

## Chapter 25

# Reflectors of Quantum Well Lasers and of Other Lasers

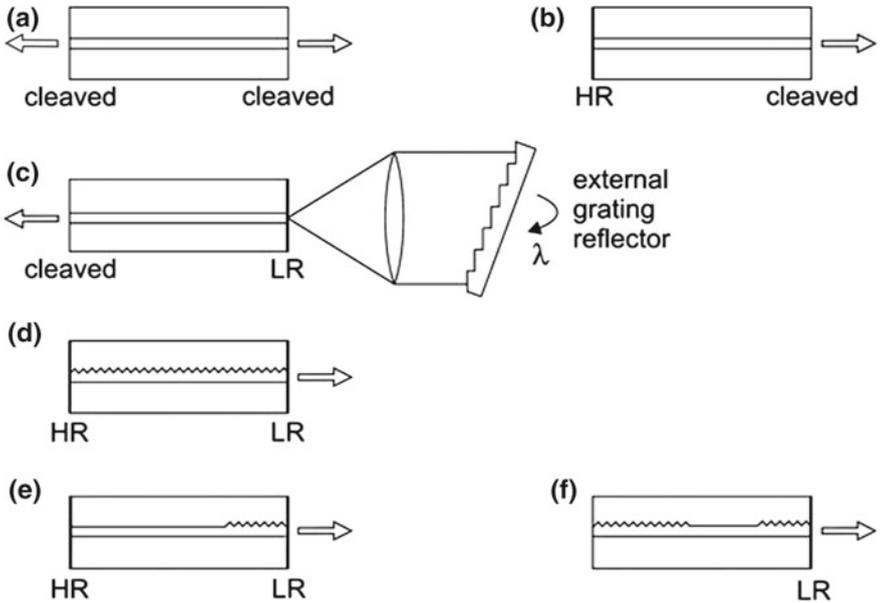
We discuss different reflectors: distributed feedback reflector; Bragg reflector and photonic crystal reflector; total internal reflector leading to whispering gallery modes. The reflectors are suited as reflectors not only in quantum well lasers but also in quantum wire and quantum dot lasers (Chap. 27). Depending on the type of reflector, it is possible to design semiconductor lasers of submillimeter size down to  $(10 \mu\text{m})^3$ . In connection with photonic crystals, we mention the photonic crystal fiber as a dielectric light guiding structure. The one-dimensional photonic crystal reflector (= Bragg reflector = multilayer reflector) is in use for almost all types of lasers.

We consider propagation of electromagnetic waves in layered materials (stratified media) in the special—but important—case that radiation is propagating in the direction perpendicular to the layers. We introduce the plane-wave transfer matrix that describes transfer of a wave from one side of an interface to the other side—and the propagation matrix, which characterizes propagation of a wave within a medium. The plane-wave matrix method is based on the boundary conditions for fields at an interface. We treat: thin film between two media; dielectric multilayer; one-dimensional photonic crystal. We will apply (Sect. 30.3) the plane-wave matrix method also to investigate electron waves passing through an interface of two semiconductor media.

### 25.1 Plane Surface

We have already discussed the edge-emitting quantum well laser with two uncovered crystal surfaces as reflectors. Cleaving a substrate (together with the layers on the substrate) results in a plane surface and cleaving along two parallel planes results in a resonator. Disadvantages and advantages of an edge-emitting laser with cleaved surfaces (Fig. 25.1a) are the following:

- The reflectivity, determined by the refractive index of the cleaved semiconductor material has a fixed value.



**Fig. 25.1** Resonators of semiconductor lasers. **a** Resonator with cleaved surfaces. **b** Resonator with a coated and an uncoated surface. **c** Resonator with an external reflector. **d** Resonator with distributed feedback. **e** Resonator with a distributed Bragg reflector. **f** Resonator with two distributed Bragg reflectors

- Emission occurs into two directions.
- The laser beam has a large angle of aperture in the plane perpendicular to the active layer.
- It is easy to prepare a cleaving surface.

We now will discuss other possibilities to realize laser resonators of semiconductor lasers.

### 25.2 Coated Surface

By deposition of a dielectric coating on a surface (Fig. 25.1b), one of the reflectors has a high reflectivity (HR). The other surface can remain without coating. Thus, the laser emits radiation in one direction only.

## 25.3 External Reflector

An external reflector (Fig. 25.1c) makes it possible to realize a tunable semiconductor laser. The reflector is an echelette grating in Littrow arrangement. Rotation of the grating results in a change of the wavelength of the laser radiation. Resonances between the surface and the grating are avoided by the use of an antireflecting coating (a coating with a low reflectivity, LR) on one of the surfaces of the laser diode. The length of the external resonator limits the tuning range—up to about 50 GHz—for tuning on one mode.

## 25.4 Distributed Feedback Reflector

The integration of a grating into the light guiding structure (Fig. 25.1d) leads to *distributed feedback*. The wavelength of the laser radiation is mainly determined by the period of the distributed reflector. Distributed feedback together with coated cleaved surfaces makes it possible to optimize the laser output of distributed feedback edge-emitting semiconductor lasers.

Distributed feedback reflectors are also suitable as reflectors of solid state dye lasers and organic and polymer lasers (Sect. 34.4).

## 25.5 Distributed Bragg Reflector

A distributed reflection grating separated from the gain region is a distributed Bragg reflector (Fig. 25.1e). A distributed Bragg reflector acts as output coupler. The surface opposite to the Bragg reflector is highly reflecting.

By the use of two distributed Bragg reflectors (Fig. 25.1f), a high reflectivity at one end of the active region and an optimized output coupler at the other end can be realized.

Distributed Bragg reflectors are well suitable as reflectors of bipolar semiconductor lasers.

## 25.6 Total Reflector

Internal total reflection in a circularly shaped solid results in a resonator with a very high Q factor. Light is propagating in a whispering gallery mode (Fig. 25.2). Output coupling of radiation is possible by positioning a prism near the surface, resulting in frustrated total reflection in a small region of the resonator. The distance  $d$  between the prism and the surface of the resonator regulates the output power. The quality factor can be of the order of  $10^6$ .

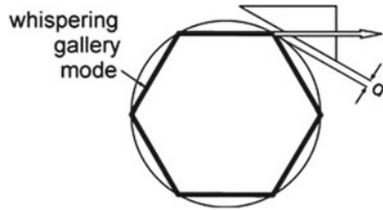


Fig. 25.2 Whispering gallery mode

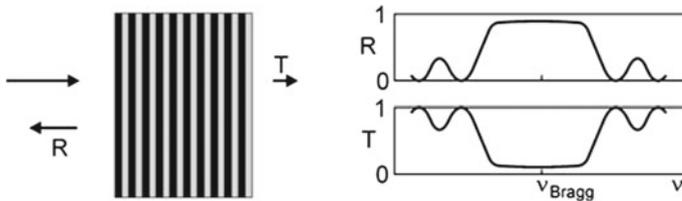


Fig. 25.3 Bragg reflector

## 25.7 Bragg Reflector

A very successful type of reflector is the Bragg reflector (=dielectric multilayer reflector = 1D photonic crystal reflector); it can consist of a multilayer coating on a transparent substrate.

A Bragg reflector (Fig. 25.3, left) consists of dielectric layers of two different materials of different refractive indices. Each layer has a thickness that is equal to a fourth of the wavelength of radiation in the corresponding material; a material (refractive index  $n_1$ ) has the thickness  $\lambda/(4n_1)$  and the other material ( $n_2$ ) has the thickness  $\lambda/(4n_2)$ . With increasing number of quarter-wavelength layers (of two materials in turn) the reflectivity increases and can reach a value very near unity. A Bragg reflector can have a high reflectivity  $R$  over nearly one octave of the spectrum (Fig. 25.3, right) around a central frequency  $\nu_{\text{Bragg}}$ . The transmissivity  $T$  of the Bragg reflector is  $T = 1 - R$ . Radiation incident on a Bragg reflector is either reflected or transmitted. Bragg reflectors are essential for operation of vertical-cavity surface-emitting lasers (VCSELs). A Bragg reflector can consist of quarter-wavelength layers of two semiconductors—for instance, of GaAs and AlAs for red and infrared lasers.

Bragg reflectors (=dielectric mirrors) consisting of other materials (e.g., layers of two glass types with different refractive indices) can be used as reflectors and as partial reflectors of almost all types of lasers (Chaps. 14–19).

## 25.8 Photonic Crystal

A medium that has a spatially periodic dielectric constant is a photonic crystal. A medium consisting of a periodic metal structure with holes or dielectric inclusions can be a photonic crystal too. All photonic crystals have in common that the light propagation is anisotropic and that there can be a photonic bandgap—radiation of a frequency that lies in the bandgap cannot propagate in a photonic crystal. There are three types of photonic crystals (Fig. 25.4, upper row):

- 1D (one-dimensional) photonic crystal. A 1D photonic crystal is, with respect to the optical properties, periodic in one direction and has no structure along the two other directions; the 1D photonic crystal is a three-dimensional medium. The frequency spectrum  $\omega(\mathbf{k})$  can have a gap for electromagnetic waves propagating along the direction of periodicity (Sect. 25.14).
- 2D (two-dimensional) photonic crystal. The 2D photonic crystal is periodic in two directions and has no structure along the third direction; the 2D photonic crystal is a three-dimensional medium. The frequency spectrum  $\omega(\mathbf{k})$  can show gaps for electromagnetic waves with wave vectors in the plane, which contains the two directions of periodicity.
- 3D (three-dimensional) photonic crystal. The 3D photonic crystal is periodic in three directions. The frequency spectrum  $\omega(\mathbf{k})$  can show gaps for electromagnetic waves in all three spatial directions.

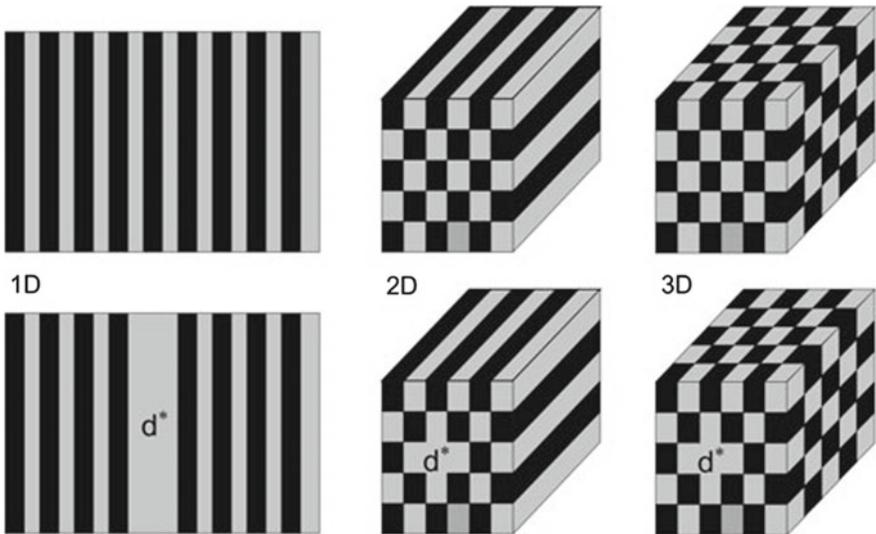


Fig. 25.4 Photonic crystals without and with a defect

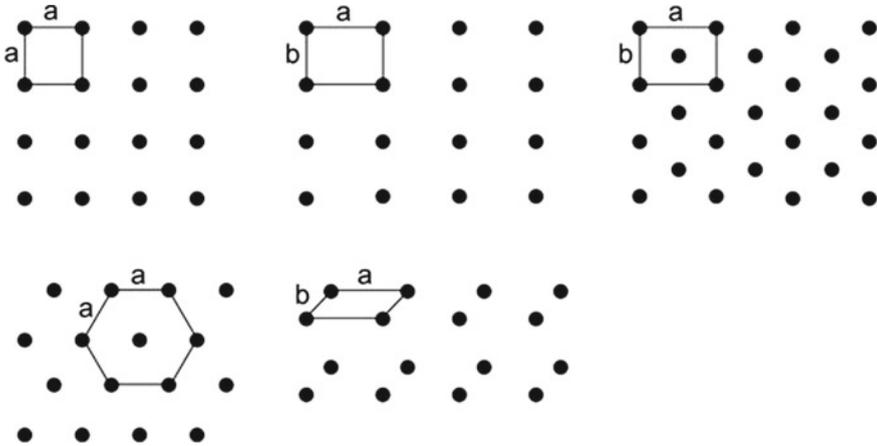


Fig. 25.5 Elementary cells of 2D photonic crystal lattices

A photonic crystal of finite length can act as a partial reflector for radiation of a photon energy in the photonic gap: radiation incident on a photonic crystal of finite thickness is partly reflected and partly transmitted.

If a single layer of a one-dimensional crystal is missing (Fig. 25.4, lower row), the photonic crystal contains a defect ( $d^*$ ). A photonic crystal with a defect represents a resonator—it is a Fabry–Perot resonator. A two-dimensional photonic crystal with a defect can act as a light guide. A three-dimensional crystal with a defect represents a cavity-like resonator. In all the three cases (1D, 2D, or 3D photonic crystal), a defect can also consist in the modification of more than one structural element.

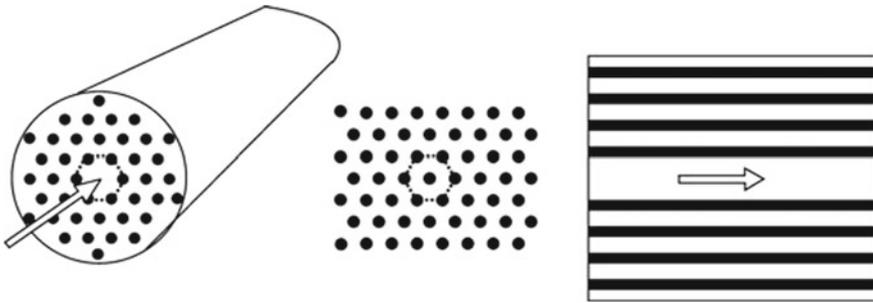
Common to all photonic crystals (assumed to have infinite extensions in all three directions) is the translational symmetry. A 1D photonic crystal contains a structural element that periodically repeats in one direction. There is no structure in the lateral directions. A 2D photonic crystal has two axes of periodicity and a 3D photonic crystal has three axis of periodicity.

A 2D photonic crystal can have one of the five different types of lattices (Fig. 25.5): square lattice; rectangular lattice; centered rectangular lattice; hexagonal lattice and parallelogram lattice.

A 3D photonic crystal can have one of 14 different lattice types. The simplest three-dimensional lattice is the (primitive) cubic lattice.

## 25.9 Photonic Crystal Fiber

A photonic crystal fiber (Fig. 25.6) can consist of a fiber with an internal two-dimensional photonic crystal (e.g., a hexagonal two-dimensional lattice). The structure is composed of two different glass materials, one with a higher dielectric con-



**Fig. 25.6** Photonic crystal fiber

stant than the other. A defect allows for propagation of light (along the defect). A photonic bandgap for propagation of radiation along a direction perpendicular to the axis of the two-dimensional photonic crystal avoids spreading of the radiation.

## 25.10 Remark About Photonic Crystals

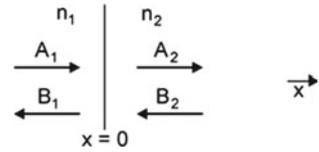
In 1887, Lord Rayleigh explained an experiment, which indicated that a periodic dielectric multilayer stack showed a spectral range of high reflectivity [217], corresponding to a stop-band of radiation. Such multilayer stacks (later called Bragg reflectors or photonic crystals) are widely studied and applied as reflectors or partial reflectors. The term “photonic crystal” describing inhomogeneous but periodic structures was introduced by E. Yablonovitch [218] in 1987.

Reflectors of the two-dimensional photonic crystal type were first used as reflectors of microwave Fabry–Perot interferometers in 1957 [219] and of far infrared Fabry–Perot interferometers up to frequencies of several THz in 1962 [220]; a far infrared reflector consists of a thin metal mesh and a Fabry–Perot interferometer of two meshes in parallel.

## 25.11 Plane-Wave Transfer Matrix Method Characterizing an Optical Interface

We consider the interface of two optically isotropic, nonabsorbing, and nonmagnetic materials (Fig. 25.7). The refractive indices of the two media are  $n_1$  and  $n_2$ . We study the special case of monochromatic radiation (frequency  $\omega$ ) propagating along the  $x$  or  $-x$  direction. The field in medium 1 is

**Fig. 25.7** Amplitudes of electromagnetic plane waves at an interface



$$E_1 = E_1^+ + E_1^- = A_1 e^{i(\omega t - k_1 x)} + B_1 e^{i(\omega t + k_1 x)} \tag{25.1}$$

and the field in medium 2 is

$$E_2 = E_2^+ + E_2^- = A_2 e^{i(\omega t - k_2 x)} + B_2 e^{i(\omega t + k_2 x)}. \tag{25.2}$$

$A_1, B_1, k_1, -k_1$  and  $A_2, B_2, k_2, -k_2$  are the amplitudes and the wave vectors of the waves in medium 1 and medium 2 in  $x$  and  $-x$  direction, respectively.

We assume that the electric field is oriented along the  $y$  axis and that, according to Maxwell's equations,  $H$  is therefore oriented along the  $z$  direction. It follows from Maxwell's equation

$$\mathbf{H} = \frac{i}{\mu_0 \omega} \nabla \times \mathbf{E} \tag{25.3}$$

that the magnetic field strength is given by

$$H_n = \frac{i}{\mu_0 \omega} \frac{dE}{dx}. \tag{25.4}$$

The boundary conditions for the electromagnetic fields (continuity of  $E_t$  and  $H_n$ ) require that  $E$  and  $dE/dx$  are continuous at the boundary ( $x = 0$ ),

$$E_1 = E_2 \quad \text{at } x = 0, \tag{25.5}$$

$$dE_1/dx = dE_2/dx \quad \text{at } x = 0, \tag{25.6}$$

or

$$A_1 + B_1 = A_2 + B_2, \tag{25.7}$$

$$k_1 A_1 - k_1 B_1 = k_2 A_2 - k_2 B_2. \tag{25.8}$$

We can write

$$M_1 \begin{pmatrix} A_1 \\ B_1 \end{pmatrix} = M_2 \begin{pmatrix} A_2 \\ B_2 \end{pmatrix}, \tag{25.9}$$

where, with  $l = 1, \text{ or } 2$ ,

$$M_l = \begin{pmatrix} 1 & 1 \\ k_l & -k_l \end{pmatrix}, \tag{25.10}$$

$$\begin{pmatrix} A_1 \\ B_1 \end{pmatrix} = M_1^{-1} M_2 \begin{pmatrix} A_2 \\ B_2 \end{pmatrix} = M_{12} \begin{pmatrix} A_2 \\ B_2 \end{pmatrix}, \tag{25.11}$$

and

$$M_{12} = \begin{pmatrix} \frac{1}{2}(1 + k_2/k_1) & \frac{1}{2}(1 - k_2/k_1) \\ \frac{1}{2}(1 - k_2/k_1) & \frac{1}{2}(1 + k_2/k_1) \end{pmatrix}. \tag{25.12}$$

The matrix  $M_{12}$  is the *plane-wave transfer matrix*. It relates the amplitudes of electromagnetic plane waves in medium 1 and the amplitudes of electromagnetic plane waves in medium 2.

### 25.12 Thin Film Between Two Media

A thin film (thickness  $a$ ) located between two media (Fig. 25.8) has two boundaries. The boundary conditions lead to

$$\begin{pmatrix} A_1 \\ B_1 \end{pmatrix} = M_1^{-1} M_2 \begin{pmatrix} A_2 \\ B_2 \end{pmatrix} = M_{12} \begin{pmatrix} A_2 \\ B_2 \end{pmatrix}, \tag{25.13}$$

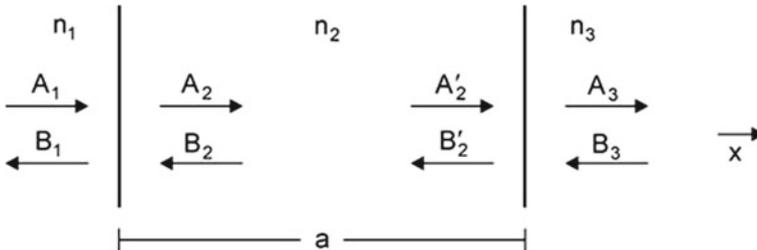
$$\begin{pmatrix} A'_2 \\ B'_2 \end{pmatrix} = P_2 \begin{pmatrix} A_2 \\ B_2 \end{pmatrix} = \begin{pmatrix} e^{i\varphi_2} & 0 \\ 0 & e^{-i\varphi_2} \end{pmatrix} \begin{pmatrix} A_2 \\ B_2 \end{pmatrix}, \tag{25.14}$$

$$\begin{pmatrix} A_2 \\ B_2 \end{pmatrix} = M_2^{-1} M_3 \begin{pmatrix} A_3 \\ B_3 \end{pmatrix} = M_{23} \begin{pmatrix} A_3 \\ B_3 \end{pmatrix}, \tag{25.15}$$

where  $\varphi_2 = k_2 a$  and  $K_2 = n_2 \omega / c$ . The matrix

$$P_2 = \begin{pmatrix} e^{i\varphi_2} & 0 \\ 0 & e^{-i\varphi_2} \end{pmatrix} \tag{25.16}$$

is the propagation matrix taking into account the phase change due to propagation.



**Fig. 25.8** Thin film (refractive index  $n_2$ , thickness  $a$ ) between two extended media

### 25.13 Dielectric Multilayer

We study a dielectric multilayer system (Fig. 25.9, upper part) for radiation propagating along the axis ( $x$  axis) that is perpendicular to the layers. We consider a system of  $N$  layers ( $n_l =$  refractive index and  $d_l =$  thickness of the  $l$ th layer) on a substrate (refractive index  $n_s$ ). The multilayer system is covered with a medium of refractive index  $n_0$ . We apply the matrix method and find:

$$\begin{pmatrix} A_l \\ B_l \end{pmatrix} = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \begin{pmatrix} A_{l+1} \\ B_{l+1} \end{pmatrix}, \tag{25.17}$$

$$\begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} = M_0^{-1} \left[ \prod_{l=1}^N M_l P_l M_l^{-1} \right] M_s, \tag{25.18}$$

$$P_l = \begin{pmatrix} e^{i\varphi_l} & 0 \\ 0 & e^{-i\varphi_l} \end{pmatrix}, \tag{25.19}$$

$$\varphi_l = k_l a_l. \tag{25.20}$$

Making use of the dispersion relations

$$k_l = n_l \omega / c, \tag{25.21}$$

and with  $B_s = 0$ , we can determine the reflectivity and the transmissivity of the multilayer system ( $A_0, B_0, A_s =$  amplitudes of incident, reflected, transmitted field):

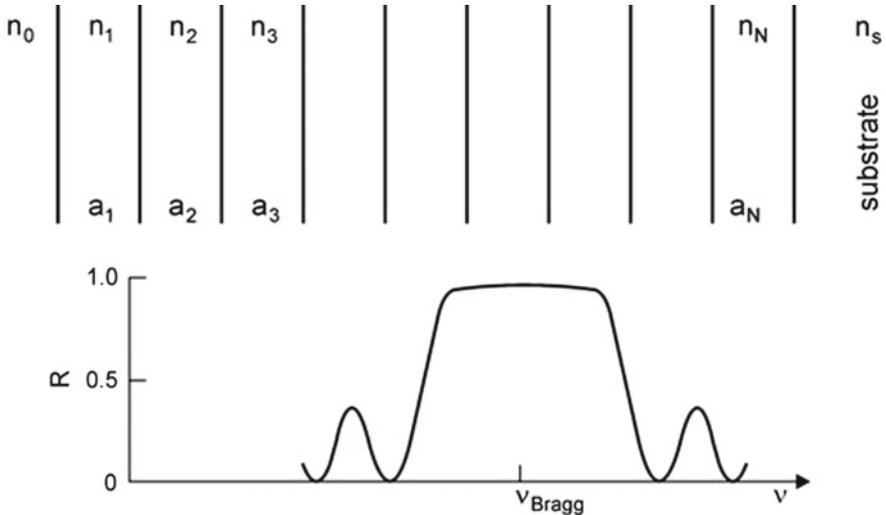


Fig. 25.9 Dielectric multilayer reflector (Bragg reflector) and reflectivity

$$R = (B_0/A_0)^2, \tag{25.22}$$

$$T = (A_s/A_0)^2. \tag{25.23}$$

In the special case that the multilayer system consists of a sequence of two layers (refractive indices  $n_1$  and  $n_2$  that have quarter-wavelength thicknesses  $d_1 = \lambda_0/(4n_1)$  and  $d_2 = \lambda_0/(4n_2)$  for radiation of wavelength  $\lambda_0$ , the reflectivity at  $\lambda_0$  is given by

$$R = \left( \frac{1 - (n_s/n_0)(n_2/n_1)^{2N}}{1 + (n_s/n_0)(n_2/n_1)^{2N}} \right)^2, \tag{25.24}$$

where  $N$  is the number of double-layers. The reflectivity approaches unity if  $N$  becomes very large.

Figure 25.9 (lower part) shows the reflectivity of a GaAs/AlAs Bragg reflector.

### 25.14 One-Dimensional Photonic Crystal

We consider a stratified periodic medium consisting of a series of double layers (Fig. 25.10). A double layer consists of a layer 1 (refractive  $n_1$ , thickness  $a_1$ ) and a layer 2 (refractive  $n_2$ , thickness  $a_2$ ). The stratified medium is spatially periodic, the spatial period is

$$a = a_1 + a_2. \tag{25.25}$$

The unit cell consists of a double layer. We suppose that we have an infinite number of cells, numbered  $l = \dots - 1, 0, 1, \dots$

We study the propagation of a monochromatic plane wave (frequency  $\omega$ ). Our goal is to find the dispersion relation  $\omega(k)$ . We use the ansatz (Boch theorem):

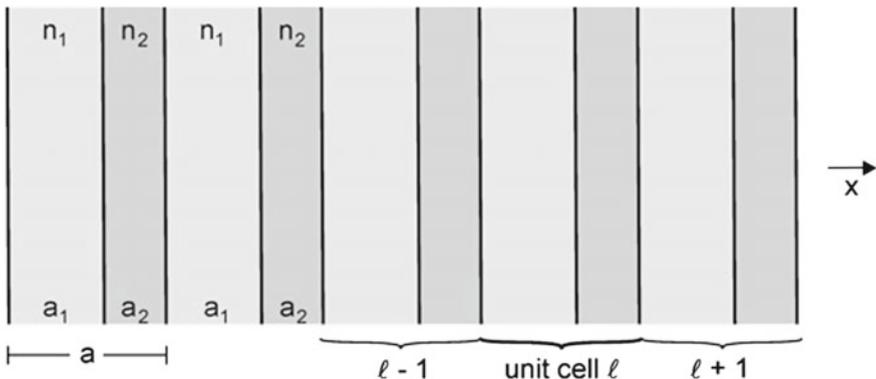


Fig. 25.10 One-dimensional photonic crystal

$$E(x) = A_k(x) e^{i(\omega t - kx)}, \quad (25.26)$$

where  $k$  is the wave vector and  $A_k(x)$  an amplitude that is a periodic function,

$$A_k(x + a) = A_k(x). \quad (25.27)$$

The field  $E(x)$  is a Bloch wave. The amplitude varies within a periodicity interval. However it is, for a particular  $k$ , lattice-periodic. The propagation of the plane wave over the distance  $x$  causes, as it is typical for plane waves, a change  $kx$ . The wave vector  $k$  of the plane wave depends on the frequency,  $k = k(\omega)$ , or

$$\omega = \omega(k). \quad (25.28)$$

The field in the  $l$ th cell is

$$E(x) = A_l e^{ik_1(x-la)} + B_l e^{ik_1(x-la)} \quad \text{in layer 1 of cell } l, \quad (25.29)$$

$$E(x) = C_l e^{ik_2(x-la)} + D_l e^{ik_2(x-la)} \quad \text{in layer 2 of cell } l, \quad (25.30)$$

with  $k_1 = n_1\omega/c$  and  $k_2 = n_2\omega/c$ .

We relate, in a first step, the electric fields in three neighboring cells (*see* Fig. 25.10):

$$\begin{pmatrix} A_l \\ B_l \end{pmatrix} = M_1^{-1} M_2 P_1 \begin{pmatrix} C_l \\ D_l \end{pmatrix}, \quad (25.31)$$

$$\begin{pmatrix} C_l \\ D_l \end{pmatrix} = M_2^{-1} M_1 P_2 \begin{pmatrix} A_{l+1} \\ B_{l+1} \end{pmatrix}, \quad (25.32)$$

$$P_1 = \begin{pmatrix} e^{ik_1 a_1} & 0 \\ 0 & e^{-ik_1 a_1} \end{pmatrix}, \quad (25.33)$$

$$P_2 = \begin{pmatrix} e^{ik_2 a_2} & 0 \\ 0 & e^{-ik_2 a_2} \end{pmatrix}. \quad (25.34)$$

$P_1$  is the propagation matrix for layer 1 and  $P_2$  is the propagation matrix for layer 2. Matrix multiplication yields

$$\begin{pmatrix} A_l \\ B_l \end{pmatrix} = \frac{1}{2} \begin{pmatrix} e^{ik_1 a_1} (1 + k_2/k_1) & e^{-ik_1 a_1} (1 - k_2/k_1) \\ e^{ik_1 a_1} (1 - k_2/k_1) & e^{-ik_1 a_1} (1 + k_2/k_1) \end{pmatrix} \begin{pmatrix} C_l \\ D_l \end{pmatrix} \quad (25.35)$$

and

$$\begin{pmatrix} C_l \\ D_l \end{pmatrix} = \frac{1}{2} \begin{pmatrix} e^{ik_2 a_2} (1 + k_1/k_2) & e^{-ik_2 a_2} (1 - k_1/k_2) \\ e^{ik_2 a_2} (1 - k_1/k_2) & e^{-ik_2 a_2} (1 + k_1/k_2) \end{pmatrix} \begin{pmatrix} A_{l+1} \\ B_{l+1} \end{pmatrix}. \quad (25.36)$$

We write

$$\begin{pmatrix} A_l \\ B_l \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} A_{l+1} \\ B_{l+1} \end{pmatrix}, \quad (25.37)$$

where

$$A = e^{ik_1 a_1} \left[ \cos k_1 a_1 + \frac{1}{2}i \left( \frac{k_2}{k_1} + \frac{k_2}{k_2} \right) \sin k_1 a_1 \right], \quad (25.38)$$

$$B = e^{-ik_1 a_1} \left[ \frac{1}{2}i \left( \frac{k_2}{k_1} - \frac{k_1}{k_2} \right) \sin k_1 a_1 \right], \quad (25.39)$$

$$C = e^{ik_2 a_2} \left[ -\frac{1}{2}i \left( \frac{k_2}{k_1} - \frac{k_1}{k_2} \right) \sin k_2 a_2 \right], \quad (25.40)$$

$$D = e^{-ik_2 a_2} \left[ \cos k_2 a_2 - \frac{1}{2}i \left( \frac{k_2}{k_1} + \frac{k_1}{k_2} \right) \sin k_2 a_2 \right]. \quad (25.41)$$

We have the relation

$$AD - BC = 1. \quad (25.42)$$

It follows that the amplitude in the  $l$ th cell and the amplitude in the zeroth cell are related:

$$\begin{pmatrix} A_l \\ B_l \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1} \begin{pmatrix} A_0 \\ B_0 \end{pmatrix} \quad (25.43)$$

or

$$\begin{pmatrix} A_l \\ B_l \end{pmatrix} = \begin{pmatrix} D & -B \\ -C & A \end{pmatrix} \begin{pmatrix} A_0 \\ B_0 \end{pmatrix}. \quad (25.44)$$

If we specify  $A_0$  and  $B_0$ , all amplitudes can be calculated.

We make use of the periodicity of the multilayer system and write

$$\begin{pmatrix} A_{l+1} \\ B_{l+1} \end{pmatrix} = e^{-ika} \begin{pmatrix} A_l \\ B_l \end{pmatrix}, \quad (25.45)$$

leading to

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} A_{l+1} \\ B_{l+1} \end{pmatrix} = e^{ika} \begin{pmatrix} A_{l+1} \\ B_{l+1} \end{pmatrix}. \quad (25.46)$$

The phase factor  $\exp(ika)$  is the eigenvalue of the matrix ABCD. We find

$$e^{ika} = \frac{1}{2}(A + D) \pm \sqrt{\frac{1}{4}(A + D)^2 - 1}. \quad (25.47)$$

The sum  $A + D$  is real. It follows, with

$$e^{ika} = \cos ka + i \sin ka, \quad (25.48)$$

that

$$\cos ka = \frac{1}{2}(A + D) \quad (25.49)$$

and

$$\sin ka = \pm \sqrt{1 - \frac{1}{4}(A + D)^2}. \quad (25.50)$$

We obtain, with

$$\xi = k_1/k_2 = n_1/n_2, \quad (25.51)$$

the dispersion relation

$$\cos ka = \cos(k_1 a_1) \cos(k_2 a_2) - \frac{1}{2} \left( \xi + \frac{1}{\xi} \right) \sin(k_1 a_1) \sin(k_2 a_2) \quad (25.52)$$

or

$$\cos ka = \cos \left( n_1 \frac{\omega}{c} a_1 \right) \cos \left( n_2 \frac{\omega}{c} a_2 \right) - \frac{1}{2} \left( \frac{n_2}{n_1} + \frac{n_1}{n_2} \right) \sin \left( n_1 \frac{\omega}{c} a_1 \right) \sin \left( n_2 \frac{\omega}{c} a_2 \right). \quad (25.53)$$

We discuss the dispersion relation in the special case that the optical paths in layer 1 and layer 2 are equal,

$$n_1 a_1 = n_2 a_2. \quad (25.54)$$

The dispersion relation has the form

$$\cos ka = \cos^2 \left( n_1 \frac{\omega}{c} a_1 \right) - \frac{1}{2} \left( \frac{n_2}{n_1} + \frac{n_1}{n_2} \right) \sin^2 \left( n_1 \frac{\omega}{c} a_1 \right) \quad (25.55)$$

or

$$k = \frac{1}{a} \cos^{-1} \left[ \cos^2 \left( n_1 \frac{\omega}{c} a_1 \right) - \frac{1}{2} \left( \frac{n_2}{n_1} + \frac{n_1}{n_2} \right) \sin^2 \left( n_1 \frac{\omega}{c} a_1 \right) \right]. \quad (25.56)$$

The frequency increases proportionally to the wave vector  $k$  at long waves ( $ka \ll 1$ ):

$$k = n_{\text{eff}} \frac{\omega}{c}, \quad (25.57)$$

where

$$n_{\text{eff}} = \sqrt{n_1 n_2} \quad (25.58)$$

is an effective refractive index. The appearance of an effective refractive index with a value between  $n_1$  and  $n_2$  is a consequence of the reflection of the radiation at the interfaces. There are frequency gaps for

$$k = \pm \frac{\pi}{a}, \pm \frac{3\pi}{a}, \dots \quad (25.59)$$

Because of the periodicity,

$$\omega\left(k + \frac{2\pi}{a}\right) = \omega(k), \quad (25.60)$$

we can restrict the  $k$  values to the first Brillouin zone,

$$-\frac{\pi}{a} < k \leq \frac{\pi}{a}. \quad (25.61)$$

The values  $-\pi/a$  and  $\pi/a$  are the Brillouin zone boundaries.

The curve  $k = n_{\text{eff}}\omega/c$  reaches the Brillouin zone boundary at the Bragg frequency

$$\omega_{\text{Bragg}} = \frac{\pi c}{n_{\text{eff}}a}, \quad (25.62)$$

which corresponds to the Bragg wavelength

$$\lambda_{\text{Bragg}} = 2n_{\text{eff}}a. \quad (25.63)$$

For radiation of this vacuum-wavelength, the multilayer system represents a stack of quarter-wavelength layers. Bragg reflection of the radiation at wavelengths around  $\lambda_{\text{Bragg}}$  is responsible for the occurrence of a frequency gap.

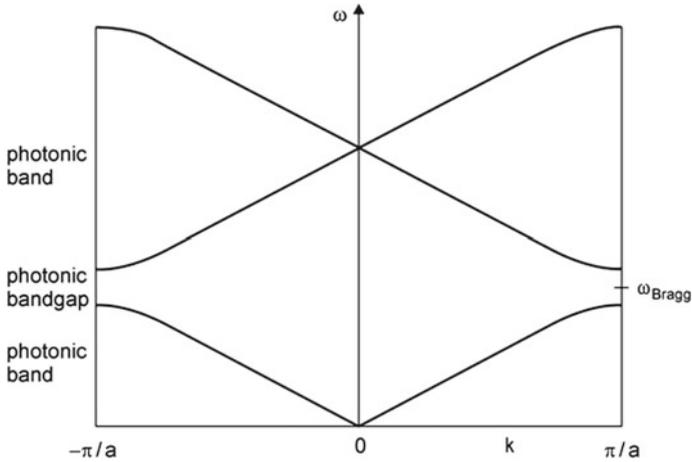
It follows that the field in layer 1 of the  $l$ th cell is given by

$$E(x) = [A_0 e^{in_1(\omega/c)(x-la)} + B_0 e^{in_1(\omega/c)(x-la)}] e^{i(\omega t - lka)}. \quad (25.64)$$

*Example* Dispersion relation of radiation in a GaAs/AlAs photonic crystal (Fig. 25.11);  $n_1 = 3.3$ ;  $n_2 = 2.9$ ;  $a_1 = 152$  nm;  $a_2 = 173$  nm;  $a = 325$  nm;  $n_1a = n_2b$ ;  $n_{\text{eff}} = 3.09$ .

## 25.15 Bragg Reflection as Origin of Energy Gaps

The occurrence of energy gaps is a consequence of the ability of radiation to undergo Bragg reflection. Bragg reflection occurs for radiation with discrete values of the wave vector, namely for  $k = k_{\text{Bragg}}$ . In the case of a one-dimensional crystal,  $k_{\text{Bragg}} = \pi/a$ . A two-dimensional photonic crystal has Bragg vectors that lie on a plane. A two-dimensional photonic crystal can show a photonic bandgap (frequency gap) for radiation of all  $k$  vectors in a plane. A three-dimensional photonic crystal has Bragg vectors in the three-dimensional  $k$  space. A three-dimensional photonic crystal can have energy gaps for  $k$  vectors of all spatial directions.



**Fig. 25.11** One-dimensional photonic crystal: dispersion relation of radiation, with the propagation direction parallel to the axis of a periodic multilayer structure

We come back to Fig. 25.11. The speed of light is equal to  $c/n_{\text{eff}}$  for long wavelengths, i.e., for small frequencies. The speed of light is almost constant at small wave vectors, decreases at large wave vectors and becomes zero for  $k = \pi/a$ . A photonic crystal is thus able to slow down an electromagnetic wave.

References [26, 28, 177, 207–220].

## Problems

**25.1 Bragg reflection.** Formulate the conditions for the occurrence of Bragg reflection of electromagnetic radiation in different systems. (a) A 1D photonic crystal, (b) 2D photonic crystal, (c) 3D photonic crystal.

### 25.2 Bragg reflection of X-rays.

- Formulate the conditions for the occurrence of Bragg reflection of X-rays.
- Why are Bragg peaks at X-rays extremely sharp?
- Estimate the width of an energy gap expected for X-rays. [*Hint*: estimate the refractive index of X-rays—it is slightly smaller than unity—and describe a crystal (e.g., with respect to the 100 direction) as a 1D photonic crystal with the electrons distributed in thin layers perpendicular to the propagation direction of the X-rays.]

### 25.3 One-dimensional photonic crystal.

- (a) Estimate the widths of forbidden frequency bands in the case that  $n_2 - n_1 \ll 1$ .
- (b) Estimate, for radiation of the vacuum wavelength  $1 \mu\text{m}$ , the widths of forbidden frequency bands in the case that the photonic crystal consists of a stack of GaAs/AlAs quarter-wavelength films.

### 25.4 One-dimensional photonic crystal consisting of freestanding plates.

- (a) Determine the effective refractive index, the Bragg frequency and the Bragg wavelength of thin freestanding silicon plates (thickness  $1 \mu\text{m}$ , refractive index  $n = 4$ ) separated by air under the assumption that the plates and the space between two plates have the same optical thickness.
- (b) Calculate the dispersion relation of radiation in such a one-dimensional photonic crystal.

**25.5** How many quarter-wavelength films of GaAs and AlAs films on a GaAs substrate are necessary to obtain reflectivities  $R \sim 70, 80, 90, 95$  or,  $99, 99.9\%$ ?

### 25.6 Antireflecting coating.

- (a) Show, by use of the matrix method, that the reflectivity of the surface of an optical substrate (refractive index  $n_s$ ) covered with a quarter-wavelength film, thickness  $\lambda/(4n)$ , is zero if the refractive index  $n$  of the film satisfies the condition  $n_s - n^2 = 0$ . [*Hint*: assume that the substrate has infinite thickness, so that no reflection from the end surface of the substrate occurs.]
- (b) Show that the multiple beam method (introduced in Sect. 3.5) yields the same result. [*Hint*: add all beams reflected by the two surfaces of the film, taking multiple reflection into account.]

**25.7** Determine the Airy formula (Sect. 3.5) by use of the matrix method.

**25.8 Double-resonator.** We consider a double-resonator (Fabry–Perot resonator) with three lossless mirrors of equal reflectivity  $R$ . The distance between mirror 1 and mirror 2 is  $L_1$  and the distance between mirror 2 and mirror 3 is  $L_2$ . Derive, by the use of the matrix method, the transmission curve of a double resonator for (a)  $L_1 = L_2$ ; (b)  $L_2 \ll L_1$ , (c)  $L_1 = \lambda/2$ . (d) Choose  $R = 0.95$  and  $\lambda = 1 \mu\text{m}$  for a discussion of the results.

### 25.9 Boundary between two dielectric media.

- (a) Show that the boundary conditions for normal incidence are consistent with the requirement that the energy flux density is the same in medium 2 as in medium 1. [*Hint*: describe the energy flux density by the Poynting vector  $\mathbf{P} = \mathbf{E} \times \mathbf{H}$ .]
- (b) Derive the Fresnel coefficient of reflection for normal incidence by the use of the matrix method.

**25.10 Bloch theorem.** Derive the Bloch theorem for the one-dimensional photonic crystal, i.e., justify the ansatz (25.26). [*Hint*: make use of periodic boundary conditions.]

**25.11 Propagating of radiation in a one-dimensional crystal.** Discuss the dependence of group and phase velocity on the wave vector of radiation belonging to the two lowest branches of the dispersion curves shown in Fig. 25.11.

**25.12** Determine, by use of the matrix method, the halfwidth of the resonance curve of a Fabry–Perot resonator (Sect. 3.6) that has a reflector of a reflectivity of unity and a partial reflector.

**25.13** Derive the Airy formula for a Fabry–Perot resonator containing an active medium (Sect. 3.7), by the use of the matrix method. [*Hint*: assume that one of the mirrors has a reflectivity of unity.]

**25.14 Reflection of radiation by a perfect conductor.**

- Show that the reflectivity of a perfect conductor is 1.
- The radiation penetrates into the conductor. Derive an expression of the penetration depth of the electric field and of the radiation energy.
- Calculate the penetration depth of radiation reflected by a perfect conductor, which contains electrons of a concentration  $N = 10^{28} \text{ m}^{-3}$ , for radiation of 1 mm and of  $0.5 \text{ }\mu\text{m}$  wavelength.

**25.15 A perfect mirror.** A thin film consisting of a perfectly conducting material can act as a partial mirror. [*Hint*: a perfect conductor for currents at microwave frequencies is superconducting lead at a temperature well below the superconducting transition temperature of 7 K.]

- Determine the complex transmission coefficient  $\tilde{t}$ , the complex reflection coefficient  $\tilde{r}$ , the phase  $\varphi$  of the reflected beam, the phase  $\varphi_t$  of the transmitted beam, transmissivity  $T$  and the reflectivity  $R$  (see Sect. 3.4). [*Hint*: make use of the matrix method; treat the film as a free-standing film surrounded by air].
- Design partial mirrors that have reflectivities  $R \sim 70, 80, 90, 95, 99, 99.9\%$  for radiation of 1 mm wavelength, assuming that the mirror is perfectly conducting and contains electrons of a concentration  $N = 10^{28} \text{ m}^{-3}$ .
- Calculate, for a Fabry–Perot resonator formed by two (perfect) partial mirrors as reflectors, the change of phase per round trip transit of radiation of 1 mm wavelength in the case that the reflectivity of each mirror is  $R = 0.9$ .

**25.16 Methods of describing the field in a resonator.** Show that the three methods of describing a field in a resonator lead to the same result:

- The method of multiple reflection (Sect. 3.5).
- The method directly based on the boundary conditions (this chapter).
- A method directly based on the boundary conditions but that immediately introduces the complex transmission coefficient  $\tilde{t} = B_1/A_1$  and the complex reflection coefficient  $\tilde{r} = B_2/A_1$  of a mirror; use this method to derive the Airy formula.

**25.17 Bulk metal.** We study the optical properties of a metal like copper (free-electron concentration  $N = 10^{28} \text{ m}^{-3}$ , relaxation time  $\tau = 10^{-13} \text{ s}$ ).

- (a) Determine, by use of the complex optical constants, the frequency dependence of the reflectivity.
- (b) Compare the reflectivity of the metal with the reflectivity of a perfect conductor that contains electrons of the same density.
- (c) Determine the optical constants and the reflectivity of a metal for radiation of long wavelengths (i.e., for  $\omega \ll \omega_p = \sqrt{Ne^2/\epsilon_0 m_0}$  = plasma frequency).

**25.18 Metal film.** Study optical properties of a metal film (e.g., a copper film). Restrict the discussion to long wavelengths.

- (a) Determine the dependence of transmissivity  $T$ , reflectivity  $R$  and absorptivity  $A$  of a metal film on the thickness of the film by use of the matrix method.
- (b) Show that there is a film thickness where  $T = R = 0.25$  and  $A = 0.5$ , and that  $T \ll A$  for thicker films.