polarization $P = (N_1 - N_2)p$ is equal to

$$W = < E\,\frac{\mathrm{d}P}{\mathrm{d}t} >_t = \frac{1}{2}\varepsilon_0 \chi_2(\omega_0) A^2 = (N_1 - N_2)\frac{\pi\,\varepsilon_0}{\omega_0} b_{21}(\omega_0) A^2. \tag{9.86}$$

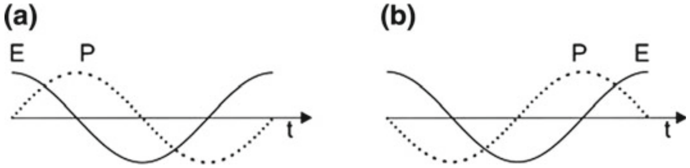


Fig. 9.2 Polarization and electric field at resonance **a** of an inactive medium and **b** of an active medium

The power is negative, $W < 0$, if a medium is active: then the polarization delivers energy to the field. The polarization is maintained by pumping of the active medium. Thus, the polarization mediates gain of the field. Pump power is converted—via the polarization—to power of the high frequency field.

If an active medium interacts with radiation at a frequency that is not the resonance frequency ($\omega \neq \omega_0$), the phase shift between field and polarization is

$$\tan\varphi(\omega) = \chi_2(\omega)/\chi_1(\omega) \tag{9.87}$$

and the power is

$$W(\omega) = W_0 \sin\varphi, \tag{9.88}$$

where

$$W_0 = (N_1 - N_2)\,\frac{\pi\varepsilon_0}{\omega_0}\,b_{21}(\omega_0)A^2 \tag{9.89}$$

is the power delivered by the field for $\omega = \omega_0$. Accordingly, the power transfer is smaller at frequencies outside the resonance frequency, that is, the gain in a laser medium is largest for radiation at the resonance frequency.

In an active medium of a laser, after a sudden turning on of the population difference (at $t = 0$), the atomic dipole moments oscillate with arbitrary phases relative to each other and therefore the polarization is zero (Fig. 9.3, left). An electric field of small amplitude $A(t = 0)$ produces a weak polarization and this enhances the field. The amplitude of the field and the amplitude of the polarization grow together by the mutual interaction of field and polarization until a steady state oscillation is established (right).

The interplay of the radiation and the atomic dipoles results in the growth of both the field and the polarization. With increasing field, the atomic dipole oscillations become more and more synchronized to the field. Accordingly, the polarization becomes more and more able to deliver energy to the field. The energy necessary for the buildup of the field originates from the excitation energy of the ensemble of two-level atomic systems. The initial field that starts oscillation can be due to spontaneous emission of radiation by the ensemble of two-level atomic systems (in the visible, UV, and X-ray range) or due to thermal radiation (in the far infrared spectral region).

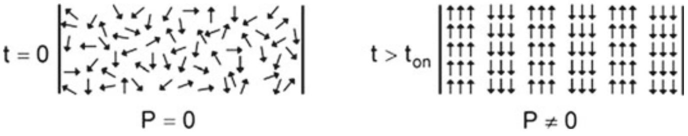


Fig. 9.3 Polarization of an active medium in a laser immediately after generation of population inversion and at steady state oscillation of the laser

At steady state oscillation, synchronization of the field and the polarization is maintained. Loss of polarization is compensated by pumping. The field synchronizes the atomic dipole oscillations that are produced by the pumping.

A high frequency polarization is characteristic of a large variety of excitations used in lasers:

- Electronic excitations of atoms, molecules, or ions.
- Electronic excitations by interband transitions in bipolar semiconductor lasers.
- Electronic excitations by intersubband transitions in quantum cascade lasers.
- Vibrational excitations of molecules.
- Rotational excitations of molecules.

The origin of electric dipole moments are oscillating charges in electronic and vibrational excitations and rotating charges in rotational excitations.

Before we treat the question how the amplitudes of the field and of the polarization build up during the onset of laser oscillation, we introduce the polarization current.

9.6 Polarization Current

If transverse relaxation is absent, we can introduce the *polarization current* characterized by the polarization current density

$$j_{\mathrm{pol}} = Nq\dot{x} = \mathrm{d}P/\mathrm{d}t. \tag{9.90}$$

The polarization current density is equal to the rate of change of the polarization. By differentiation of (9.53) with respect to time and multiplication of the equation by Nq, we obtain (with $q = -e$)

$$\frac{\mathrm{d}^2 j_{\mathrm{pol}}}{\mathrm{d}t^2} + \beta \frac{\mathrm{d}j_{\mathrm{pol}}}{\mathrm{d}t} + \omega_0^2 j_{\mathrm{pol}} = \mathrm{Re}\left[\mathrm{i}\omega \frac{Ne^2}{m_0} A\mathrm{e}^{\mathrm{i}\omega t}\right]. \tag{9.91}$$

With $j_{\mathrm{pol}} = \mathrm{Re}[\tilde{j}_{\mathrm{pol}}]$ and the ansatz

$$\tilde{j}_{\mathrm{pol}} = \tilde{\sigma}^{\mathrm{pol}} \tilde{E}, \tag{9.92}$$

we find the complex *polarization conductivity*

$$\tilde{\sigma}^{\mathrm{pol}} = \frac{Ne^2}{2m_0} \frac{1}{\omega_0 - \omega + \mathrm{i}\Delta\omega_0/2}. \tag{9.93}$$

We write

$$\tilde{\sigma}^{\mathrm{pol}} = \sigma_1^{\mathrm{pol}} - \mathrm{i}\sigma_2^{\mathrm{pol}}. \tag{9.94}$$

Making use of (9.69), we obtain quantum mechanical expressions of the real and imaginary parts of the polarization conductivity,

$$\sigma_1^{\mathrm{pol}}(\omega) = \varepsilon_0 \omega \chi_2(\omega) = (N_1 - N_2) \times 2\pi \varepsilon_0 b_{21}(\omega), \tag{9.95}$$

$$\sigma_2^{\mathrm{pol}}(\omega) = \varepsilon_0 \omega \chi_1(\omega) = \frac{\omega_0 - \omega}{\Delta\omega_0/2}\, \sigma_1(\omega). \tag{9.96}$$

The polarization conductivities on the frequency scale are

$$\sigma_1^{\mathrm{pol}}(\nu) = (N_1 - N_2) \times \varepsilon_0 b_{21}(\nu), \tag{9.97}$$

$$\sigma_2^{\mathrm{pol}}(\nu) = \frac{\nu_0 - \nu}{\Delta\nu_0/2}\, \sigma_1(\nu). \tag{9.98}$$

We can also write

$$\sigma_1^{\mathrm{pol}}(\omega) = 2\pi \varepsilon_0 \omega_0 B_{21}^{\omega} (N_1 - N_2)\, g_{\mathrm{L,res}}(\omega), \tag{9.99}$$

$$\sigma_2^{\mathrm{pol}}(\omega) = \frac{\omega_0 - \omega}{\Delta\omega_0/2}\, \sigma_1(\omega) = 2\pi \varepsilon_0 \omega B_{21}^{\omega} (N_1 - N_2)\, g_{\mathrm{L,disp}}(\omega), \tag{9.100}$$

or

$$\sigma_1^{\mathrm{pol}}(\nu) = 2\pi \varepsilon_0 \nu_0 B_{21} (N_1 - N_2)\, g_{\mathrm{L,res}}(\nu), \tag{9.101}$$

$$\sigma_2^{\mathrm{pol}}(\nu) = \frac{\nu_0 - \nu}{\Delta\nu_0/2}\, \sigma_1(\nu) = 2\pi \varepsilon_0 \nu_0 B_{21} (N_1 - N_2)\, g_{\mathrm{L,disp}}(\nu). \tag{9.102}$$

The real part of the polarization conductivity of a nonactive medium (Fig. 9.4a) is positive and has a Lorentzian shape. The (negative) imaginary part is positive for $\omega < \omega_0$ and negative for $\omega > \omega_0$. The polarization conductivities are dynamical conductivities (=high frequency conductivities). The real part of the polarization conductivity shows a resonance curve with the maximum at ω_0. The signs of the polarization conductivities are reversed in case of an active medium (Fig. 9.4b).

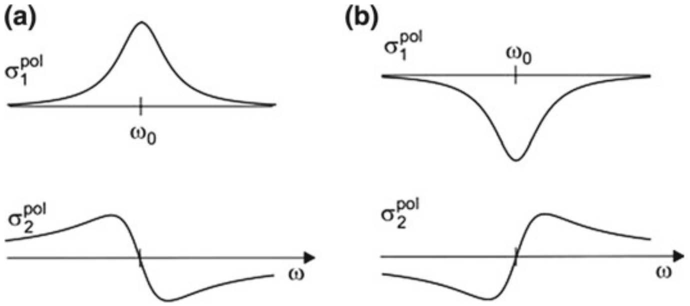


Fig. 9.4 Polarization conductivities **a** of an inactive medium and **b** of an active medium

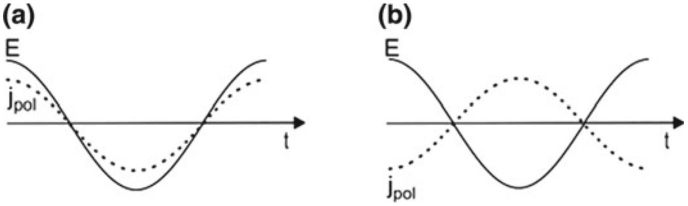


Fig. 9.5 Polarization current and field at resonance **a** of an inactive medium and **b** of an active medium

Thus, we found that the real part of the polarization conductivity of a medium with a homogeneously damped resonance transition has the shape of a Lorentz resonance function and the imaginary part has the shape of a Lorentz dispersion function.

We consider the special case that $\omega = \omega_0$ and that therefore $\sigma_2 = 0$. With $E = A\cos\omega_0 t$, we obtain

$$j_{\mathrm{pol}} = \sigma_1^{\mathrm{pol}}(\omega_0)E = (N_1 - N_2) \times 2\pi\varepsilon_0 b_{21}(\omega_0)\cos(\omega_0 t)\,A. \tag{9.103}$$

The polarization current of a nonactive medium has the same phase as the field (Fig. 9.5a) and $j_{\mathrm{pol}}E$ is positive at any moment. The field experiences damping. Power of the field is converted to heat due to relaxation of the polarization. The polarization current of an active medium has a phase of π relative to the field (Fig. 9.5b). The product $j_{\mathrm{pol}}E$ is negative at any moment. The field is amplified. Pump power is converted to power of the field. Amplification is mediated by the polarization current.

An electric field delivers to an oscillating charge q the

$$\text{power} = \frac{\text{work}}{\text{time}} = qE\dot{x} \tag{9.104}$$

and to an ensemble of electrons the power $j_{\mathrm{pol}}E$. The time average of the power transferred from the field to the polarization current is given by

$$(N_1 - N_2)\sigma_1(\omega_0)A < \cos^2\omega_0 t >_t = (N_1 - N_2) \times \pi\varepsilon_0 b_{21}(\omega_0)A^2. \tag{9.105}$$

To describe the resonance interaction of a medium with radiation, we will make use of both the polarization and the polarization current. The polarization is the fundamental quantity. The use of the polarization current density—in the case that transverse relaxation can be neglected—has the advantage that we can write some expressions in a simpler form, particularly in the special case that $\omega = \omega_0$; then the phase between current and field is either zero or π.

9.7 Laser Oscillation Driven by a Polarization

We now treat the case that the polarization is the origin of gain and that the active medium is a purely dielectric medium ($j = 0$). The wave equation has the form

$$\frac{1}{c^2}\frac{\partial^2 \boldsymbol{E}}{\partial t^2} - \nabla^2 \boldsymbol{E} = -\frac{1}{\varepsilon_0}\frac{\partial^2 \boldsymbol{P}}{\partial t^2}. \tag{9.106}$$

We assume, for simplicity, that the active medium is optically isotropic and fills a Fabry–Perot resonator completely. We also assume that the field does not vary over the cross-sectional area of the resonator. We can write

$$\frac{\partial^2 E(z,t)}{\partial t^2} - c^2\frac{\partial^2 E(z,t)}{\partial z^2} = -\frac{1}{\varepsilon_0}\frac{\partial^2 P(z,t)}{\partial t^2}. \tag{9.107}$$

The field in the resonator represents a standing wave

$$E(z,t) = E(t)\sin kz. \tag{9.108}$$

This leads, with $k^2 = \omega_{\mathrm{res}}^2/c^2$, where ω_{res} is the resonance frequency of the resonator, to the differential equation

$$\left(\frac{\mathrm{d}^2 E}{\mathrm{d}t^2} - \omega_{\mathrm{res}}^2 E\right)\sin kz = -\frac{1}{\varepsilon_0}\frac{\mathrm{d}^2 P(z,t)}{\mathrm{d}t^2}. \tag{9.109}$$

The polarization has the same z dependence as the field. Therefore, we can divide by $\sin kz$, except at the positions where $E(z) = 0$. Thus, we obtain, with $P(z,t) = P(t)\sin kz$ and $P = P(t)$, the differential equation:

$$\frac{\mathrm{d}^2 E}{\mathrm{d}t^2} - \omega_{\mathrm{res}}^2 E = -\frac{1}{\varepsilon_0}\frac{\mathrm{d}^2 P}{\mathrm{d}t^2}. \tag{9.110}$$

In our derivation of the wave equation, we did not include damping of the field that is, for instance, due to output coupling of radiation. We now introduce damping by use of the differential equation of the empty resonator:

$$\frac{\mathrm{d}^2 E}{\mathrm{d}t^2} + \kappa\frac{\mathrm{d}E}{\mathrm{d}t} + \omega_{\mathrm{res}}^2 E = 0, \tag{9.111}$$

where κ is the damping coefficient of the resonator. We assume that $\kappa \ll \omega_{\mathrm{res}}$, i.e., that the field in the empty resonator is given by

$$E = A_0 \mathrm{e}^{-\frac{1}{2}\kappa t}\cos\omega_{\mathrm{res}} t. \tag{9.112}$$

A_0 is the amplitude at $t = 0$. Now, the differential equation describing a field in a Fabry–Perot resonator containing an active medium is given by

$$\frac{\mathrm{d}^2 E}{\mathrm{d}t^2} + \kappa \frac{\mathrm{d}E}{\mathrm{d}t} + \omega_{\mathrm{res}}^2 E = -\frac{1}{\varepsilon_0} \frac{\mathrm{d}^2 P}{\mathrm{d}t^2}. \tag{9.113}$$

We now assume that transverse relaxation of the polarization is absent and treat the particular case that all three frequencies—laser frequency, transition frequency and resonance frequency of the resonator—coincide with each other,

$$\omega = \omega_{\mathrm{res}} = \omega_0. \tag{9.114}$$

We therefore can replace the second derivative of the polarization by the first derivative of the polarization current,

$$\frac{\mathrm{d}^2 P}{\mathrm{d}t^2} = \frac{\mathrm{d}j_{\mathrm{pol}}}{\mathrm{d}t}, \tag{9.115}$$

and obtain the differential equation

$$\frac{\mathrm{d}^2 E}{\mathrm{d}t^2} + \kappa \frac{\mathrm{d}E}{\mathrm{d}t} + \omega^2 E = -\frac{1}{\varepsilon_0} \frac{\mathrm{d}j_{\mathrm{pol}}}{\mathrm{d}t}. \tag{9.116}$$

At a sudden turning on of an initial population difference, the amplitude of the field is time dependent,

$$E(t) = A(t) \cos \omega t. \tag{9.117}$$

Immediately after starting the pumping, the polarization current density is

$$j_{\mathrm{pol}} = \sigma_{1,0}^{\mathrm{pol}} E, \tag{9.118}$$

where

$$\sigma_{1,0}^{\mathrm{pol}} = -\varepsilon_0 b_{21}^0 (N_2 - N_1)_0 \tag{9.119}$$

is the small-signal polarization conductivity, $(N_2 - N_1)_0$ is the initial population difference, and

$$b_{21}^0 = b_{21}(\omega_0) = \hbar\omega_0 B_{21}^{\omega} g_{\mathrm{L,res}}(\omega_0) \tag{9.120}$$

is the growth rate constant at the transition frequency ω_0. It follows that

$$\frac{\mathrm{d}^2 E}{\mathrm{d}t^2} + (-\gamma_0 + \kappa) \frac{\mathrm{d}E}{\mathrm{d}t} + \omega_0^2 E = 0, \tag{9.121}$$

where

$$\gamma_0 = \gamma(\omega_0) = -\frac{1}{\varepsilon_0}\sigma_{1,0}^{\mathrm{pol}} = b_{21}^0 (N_2 - N_1)_0 \tag{9.122}$$

is the growth coefficient of the active medium at ω_0 and

$$\alpha_0 = \alpha(\omega_0) = \frac{\gamma_0}{c/n} = -\frac{n}{c\varepsilon\varepsilon_0}\sigma_{1,0}^{\mathrm{pol}} = \frac{n}{c}\, b_{21}^0\, (N_2 - N_1)_0 \tag{9.123}$$

is the gain coefficient at ω_0. If $\gamma_0 \ll \omega$, the solution of (9.121) is given by

$$E = A_0\, \mathrm{e}^{\frac{1}{2}(\gamma_0 - \kappa)t} \cos \omega t. \tag{9.124}$$

A_0 is the amplitude of the starting field. A very small field e.g., a field corresponding to one photon in the resonator mode, can initiate the oscillation. The amplitude of the field increases exponentially.

We now investigate the large-signal behavior. We assume that the amplitude $A(t)$ is a slowly varying function, i.e., that the envelope of the function $E(t)$ varies slowly (*slowly varying envelope approximation*). This means: the temporal change of the amplitude during one period of the oscillation period is negligibly small,

$$|\mathrm{d}A/\mathrm{d}t| \ll \omega |A(t)|, \tag{9.125}$$

$$|\mathrm{d}^2 A/\mathrm{d}t^2| \ll \omega |\mathrm{d}A/\mathrm{d}t| \ll \omega^2 |A(t)|. \tag{9.126}$$

Thus, we obtain

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \frac{\mathrm{d}A}{\mathrm{d}t}\cos \omega t - \omega A \sin \omega t, \tag{9.127}$$

$$\frac{\mathrm{d}^2 E}{\mathrm{d}t^2} = -2\omega \frac{\mathrm{d}A}{\mathrm{d}t}\sin \omega t - \omega^2 A \cos \omega t. \tag{9.128}$$

In the last equation, we omitted the term $(\mathrm{d}^2 A/\mathrm{d}t^2)\cos \omega t$. Making use of the last two equations, we find from (9.116) the differential equation

$$-2\omega \frac{\mathrm{d}A}{\mathrm{d}t}\sin \omega t - \kappa\omega A \sin \omega t = -\frac{1}{\varepsilon_0}\frac{\mathrm{d}j_{\mathrm{pol}}}{\mathrm{d}t}. \tag{9.129}$$

We neglected the term $\kappa(\mathrm{d}A/\mathrm{d}t)\cos \omega t$ in accordance with the condition $|\mathrm{d}A/\mathrm{d}t| \ll \omega |A(t)|$.

We write the polarization current density in the form

$$j_{\mathrm{pol}} = -J(t) \cos \omega t. \tag{9.130}$$

$J(t)$ is the time-dependent amplitude of the polarization current density. This leads to the differential equation

$$\frac{\mathrm{d}A}{\mathrm{d}t} + \frac{\kappa}{2}A = \frac{1}{2\varepsilon_0}J. \tag{9.131}$$

This differential equation relates the amplitude of the field and the amplitude of the polarization current density. From $j_{\mathrm{pol}} = \sigma_1^{\mathrm{pol}} E$, we find immediately a relation between the amplitude of the current density and the population difference,

$$J = \varepsilon_0 b_{21}^0 (N_2 - N_1) A. \tag{9.132}$$

Another relation follows from the energy conservation law: the change of the population difference by stimulated emission and by relaxation from level 2 to level 1 compensates the power, which the polarization current transfers to the field. We assume that the population of the lower level is negligibly small compared with the population of the upper level. Then we can write

$$\left[\frac{\mathrm{d}}{\mathrm{d}t}(N_2 - N_1) + \frac{(N_2 - N_1)}{\tau_{\mathrm{rel}}^*} - \frac{(N_2 - N_1)_0}{\tau_{\mathrm{rel}}^*}\right] h\nu = j_{\mathrm{pol}} E. \tag{9.133}$$

Neglecting the rapidly varying term in $j_{\mathrm{pol}}E$ and averaging over the temporal and spatial variation,

$$< j_{\mathrm{pol}} E >_{t,z} = JA < \cos^2 \omega t >_t < \sin^2 kz >_z = \frac{1}{4} JA, \tag{9.134}$$

we obtain *three laser equations*

$$\frac{\mathrm{d}A}{\mathrm{d}t} + \frac{\kappa}{2}A = \frac{1}{2\varepsilon_0}J, \tag{9.135}$$

$$J = \varepsilon_0 b_{21}^0 (N_2 - N_1) A, \tag{9.136}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}(N_2 - N_1) + \frac{(N_2 - N_1)}{\tau_{\mathrm{rel}}^*} - \frac{(N_2 - N_1)_0}{\tau_{\mathrm{rel}}^*} = -\frac{1}{4h\nu} JA. \tag{9.137}$$

These relate the amplitude of the field, the amplitude of the current density and the population difference.

By eliminating J and $N_2 - N_1$ from the two first equations and from the first and the third equation, respectively, we obtain

$$\frac{\mathrm{d}A}{\mathrm{d}t} + \frac{\kappa}{2}A = \frac{b_{21}^0}{2}(N_2 - N_1)A, \tag{9.138}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}(N_2 - N_1) + \frac{N_2 - N_1}{\tau_{\mathrm{rel}}^*} - \frac{(N_2 - N_1)_0}{\tau_{\mathrm{rel}}^*} = -\frac{\varepsilon_0}{2h\nu} A \frac{\mathrm{d}A}{\mathrm{d}t} - \frac{\varepsilon_0 \kappa}{4h\nu} A^2. \tag{9.139}$$

From the first of these equations, we obtain

$$N_2 - N_1 = \frac{2}{b_{21}^0}\frac{1}{A}\frac{\mathrm{d}A}{\mathrm{d}t} + \frac{\kappa}{b_{21}^0} \tag{9.140}$$

and, by differentiation,

$$\frac{\mathrm{d}}{\mathrm{d}t}(N_2 - N_1) = \frac{2}{b_{21}^0}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{A}\frac{\mathrm{d}A}{\mathrm{d}t}\right). \tag{9.141}$$

By elimination of $\mathrm{d}(N_2 - N_1)/\mathrm{d}t$ and of $N_2 - N_1$ from (9.138), (9.139), and (9.141), we find

$$a\frac{\mathrm{d}A}{\mathrm{d}t} + \left[\frac{\kappa}{2} - \frac{b_{21}^0(N_2 - N_1)_0}{2}\right]A + \frac{\varepsilon_0 b_{21}^0 \tau_{\mathrm{rel}}^* \kappa}{8h\nu}A^3 = 0. \tag{9.142}$$

The abbreviation

$$a = 1 + \frac{\varepsilon_0 b_{21}^0 \tau_{\mathrm{rel}}^*}{4h\nu}A^2 + \tau_{\mathrm{rel}}^*\frac{1}{A}\frac{\mathrm{d}A}{\mathrm{d}t}, \tag{9.143}$$

contains two terms that are small compared to 1, so that we obtain

$$\frac{\mathrm{d}A}{\mathrm{d}t} + \frac{1}{2}(-\gamma_0 + \kappa)A + \frac{\varepsilon_0 b_{21}^0 \tau_{\mathrm{rel}}^* \kappa}{8h\nu}A^3 = 0. \tag{9.144}$$

The amplitude A increases at small times exponentially and approaches at large times the steady state value

$$A_\infty = 2\sqrt{\frac{(\gamma_0 - \kappa)h\nu}{\varepsilon_0 b_{21}^0 \tau_{\mathrm{rel}}^* \kappa}}. \tag{9.145}$$

The differential (9.144) has the solution

$$A(t) = \frac{A_\infty}{\sqrt{1 + (A_\infty/A_0)^2 \mathrm{e}^{-(\gamma_0 - \kappa)t}}}. \tag{9.146}$$

$A_0 = A(t = 0)$ is the initial amplitude of the field. According to (9.131), the amplitude of the polarization current density is

$$J = \varepsilon_0 \kappa A + 2\varepsilon_0 \frac{\mathrm{d}A}{\mathrm{d}t}. \tag{9.147}$$

It follows from (9.138) that the population difference is equal to

$$N_2 - N_1 = \frac{1}{\varepsilon_0 b_{21}^0} \frac{J}{A}. \tag{9.148}$$

Multiplication of the differential (9.144) by $2A$ leads to

$$\frac{\mathrm{d}}{\mathrm{d}t}(A^2) + (-\gamma_0 + \kappa)A^2 + \frac{\varepsilon_0 b_{21}^0 \tau_{\mathrm{rel}}^* \kappa}{4h\nu} A^4 = 0. \tag{9.149}$$

Taking into account that the energy density of the field is $Zh\nu = \frac{1}{4}\varepsilon_0 A^2$, we obtain

$$\frac{\mathrm{d}Z}{\mathrm{d}t} = (\gamma_0 - \kappa)Z - b_{21}^0 \tau_{\mathrm{rel}}^* \kappa Z^2. \tag{9.150}$$

The equation describes the initial exponential increase of Z as well as transition to the steady state. The photon density at the steady state is

$$Z_\infty = \frac{\gamma_0 - \kappa}{b_{21}^0 \tau_{\mathrm{rel}}^* \kappa} = (r - r_{\mathrm{th}})\tau_p, \tag{9.151}$$

where $\tau_p = \kappa^{-1}$ is the lifetime of a photon in the resonator, $r_{\mathrm{th}} = (N_2 - N_1)_{\mathrm{th}}/\tau_{\mathrm{sp}}$ is the threshold pump rate and $r = (N_2 - N_1)_0/\tau_{\mathrm{sp}}$ is the pump rate. The expression of (9.151) is the same as (8.16), derived earlier.

The differential equation (9.150) has, with $Z(t = 0) = Z_0 \ll Z_\infty$, the solution

$$Z(t) = \frac{Z_\infty}{1 + Z_\infty/Z_0\, \mathrm{e}^{-(\gamma_0 - \kappa)t}}. \tag{9.152}$$

We define the oscillation onset time as the time where

$$Z(t_{\mathrm{on}}) = Z_\infty/2 \tag{9.153}$$

and find a value,

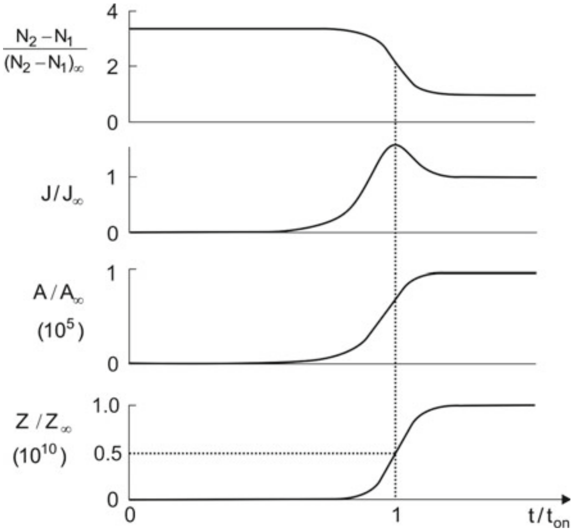
$$t_{\mathrm{on}} = \frac{\ln(Z_\infty/Z_0)}{\gamma_0 - \kappa}, \tag{9.154}$$

which we derived in Sects. 2.9 and 8.4 by simple arguments; *see* (2.85) and (8.29).

Figure 9.6 shows the buildup of laser oscillation at a sudden turning on of the population difference; the numbers concern a helium–neon laser (*see* next example). The curves of the figure indicate the following:

- The initial population difference (produced at $t = 0$) remains almost constant and decreases near t_{on} smoothly to the steady state value.
- The amplitude of the polarization current increases exponentially, shows a maximum at the time $t = t_{\mathrm{on}}$ and then decreases to the steady state value.

Fig. 9.6 Onset of laser oscillation: population difference; amplitude of the polarization current density; amplitude of the electric field; and photon density



- The amplitude of the field increases exponentially at $t < t_{\text{on}}$ and reaches, at $t > t_{\text{on}}$, the steady state value.
- The photon density reaches half the steady state value at $t = t_{\text{on}}$.

It follows from the preceding equations that the steady state amplitude of the polarization current density is equal to

$$J_\infty = \varepsilon_0 A_\infty / \tau_p \tag{9.155}$$

and that the energy density of the polarization is

$$u_{\text{pol}} = \frac{1}{4} J_\infty A_\infty \tau_p = \frac{\varepsilon_0}{4} A_\infty^2 = u. \tag{9.156}$$

At steady state oscillation, the polarization energy density is equal to the energy density of the electric field. During the buildup of laser oscillation, the polarization energy exceeds the electric field energy; the polarization energy is largest at the onset time t_{on}.

The ratio of the initial population difference and the steady state population difference is given by

$$\frac{(N_2 - N_1)_0}{(N_2 - N_1)_\infty} = 1 + b_{21}^0 \tau_{\text{rel}}^* Z_\infty. \tag{9.157}$$

During the buildup of laser oscillation, the polarization conductivity changes from the small-signal value

$$\sigma_{1,0}^{\text{pol}} = -(N_2 - N_1)_0 \varepsilon_0 b_{21}^0 \tag{9.158}$$

to the large-signal value

$$\sigma_{1,\infty}^{\text{pol}} = -(N_2 - N_1)_\infty \varepsilon_0 b_{21}^0. \tag{9.159}$$

The ratio of the small-signal and the large-signal polarization conductivities is equal to the corresponding ratios of the susceptibilities and gain coefficients,

$$\frac{\sigma_{1,0}^{\text{pol}}}{\sigma_{1,\infty}^{\text{pol}}} = \frac{\chi_{1,0}}{\chi_{1,\infty}} = \frac{\alpha_0}{\alpha_\infty} = \frac{(N_2 - N_1)_0}{(N_2 - N_1)_\infty} = 1 + b_{21}^0 \tau_{\text{rel}}^* Z_\infty. \tag{9.160}$$

Example Helium–neon laser: output power 3 mW; $\pi(d/2)^2 L = 10^{-6}\,\text{m}^3$; L = 0.5 m; gain cross section $\sigma_{21} = 1.4 \times 10^{-16}\,\text{m}^2$; $\tau_{\text{rel}}^* = 100\,\text{ns}$; $\tau_p = 1.8 \times 10^{-7}\,\text{s}$. We find the following values:

- $P_{\text{out}} = Z_\infty \pi (d/2)^2\, L\, h\nu/\tau_p$; $Z_\infty = 1.6 \times 10^{15}\,\text{m}^{-3}$.
- $Z_\infty = \varepsilon_0 A_\infty^2/4\, h\nu$; $A_\infty = 1.5 \times 10^4\,\text{V}\,\text{m}^{-1}$.
- $(N_2 - N_1)_0/(N_2 - N_1)_\infty = 1 + c\tau_{\text{rel}}^* \sigma_{21} Z_\infty = 4.1$.
- $(N_2 - N_1)_\infty = (\tau_p b_{21}^0)^{-1} = 1.2 \times 10^{14}\,\text{m}^{-3}$; $b_{21}^0 = c\sigma_{21}$.
- $\sigma_{1,0}^{\text{pol}}/\sigma_{1,\infty}^{\text{pol}} = \chi_{1,0}/\chi_{1,\infty} = \alpha_0/\alpha_\infty = (G_0 - 1)/(G_\infty - 1) = 4.1$.
- $G_\infty = 1.02$; $G_0 = 1.08$.
- $J_\infty = \varepsilon_0 A_\infty/\tau_p = 0.68\,\text{A}\,\text{m}^{-2}$.
- $Z_0 = 10^6\,\text{m}^{-3}$; $Z_\infty/Z_0 = 1.6 \times 10^9$.
- $t_{\text{on}} = T \ln(Z_\infty/Z_0)/\ln(G_0 V) = 22\, T = 720\,\text{ns}$.

A current density-field curve (Fig. 9.7a) is a straight line with a negative slope described by the relation

$$j_{\text{pol}}(t) = \sigma_1^{\text{pol}} E(t). \tag{9.161}$$

It follows from the negative slope that $j_{\text{pol}}(t)$ and $E(t)$ have opposite phases. During the onset of laser oscillation, the negative polarization conductivity varies with time (Fig. 9.7b). The variation is very slow, i.e., σ_1^{pol} is nearly constant during a cycle of the field. The absolute value of the polarization conductivity is large at $t = 0$ and decreases with increasing amplitude of the field until it reaches the steady state value $|\sigma_{1,\infty}^{\text{pol}}| = \varepsilon_0 \kappa$.

That the magnitude of the polarization conductivity σ_1^{pol} changes with time is the consequence of the *quantum mechanical origin of gain*: during the buildup of the laser field, the population difference decreases and therefore the magnitude of σ_1^{pol} decreases.

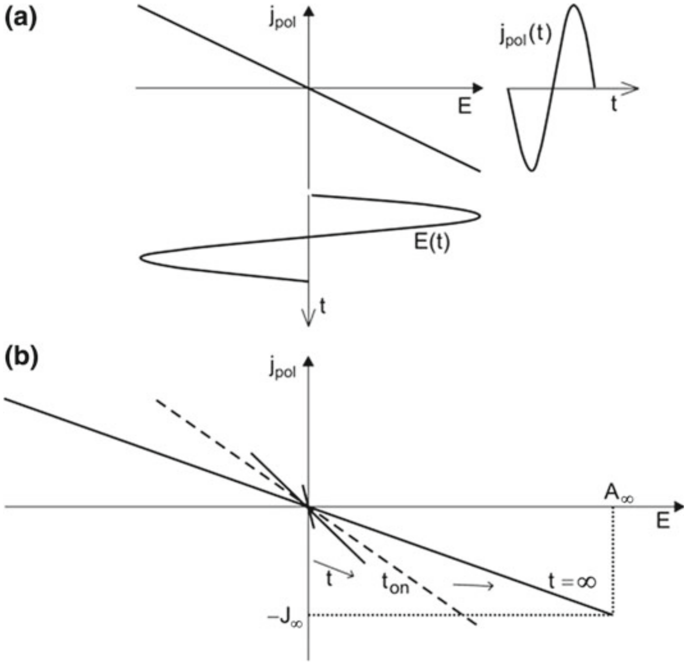


Fig. 9.7 Interplay of polarization current and field. **a** A current density-field curve and time-dependent current and field. **b** Current density-field curves during buildup of laser oscillation

9.8 Relaxation of the Polarization

Relaxation of the polarization occurs in ensembles of excited two-level systems. We consider the case that all atomic oscillators in a medium oscillate, under the action of a laser field, at the same frequency and with the same phase. We suppose that energy relaxation is absent. Then, the dipole moment of each of the oscillators is

$$p_i = p_0 \cos(\omega t + \varphi_0), \tag{9.162}$$

where

$$p_0 = q\, x_0 \tag{9.163}$$

is the amplitude of the dipole moment, +q and –q the oscillating charges, and x_0 the amplitude of the oscillation. The polarization is defined as the dipole moment density,

$$P = \sum_{i=1}^{N_0} p_i = P_0 \cos(\omega t + \varphi_0). \tag{9.164}$$

N_0 is the number of atomic oscillators per unit volume and

$$P_0 = N_0\, p_0 = N_0\, q_0 x_0 \tag{9.165}$$

is the amplitude of the polarization.

We now assume that we suddenly turn off the laser field. Then, dephasing processes can destroy the polarization. We assume that dephasing processes occur randomly in time. The number of oscillators that undergo dephasing in a small time interval dt is

$$\mathrm{d}N = \frac{N}{T_2}\,\mathrm{d}t. \tag{9.166}$$

N is the number of coherently oscillating dipoles per unit volume and T_2 is the dephasing time. It follows that

$$N(t) = N_0\,\mathrm{e}^{-\,t/T_2}. \tag{9.167}$$

Accordingly, the polarization decreases exponentially,

$$P(t) = P_0\,\mathrm{e}^{-\,t/T_2}. \tag{9.168}$$

The polarization decays with the dephasing time T_2, which is called, in connection with the decay of the polarization, *transverse relaxation time* or *phase relaxation time*.

We assumed that the population of excited two-level systems remained constant, i.e., that T_2 was much smaller than the energy relaxation time T_1 of the upper energy level, $T_2 \ll T_1$. In the case that $T_2 \gg T_1$, the change of coherently oscillating dipoles is

$$\mathrm{d}N = \frac{N}{2T_1}\,\mathrm{d}t. \tag{9.169}$$

Now, the polarization is related to the amplitude of the oscillation that decays with the time constant $2T_1$ (while the energy content in the oscillator decays with T_1).

In the intermediate case, $T_2 \approx T_1$, the polarization decays according to

$$P(t) = P_0 \exp\left[\left(-\frac{1}{2T_1} + \frac{1}{T_2}\right) t\right]. \tag{9.170}$$

In connection with the polarization, the time T_1 is called *longitudinal relaxation time* (of the polarization).

9.9 Laser Equations

We study the laser oscillation of a more general case (assuming again that the population difference is suddenly turned on):

- $\omega \neq \omega_{\text{res}} \neq \omega_0$; i.e., laser frequency, resonance frequency of the resonator and transition frequency have different values.
- The polarization undergoes both longitudinal and transverse relaxation.

Our goal is to find differential equations that relate the field, the polarization and the population difference. We make use of the quantities:

- ω = laser frequency.
- ω_{res} = resonance frequency of the laser resonator.
- ω_0 = transition frequency; $\omega_0 = (E_2 - E_1)/\hbar$.
- $\tilde{A}$ = amplitude of the field.
- $\tilde{B}$ = amplitude of the polarization.
- $\Delta N = N_2 - N_1$ = population difference.
- $\Delta N_0 = (N_2 - N_1)_0$ = initial population difference.
- $\kappa/2 = 1/(2\tau_p)$; $2/\kappa$ = lifetime of the field in the laser resonator.
- T_1 = spontaneous lifetime of the population difference = longitudinal relaxation time.
- T_2 = transverse relaxation time.
- $\Delta\omega_0/2 = 1/(2T_1) + 1/T_2$; $2/\Delta\omega_0$ = relaxation time of the polarization.
- $b_{21}^0 = \hbar\omega_0 B_{21}^{\omega} g(\omega_0)$ = growth rate constant at ω_0.

The energy conservation law requires: the change of the population difference times the quantum energy of a photon is equal to the change of the average density of the energy contained in the field and the polarization,

$$\left(\frac{\mathrm{d}\Delta N}{\mathrm{d}t} + \frac{\Delta N - \Delta N_0}{T_1}\right)\hbar\omega = -< E\frac{\mathrm{d}P}{\mathrm{d}t} >_{\mathrm{t,z}}; \tag{9.171}$$

the average is taken over a temporal and a spatial period of the field. Atomic excitation energy is converted to field and polarization energy.

We have seen (in Sect. 9.7) that a polarization can drive a laser oscillation. We now assume that we inject into the laser resonator of an oscillating laser an external high frequency electric field E_1. The external field acts as an additional source term $\kappa_1 \mathrm{d}E_1/\mathrm{d}t$, where κ_1 is a coupling constant.

We make use of laser equations of second order, namely of (9.113) with the additional source term and of (9.76). We then have, together with (9.171), the laser equations:

$$\frac{\mathrm{d}^2 E}{\mathrm{d}t^2} + \kappa \frac{\mathrm{d}E}{\mathrm{d}t} + \omega_{\mathrm{res}}^2 E = -\frac{1}{\varepsilon_0}\frac{\mathrm{d}^2 P}{\mathrm{d}t^2} + \kappa_1 \frac{\mathrm{d}E_1}{\mathrm{d}t}, \tag{9.172}$$

$$\frac{\mathrm{d}^2 P}{\mathrm{d}t^2} + \Delta\omega_0 \frac{\mathrm{d}P}{\mathrm{d}t} + \omega_0^2 P = -\frac{2}{\pi}\varepsilon_0 \hbar\omega B_{21}^{\omega}(N_2 - N_1)E, \tag{9.173}$$

$$\left(\frac{\mathrm{d}\Delta N}{\mathrm{d}t} + \frac{\Delta N - \Delta N_0}{T_1}\right) = -\frac{1}{\hbar\omega} < E\frac{\mathrm{d}P}{\mathrm{d}t} >_{\mathrm{t,z}} . \tag{9.174}$$

The equations are the *semiclassical laser equations* (also called neoclassical laser equations): the atomic states are quantized while the field is treated classically. The equations are suited to describe the dynamics of a laser oscillator.

We make use of the ansatz:

$$E = \frac{1}{2}[\tilde{A}\,\mathrm{e}^{\mathrm{i}\omega t} + c.c.], \quad E_1 = \frac{1}{2}[\tilde{F}\,\mathrm{e}^{\mathrm{i}\omega_1 t} + c.c.], \quad P = \frac{1}{2}[\tilde{B}\,\mathrm{e}^{\mathrm{i}\omega t} + c.c.]. \tag{9.175}$$

The slowly varying envelope approximation

$$\left|\frac{\mathrm{d}\tilde{A}}{\mathrm{d}t}\right| \ll \omega \left|\tilde{A}\right| \quad \text{and} \quad \left|\frac{\mathrm{d}^2\tilde{A}}{\mathrm{d}t^2}\right| \ll \omega \left|\frac{\mathrm{d}\tilde{A}}{\mathrm{d}t}\right|, \tag{9.176}$$

$$\frac{\mathrm{d}\tilde{B}}{\mathrm{d}t} \ll \omega \left|\tilde{B}\right| \quad \text{and} \quad \left|\frac{\mathrm{d}^2\tilde{B}}{\mathrm{d}t^2}\right| \ll \omega \left|\frac{\mathrm{d}\tilde{B}}{\mathrm{d}t}\right|, \tag{9.177}$$

and the restriction to frequencies around ω_0 so that

$$\omega^2 - \omega_0^2 = (\omega + \omega_0)(\omega - \omega_0) \approx 2\omega(\omega - \omega_0), \tag{9.178}$$

leads to the laser equations:

$$\frac{\mathrm{d}\tilde{A}}{\mathrm{d}t} + \left[\frac{\kappa}{2} + \mathrm{i}(\omega - \omega_{\mathrm{res}})\right]\tilde{A} = -\frac{i\omega}{2\varepsilon_0}\tilde{B} + \frac{\kappa_1}{2}\tilde{F}, \tag{9.179}$$

$$\frac{\mathrm{d}\tilde{B}}{\mathrm{d}t} + \left[\frac{\Delta\omega_0}{2} + \mathrm{i}(\omega - \omega_0)\right]\tilde{B} = \varepsilon_0 b_{21}^0 \Delta N \tilde{A}, \tag{9.180}$$

$$\frac{\mathrm{d}\Delta N}{\mathrm{d}t} + \frac{\Delta N - \Delta N_0}{T_1} = -\frac{1}{4\hbar}(\tilde{A}\tilde{B}^* - \tilde{A}^*\tilde{B}). \tag{9.181}$$

The laser equations are coupled differential equations relating: amplitude of the field, phase of the field; amplitude of the polarization, phase of the polarization; population difference. The equations take into account the dephasing of the polarization by transverse relaxation.

We introduce amplitudes and phases,

$$\tilde{A}(t) = A(t)\mathrm{e}^{\mathrm{i}\varphi(t)}, \quad \tilde{F}(t) = F(t)\mathrm{e}^{\mathrm{i}\varphi_1(t)}, \quad \tilde{P}(t) = [C(t) - \mathrm{i}S(t)]\mathrm{e}^{\mathrm{i}\varphi(t)}. \tag{9.182}$$

and obtain five laser equations (in slowly varying envelope approximation):

$$\frac{\mathrm{d}A}{\mathrm{d}t} + \frac{\kappa}{2}A = \frac{\omega}{2\varepsilon_0}S + \frac{\kappa_1}{2}F\cos(\varphi - \varphi_1), \tag{9.183}$$

$$\frac{\mathrm{d}\varphi}{\mathrm{d}t} + \omega - \omega_0 = -\frac{\omega}{2\varepsilon_0}\frac{C}{A} - \frac{\kappa_1}{2}\frac{F}{A}\sin(\varphi - \varphi_1), \tag{9.184}$$

$$\left(\frac{\mathrm{d}}{\mathrm{d}t} + \frac{\Delta\omega_0}{2}\right)C + \left(\frac{\mathrm{d}\varphi}{\mathrm{d}t} + \omega - \omega_0\right)S = 0, \tag{9.185}$$

$$\left(\frac{\mathrm{d}}{\mathrm{d}t} + \frac{\Delta\omega_0}{2}\right)S - \left(\frac{\mathrm{d}\varphi}{\mathrm{d}t} + \omega - \omega_0\right)C = -\,\varepsilon_0 b_{21}^0 \Delta N A, \tag{9.186}$$

$$\frac{\mathrm{d}\Delta N}{\mathrm{d}t} + \frac{\Delta N - \Delta N_0}{T_1} = -\,\frac{1}{4\hbar}AS. \tag{9.187}$$

These five equations describe dynamical processes in laser oscillators.

Example Injection locking of a laser. Injection locking (=frequency locking = phase locking) of a self-excited oscillator means that an external high frequency field, injected into the resonator of the oscillator, forces the oscillator to assume the frequency of the external field rather than to execute an oscillation at its "natural" frequency of the free running laser. At the same time, the field in the resonator has a fixed phase relative to the external field. Injection of an external field (frequency ω_L) of a small amplitude can force a laser (natural frequency ω_L) to oscillate at ω_1. The natural frequency is not necessarily the frequency of maximum gain but is the frequency that is determined by the corresponding condition (2.80). The small-power laser acts as "seed" laser (=master oscillator) of the large-power laser (=slave laser) as shown in Fig. 9.8a. The frequency of the seed laser can be chosen in a range around the frequency of maximum gain (Fig. 9.8b). The power of the frequency-locked laser is the same as that of the free running laser. Injection of a monochromatic field results in a large occupation number of photons of frequency ω_1. Thus, the laser starts oscillation at ω_1 rather than at ω_0. If a laser is already oscillating at ω_L and the seed laser starts, the laser frequency "jumps" to ω_1. Injection locking by use of a frequency stabilized small-power laser can result in a stabilization of the large-power laser with respect to frequency and output power. The minimum power necessary for phase locking and the locking range can be derived by use of the five laser equations; a detailed treatment of injection locking can be found in [1].

Frequency locking of an oscillator is a general phenomenon. Two mechanical oscillators that are weakly coupled can force themselves to oscillate at a fixed phase. In 1865, Huygens observed phase locking of two pendulum clocks fixed at a wall and mechanically coupled via the wall. Injection locking of an electrical classical self-excited oscillator (Sect. 31.8) is well-known: a high frequency current flowing through the active element of a self-excited classical oscillator is superimposed with

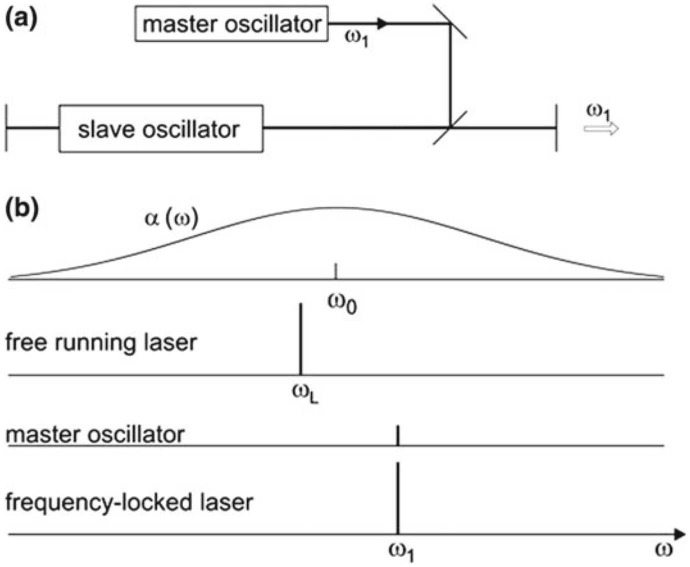


Fig. 9.8 Injection locking. **a** Seed laser (*master oscillator*) and laser. **b** Gain curve, free-running laser, external source, and frequency-locked laser

an external high frequency current. Frequency locking of a laser oscillator occurs via the influence of the electric field on the polarization of the active medium according to (9.172).

In the following, we consider the case that an external field is absent and that the laser frequency is equal to the resonance frequency of the resonator ($\omega = \omega_{\mathrm{res}}$) and, furthermore, that the phase relaxation time is much smaller than the energy relaxation time. We obtain the equations

$$\frac{\mathrm{d}\tilde{A}}{\mathrm{d}t} = -\frac{\tilde{A}}{2\tau_p} - \frac{\mathrm{i}\omega}{2\varepsilon_0}\tilde{B}, \tag{9.188}$$

$$\frac{\mathrm{d}\tilde{B}}{\mathrm{d}t} = -\frac{\tilde{B}}{T_2} + \mathrm{i}(\omega - \omega_0)\tilde{B} + \varepsilon_0 b_{21}^0 \Delta N \tilde{A}, \tag{9.189}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\Delta N = -\frac{1}{T_1}(\Delta N - \Delta N_0) - \frac{\mathrm{i}}{4\hbar}(\tilde{A}\tilde{B}^* - \tilde{A}^*\tilde{B}). \tag{9.190}$$

We introduce dimensionless variables.

- $\tau = t/T_2$ = dimensionless time.
- $\tilde{x} = \tilde{A}/K_{\mathrm{E}}$ = dimensionless amplitude of the field.
- $\tilde{y} = \tilde{B}/K_{\mathrm{P}}$ = dimensionless amplitude of the polarization.
- $z = (\Delta N_0 - \Delta N)/K_{\mathrm{N}}$; $\Delta N_0/K_{\mathrm{N}}$ = dimensionless initial population inversion; $\Delta N/K_3$ = dimensionless population inversion.

With

$$K_\mathrm{E} = \frac{1}{T_2\sqrt{\varepsilon_0 B_{21}}}, \quad K_\mathrm{P} = \frac{1}{\omega\tau_p T_2}\sqrt{\frac{\varepsilon_0}{B_{21}}}, \quad K_\mathrm{N} = \frac{1}{\hbar\omega T_2 \tau_p B_{21}}, \tag{9.191}$$

$b = T_2/T_1$, $\sigma = T_2/2\tau_p$, $\delta = \mathrm{i}(\omega - \omega_0)T_2$, and $r = \omega T_2 \tau_p b_{21}^0 \Delta N_0$, we find the *Lorenz-Haken equations*

$$\frac{\mathrm{d}\tilde{x}}{\mathrm{d}\tau} = -\sigma\tilde{x} + \sigma\tilde{y}, \tag{9.192}$$

$$\frac{\mathrm{d}\tilde{y}}{\mathrm{d}\tau} = -(1 - \mathrm{i}\delta)\tilde{y} + r\tilde{x} - \tilde{x}z, \tag{9.193}$$

$$\frac{\mathrm{d}z}{\mathrm{d}\tau} = -bz + \mathrm{Re}[\tilde{x}^*\tilde{y}]. \tag{9.194}$$

Edward Lorenz derived the equations to describe the dynamics of a convective fluid of the atmosphere and Hermann Haken derived the equations to describe laser dynamics. The equations are the basis of studies of laser dynamics, including chaotic behavior [41–45].

9.10 Laser-van der Pol Equation

According to the last section, we can characterize laser oscillation (if transverse relaxation is absent) in slowly varying envelope approximation by the equations:

$$E(t) = A(t)\cos\omega t, \tag{9.195}$$

$$\frac{\mathrm{d}A}{\mathrm{d}t} - \frac{1}{2}(\gamma_0 - \kappa)A + \frac{\varepsilon_0 b_{21}^0 \tau_\mathrm{rel}^* \kappa}{8h\nu}A^3 = 0. \tag{9.196}$$

This is the *laser-van der Pol equation*. It is a nonlinear differential equation of first order for the amplitude A.

We have found in Sect. 9.7 that the laser equations lead, in slowly varying envelope approximation, to analytical expressions for the time dependences of the amplitude of field, (9.161) and, furthermore of the amplitude of the polarization and of the polarization-current density

$$j_\mathrm{pol} = -\varepsilon_0 \kappa A(t)\cos\omega t \tag{9.197}$$

and of the population difference and the population difference

$$N_2 - N_1 = \frac{\kappa}{b_{21}^0} \frac{1}{A(t)}. \tag{9.198}$$

Thus, the analytical solution of (9.169) for $A(t)$, equation (9.146), provides also analytical expressions for the polarization current (and therefore of the polarization) and of the population difference.

The laser-van der Pol equation is always applicable for description of a laser oscillation (if transverse relaxation is absent). The reason is the following: for all lasers, the gain per period T of the laser field is small compared with unity, $(\gamma_0 - \kappa)T \ll 1$. This is the condition for the applicability of the slowly varying envelope approximation.

We have derived the laser-van der Pol equation by using the condition that laser frequency, atomic transition frequency, and resonance frequency of the resonator coincide with each other. The equation is, however, also applicable for a high frequency field composed of fields of different frequencies if the frequencies have values near the frequency of maximum gain and near the eigenfrequency of the laser resonator, i.e., if the gain curve is sufficiently broad and the Q value of the laser resonator is sufficiently small. The van der Pol equation of the laser is suitable to describe, for instance, mode beating in a laser that oscillates on two modes at the same time; *see*, for instance, [5, 40–42]. It can also be applied to treat frequency locking.

The van der Pol equation in slowly varying amplitude approximation,

$$\frac{\mathrm{d}A}{\mathrm{d}t} - \frac{1}{2}(\gamma_0 - \kappa)A + \beta A^3 = 0, \tag{9.199}$$

is a differential equation of first order. It describes an oscillation of the quantity $y = A(t)\cos \omega t$, where $A(t)$ is a slowly varying amplitude, γ_0 is a growth coefficient, κ is a damping term, and β is a term that follows from the mechanism responsible for oscillation. The equation has two solutions. One of the solutions is $A = 0$. It follows that the occurrence of oscillation supposes an initial amplitude that is nonzero. A small initial amplitude causes exponential growth of the amplitude. The A^3 term is responsible for limitation of the amplitude. In a laser, the initial amplitude of the laser field stems from spontaneous emission of radiation by the active medium. The equation holds only if the net gain per oscillation period is small compared with 1, that is, if $(\gamma_0 - \kappa)T \ll 1$, as already mentioned.

The three equations (9.196) through (9.198) are the most simple laser equations but represent a complete set of laser equations.

The differential equation (9.196) is also suitable, to analyze oscillation of a classical model oscillator if $(\gamma_0 - \kappa)T \ll 1$. In a classical oscillator model, the parameters γ_0, κ, and β represent quantities, which characterize mechanisms that differ completely from the mechanisms in a laser (*see* Chap. 31). The equation (9.196) is the slowly varying envelope approximation of the (general) van der Pol equation that is a nonlinear differential equation of second order (*see* Sect. 31.8 and Problems to this chapter). In contrast to laser oscillators, classical oscillators work in most cases with a much larger gain per period. Thus, a laser oscillator is *the* application of the classical van der Pol equation in slowly varying envelope approximation. (The van der Pol oscillator has been reported by *van der Pol* [243] long before the invention of the laser.)

9.11 Kramers–Kronig Relations

The Kramers–Kronig relations relate the real part of a linear response function and the imaginary part. If the real part of a linear response function is known for all frequencies, then the imaginary part can be calculated for all frequencies. And if the imaginary part of a quantity is known for all frequencies, the real part can be calculated for all frequencies. The Kramers–Kronig relations for the complex susceptibility are

$$\chi_2(\omega) = -\frac{2\omega}{\pi}\int_0^\infty \frac{\chi_1(\omega')}{\omega'^2-\omega^2}\,\mathrm{d}\omega', \tag{9.200}$$

$$\chi_1(\omega) = -\frac{2}{\pi}\int_0^\infty \frac{\omega'\chi_2(\omega')}{\omega'^2-\omega^2}\,\mathrm{d}\omega'. \tag{9.201}$$

Example If a susceptibility has the form of a complex (general) Lorentz function

$$\tilde{\chi}(\omega) = \chi_1 - \mathrm{i}\chi_2 = a\omega\tilde{G}_\mathrm{L}(\omega), \tag{9.202}$$

where a is a measure of the strength of the corresponding transition and

$$\begin{aligned}\tilde{G}_\mathrm{L}(\omega) &= \frac{1}{(\omega_0^2-\omega^2)+\mathrm{i}\omega\Delta\omega_0}\\ &= \frac{\omega_0^2-\omega^2}{(\omega_0^2-\omega^2)^2+(\omega\Delta\omega_0)^2}+\mathrm{i}\frac{\omega\Delta\omega_0}{(\omega_0^2-\omega^2)^2+(\omega\Delta\omega_0)^2},\end{aligned} \tag{9.203}$$

is the complex general Lorentz function. An electric field $\tilde{E} = A\mathrm{e}^{\mathrm{i}\omega t}$ causes a polarization $\tilde{P} = \varepsilon_0(\chi_1 - \mathrm{i}\chi_2)\tilde{E}$. The susceptibilities χ_1 and χ_2 are related according to (9.200) and (9.201).

The Kramers–Kronig relations are a consequence of causality. The Dutch physicists Kramers [46] and Kronig [47] derived the relations independently from each other. The relations are treated in many textbooks on Solid State Physics and Optics; *see*, for instance, [59, 177, 179, 180, 184, 297, 302].

9.12 Lorentz Functions: A Survey

As already mentioned, in mathematics, the Lorentzian function is defined as

$$f_{\mathrm{L}}(x) = \frac{1}{\pi} \frac{\Delta x_0/2}{(x_0 - x)^2 + \Delta x_0^2/4}, \tag{9.204}$$

$$\int_{-\infty}^{\infty} f_{\mathrm{L}}(x)\mathrm{d}x = 1. \tag{9.205}$$

Besides the normalization condition, the values of x_0 and Δx_0 are not restricted.

For a narrow resonance function, with $\Delta x_0 \ll x_0$, the integral from zero to infinity is approximately unity. We use this approximation of the Lorentzian function to describe narrow resonances in physical systems and designate the function as *Lorentz resonance function*, $g_{\mathrm{L,res}}(x)$.

In the case that a resonance is **not** narrow, we use the function that we call *general Lorentz resonance function*. The Kramers–Kronig relations connect the general Lorentz resonance function with the *general Lorentz dispersion function*.

In the book, we use the following functions.

- $\tilde{G}_{\mathrm{L}} = G_{\mathrm{L,res}} - \mathrm{i}\, G_{\mathrm{L,disp}}$, general complex Lorentz function.
- $G_{\mathrm{L,res}}$, general Lorentz resonance function (=real part of the general Lorentz function).
- $G_{\mathrm{L,disp}}$, general Lorentz dispersion function (=imaginary part of the general Lorentz function).
- $\bar{G}_{\mathrm{L,res}}$, normalized general Lorentz resonance function.
- $\bar{G}_{\mathrm{L,disp}}$, "normalized" general Lorentz dispersion function; normalized is the corresponding real part.
- $\tilde{g}_{\mathrm{L}} = g_{\mathrm{L,res}} - \mathrm{i}\, g_{\mathrm{L,disp}}$, complex Lorentz function.
- $g_{\mathrm{L,res}}$, Lorentz resonance function.
- $g_{\mathrm{L,disp}}$, Lorentz dispersion function.
- $\bar{g}_{\mathrm{L,res}}$, normalized Lorentz resonance function.
- $\bar{g}_{\mathrm{L,disp}}$, "normalized" Lorentz dispersion function; normalized is the corresponding Lorentz resonance function.

Table 9.1 gives the functions on the ω scale.

The Lorentz resonance function describes the frequency dependence of the gain coefficient of active media based on dipole oscillators (Table 9.2), i.e., of conventional lasers containing two-level atomic systems as the elementary systems. The Lorentz dispersion function describes the gain coefficient of an active medium in a free-electron laser containing monopole oscillators, or in a quantum mechanical description, containing energy-ladder systems. The monopole oscillations in a free-electron laser (Chap. 19) are narrow-band oscillations.

The elementary systems in a Bloch laser (Chap. 32) are energy-ladder systems, or, in a classical description, monopole oscillations of electrons. The gain function

Table 9.1 Lorentz functions

	Resonance function	Dispersion function
G_L	$\frac{\omega\,\Delta\omega_0}{(\omega_0^2-\omega^2)^2+(\omega\,\Delta\omega_0)^2}$	$\frac{\omega_0^2-\omega^2}{(\omega_0^2-\omega^2)^2+(\omega\,\Delta\omega_0)^2}$
$\bar{G}_L$	$\frac{\omega^2\Delta\omega_0^2}{(\omega_0^2-\omega^2)^2+(\omega\,\Delta\omega_0)^2}$	$\frac{(\omega_0^2-\omega^2)\,\omega\,\Delta\omega_0}{(\omega_0^2-\omega^2)^2+(\omega\,\Delta\omega_0)^2}$
g_L	$\frac{1}{\pi}\,\frac{\Delta\omega_0/2}{(\omega_0-\omega)^2+\ \Delta\omega_0^2/4}$	$\frac{1}{\pi}\,\frac{(\omega_0-\omega)}{(\omega_0-\omega)^2+\ \Delta\omega_0^2/4}$
$\bar{g}_L$	$\frac{\Delta\omega_0^2/4}{(\omega_0-\omega)^2+\ \Delta\omega_0^2/4}$	$\frac{(\omega_0-\omega)\,\Delta\omega_0/2}{(\omega_0-\omega)^2+\ \Delta\omega_0^2/4}$

Table 9.2 Shape of gain curves of different types of lasers

	Type of laser	Elementary system	Classical model
$g_{L,res}$	Conventional lasers	Two-level systems	Dipole oscillation
$\bar{g}_{L,disp}$	free-electron laser	Energy-ladder system	Monopole oscillations of electrons
$\bar{g}_{L,disp} - K(\omega)$	Bloch laser	Energy-ladder system	Monopole oscillations of electrons

is a modified Lorentz dispersion function, $\bar{g}_{L,\,disp} - K(\omega)$. This function takes into account that the response of an active medium of a Bloch laser is composed of a dynamic part, described by $\bar{g}_{L,\,disp}$, and a term, $K(\omega)$, which takes into account that a direct current is always present in a Bloch laser.

9.13 A Third Remark About the History of the Laser

We come back to the question: why did it take—after the discovery of stimulated emission by Einstein—40 years until the maser and the laser were invented? The laser is an apparatus developed by experimentalists. We consider the development of spectroscopy after stimulated emission became known. In the time from 1900 to 1930, Berlin was a center of spectroscopy, with two outstanding spectroscopists: Heinrich Rubens (1865–1922) and Rudolf Ladenburg (1882–1952). Rubens developed methods suited to study the far infrared spectral range. His result (1900) with respect to the spectral distribution of radiation emitted by a thermal radiation source was a basis of the derivation of Planck's radiation law. One of Rubens' Ph.D students, Marianus Czerny (1896–1985), studied the (far infrared) rotational spectrum of HCl [49]; he found that the positions of the absorption lines did not agree with predictions of classical physics. Richard Tolman (Caltech, Pasadena) analyzed in 1924 [50] Czerny's data (that he knew before publication by Czerny) with respect to the strength of absorption, taking into account thermal populations of energy levels, which had transition energies of the order of kT. Tolman included in his analysis stimulated emission (then called "negative absorption"). Tolman discussed in his lectures about quantum mechanics the three processes described by Einstein. Thus, stimulated emission was known in the physics community (*see* [22]).

Based on the quantum theory of Einstein and making use of the correspondence principle, Hendrik A. Kramers [51] developed a theory of the refractive index of gases taking account of stimulated emission. He showed that the refractive index in the vicinity of a resonance line of an atomic gas is expected to decrease if a portion of atoms is in the excited state, i.e., if the population difference is reduced. The effect that is due to stimulated emission was called "negative dispersion". In 1928, Rudolf Ladenburg and coworkers [52–56] found experimentally that stimulated emission resulted in a reduction of the refractive index of excited neon. Making use of an interferometric technique combined with a spectral analysis, Ladenburg and coworkers measured the change of refractive index of a gas (neon), contained in a long tube, that was excited by a gas discharge. The refractive index in the neighborhood of lines decreased, at strong current, with increasing discharge current, i.e., with increasing excitation. Thus, Ladenburg and coworkers performed experimental studies of energy levels of gases in nonequilibrium states. Ladenburg studied (beginning in 1908) atomic gases by analyzing emission spectra excited by gas discharges. He emigrated in 1928 to the USA (becoming professor at Princeton University).

In Germany, the activities in the field of experimental spectroscopy were strongly reduced after the stock market crash in 1929 and when Hitler came to power. In the U.S.A. and other countries, the Great Depression resulted in a reduced investment in physics. It seems that atomic physics was in principal understood at the end of the 1920s. During the 1930 s and especially after the discovery of nuclear fission, the field of nuclear physics became most attractive for physicists. Great interest in spectroscopy (including microwave spectroscopy) began with the discovery of masers and lasers.

Townes writes in his memoirs [22]: "By the 1950s, then, the idea of getting amplification by stimulated emission of radiation was already recognized here and there, but for one reason or another, nobody really saw the idea's potency or published it, except for me and the Russians [Basov and Prokhorov], whose work was then unknown for me." The essential new idea, besides the idea to make use of atomic transitions in a system with population inversion, was the idea—introduced by Townes—to use a resonator in order to realize a self-excited oscillator. The next step, toward the optical maser (laser), was the idea to use an optical resonator, i.e., a resonator without sidewalls.

Was there a chance to invent the laser already in the late 1920s? In his first paper mentioning negative dispersion [52], Ladenburg reported the formula of the refractive index of a gas near a resonance line but outside the range of absorption,

$$n_1 - 1 = \frac{\lambda_0^5}{\lambda - \lambda_0}\,\frac{1}{16\pi c}\,A_{21}\left(\frac{g_2}{g_1}N_1 - N_2\right), \tag{9.206}$$

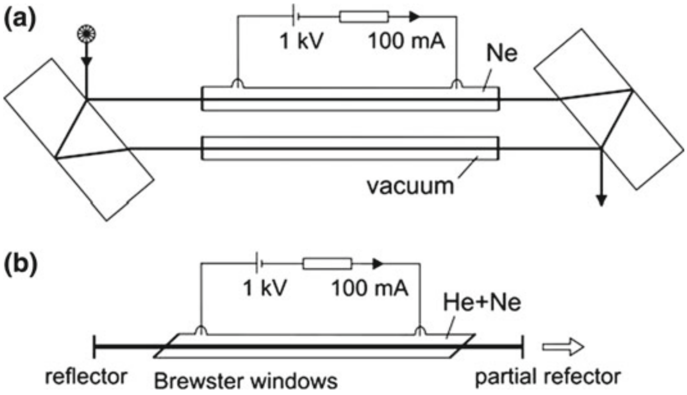


Fig. 9.9 A comparison. **a** Ladenburg's arrangement used for measurement of negative dispersion of a neon gas. **b** Helium–neon laser

where g_1 is the degeneracy of the lower level, g_2 the degeneracy of the upper level, λ the wavelength of the radiation, and $\lambda_0 = c/\nu_0$. Equation (9.206) follows from (9.77) if degeneracy of energy levels is taken into account. Ladenburg wrote (translated from German): "It is one of the most important tasks to detect experimentally Kramers' negative dispersion, whose theoretical importance is unquestionable."

Ladenburg and coworkers used an arrangement [53] shown in Fig. 9.9a. The optical arrangement consisted of a *Jamin interferometer*. One arm of the interferometer contained a tube filled with neon gas (at low pressure) and the other arm contained an evacuated reference tube. A beam of white light from an arc discharge lamp was divided into two beams (one traversing the tube with the gas and the other traversing the empty tube). Then the two beams were superimposed and passed a diffraction grating. The interference pattern on a photo plate contained information on the frequency dependence of the refractive index. At large current, both the lower and the upper level of a resonance transition were occupied. The results [54–56] indicated that the refractive index decreased as predicted by theory. The relative population difference $(N_1 - N_2 g_1/g_2)/N_1$ was most likely between ten and forty percent. The arrangement had similarities to that of a helium–neon laser (Fig. 9.9b). The dimension of the helium tube, the gas pressure, and the strength of current were similar. There are two important differences: the helium–neon laser contains, in addition to neon, also helium and the gas tube is enclosed between highly reflecting mirrors; additionally, the gas tube of a laser is closed by Brewster windows. The helium is essential to obtain a population inversion and the high-reflectivity mirrors are necessary to reach laser threshold. The effect of energy transfer from helium to neon atoms was not known at Ladenburg's time and highly reflecting dielectric mirrors were not available. A helium–neon laser generating visible light was realized in 1962. If Ladenburg or somebody else would have had the idea of a neon laser, a more elaborated investigation of gas discharges and of laser mirrors would have been necessary.

In the 1920 s and the beginning 1930s, there was not yet much knowledge about optical properties of dielectric crystals, of doped dielectric crystals and of semiconductors. Therefore, a solid state laser would not have been a reachable goal at the time. Research in the field of solid state physics grew strongly between 1920 and 1960 [57].

References [1, 4–6, 12, 13, 22, 35, 36, 40–58, 177–181, 184, 243, 297, 302].

Problems

9.1 Susceptibilities and polarization conductivities. Instead of our ansatz of an electromagnetic wave, $\tilde{E} = A\mathrm{e}^{\mathrm{i}(\omega t - kz)}$, we could use the ansatz $\tilde{E} = A\mathrm{e}^{\mathrm{i}(kz - \omega t)}$. Show that the imaginary part of the susceptibility changes sign but that the real field and the real polarization are the same in both cases. Discuss the corresponding polarization conductivity.

9.2 Linear dispersion.

(a) Determine the linear dispersion $\mathrm{d}n/\mathrm{d}\omega$ of titanium–sapphire at a population difference $N_2 - N_1 = 10^{24}\,\mathrm{m}^{-3}$.
(b) Determine the shift of the resonance frequencies of a Fabry–Perot resonator (length 0.5 m) due to optical pumping of a crystal of 1 cm length, i.e., at a change of the population difference $N_2 - N_1 = 0$ to $N_2 - N_1 = 10^{24}\,\mathrm{m}^{-3}$.

9.3 Nonlinear dispersion of optically pumped titanium–sapphire.

(a) Determine the nonlinear dispersion $\mathrm{d}^2 n/\mathrm{d}\upsilon^2$ around the center frequency ω_0.
(b) Determine the nonlinear dispersion in the case that $N_2 - N_1 = 10^{24}\,\mathrm{m}^{-3}$.
(c) How large is the change of the refractive index in the frequency range $\nu_0 - \Delta\nu_0/2$, ν_0 and in the range ν_0, $\nu_0 + \Delta\nu_0/2$?
(d) Determine the shift of the resonance frequencies (due to nonlinear dispersion of a crystal of 1 cm length of a Fabry-Perot resonator (length 0.5 m) due to optical pumping, i.e., at a change of the population difference $N_2 - N_1 = 0$ to $N_2 - N_1 = 10^{24}\,\mathrm{m}^{-3}$.

9.4 Drude theory. We obtain the Drude theory of the electric transport if we treat the electrons in a solid as free-electrons, i.e., if we set $\omega_0 = 0$ in (9.42) and introduce the electron velocity $\mathrm{v} = \mathrm{d}x/\mathrm{d}t$. Then $\beta^{-1} = \tau$ is the relaxation time of an electron: an electron (accelerated at time $t = 0$) by an electric field loses its energy after the time τ.

(a) Derive the high frequency conductivities $\sigma_1(\omega)$ and $\sigma_2(\omega)$.
(b) Determine the real part and imaginary of the high frequency mobility $\tilde{\mu}(\omega)$; $\tilde{\mathrm{v}}(\omega) = \tilde{\mu}(\omega)\tilde{E}(\omega)$.

(c) Determine the corresponding frequency-dependent susceptibilities and dielectric constants (=dielectric functions).

9.5 Perfect conductor of high frequency currents. We define a perfect conductor of high frequency currents as a conductor with free-electrons that have an infinitely long relaxation time. [A superconductor at temperatures that are small compared to its superconducting transition temperature T_c can be a perfect conductor of high frequency currents at frequencies where $h\nu < 2\Delta$; 2Δ is the superconducting energy gap; $T_c = 7\,\mathrm{K}$ for lead and $90\,\mathrm{K}$ for the high temperature superconductor $\mathrm{YBa_2Cu_3O_7}$.]

(a) Derive the high frequency conductivity of a perfect conductor.
(b) Determine the dielectric function.
(c) Calculate the values of σ_2 of an ideal conductor that contains free-electrons of a density $N = 10^{28}\,\mathrm{m}^3$; $N = 10^{25}\,\mathrm{m}^3$; $N = 10^{22}\,\mathrm{m}^3$.

9.6 Show that the slowly varying amplitude approximation is valid if the change of the amplitude within a quarter of the period of a high frequency field is small compared to the amplitude of a high frequency field.

9.7 Rabi oscillation. An ensemble of two-level atomic systems that interact with a strong electric field can show an oscillation of the population inversion and, synchronously, an oscillation of the polarization. We assume that the frequency of the field is equal to the atomic resonance frequency and that transverse relaxation is absent. We furthermore assume that the only relaxation process is spontaneous emission of radiation but that the spontaneous lifetime T_1 is much larger than the period of the field. We describe the dynamics of the polarization by,

$$\frac{\mathrm{d}^2P}{\mathrm{d}t^2} + \Delta\omega_0\frac{\mathrm{d}P}{\mathrm{d}t} + \omega_0^2 P = \frac{2}{\pi}\varepsilon_0\hbar\omega_0 B_{21}^{\omega}\Delta N E. \tag{9.207}$$

An electric field $E = A\cos\omega_0 t$ causes a polarization $P = B\sin\omega_0 t$ that is 90° phase shifted relative to the field. We find, in slowly varying amplitude approximation, the equation

$$\frac{\mathrm{d}B}{\mathrm{d}t} + \frac{\Delta\omega_0}{2}B(t) = \frac{1}{\pi}\varepsilon_0\hbar\omega_0 B_{21}^{\omega} A\Delta N(t). \tag{9.208}$$

The time dependence of the population difference is determined by the differential equation:

$$\frac{\mathrm{d}\Delta N(t)}{\mathrm{d}t} + \frac{\Delta N(t) - \Delta N_0}{T_1} = \frac{1}{2\hbar\omega_0}AB(t); \tag{9.209}$$

the change of the population difference averaged over a period of the field, multiplied by the energy of a photon, is equal to $AB/2$.

(a) Show that, under certain conditions, these two equations are equivalent to two second-order differential equations,

$$\mathrm{d}^2 B/\mathrm{d}t^2 + \omega_\mathrm{R}^2 B = 0, \tag{9.210}$$

$$\mathrm{d}^2 \Delta N/\mathrm{d}t^2 + \omega_\mathrm{R}^2 \Delta N = 0, \tag{9.211}$$

where

$$\omega_\mathrm{R}^2 = \varepsilon_0 b_{21}^0/(4\hbar\omega_0)A^2 \tag{9.212}$$

and ω_R is the Rabi frequency and, fustheremore, $\omega_\mathrm{R} \ll \omega_0$; this condition allows for application of the slowly varying envelope approximation. The differential equations are approximately valid if $\omega_\mathrm{R} \gg \omega_0$. The solutions are

$$B = B_0 \sin \omega_\mathrm{R} t, \tag{9.213}$$

$$\Delta N = \Delta N_0 \cos \omega_\mathrm{R} t. \tag{9.214}$$

The amplitude of the polarization and the population difference oscillate with the Rabi frequency. The Rabi frequency is proportional to the amplitude of the electric field.

(b) Make a draft of the time dependences of the population difference ΔN and the amplitude of the polarization.

(c) Calculate the Rabi frequencies for a medium with a naturally broadened line, with $T_1 = 10^{-2}$ s. What is the minimum field amplitude necessary for the occurrence of a Rabi oscillation? (For more information about Rabi oscillations, *see*, for instance, [1, 5, 40]).

9.8 Start of laser oscillation.

Show, by use of the van der Pol (vdP) equation of a laser, that laser oscillation cannot start without an initial field [*Hint*: the vdP equation has two different solutions, depending on the initial conditions].

9.9 Write the van der Pol equation of a laser in dimensionless units, as well as the solution for the electric field.

9.10 Derive the van der Pol equation from the Lorenz-Haken equations.

9.11 The van der Pol equation.

Derive the van der Pol equation of a laser from the (general) van der Pol equation

$$\frac{\mathrm{d}^2 y}{\mathrm{d}\tau^2} + \varepsilon(-1 + y^2)\frac{\mathrm{d}y}{\mathrm{d}\tau} + y = 0,$$

where y is the dimensionless field, ε (>0) is a parameter, and τ the dimensionless time [*Hint*: Make use of SVEA; *see* Problem 31.4].

9.12 Determine the amplitude of the polarization of a helium–neon laser medium that carries a polarization-current of 0.68 A m^{-2} (*see Example* to Fig. 9.6).

9.13 Phase portrait.

(a) Characterize onset of laser oscillation by a phase portrait. *Hint*: Make use of the solution $A(t)$ of the van der Pol equation of a laser. The phase portrait is obtained for a plot of $\dot{A}$ (on the y axis) versus A (on the x axis), with the time t as a parameter that varies from $t = 0$ to $t \rightarrow \infty$.

(b) Draw the phase portrait of a laser oscillation for the case that the gain is suddenly turned off.