Chapter 2
Laser Principle

A laser (=laser oscillator) is a self-excited oscillator. A self-excited oscillator starts oscillation by itself and maintains an oscillation. Laser radiation is generated by stimulated transitions in an active medium. The active medium is a gain medium—propagation of radiation in the active medium results in an increase of the energy density of the radiation. The active medium in a laser experiences feedback from radiation stored in a laser resonator. A portion of radiation coupled out of the resonator represents the useful radiation.

In this chapter, we characterize an active medium by a population inversion in an ensemble of two-level atomic systems. We formulate the threshold condition of laser oscillation. We solve the resonator eigenvalue problem and find possible frequencies of laser oscillation. We also show that the buildup of steady state oscillation of a laser takes time—the oscillation onset time.

To describe a coherent electromagnetic wave, we make use of the model of a quasiplane wave—a parallel beam of coherent radiation. A quasiplane wave is characterized by a well-defined propagation direction, a finite spatial extension perpendicular to the propagation direction, and a constant field amplitude within the beam. The model is very useful for a basic description of the field in a laser oscillator. Later (in Chap. 11) we will introduce a modified description of a coherent electromagnetic wave.

In later chapters, we will specify the two-level atomic systems. A two-level atomic system can belong to various states: electronic states of an atom or an ion (in a gas, crystal, glass, or liquid); electronic, vibrational, or rotational states of a molecule; electronic states of electrons in a semiconductor or a semiconductor heterostructure.

We will introduce (Chap. 4) still another type of active medium—an active medium containing energy-ladder systems rather than two-level systems.

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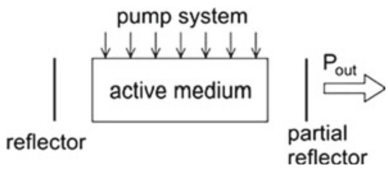
Fig. 2.1 A laser



2.1 A Laser

A laser (Fig. 2.1) emits coherent radiation of an output power P_{out}. A laser has the following parts.

- *Active medium* (=*gain medium* = *laser medium*). The active medium is able to amplify electromagnetic radiation. The active medium, located inside a resonator, fills out a resonator partly or completely.
- *Pump system*. It "pumps" the active medium. Methods of pumping are: optical pumping with another laser or a lamp; pumping with a gas discharge; pumping with a current through a semiconductor or a semiconductor heterostructure; chemical pumping.
- *Laser resonator*. The laser resonator has the task to store a coherent electromagnetic field and to enable the field to interact with the active medium—the active medium experiences feedback from the coherent field. We will describe resonators that consist of two mirrors—one is a reflector of a reflectivity R_1 near 1, and the other is a *partial reflector* serving as *output coupler*. The output coupling mirror has a reflectivity (R_2) that also can have a value near 1 but that can be much smaller; semiconductor lasers can have reflectors with $R_1 = R_2 \sim 0.3$. Each type of laser requires its own resonator design. There is a main criterion concerning reflectivities of resonators: a laser should be able to work at all. Depending on the task of a laser, other criteria can be chosen—for instance, that a laser should have optimum efficiency of conversion of pump power to laser output power.

2.2 Coherent Electromagnetic Wave

We describe a coherent electromagnetic wave generated by a continuous wave laser as a *quasiplane wave* (=parallel beam of coherent light),

$$E(z, t) = A \cos[\omega(t - t_0) - k(z - z_0)]. \tag{2.1}$$

E is the electric field at time t and location z, A is the amplitude, $\omega = 2\pi\nu$ the angular frequency, ν the frequency, and k the wave vector of the wave; t_0 defines a time coordinate and z_0 a spatial coordinate. The direction of E (and A) is perpendicular to the direction of the propagation direction (z direction). The dispersion relation,

$$\omega = ck, \tag{2.2}$$

relates the frequency and the wave vector. The quasiplane wave has a finite lateral extension. We suppose that the amplitude of the field does not vary, at a fixed z, over the cross section and that it is independent of z. The quasiplane wave is a section of a plane wave (which has infinite extensions in the plane perpendicular to the direction of propagation).

If we choose $t_0 = 0$ and $z_0 = 0$ to describe a quasiplane wave propagating in free space, we can write

$$E(z,t) = A\cos(\omega t - kz). \tag{2.3}$$

The instantaneous energy density, u_{inst}, in the electromagnetic field is

$$u_{\text{inst}} = \varepsilon_0 A^2 \cos^2(\omega t - kz), \tag{2.4}$$

where ε_0 is the electric field constant. The *energy density* u of the electromagnetic field, that is, the instantaneous energy density averaged over a temporal period $T = 2\pi/\omega$, is equal to

$$u = \frac{1}{2}\varepsilon_0 A^2. \tag{2.5}$$

The quasiplane wave transports energy in z direction. The power P of the wave is

$$P = \frac{1}{2}c\varepsilon_0 A^2 a_1 a_2, \tag{2.6}$$

where $a_1 a_2$ is the cross-sectional area of a beam of rectangular shape. The *intensity* (= power per unit area = energy flux density) is

$$I = \frac{P}{a_1 a_2} = \frac{1}{2}c\varepsilon_0 A^2. \tag{2.7}$$

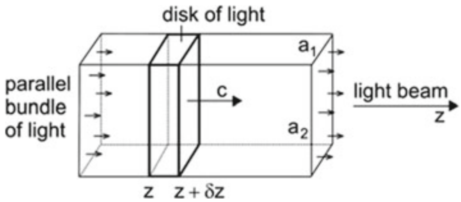
We interpret the transport of radiation energy as a flux of photons along the z direction and introduce the average number of photons per unit of volume, the *photon density* Z, by the relation

$$u = Zh\nu = Z\hbar\omega. \tag{2.8}$$

The energy density is equal to the photon density times the photon energy $h\nu$.

Simplifying further, we describe a light beam of laser light as a *parallel bundle of light rays* (Fig. 2.2). To characterize the propagation of light within a parallel light bundle, we introduce the *disk of light*. It is a section of a light bundle and has the length δz; we assume that δz is much larger than the wavelength of the radiation,

Fig. 2.2 Parallel light bundle and disk of light



$\delta z \gg \lambda$. The disk of light propagates along the z direction with the speed of light. The energy density in a disk of light is $u(\nu, z)$ and the photon density is equal to

$$Z(\nu, z) = \frac{u(\nu, z)}{h\nu}. \tag{2.9}$$

We will make use of the complex notation of the field. A complex field $\tilde{E}$ corresponds to a real field according to the relation

$$E = \mathrm{Re}[\tilde{E}] = \frac{1}{2}(\tilde{E} + \tilde{E}^*) = \frac{1}{2}\tilde{E} + c.c., \tag{2.10}$$

where $\mathrm{Re}[\tilde{E}]$ is the real part of $\tilde{E}$. The real field is equal to the sum of $\tilde{E}/2$ and its conjugate complex ($c.c.$) $\tilde{E}^*/2$.

The real part of a complex field, which is the product of a complex quantity $\tilde{A}$ and another complex quantity $\tilde{K}$, is

$$E = \mathrm{Re}[\tilde{E}] = \frac{1}{2}(\tilde{A}^*\tilde{K} + \tilde{A}\tilde{K}^*) = \frac{1}{2}\tilde{A}^*\tilde{K} + c.c. \tag{2.11}$$

The complex field $\tilde{E} = A\,\mathrm{e}^{\mathrm{i}(\omega t - kz)}$, with the real amplitude A, corresponds to the real field $E = A\cos(\omega t - kz)$. It follows that the energy density in an electromagnetic field is

$$u = \frac{\varepsilon_0}{2}\tilde{E}\tilde{E}^* = \frac{\varepsilon_0}{2}|\tilde{E}|^2 = \frac{\varepsilon_0}{2}A^2. \tag{2.12}$$

The photon density is given by

$$Z = \frac{\varepsilon_0}{2h\nu}\tilde{E}\tilde{E}^* = \frac{\varepsilon_0}{2h\nu}|\tilde{E}|^2 = \frac{\varepsilon_0}{2h\nu}A^2. \tag{2.13}$$

Accordingly, the amplitude of the field is

$$A = \sqrt{2h\nu Z/\varepsilon_0}. \tag{2.14}$$

More generally, we can characterize a quasiplane wave by

$$E = A\cos[\omega(t - t_0) - kz + \varphi_0], \tag{2.15}$$

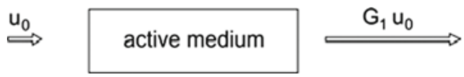


Fig. 2.3 Amplification of radiation in an active medium

where the time t_0 defines the time axis and φ_0 the z axis. The corresponding complex field is

$$\tilde{E} = A\,\mathrm{e}^{\mathrm{i}[\omega(t-t_0)-kz+\varphi_0]}. \tag{2.16}$$

If a wave propagates in a dielectric medium (dielectric constant ε), then ε_0 has to be replaced by $\varepsilon\varepsilon_0$ in the expressions concerning u, Z, A, and intensity.

The transit of coherent radiation through an active medium (Fig. 2.3) results in an increase of the photon density,

$$Z = G_1 Z_0, \tag{2.17}$$

and of the energy density,

$$u = G_1 u_0. \tag{2.18}$$

G_1 (>1) is the single-pass *gain factor*. Z_0 is the photon density and u_0 the energy density in the incident beam. Z is the photon density and u the energy density in the beam after passing through the active medium. We write

$$G_1 = \mathrm{e}^{\alpha L}, \tag{2.19}$$

where α is the *gain coefficient* of the active medium and L the length of the active medium. It follows that

$$\alpha = \frac{1}{L}\ln G_1 = \frac{1}{\ln 10}\,\frac{1}{L}\,\log G_1 = 0.43\,\frac{1}{L}\,\log G_1. \tag{2.20}$$

The transit of radiation through an absorbing medium results in a decrease of energy density and of photon density,

$$u = \bar{G}_1 u_0, \tag{2.21}$$
$$Z = \bar{G}_1 Z_0, \tag{2.22}$$

where $\bar{G}_1 < 1$ is the absorption factor ($=Z/Z_0$). We write

$$\bar{G}_1 = \mathrm{e}^{-\alpha_{\mathrm{abs}} L}. \tag{2.23}$$

α_{abs} is the *absorption coefficient* of a medium. It follows that

$$\alpha_{\mathrm{abs}} = \frac{1}{L}\ln \bar{G}_1 = 0.43\,\frac{1}{L}\,\log \bar{G}_1. \tag{2.24}$$

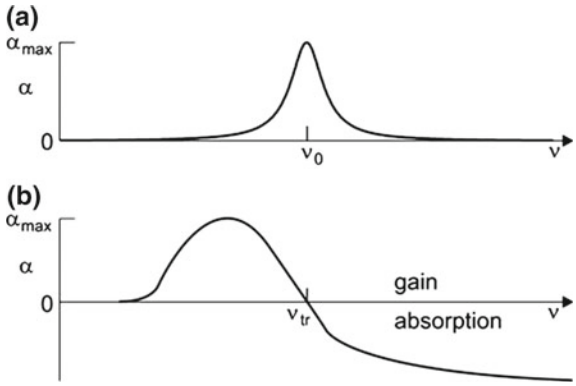


Fig. 2.4 Frequency dependence of the gain coefficient of an active medium. **a** Gain coefficient at frequencies around a resonance frequency. **b** Gain coefficient at frequencies around a transparency frequency

We can interpret the absorption coefficient as a negative gain coefficient; for units of gain, *see* Sect. 16.11.

An active medium can have a gain coefficient that has a maximum α_{max} at a resonance frequency ν_0 (Fig. 2.4a). The gain coefficient decreases toward smaller and larger frequencies and remains positive. However, an active medium can have a gain coefficient that changes sign (Fig. 2.4b). Such a medium has a transparency frequency ν_{tr}— the active medium is amplifying at frequencies $\nu < \nu_{\text{tr}}$ but absorbing at frequencies $\nu > \nu_{\text{tr}}$.

2.3 An Active Medium

An atom (or molecule) used in a laser has *two laser levels*, besides other energy levels. We ignore for the moment the other levels and describe an atom as a *two-level atomic system* (Fig. 2.5) and accordingly an ensemble of atoms as an *ensemble of two-level atomic systems*. We introduce the following notation.

- Level 2 (energy E_2) = upper laser level.
- Level 1 (energy E_1) = lower laser level.
- $E_{21} = E_2 - E_1$ = energy difference between the two laser levels = transition energy.
- N_2 = population of level 2 = density of excited two-level atomic systems = number density (number per unit volume) of excited two-level atomic systems.
- N_1 = population of level 1 = density of unexcited two-level atomic systems.
- $N_2 - N_1$ = *population difference*.
- $N_1 + N_2$ = density of two-level atomic systems.

Monochromatic electromagnetic radiation of frequency ν can interact with a two-level atomic system if Bohr's energy-frequency relation

$$h\nu = E_{21} = E_2 - E_1 \tag{2.25}$$

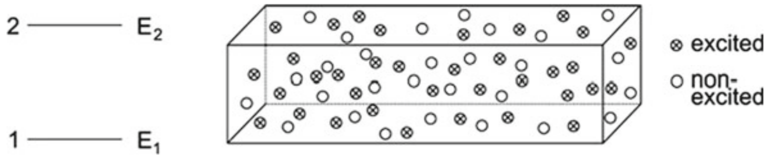
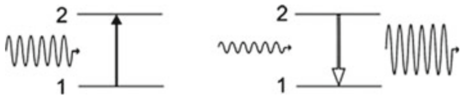


Fig. 2.5 Two-level atomic system and ensemble of two-level atomic systems

Fig. 2.6 Absorption and stimulated emission



holds, that is, if the photon energy $h\nu$ of the photons of the radiation field is equal to the energy difference E_{21}. But because of lifetime broadening of the upper level, a two-level atomic system can also interact if $h\nu$ is unequal to E_{21}.

Two processes are competing with each other in a laser, *absorption* and *stimulated emission* of radiation. In an absorption process (Fig. 2.6), a photon is converted to excitation energy of a two-level atomic system by a $1 \rightarrow 2$ transition. An excited two-level atomic system transfers by a stimulated emission process its excitation energy to the light field. Einstein showed: *Radiation created by stimulated emission has the same frequency, direction, polarization and phase as the stimulating radiation.*

If the active medium is an ensemble of two-level systems, the strength of stimulated emission is proportional to N_2, and the strength of absorption is proportional to N_1. We will later (Sect. 6.5) show, for an ensemble of identical two-level systems, that the factor of proportionality is the same for both processes. The net effect is proportional to the population difference $N_2 - N_1$. Stimulated emission prevails if $N_2 - N_1 > 0$ while absorption prevails if $N_2 - N_1 < 0$. In an active medium, the population difference $N_2 - N_1$ is larger than zero,

$$N_2 - N_1 > 0. \tag{2.26}$$

Alternatively, we can write:

$$N_2 \; > \; N_1; \tag{2.27}$$

in an active medium the population of the upper laser level is larger than the population of the lower laser level. Stimulated emission and absorption compensate each other if $N_2 = N_1$. In this case, the medium is transparent. We can formulate the *transparency condition*:

$$N_2 - N_1 = 0. \tag{2.28}$$

(If the lower laser level has the degeneracy g_1 and the upper laser level the degeneracy g_2, the criterion of population inversion is

$$g_2 N_2 - g_1 N_1 > 0. \tag{2.29}$$

We assume, for convenience, in the following that $g_1 = g_2$.)

It is useful, in particular with respect to the treatment of semiconductor lasers, to make use of *occupation numbers*. We introduce the *(relative) occupation number*

$$f_i = \frac{N_i}{\sum N_i}, \tag{2.30}$$

where N_i is the population of level i and $\sum N_i$ is the sum of the populations of all levels of an ensemble of atomic systems. The sum of the relative occupation numbers of an ensemble is unity, $\sum f_i = 1$. The relative occupation number f_i is equal to the probability that level i is occupied.

The relative occupation number of the upper laser level (Fig. 2.7) is equal to

$$f_2 = \frac{N_2}{N_1 + N_2}, \tag{2.31}$$

and the relative occupation number of the lower laser level is

$$f_1 = \frac{N_1}{N_1 + N_2}. \tag{2.32}$$

The sum of the relative occupation numbers is unity,

$$f_2 + f_1 = 1. \tag{2.33}$$

The *occupation number difference* (that is the difference between two probabilities) is

$$f_2 - f_1 = \frac{N_2 - N_1}{N_1 + N_2}. \tag{2.34}$$

The occupation number difference is the ratio of the population difference $N_2 - N_1$ and the density $N_1 + N_2$ of two-level atomic systems. Thus, the population difference is equal to the occupation number difference times the density of two-level atomic systems,

$$N_2 - N_1 = (f_2 - f_1)(N_1 + N_2). \tag{2.35}$$

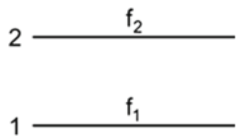


Fig. 2.7 Relative occupation numbers of a two-level atomic system

It follows that the strength of stimulated emission of radiation is proportional to f_2, and that the strength of absorption of radiation is proportional to f_1. The net effect—the difference between the strength of stimulated emission and absorption—is proportional to $f_2 - f_1$. Stimulated emission prevails if

$$f_2 - f_1 > 0, \tag{2.36}$$

while absorption prevails if $f_2 - f_1 < 0$. The condition $f_2 - f_1 > 0$ is again the condition of gain. We can write the transparency condition in the form:

$$f_2 - f_1 = 0; \tag{2.37}$$

a medium is transparent if the occupation number difference (more accurately: the relative occupation number difference) is zero. The corresponding density of two-level atomic systems in the upper laser level is the *transparency density* N_{tr} $(= N_{2,\mathrm{tr}})$.

A population inversion corresponds to a nonequilibrium state of an ensemble of two-level atomic systems. At thermal equilibrium, the population N_2 of an ensemble of two-level atomic systems is always smaller than the population N_1.

Thermal equilibrium of many media containing an ensemble of atomic systems is governed by *Boltzmann's statistics*. If Boltzmann's statistics holds, the ratio of the population of the upper level and the population of the lower level is given by

$$\frac{N_2}{N_1} = \mathrm{e}^{-(E_2-E_1)/kT}, \tag{2.38}$$

where k $(= 1.38 \times 10^{-23}\ \mathrm{J\,K^{-1}})$ is Boltzmann's constant and T the temperature of the ensemble. At thermal equilibrium, the population difference is always negative, $N_2 - N_1 < 0$; the net effect of stimulated emission and absorption of radiation results in damping of radiation at the frequency $\nu \sim (E_2 - E_1)/h$. For an ensemble of two-level atomic systems, which obeys Boltzmann's statistics, the occupation number of the upper level is equal to

$$f_2^{\mathrm{Boltz}} = \frac{N_2}{N_1 + N_2} = \frac{1}{\exp[(E_2 - E_1)/kT] + 1} \tag{2.39}$$

and the occupation number of the lower level is

$$f_1^{\mathrm{Boltz}} = \frac{N_1}{N_1 + N_2} = \frac{1}{\exp[-(E_2 - E_1)/kT] + 1}. \tag{2.40}$$

At thermal equilibrium, the relative occupation number of the lower level is always larger than the relative occupation number of the upper level, $f_1^{\mathrm{Boltz}} - f_2^{\mathrm{Boltz}} > 0$, that is, at thermal equilibrium, absorption always exceeds stimulated emission.

In the case that the energy levels of an ensemble governed by Boltzmann's statistics are degenerate, the relative occupation number of level i is given by

$$f_i^{\mathrm{Boltz}} = \frac{g_i \exp\left[E_i/kT\right]}{\sum g_i \exp\left[E_i/kT\right]}. \tag{2.41}$$

In atomic physics and thermodynamics, the occupation number of an atomic level concerns the total number of atoms. To describe laser media, it is convenient to make use of number densities and of relative occupation numbers. In order to avoid a confusion, we mark total numbers by the suffix "tot." If an ensemble of two-level systems is distributed in the volume V, we are dealing with the following quantities:

- N_1 = density of atoms in level 1.
- N_2 = density of atoms in level 2.
- $N_{1,\mathrm{tot}} = N_1 \times V$ = occupation number of level 1 = total number of two-level systems in level 1.
- $N_{2,\mathrm{tot}} = N_2 \times V$ = occupation number of level 2 = total number of two-level systems in level 2.
- $N_{\mathrm{tot}} = (N_1 + N_2) \times V$ = total number of two-level atomic systems.
- $f_2 = N_2/(N_2 + N_1) = N_{2,\mathrm{tot}}/N_{\mathrm{tot}} \times V$ = relative occupation number of level 2.
- $f_1 = N_1/(N_2 + N_1) = N_{1,\mathrm{tot}}/N_{\mathrm{tot}} \times V$ = relative occupation number of level 1.
- $f_2 - f_1 = (N_2 - N_1)/(N_2 + N_1) = (N_{2,\mathrm{tot}} - N_{1,\mathrm{tot}})/N_{\mathrm{tot}}$ = *occupation number difference* (=difference of the relative occupation numbers).

We will use the notation "occupation number" instead of "relative occupation number."

2.4 Laser Resonator

The *Fabry–Perot resonator* (Fig. 2.8) consists of two plane mirrors arranged in parallel at a distance L; the Fabry–Perot resonator is an open resonator—it has no sidewalls. We consider a Fabry–Perot resonator with reflectors of rectangular shape. We choose cartesian coordinates with the z axis parallel to the resonator axis; laser

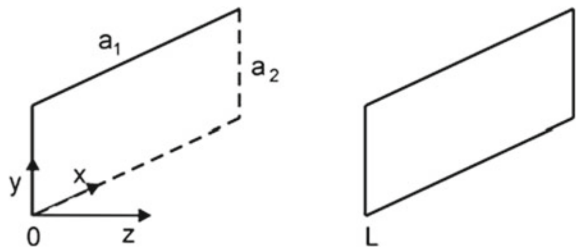


Fig. 2.8 Fabry–Perot resonator

radiation propagates along z. Characteristic quantities of a Fabry–Perot resonator are:

- a_1 = width of the resonator (along x).
- a_2 = height of the resonator (along y).
- L = length of the resonator (along z).
- $z = 0$ = location of mirror 1.
- $z = L$ = location of mirror 2.
- R_1 = reflectivity of mirror 1.
- R_2 = reflectivity of mirror 2.

We assume for the mirrors are perfectly reflecting ($R_1 = R_2 = 1$) and describe the laser field within a resonator as a *quasiplane standing wave* composed of two waves of equal amplitude and opposite propagation directions:

$$E = \frac{1}{2}A\cos[\omega t - (kz - \varphi_0)] + \frac{1}{2}A\cos[\omega t + (kz - \varphi_0)]. \tag{2.42}$$

The field E and the amplitude A have an orientation along a direction (e.g., the x direction) perpendicular to the z direction. Using the relations $\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$, we can write (2.42) in the form

$$E = A\cos(kz - \varphi_0)\cos\omega t, \tag{2.43}$$

which describes a standing wave. To find k, ω, and φ_0, we make use of three conditions:

- The solution of the *resonator eigenvalue problem* provides discrete values of the wave vector.
- The dispersion relation for electromagnetic radiation then yields the resonance frequencies of a resonator.
- Two *boundary conditions* for electromagnetic fields provide the phase.

The *resonator eigenvalue problem reads*: after a round trip transit through the resonator, the field at a location z at time $t + T$ is the same as the field at time t,

$$E(z, t + T) = E(z, t). \tag{2.44}$$

This leads to the condition

$$2kL = l \times 2\pi; \quad l = 1, 2, 3, \ldots. \tag{2.45}$$

The integer l is the order of a resonance. The change of phase per round trip transit is $2kL = 2\pi$. Accordingly, the wave vector has discrete values,

$$k_l = l \times \frac{2\pi}{2L}. \tag{2.46}$$

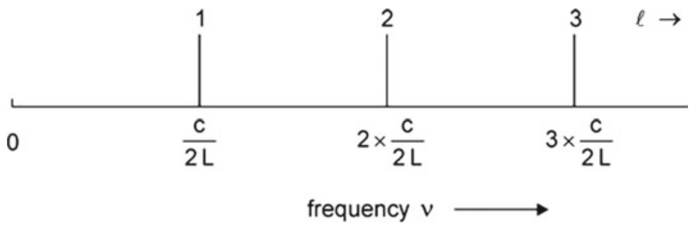


Fig. 2.9 Resonance frequencies of the Fabry–Perot resonator

We obtain, with $k = \omega/c$,

$$\omega_l = l \times \frac{2\pi c}{2L}, \tag{2.47}$$

or

$$\nu_l = l \times \frac{c}{2L}. \tag{2.48}$$

The resonance frequencies (= eigenfrequencies) ν_l of a Fabry–Perot resonator are multiples of $c/2L$. The resonance frequencies ν_l are equidistant. Next near resonance frequencies have the frequency distance (Fig. 2.9)

$$\nu_l - \nu_{l-1} = \frac{c}{2L}. \tag{2.49}$$

The round trip transit time—the time it takes the radiation to perform a round trip transit through the resonator—is

$$T = 1/\nu_1 = 2L/c. \tag{2.50}$$

The resonance wavelengths of radiation in a Fabry–Perot resonator are given by the relation

$$l \times \lambda_l/2 = L; \tag{2.51}$$

the length of the Fabry–Perot resonator is a multiple of $\lambda_l/2$. To determine the resonance wavelengths of a Fabry–Perot resonator containing a medium of refractive index n, we have to take into account that the speed of light in a medium is c/n and the wavelength is λ/n, where λ is the wavelength of light in vacuum.

Taking account of the boundary conditions and the dispersion relation, we obtain:

$$E = A\cos(k_l z - \varphi_0)\cos\omega_l t. \tag{2.52}$$

The *boundary conditions are*: $E = 0$ at $z = 0$ and $z = L$. We obtain the phase $\varphi_0 = \pi/2$. Thus, the standing wave has the form

$$E = A\sin k_l z\ \cos\omega_l t. \tag{2.53}$$

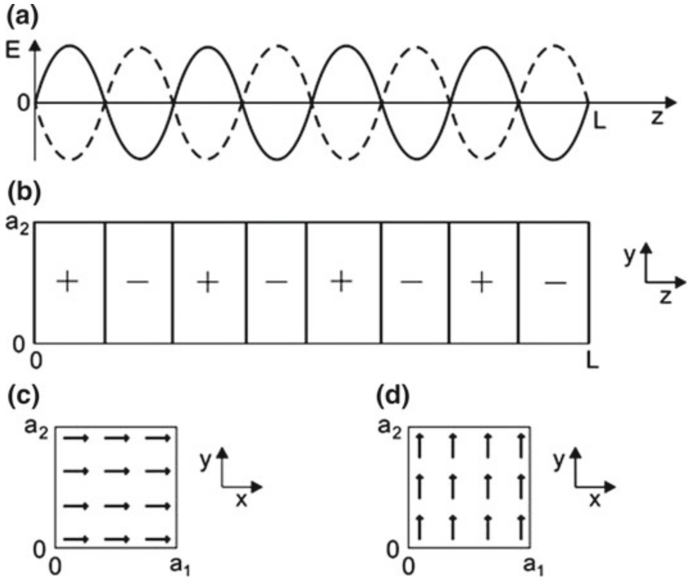


Fig. 2.10 Standing wave in a Fabry–Perot resonator. **a** Field $E(z)$. **b** Phase of the field in the zy plane. **c** Field lines in the xy plane for $E \parallel x$. **d** Field lines in the xy plane for $E \parallel y$

A resonance characterized by a frequency $\omega_l = 2\pi \nu_l$ and a wave vector k_l corresponds to a *resonator mode*, that is, to a particular pattern of the amplitude of the electromagnetic wave within the resonator. Figure 2.10a shows the electric field $E(z)$ for $t = 0$ and $T/2$. At a fixed time, the field varies sinusoidally along the resonator axis according to the variation of the phase kz. The sign of the field varies in z direction (Fig. 2.10b). The polarization of the electric field has a direction perpendicular to the z axis—along the x axis (Fig. 2.10c) or along the y axis (Fig. 2.10d). We now summarize the main properties of a quasiplane standing wave in a Fabry–Perot resonator.

- *Amplitude.* The amplitude A is a constant everywhere within the Fabry–Perot resonator.
- *Phase variation along the z axis.* The phase varies along the z axis.
- *Phase variation perpendicular to the z axis.* The phase does not vary in directions perpendicular to the z axis.
- *Polarization of the radiation.* The field is oriented perpendicular to z.

Two waves propagating in opposite directions add to the field, now written in complex form,

$$\tilde{E} = \frac{1}{2} A\, e^{i[\omega t - (kz - \varphi_0)]} + \frac{1}{2} A\, e^{i[\omega t + (kz - \varphi_0)]}. \tag{2.54}$$

The energy density of the field at a location z in the resonator averaged over a period of time is equal to

$$u(z) = \frac{1}{2}\varepsilon_0 A^2 \sin^2 k_l z \tag{2.55}$$

and the photon density (also averaged over a period of time) is

$$Z(z) = \frac{\varepsilon_0 A^2}{2h\nu} \sin^2 k_l z. \tag{2.56}$$

The average taken over a wavelength of the radiation yields the average energy density u of the electromagnetic field in the resonator

$$u = \frac{1}{4}\varepsilon_0 A^2 \tag{2.57}$$

and the average photon density

$$Z = \frac{u}{h\nu} = \frac{\varepsilon_0}{4h\nu} A^2. \tag{2.58}$$

If a resonator has two reflectors both with $R = 1$, light within the resonator travels without loss; it performs an infinite number of round trip transits. But the number of round trip transits is finite if one of the reflectors is a partial reflector acting as output coupling mirror. Then, a reflection at the output coupling mirror corresponds to a reduction of the energy density within the resonator. How long does a photon remain in a resonator? We consider the energy density u at a fixed location within the resonator (Fig. 2.11). After one round trip of the radiation, the energy density is Vu, where the V factor describes how much of the energy remained in the resonator after one round trip transit; accordingly, the photon density Z is reduced to VZ after one round trip of the radiation. The V factor is a measure of loss.

- *V factor* = fraction of radiation energy that remains in the resonator after a round trip transit = fraction of the number of photons remaining in the resonator after one round trip = survival probability of a photon after a round trip transit through the resonator.
- $V = 1$, there is no loss.
- $V < 1$, there is loss.

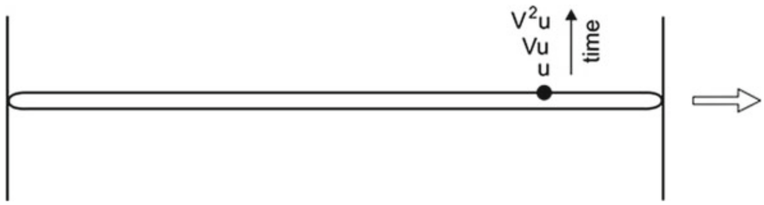


Fig. 2.11 Resonator with loss

The photon density develops as follows:
$t = 0$; Z_0.
$t = T$; one round trip transit, $Z = VZ_0$.
$t = sT$; s round trip transits,

$$Z(s) = V^s Z_0. \tag{2.59}$$

Replacing s by the continuous variable t/T, we write

$$Z(t) = Z_0 V^{t/T}. \tag{2.60}$$

Using the identity $a^x = \mathrm{e}^{x \ln a}$, we obtain

$$Z(t) = Z_0\,\mathrm{e}^{-\kappa t} = Z_0\,\mathrm{e}^{-t/\tau_p}, \tag{2.61}$$

where

$$\kappa = \frac{1}{\tau_p} = \frac{-\ln V}{T} \tag{2.62}$$

is the *loss coefficient* of the resonator and

$$\tau_\mathrm{p} = \frac{T}{-\ln V} \tag{2.63}$$

is the *photon lifetime* (= average lifetime of a photon in the resonator = decay time of the energy density of radiation in the resonator). We write the V factor as

$$V = \mathrm{e}^{-T/\tau_\mathrm{p}} = \mathrm{e}^{-\kappa T}. \tag{2.64}$$

The energy density decreases exponentially with the same decay constant as the photon density,

$$u = u_0\,\mathrm{e}^{-t/\tau_\mathrm{p}} = u_0\,\mathrm{e}^{-\kappa t}, \tag{2.65}$$

where u_0 is the initial energy density.

If the loss is due to both output coupling loss (described by V_out) and internal loss in the resonator (V_i), the total V factor is equal to

$$V = V_\mathrm{out} V_\mathrm{i}. \tag{2.66}$$

Then the loss coefficient of a resonator is

$$\kappa = \kappa_\mathrm{out} + \kappa_i, \tag{2.67}$$

where

- κ_{out} is the decay coefficient due to output coupling of radiation and
- κ_i is the loss coefficient due to internal loss.

Diffraction at the reflectors, for instance, causes internal loss.

The relative decrease of the energy density after one round trip transit is

$$(u - Vu)/u = 1 - V. \tag{2.68}$$

The quantity $1 - V$ is the *loss per round trip transit*.

Example A resonator has the V factor V = 0.9. This means:

- There remain, after one round trip of the radiation, 90% of the photons in the resonator.
- The loss per round trip is 10%.
- The photon lifetime is $\tau_p = T/(-\ln V) = 9.5T$.
- The survival probability of a photon after a round trip through the resonator is 0.9.

2.5 Laser = Laser Oscillator

The laser (=laser oscillator) is a *self-excited oscillator* (=self-sustained oscillator). It is characteristic of a self-excited oscillator that it starts oscillation itself and maintains oscillation as long as pump energy is supplied by an external energy source.

We mention a classical self-excited oscillator. A string of a violin is excited to an oscillation during a continuous motion of the bow. The string together with the bow, which steadily delivers energy to the oscillation, is a self-excited oscillator. The length of the string determines the fundamental frequency.

We will now formulate the condition of laser oscillation and also show that the buildup of a steady state oscillation takes time.

2.6 Radiation Feedback and Threshold Condition

Radiation in a resonator containing an active medium is repeatedly propagating through the active medium. The active medium experiences feedback from the radiation that is stored in the resonator.

We now assume that population inversion and gain are suddenly turned on at time $t = 0$. At the start of a laser oscillation, the energy density of the radiation is u. The energy density is equal to VGu after one round trip (Fig. 2.12). G is the *gain*

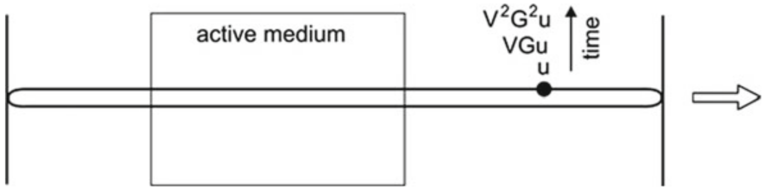


Fig. 2.12 Balance of energy in a laser

factor per round trip transit. It is equal to the product of the single-pass gain factors ($G = G_1^2$). The energy density increases if

$$VGu > u \tag{2.69}$$

or

$$GV > 1. \tag{2.70}$$

The energy in a resonator increases with time if the product of the gain factor and the V factor is larger than unity. Without loss ($V = 1$), the energy density after one round trip is Gu. The relative increase of the energy density after one round trip is

$$\frac{Gu - u}{u} = G - 1. \tag{2.71}$$

The quantity $G - 1$ is the *gain per round trip*. If gain and loss are present, we can write the condition of net gain in the form

$$\frac{Gu - u}{u} > \frac{u - Vu}{u} \tag{2.72}$$

or

$$G - 1 > 1 - V, \tag{2.73}$$

gain per round trip > loss per round trip. We did not differ between radiation propagating in the resonator in clockwise or counterclockwise direction: we supposed that propagation in both directions leads to gain.

We assume that V does not change with time. Can we also assume that the gain factor G is independent of time? If G would be constant, the energy in the laser would permanently increase and reach an infinitely large value. But this is not the case if $V < 1$. Then the energy in a laser resonator becomes finite—G decreases during the buildup of the light field in the laser resonator. We have the condition that $VG_\infty u = u$ at steady state or

$$G_\infty V = 1; \tag{2.74}$$

at steady state oscillation, the product of the gain factor G_∞ and the V factor is 1.

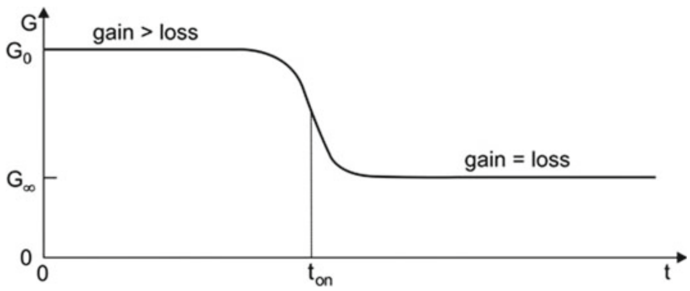


Fig. 2.13 Gain during onset of laser oscillation

We describe the following case: an active medium is suddenly turned on at time $t = 0$. The initial gain factor is G_0 (Fig. 2.13). The gain factor remains nearly constant and then decreases to the steady state value G_∞. The transition from G_0 to G_∞ occurs at the *onset time* t_{on}, which is a measure of the time it takes to build up a steady state oscillation. G_0 is the small-signal gain factor and G_∞ the large-signal gain factor. The two conditions lead to the *threshold condition of laser oscillation* (= laser condition):

$$GV \geq 1. \tag{2.75}$$

The condition implies that during the buildup of a laser field, G is larger than at steady state oscillation. We can also interpret the threshold condition as follows: an oscillation builds up if G_0 is only slightly larger than G_∞. In the extreme case that $G_0 \to G_\infty$, reaching a steady state takes infinitely long time ($t_{\text{on}} \to \infty$). In this sense, we introduce the threshold gain factor,

$$G_{\text{th}} = G_\infty. \tag{2.76}$$

The small-signal gain factor G_0 is always larger than the large-signal gain factor G_∞ ($= G_{\text{th}}$). At steady state oscillation, the gain factor is clamped at

$$G_\infty = V^{-1}. \tag{2.77}$$

We will treat onset of oscillation in more detail (Sects. 2.9, 8.4, and 9.7; Fig. 9.6). A laser is a regenerative amplifier: at steady state oscillation, radiation lost during a round trip transit through the resonator is regenerated after the round trip.

The energy density of radiation in a lossless resonator of a continuously pumped laser increases to infinitely large values—but optical damage limits the energy density (Sect. 16.10).

2.7 Frequency of Laser Oscillation

At steady state oscillation, the electric field at a fixed location in the laser resonator reproduces itself after each round trip transit through the resonator; this is the resonator eigenvalue problem in the case that a resonator contains an active medium. The electric field in a resonator (Fig. 2.14) has to obey the condition

$$\tilde{G}_{\mathrm{E}}\tilde{V}_{\mathrm{E}}\exp(-\mathrm{i}(2kL-\Delta\phi-\varphi_3+\varphi_{\mathrm{R1}}+\varphi_{\mathrm{R2}}))\,\tilde{E}=\tilde{E}. \tag{2.78}$$

The quantities concern a round trip transit through the active medium.

- $\tilde{G}_{\mathrm{E}} = G_{\mathrm{E}}\mathrm{e}^{\mathrm{i}\varphi_1}$ = complex gain factor with respect to the field.
- $\tilde{V}_{\mathrm{E}} = V_{\mathrm{E}}\mathrm{e}^{\mathrm{i}\varphi_2}$ = complex loss factor with respect to the field.
- 2kL = geometric phase shift.
- φ_{R1} = phase shift due to reflection at one of the mirrors.
- φ_{R2} = phase shift due to reflection at the other mirror.
- φ_3 = phase shift due to dispersion in the resonator (Sect. 13.3).
- $\Delta\phi$ = *Gouy phase shift* = additional phase shift occuring for radiation propagating in a resonator with curved mirrors (Sect. 11.7); the Gouy phase shift of radiation in a Fabry–Perot resonator is zero.

We obtain the condition

$$G_{\mathrm{E}}V_{\mathrm{E}}\exp{-\mathrm{i}}(2kL-\Delta\phi-\varphi_3-\varphi_1-\varphi_2+\varphi_{\mathrm{R1}}+\varphi_{\mathrm{R2}})=1. \tag{2.79}$$

The factor to the exponential has to be equal to unity, and the sum of all phases has to be a multiple of 2π. It follows that $G_{\mathrm{E}}V_{\mathrm{E}} = 1$ and, with $G_{\mathrm{E}}^2 = G$ and $V_{\mathrm{E}}^2 = V$, that $GV = 1$ as already derived in the preceding section. The second condition is

$$2kL-\Delta\phi-\varphi_3+\varphi_{\mathrm{R1}}+\varphi_{\mathrm{R2}}=l\times 2\pi, \tag{2.80}$$

where l is an integer. The sum of all changes of phase after a round trip transit has to be a multiple of 2π. Since $k = \omega/c$, the condition provides the eigenfrequencies $\omega_l = 2\pi\nu_l$. In the special case that all additional phases—but not the geometric phase shift kz—are zero, we obtain the eigenfrequencies $\nu_l = lc/(2L)$, with l = 1, 2, …; otherwise the resonance frequencies are shifted.

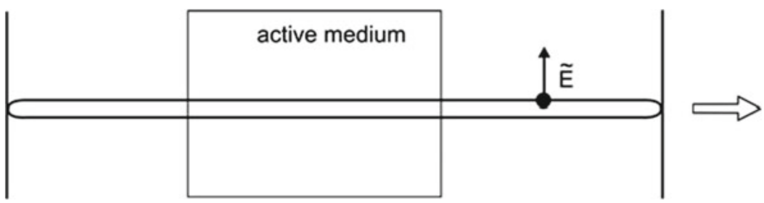


Fig. 2.14 Field in a laser

Electromagnetic radiation propagating in a medium of refractive index n has the wave vector $k = n\omega/c$. It depends on the properties of the resonator and of the active medium at which resonance frequency (or frequencies) a laser oscillates.

We will discuss the origin of phase shifts of electromagnetic wave propagating in an active medium in Chap. 9 and phase shifts of an electromagnetic wave in a laser resonator in Chap. 11 and Sect. 13.4.

2.8 Data of Lasers

Table 2.1 shows data of continuous wave lasers. The data concern the quantities (*see* also Fig. 2.15):

- L = resonator length.
- d = diameter in case of a circular resonator.
- a_1 = width and a_2 = height of a rectangular resonator.
- Resonator volume $= \pi(d/2)^2 L$ of a circular-mirror resonator and $a_1 a_2 L$ of a rectangular resonator.
- G = gain factor per round trip of the radiation; G_1, per single transit.
- $V = V$ factor per round trip; it indicates the reduction of the photon density in the resonator per round trip transit, and V_1 loss per single transit.
- P_{out} = output power.

By modifying a laser (e.g., by choosing an other cross sectional areas), it can be possible to obtain a larger or a smaller output power. The lasers generate radiation at wavelengths listed in the table but also at other wavelengths. The length of the active medium of a gas laser is about equal to the length of the resonator. The length of the active medium of a solid state laser is smaller than the resonator length. The length of the active medium of a semiconductor laser is about equal to the resonator length. Semiconductor lasers have much smaller sizes than other lasers.

- *Helium–neon laser*. The gain is small. Therefore, the V factor has to be close to unity—the reflectivities of the resonator mirrors have to be near unity

Table 2.1 Data of lasers

Laser	λ	L(m)	Resonator d(m), or a_1 (m); a_2 (m)	Volume (m^3)	G [or G_1]	V [or V_1]	P_{out} (W)
HeNe	633 nm	0.5	2×10^{-3}	5×10^{-5}	1.02	0.99	10^{-2}
CO_2	10.6 µm	0.5	2×10^{-2}	5×10^{-3}	3	0.95	70
Nd:YAG	1.06 µm	0.5	2×10^{-2}	5×10^{-3}	50	0.9	2
TiS	830 nm	0.5	2×10^{-2}	5×10^{-3}	50	0.9	5
Fiber	1.5 µm	10	10^{-5}	10^{-9}	100	0.5	1
SC	810 nm	10^{-3}	10^{-6}; 10^{-4}	10^{-13}	[12]	[0.3]	10^{-1}
QCL	5 µm	10^{-3}	10^{-5}; 10^{-4}	10^{-12}	10	0.9	10^{-3}

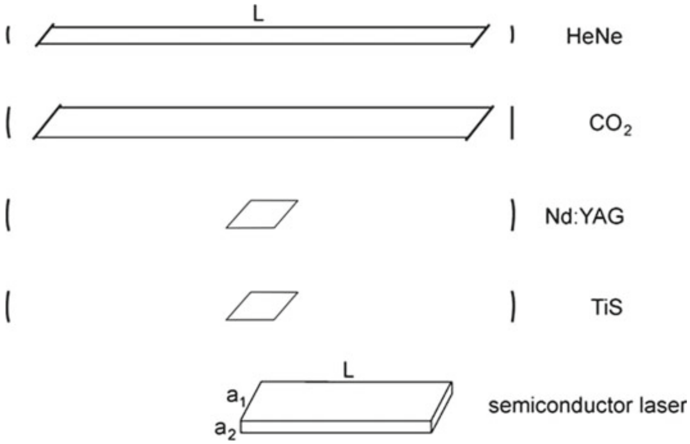


Fig. 2.15 Various lasers

(e.g., $R_1 = 0.998$ and $R_2 = 0.99$). A glass plate covered on the front surface with a highly-reflecting multilayer coating and on the back surface with anantireflecting dielectric coating is a high-reflectivity mirror (Sect. 25.7). The front surface with its coating acts as resonator mirror while the back surface is outside the resonator. The glass tube that contains the laser gas is closed by Brewster windows.

- *CO_2 laser*. The gain is large. One of the reflectors is a metal mirror; a metal mirror has a reflectivity near unity for radiation at wavelengths larger than about 5 μm. The output coupling mirror has a reflectivity that is noticeably smaller than unity (e.g., $R_1 = 1$; $R_2 = 0.95$). The output coupling mirror is a dielectric plate (e.g., a germanium plate) covered on the resonator side with a dielectric multilayer coating and on the other side with a dielectric antireflecting coating; a metal film is not suitable as a partial reflector because of a very high absorptivity for radiation passing through a metal film (Problem 25.18).
- *Neodymium YAG laser* (Nd:YAG laser). The gain is large. The length of the active medium is much smaller than the length of the resonator. Both mirrors (e.g., $R_1 \sim 1$ and $R_2 = 0.95$) consist of dielectric multilayers on glass plates. The Nd:YAG crystal surfaces are obliquely oriented relative to the beam axis so that the angle of incidence of the radiation is the Brewster angle and radiation traverses the crystal surfaces without loss.
- *Titanium–sapphire laser*. The active medium also fills a small portion of the resonator. The gain is large (mirror reflectivities are, e.g., $R_1 \sim 1$ and $R_2 \sim 0.95$). The titanium–sapphire crystal surfaces are obliquely oriented relative to the beam axis so that the angle of incidence of the radiation is the Brewster angle and radiation traverses the crystal surfaces without loss.
- *Fiber laser*. The active medium is a doped fiber of small diameter and large length. The output power can reach several hundred watt.
- *Bipolar semiconductor laser* (SC). The gain can be large already at a small length of an active medium. It is possible to use the semiconductor surfaces as reflectors.

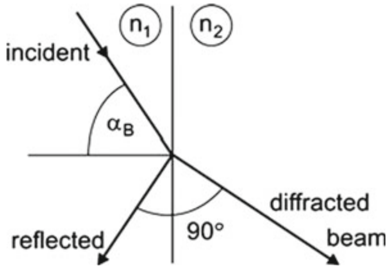


Fig. 2.16 Brewster angle

Then each of the surfaces has the reflectivity $R_1 = R_2 = R = (n-1)^2/(n+1)^2$, where n is the refractive index of the semiconductor laser material. The resonator material of a GaAs semiconductor laser has the refractive index $n = 3.6$, and the reflectivity has a value ($R = 0.32$) that is markedly smaller than unity.

- *Quantum cascade laser*. The gain is also large at small length of the gain medium.

Radiation passing at normal incidence through an interface between air and a medium experiences loss due to reflection, while radiation of the appropriate polarization direction passes the interface under the Brewster angle without reflection loss. The Brewster angle follows from *Snell's law*

$$\frac{\sin\,\alpha_2}{\sin\,\alpha_1} = \frac{n_1}{n_2}, \tag{2.81}$$

where α_1 is the angle of incidence, α_2 the angle of the transmitted beam, n_1 (~ 1) the refractive index of air, and n_2 the refractive index of the dielectric material. The reflectivity is zero if the electric field vector lies within the plane of incidence (p polarization) and if the angle of incidence is equal to the Brewster angle α_B (Fig. 2.16). The Brewster angle is determined by the relation

$$\tan\,\alpha_B = n_2/n_1. \tag{2.82}$$

The diffracted and the reflected beam (that has no power) are perpendicular to each other. Radiation of a polarization perpendicular to the plane of incidence (s polarization) is partly reflected and is therefore attenuated by a Brewster window. Accordingly, the radiation of a laser that contains Brewster windows is polarized.

2.9 Oscillation Onset Time

The threshold condition does not specify the value of the initial energy density of the electromagnetic field. However, there is a physical limit: if there is no electromagnetic energy in the resonator, nothing can be amplified. Already one photon in the resonator can initiate laser oscillation if the threshold condition is fulfilled. One photon in a resonator corresponds to a photon density of $Z_0 = (a_1 a_2 L)^{-1}$, where $a_1 a_2 L$ is the volume of the resonator. After the round trip transit time, the density of the photons is VG_0 and after s round trip transits, the density of the photons in the resonator is

$$Z(s) = Z_0 (VG_0)^s. \tag{2.83}$$

To estimate the onset time, we assume that the gain, described by the round trip gain factor G_0, is turned on at $t = 0$, then remains constant, and suddenly decreases at $t = t_{on}$ to G_∞ (Fig. 2.17, upper part). It follows that the photon density increases exponentially from the initial value Z_0 until it reaches at $t = t_{on}$ the steady state value Z_∞ (= density of photons in the resonator at steady state oscillation). We write

$$(VG_0)^{s_{on}} = Z_\infty / Z_0, \tag{2.84}$$

where s_{on} is the number of round trip transits necessary to reach the steady state. It follows that the oscillation onset time, $t_{on} = s_{on} T$, is given by

$$t_{on} = T \, \frac{\ln(Z_\infty / Z_0)}{\ln(VG_0)}. \tag{2.85}$$

The onset time is proportional to the round trip transit time and to the natural logarithm of Z_∞ / Z_0. And it is inversely proportional to the natural logarithm of the product VG_0.

We replace s by the continuous variable t/T and write

$$Z(t) = Z_0 \, (VG_0)^{t/T}. \tag{2.86}$$

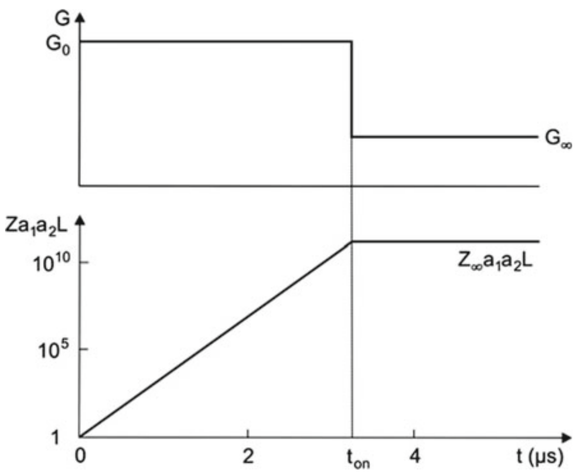


Fig. 2.17 Onset of laser oscillation: gain factor and photon number

With the identity $a^x = e^{x \ln a}$, we obtain

$$Z = Z_0\, e^{\ln(VG_0)t/T}. \tag{2.87}$$

The number of photons increases exponentially until, at $t = t_{on}$, a steady state oscillation is established.

It follows, for $1 - V \ll 1$ and $G_0 - 1 \ll 1$, that $\ln(VG_0) = \ln V + \ln G_0 = (G_0 - 1) - (1 - V)$ = gain minus loss per round trip. Then we can write

$$t_{on} = \frac{T}{(G_0 - 1) - (1 - V)} \ln(Z_\infty/Z_0). \tag{2.88}$$

If the gain is small compared to unity, $(G_0 - 1) \ll 1$, the oscillation onset time is large compared to the round trip time, $t_{on} \gg T$.

Example A helium–neon laser (length 0.5 m; cross-sectional area 1 mm^2; output power 1 mW; gain $= G - 1 = 0.02$; loss $1 - V = 0.01$) starts with one photon in the laser mode ($Z_0 a_1 a_2 L = 1$) and contains, at steady state oscillation, $Z_\infty a_1 a_2 L = P_{out}\tau_p = 10^{10}$ photons, where $\tau_p = T/(1 - V) = 3.3 \times 10^{-7}$ s is the photon lifetime. The density of photons in the resonator at $t = 0$ is $Z_0 = (a_1 a_2 L)^{-1} = 10^6$ m^{-3}. The density of photons at steady state oscillation is $Z_\infty \sim 10^{16}$ m^{-3}. The round trip transit time is $T = 3.3 \times 10^{-9}$ s. It follows that the onset time is $t_{on} \sim 8$ μs. The buildup of steady state oscillation of a helium–neon laser requires that the radiation performs about thousand round trip transits through the resonator. The photon density (Fig. 2.17, lower part) increases exponentially during the onset time ($t < t_{on}$) and has a constant value for $t > t_{on}$.

References [1–11, 26–28].

Problems

2.1 Photon density. Calculate the density Z of photons in a radiation field (wavelength 1 μm, 1 nm, or 1 mm) of an energy density of 1 J m^{-3}.

2.2 Amplitude of a field in a resonator containing a medium of the dielectric constant $\varepsilon = 1$ at the laser wavelength.

(a) Determine the amplitude of a field that corresponds to radiation of an energy density of 1 J/m^3.
(b) Evaluate the photon density Z, the field amplitude, and the energy density in a laser resonator (size 1 cm^3) if the resonator contains 1 photon and if the energy of a photon corresponds to a wavelength of 1 μm.
(c) Evaluate the photon density, the field amplitude, and the energy density in a laser resonator (size 0.4 μm × 100 μm × 500 μm) if it contains 1 photon (photon wavelength 1 μm).

2.3 Thermal occupation number of an atomic system governed by Boltzmann statistics.

(a) Show that $f_1^{\text{Boltz}} - f_2^{\text{Boltz}} > 0$ for an ensemble of two-level atomic systems in thermal equilibrium.
(b) Estimate the thermal occupation number difference $f_2^{\text{Boltz}} - f_1^{\text{Boltz}}$ for an ensemble of two-level atomic systems (at 300 K) at level separations that correspond to visible radiation (wavelength of 600 nm).
(c) Calculate the occupation number difference at level separations that correspond to far infrared radiation ($\lambda = 300\ \mu$m).

2.4 Threshold condition of laser oscillation.

(a) Show that the threshold condition (expressed for the power of radiation) is the same for light propagating in $-z$ direction as for light propagating in $+z$ direction.
(b) Show that the threshold condition is also the same for two electromagnetic fields propagating in $\pm$ z directions if $G_2 \neq G_1$, where the gain factors correspond to the single-pass gain factors for the two propagation directions.

2.5 Brewster angle. Determine the Brewster angles of materials used in lasers as windows or as active materials.

(a) Helium–neon laser (633 nm); quartz glass, $n = 1.4$.
(b) CO_2 laser; NaCl crystal, $n = 1.5$.
(c) Nd:YAG laser; YAG, $n = 1.82$.
(d) Titanium–sapphire laser; sapphire, $n = 1.76$.

2.6 Photon lifetime and oscillation onset time. Determine the photon lifetime and the oscillation onset time of lasers mentioned in Table 2.1.

2.7 Fresnel coefficients.
Derive the Brewster angle by use of the *Fresnel coefficients*:

$$r_\perp = \frac{E_\perp^{(r)}}{E_\perp^{(i)}} = \frac{n_1 \cos\theta_1 - n_2 \cos\theta_2}{n_1 \cos\theta_1 + n_2 \cos\theta_2}, \tag{2.89}$$

$$t_\perp = \frac{E_\perp^{(t)}}{E_\perp^{(i)}} = \frac{2n_1 \cos\theta_1}{n_1 \cos\theta_1 + n_2 \cos\theta_2}, \tag{2.90}$$

$$r_{||} = \frac{E_{||}^{(r)}}{E_{||}^{(i)}} = \frac{n_2 \cos\theta_1 - n_1 \cos\theta_2}{n_1 \cos\theta_1 + n_2 \cos\theta_2}, \tag{2.91}$$

$$t_{||} = \frac{E_{||}^{(t)}}{E_{||}^{(i)}} = \frac{2n_1 \cos\theta_1}{n_1 \cos\theta_1 + n_2 \cos\theta_2}. \tag{2.92}$$

- $r_\perp$ = Fresnel coefficient of reflection, with the electric field direction perpendicular to the plane of incidence.

- $r_{||}$ = Fresnel coefficient of reflection, with the electric field direction in the plane of incidence.
- θ_1 = angle of incidence.
- θ_2 = angle of the refracted beam.
- n_1 = refractive index of medium 1.
- n_2 = refractive index of medium 2.

The coefficients $r_\perp$ and $r_{||}$ are the corresponding Fresnel coefficients of transmission.

2.8 Fresnel coefficients of normal incidence.

(a) Show that $r_{||} = r_\perp = r = (n_1 - n_2)/(n_1 + n_2)$ and $t_{||} = t_\perp = t = 2n_1/(n_1 + n_2)$ and determine the reflectivity R and the transmissivity T.
(b) Show that $r_{21} = -r_{12}$ and that $t_{12}t_{21} - r_{12}r_{21} = 1$.

2.9 Relate the intensity of radiation to the photon density.

2.10 Photon flux. The photon flux is equal to the number of photons per second per unit area.

(a) Relate to other quantities that characterize a plane wave, namely photon density, energy density, intensity, and amplitude.
(b) Determine the photon flux for the radiation fields mentioned in Problem 2.1.
(c) Determine the photon flux for the output of the lasers mentioned in Table 2.1.

2.11 Determine the Brewster angle for laser materials mentioned in Table 6.1.

2.12 Radiation of a helium-neon laser (power 1 mW, wavelength 633 nm) is focused to an area of diameter 10 μm^2. Determine the intensity, the photon density, the energy density and the amplitude of the electric field in the focus.

2.13 Circularly polarized radiation.

(a) Characterize the field of circularly polarized radiation.
(b) Show that circularly polarized radiation can be obtained by sending a plane wave through a quarter-wave plate; a quarter-wave plate consists of an anisotropic crystal with different refractive indices for the ordinary and the extraordinary beam propagating in the same direction.