

Chapter 30

Electron Waves in Semiconductor Heterostructures

We study electron waves in one-dimensional potentials and in semiconductor heterostructures.

We will begin with the discussion of the one-dimensional square well potential, then describe the origin of energy bands for electrons in a one-dimensional periodic potential. We make use of the tight binding method.

We will introduce the plane-wave transfer matrix method to describe, for an interface of two semiconductors, how a wave function of a semiconductor continues in the other semiconductor. The requirement that the energy flux through a boundary is steady provides the boundary conditions for electron waves at an interface of two different semiconductors. The plane-wave transfer matrix method allows for determination of the energy bands (minibands) of a superlattice. Finally, we will treat the quantum well and the double quantum well.

The plane-wave transfer matrix method is the same we used to describe electromagnetic plane waves in layered systems (Sect. 25.11). The difference of the results comes from the different dispersion relations: the wave vector of a free-electron wave in vacuum (or in a semiconductor) varies with the square root of energy while the wave vector of an electromagnetic wave in vacuum (or in a homogeneous medium) shows a linear dependence on frequency.

30.1 Electron in a One-Dimensional Square Well Potential

An electron wave with the wave vector k obeys the dispersion relation

$$E = \frac{\hbar^2 k^2}{2m_0}. \quad (30.1)$$

E is the energy of an electron and m_0 the electron mass. The energy E increases quadratically with k . We describe free-electrons (in a bulk semiconductor) propagating in x direction by the use of the time-independent Schrödinger equation

$$-\frac{\hbar^2}{2m_0} \frac{d^2\varphi}{dx^2} = E\varphi, \quad (30.2)$$

where $\varphi(x)$ is the wave function. A solution is

$$\varphi(x) = Ae^{ikx}; \quad k = +\sqrt{2m_0E/\hbar^2}. \quad (30.3)$$

We consider an electron in a one-dimensional square well potential (width a) with rigid walls (Fig. 30.1a). The Schrödinger equation for $|x| < a/2$,

$$-\frac{\hbar^2}{2m_0} \frac{d^2\varphi}{dx^2} = E\varphi, \quad (30.4)$$

has the general solution

$$\varphi(x) = A \sin kx + B \cos kx; \quad k = +\sqrt{2m_0E/\hbar^2}. \quad (30.5)$$

The boundary conditions require that $\varphi(\pm a/2) = 0$ or

$$A \sin(ka/2) + B \cos(ka/2) = 0, \quad (30.6)$$

$$-A \sin(ka/2) + B \cos(ka/2) = 0. \quad (30.7)$$

Solutions are

- $A = 0$ and $\cos(ka/2) = 0$ leading to

$$\varphi(x) = B \cos \frac{n\pi x}{a}, \quad n = 1, 3, \dots \quad (\text{even solution}); \quad (30.8)$$

- $B = 0$ and $\sin(ka/2) = 0$ leading to

$$\varphi(x) = A \sin \frac{n\pi x}{a}, \quad n = 2, 4, \dots \quad (\text{odd solution}). \quad (30.9)$$

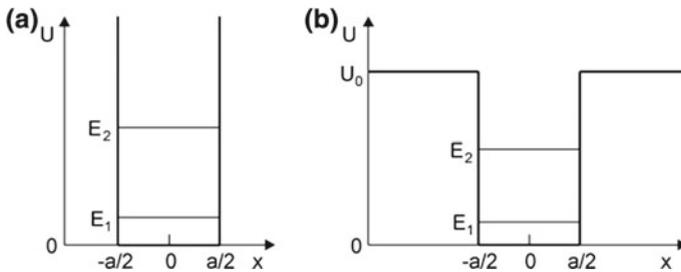


Fig. 30.1 One-dimensional square well potential **a** with infinitely high walls and **b** with walls of finite height

The energy eigenvalues are

$$E_n = \frac{\pi^2 \hbar^2 n^2}{2m_0 a^2}. \quad (30.10)$$

For a square well potential with finite potential steps (Fig. 30.1b), the Schrödinger equation is unaltered for $|x| < a/2$. The wave equation for $|x| > a/2$,

$$-\frac{\hbar^2}{2m_0} \frac{d^2\varphi}{dx^2} + U_0\varphi = E\varphi, \quad (30.11)$$

has the solution

$$\varphi(x) = Ce^{-\kappa x} + De^{\kappa x}; \quad \kappa = +\sqrt{2m_0(U_0 - E)/\hbar^2}. \quad (30.12)$$

The boundary conditions require that $\varphi(x)$ and $d\varphi/dx$ are continuous for $x = \pm a/2$. The application of the boundary conditions would allow for determination of A, B, C, D and of the eigenvalues. Instead, we make use of the symmetry of the potential. The ansatz of the even solutions

$$\varphi(x) = B \cos kx \quad \text{for } |x| < a/2, \quad (30.13)$$

$$\varphi(x) = Ce^{-\kappa x} \quad \text{for } |x| > a/2, \quad (30.14)$$

and the condition of continuity of φ and $d\varphi/dx$ at $|x| = a/2$ lead to

$$B \cos(ka/2) = Ce^{-\kappa a/2}, \quad (30.15)$$

$$kB \sin(ka/2) = \kappa Ce^{-\kappa a/2} \quad (30.16)$$

or

$$k \tan(ka/2) = \kappa. \quad (30.17)$$

The odd solutions are

$$\varphi(x) = A \sin kx \quad \text{for } |x| < a/2, \quad (30.18)$$

$$\varphi(x) = Ce^{-\kappa x} \quad \text{for } |x| > a/2. \quad (30.19)$$

The boundary conditions of the odd solutions require that

$$k \cot(ka/2) = \kappa. \quad (30.20)$$

The conditions $k \tan(ka/2) = \kappa$ and $k \cot(ka/2) = \kappa$ provide a finite number of discrete energy eigenvalues E_n .

30.2 Energy Bands of Electrons in a Periodic Square Well Potential

We describe a model (tight binding model) that illustrates the occurrence of energy bands and of dispersion for electron waves in a periodic potential.

A single isolated square well potential at position x_l (Fig. 30.2, upper part) is characterized by the wave equation

$$\left[-\frac{\hbar^2}{2m_0} \frac{d^2}{dx^2} + U_l(x - x_l) \right] \varphi_l(x - x_l) = E_0 \varphi_l(x - x_l). \tag{30.21}$$

We regard E_0 as the energy E_1 of the lowest state in a square well potential (Sect. 30.1) and $\varphi_l = \varphi_l(x - x_l)$ as the corresponding wave function. The wave function is normalized,

$$\int_{-\infty}^{\infty} \varphi_l^*(x - x_l) \varphi_l(x - x_l) dx = 1. \tag{30.22}$$

The wave equation of a periodic sequence of identical square well potentials (Fig. 30.2, center) is

$$\left(-\frac{\hbar^2}{2m_0} \frac{d^2}{dx^2} + U(x) \right) \psi(x) = E \psi(x). \tag{30.23}$$

The potential energy is a periodic function,

$$U(x + a) = U(x), \tag{30.24}$$

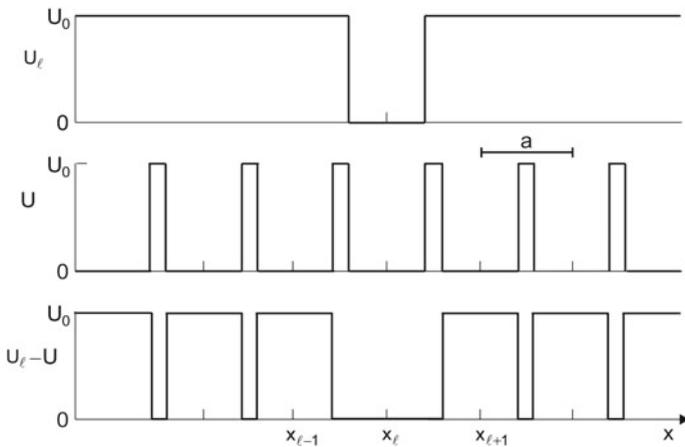


Fig. 30.2 A single square well potential, an infinite series of square well potentials and the difference between the two potentials

where a is the period. We describe the wave function of the periodic system by a linear combination of the wave functions of the single wells,

$$\psi(x) = \sum_l c_l \varphi_l(x - x_l). \quad (30.25)$$

Making use of the periodicity, we write

$$\psi(x) = \frac{1}{\sqrt{N}} \sum_{l=0}^{N-1} e^{ikla} \varphi_l(x - la). \quad (30.26)$$

N is the number of quantum wells in a periodicity interval. We apply periodic boundary conditions, $\psi(x + Na) = \psi(x)$, and find

$$k = \frac{2\pi l}{Na}; \quad l = 0, 1, \dots, N-1. \quad (30.27)$$

We restrict the k values to the first Brillouin zone

$$-\frac{\pi}{a} < k \leq \frac{\pi}{a}. \quad (30.28)$$

Inserting $\psi(x)$ into the wave equation provides

$$\sum_l e^{ikla} [U(x) - E] \varphi_l = - \sum_l e^{ikla} \left(-\frac{\hbar^2}{2m_0} \frac{d^2}{dx^2} \right) \varphi_l. \quad (30.29)$$

We add on both sides the term $(-U_l(x - x_l) + E_0)\varphi_l$ and obtain

$$\begin{aligned} & \sum_l e^{ikla} [U(x) - U_l(x - x_l) - E + E_0] \varphi_l \\ &= - \sum_l e^{ikla} \left[-\frac{\hbar^2}{2m_0} \frac{d^2}{dx^2} - U_l(x - x_l) + E_0 \right] \varphi_l. \end{aligned} \quad (30.30)$$

The right side is zero. We find

$$(E - E_0) \sum_l e^{ikla} \varphi_l = \sum_l e^{ikla} [U(x) - U_l(x - x_l)] \varphi_l. \quad (30.31)$$

We multiply the equation by

$$\psi^*(x) = \frac{1}{\sqrt{N}} \sum_{m=0}^{N-1} e^{-ikma} \varphi_m^*(x - ma) \quad (30.32)$$

and integrate over the length L of the periodicity interval. We obtain

$$(E - E_0) \sum_{l,m} e^{ik(l-m)a} \int_0^L \varphi_m^* \varphi_l dx = \sum_{l,m} e^{ik(l-m)a} \int \varphi_m^* [U(x) - U_l(x - x_l)] \varphi_l dx. \quad (30.33)$$

Neglecting the weak overlap of φ_l^* and φ_l for $l \neq m$, we obtain for the term on the left side $N(E - E_0)$. Because of the large values of $U - U_l$ (Fig. 30.2, lower part) at positions of the cells $m \neq l$, we cannot neglect the terms with $m \neq l$ on the right side of the equation. We assume that $\varphi_l(x_l - la)$ decreases strongly at large distance $|x - x_l|$. Then we can restrict the double sum to terms that correspond to neighboring cells. We obtain

$$\begin{aligned} N \times (E - E_0) &= N \times \int \varphi_l^* [U(x) - U_l(x - x_l)] \varphi_l dx \\ &\quad + N \times e^{-ika} \int \varphi_{l-1}^* [U(x) - U(x - x_l)] \varphi_l dx \\ &\quad + N \times e^{ika} \int \varphi_{l+1}^* [U(x) - U(x - x_l)] \varphi_l dx. \end{aligned} \quad (30.34)$$

It follows, with

$$\alpha = \int \varphi_l^* [U_l(x - x_l) - U(x)] \varphi_l dx \quad (30.35)$$

and

$$\gamma = \int \varphi_{l-1}^* [U_l(x - x_l) - U(x)] \varphi_l dx = \int \varphi_{l+1}^* [U_l(x - x_n) - U(x)] \varphi_l dx, \quad (30.36)$$

that the energy is equal to

$$E = E(k) = E_0 - \alpha - \gamma \cos ka. \quad (30.37)$$

We introduce

$$\epsilon(k) = E_0 - E(k) - \alpha - \gamma. \quad (30.38)$$

With

$$\epsilon_m = -2\gamma \quad (30.39)$$

we find

$$\epsilon(k) = \epsilon_m \left(\frac{1}{2} - \frac{1}{2} \cos ka \right). \quad (30.40)$$

We obtain the lowest energy band (Fig. 30.3), with a minimum at $k = 0$ (width ϵ_m), then an energy gap, and a second band (maximum at $k = 0$ since $\gamma < 0$). We can interpret $E_0 - \alpha - \gamma$ as zero point energy of an electron in the periodic potential.

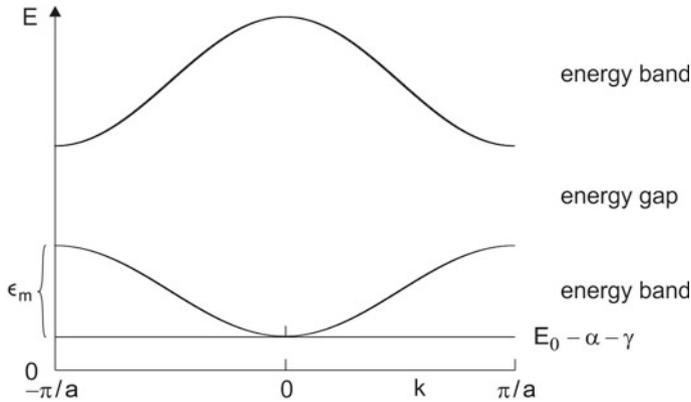
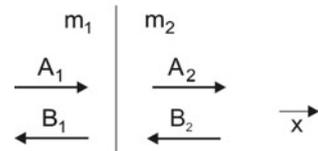


Fig. 30.3 Energy bands

Fig. 30.4 Electron wave at an interface of two semiconductors



30.3 Plane-Wave Transfer Matrix Method of Characterizing a Semiconductor Interface

We consider (Fig. 30.4) propagation of an electron wave through an interface of two semiconductor materials (for instance GaAs and AlAs). At the interface, the potential energy and the effective mass of an electron change abruptly. We describe an electron wave propagating in x direction (perpendicular to the interface) by the time-independent Schrödinger equation

$$\left(-\frac{\hbar^2}{2m(x)} \frac{d^2}{dx^2} + U(x) \right) \psi(x) = E \psi(x), \tag{30.41}$$

where $m(x)$ is the effective mass and $U(x)$ the potential energy. We look for wave functions $\psi(x)$ and energies E that satisfy the equation. We describe the wave functions in medium 1 and medium 2 by the ansatz:

$$\psi_1 = A_1 e^{ik_1 x} + B_1 e^{-ik_1 x}, \tag{30.42}$$

$$\psi_2 = A_2 e^{ik_2 x} + B_2 e^{-ik_2 x}, \tag{30.43}$$

With

$$k_1 = +\sqrt{2m_1 E/\hbar^2}, \quad (30.44)$$

$$k_2 = +\sqrt{2m_2(E - U_0)/\hbar^2} \quad (30.45)$$

being the wave vectors of the waves in medium 1 and medium 2, respectively. A_1 and B_1 are amplitudes of the waves of opposite directions. We restrict the discussion to the case that $E < U_0$. Then k_2 is imaginary and ψ_2 describes a wave with an increasing term (amplitude A_2) and a decreasing term (amplitude B_2). We use the boundary conditions

$$\psi_1 = \psi_2 \quad \text{at } x = 0, \quad (30.46)$$

$$\frac{1}{m_1} \frac{d\psi_1}{dx} = \frac{1}{m_2} \frac{d\psi_2}{dx} \quad \text{at } x = 0. \quad (30.47)$$

We write

$$M_1 \begin{pmatrix} A_1 \\ B_1 \end{pmatrix} = M_2 \begin{pmatrix} A_2 \\ B_2 \end{pmatrix} \quad (30.48)$$

and find

$$M_l = \begin{pmatrix} 1 & 1 \\ k_l & -k_l \end{pmatrix}; \quad l = 1, 2. \quad (30.49)$$

It follows that

$$\begin{pmatrix} A_1 \\ B_1 \end{pmatrix} = M_1^{-1} M_2 \begin{pmatrix} A_2 \\ B_2 \end{pmatrix} = M_{12} \begin{pmatrix} A_2 \\ B_2 \end{pmatrix}, \quad (30.50)$$

where

$$M_{12} = \begin{pmatrix} \frac{1}{2}(1 + k_2/k_1) & \frac{1}{2}(1 - k_2/k_1) \\ \frac{1}{2}(1 - k_2/k_1) & \frac{1}{2}(1 + k_2/k_1) \end{pmatrix}. \quad (30.51)$$

is the transfer matrix. It has exactly the same form as the transfer matrix for a plane electromagnetic wave at an interface; *see* (25.12).

The continuity conditions we used follow from the requirements that the probability density $\rho(x) = \psi^*(x)\psi(x)$ and the probability current density $\partial\rho/\partial t$ are continuous. The first condition is fulfilled if $\psi(x)$ is continuous. To discuss the second condition, we replace in the time-dependent Schrödinger equation

$$-\frac{\hbar}{i} \frac{\partial}{\partial t} \psi(x) = \left(-\frac{\hbar^2}{2m_0} \nabla^2 + U(x) \right) \psi(x, t) \quad (30.52)$$

the first term on the right side:

$$-\frac{\hbar^2}{2m_0} \frac{\partial^2}{\partial x^2} \psi(x) \rightarrow \frac{\hbar^2}{2} \frac{\partial}{\partial x} \left(\frac{1}{m(x)} \frac{\partial \psi(x)}{\partial x} \right). \quad (30.53)$$

Then

$$\begin{aligned} \frac{\partial}{\partial t} (\psi^* \Psi) &= \frac{i\hbar}{2} \frac{\partial}{\partial x} \left(\frac{1}{m(x)} \frac{\partial}{\partial x} \psi^* \right) \psi \\ &= \frac{i\hbar}{2} \left[\frac{\partial}{\partial x} \left(\psi^* \frac{1}{m(x)} \frac{\partial}{\partial x} \psi \right) - \psi \frac{1}{m} \frac{\partial}{\partial x} \Psi^* \right] = 0. \end{aligned} \quad (30.54)$$

This condition is satisfied at an interface (at $x = 0$) between two semiconductors if

$$\frac{1}{m_1} \frac{\partial \psi}{\partial x} = \frac{1}{m_2} \frac{\partial \psi}{\partial x} \quad \text{at } x = 0. \quad (30.55)$$

30.4 Minibands

The potential energy of an electron in a superlattice is a periodic function,

$$U(x + a) = U(x), \quad (30.56)$$

where a is the period. We describe the wave function of an electron in the periodic system as a linear combination of the wave functions of the single wells,

$$\psi(x) = \sum_l c_l \varphi_l(x - x_l). \quad (30.57)$$

Making use of the periodicity, we write

$$\psi(x) = \frac{1}{\sqrt{N}} \sum_{l=0}^{N-1} e^{ikla} \varphi_l(x - la). \quad (30.58)$$

N is the number of quantum wells in a periodicity interval. We apply periodic boundary conditions, $\psi(x + Na) = \psi(x)$, and find

$$k = \frac{2\pi l}{Na}; \quad l = 0, 1, \dots, N-1. \quad (30.59)$$

We restrict the k values to the first Brillouin zone

$$-\frac{\pi}{a} < k \leq \frac{\pi}{a}. \quad (30.60)$$

We now use the transfer matrix method. We described in the preceding section the transfer matrix of an electron wave at a boundary. Taking account of propagation, we find the same equations, (25.31)–(25.34), as for electromagnetic plane waves in a one-dimensional photonic crystal. The equations lead, as shown in Sect. 25.14, to the dispersion relation

$$\cos ka = \cos k_1 a_1 \cos k_2 a_2 - \frac{1}{2} \left(\xi + \frac{1}{\xi} \right) \sin k_1 a_1 \sin k_2 a_2. \quad (30.61)$$

For an electron wave in a superlattice, the quantities are:

$$k_1 = +\sqrt{2m_1 E/\hbar^2}, \quad (30.62)$$

$$\xi = \frac{k_1}{k_2} \frac{m_2}{m_1} = -i \frac{k_1}{\kappa} \frac{m_2}{m_1}, \quad \kappa = +\sqrt{2m_2(U_0 - E)/\hbar^2}. \quad (30.63)$$

We can write the dispersion relation of a miniband electron in the form

$$\cos ka = \cos k_1 a_1 \cosh \kappa a_2 - \frac{1}{2} \left(|\xi| + \frac{1}{|\xi|} \right) \sin k_1 a_1 \sinh \kappa a_2 = f(E). \quad (30.64)$$

This equation, $\cos ka = f(E)$, cannot be solved analytically. However, we can obtain an approximate solution. We expand $f(E)$ around the eigenvalue $E_{0,n}$ of an isolated quantum well,

$$f(E) \approx f(E_{0,n}) + \left(\frac{df}{dE} \right)_{E=E_0} \times (E - E_0). \quad (30.65)$$

We find

$$E(k) = E_0 - \alpha - \gamma \cos ka, \quad (30.66)$$

$$\alpha = f(E_0)/(df/dE)_{E=E_0}, \quad (30.67)$$

$$-\gamma = [(df/dE)^{-1}]_{E=E_0}. \quad (30.68)$$

It follows, with $-2\gamma = \epsilon_m$, that

$$\epsilon(k) = \epsilon_m \left(\frac{1}{2} - \frac{1}{2} \cos ka \right). \quad (30.69)$$

Taking into account the free motion perpendicular to the superlattice axis, we obtain, with $k = k_x$, the total energy

$$\epsilon(\mathbf{k}) = \epsilon_m \left(\frac{1}{2} - \frac{1}{2} \cos k_x a \right) + \frac{\hbar^2}{2m_e} (k_y^2 + k_z^2), \quad (30.70)$$

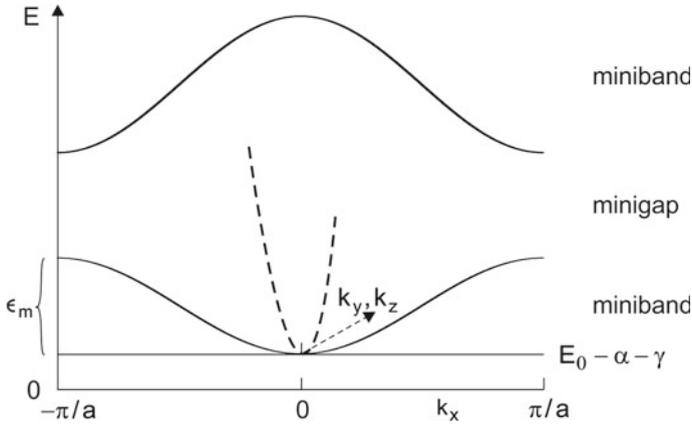


Fig. 30.5 Minibands

where m_e is the effective mass of an electron in GaAs in a GaAs/AlAs superlattice and $\mathbf{k} = (k_x, k_y, k_z)$. The energy dispersion curves (Fig. 30.5) indicate minibands (and minigaps) of electron wave vectors oriented along the superlattice axis. There is no gap for electrons that have wave vectors with components (k_y, k_z) perpendicular to the superlattice axis. The energy $E = 0$ is equal to the energy of the minimum of the conduction band of bulk GaAs.

The widths of the minibands and of the minigaps depend on the superlattice parameters:

- a_1 = thickness of a quantum well layer.
- a_2 = thickness of a barrier layer.
- $a = a_1 + a_2$ = superlattice period.

It is possible to design superlattices for a great range of values of ϵ_m , namely $\epsilon_m = 5\text{--}140$ meV for GaAs superlattices and ϵ_m up to 300 meV for GaInAs/GaAlInAs superlattices.

If we neglect the difference of the effective masses of the superlattice materials, the matrix method yields the same result as obtained via the superposition of the wave functions of the single wells (Sect. 30.2).

A remark. The method of superposition of elementary wave functions (tight binding model) was introduced by Felix Bloch in 1928 [251]. Ralph Kronig and William Penney [231] introduced (in 1931) the square well potential (Kronig-Penney potential) and derived the dispersion relation (30.40). Gerard Bastard [232, 233] extended the model (extended Kronig-Penney model) to describe energy bands of semiconductor superlattices—with different effective masses of an electron in different layers of a superlattice.

30.5 Quantum Well

Knowing the boundary conditions for wave functions at the interface of two semiconductors, we find the expression that allows for determination of the eigenvalues of electronic states of a quantum well (Problems):

$$k \tan(ka/2) = -\alpha m_2/m_1 \quad \text{for even solutions,} \tag{30.71}$$

$$k \cot(ka/2) = -\alpha m_2/m_1 \quad \text{for odd solutions.} \tag{30.72}$$

30.6 Double-Quantum Well

The energy levels of electrons in a double-well potential (Fig. 30.6) are doublets. The energy level E_1 of the lowest state of isolated potential wells splits into two levels E_1^+ and E_1^- . Correspondingly, the level E_2 splits into two levels (E_2^- and E_2^+); see Problem 30.2.

References [31, 178, 186, 226–233, 251].

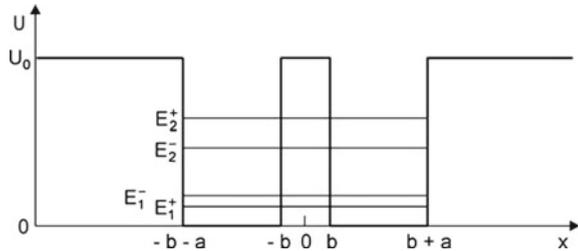
Problems

30.1 Quantum well.

Estimate the eigenvalues E_1 and E_2 of an electron in an AlAs/GaAs/AlAs quantum well (barrier height 2.2 eV; $m_{\text{GaAs}} = 0.07 m_0$; $m_{\text{AlAs}} \sim 3 m_{\text{GaAs}}$) if the well consists of films of different thickness.

- (a) Film thickness = 14 GaAs monolayers
- (b) Film thickness = 2 GaAs monolayers.

Fig. 30.6 Double-well potential



30.2 Double-quantum well.

- Determine the eigenvalues of a one-dimensional double well, which correspond to the two lowest energy levels ($s = 1, 2$) of a single one-dimensional double well. [*Hint*: make use of the symmetry.]
- Determine the energy level splitting $E_1^- - E_1^+$ for the two lowest levels.
- Sketch the wave functions that correspond to the four lowest levels.
- Calculate the level splitting occurring in an AlAs/GaAs/AlAs/GaAs/AlAs double quantum well (Fig. 30.6) for $a_1 = 10$ nm and $a_2 = 2$ nm.

30.3 Dispersion of electrons in a periodic potential.

Derive the dispersion relation of electrons in a periodic potential by the use of the matrix method.

30.4 Interface.

- Electrons (energy ϵ) propagate toward a GaAs/AlAs interface and are reflected. Determine the average penetration depth of electrons. [*Hint*: take into account the difference between the penetration depth of the wave function and of the electrons.]
- Determine the penetration depth for $\epsilon = 10$ meV and 100 meV.
- Show that the reflectivity is $R = |k_1 - i\kappa|/|k_1 - i\kappa_1|$.
- Explain the electron total reflector used in a GaN quantum well laser (Sect. 24.3, Fig. 24.3a).

30.5 Tunneling.

- Determine the transmissivity of an AlAs barrier in a GaAs/AlAs/GaAs heterostructure for electrons of energy ϵ .
- Determine the transmissivity for electrons of energy $\epsilon = 10$ meV and 100 meV at a barrier width of 2 monolayers of AlAs and for a barrier of 10 monolayers of AlAs.

30.6 Resonance state.

- Given is a GaAs/AlAs/GaAs/AlAs/GaAs heterostructure. Determine the energy dependence of the transmissivity for electron waves of different energies.
- Design a heterostructure that is transparent for electrons of $\epsilon = 10$ meV.
- Design a heterostructure that is transparent for electrons of $\epsilon = 100$ meV.

30.7 Injector of a quantum cascade laser.

- Design a quantum cascade laser of AlAs/GaAs/AlAs/GaAs/AlAs heterostructures embedded in chirped GaAs/AlAs superlattices for a quantum cascade laser that may be able to generate radiation at a frequency of 4 THz.
- Estimate the thicknesses of the different layers.
- Discuss the role of the superlattice, especially in view of the result of the preceding problem.

30.8 Semiconductor superlattice.

- (a) Determine the effective mass m^* of an electron in a superlattice (for propagation along the superlattice axis) for an electron with $k \sim 0$.
- (b) Determine m^* of a GaAs/AlAs superlattice with 14 monolayers GaAs and 2 monolayers AlAs ($\epsilon_m \sim 140 \text{ meV}$).
- (c) Determine m^* of a GaAs/AlAs superlattice with 4 monolayers GaAs and 2 monolayers AlAs ($\epsilon_m \sim 40 \text{ meV}$).
- (d) Determine the effective mass m^* of an electron in a superlattice (for propagation along the superlattice axis) for arbitrary k and discuss the slope $m^*(k)$ and $m^*(\epsilon)$.
- (e) Determine the group velocity $v_g(k)$ and the peak group velocity.
- (f) Sketch the wave functions of the lowest miniband for $k \sim 0$ and $k = \pi/a$.
- (g) Sketch the wave functions of the second miniband for $k \sim 0$ and $k = \pi/a$.