Chapter 4
The Active Medium: Energy Levels and Lineshape Functions

We present a characterization of active media with respect to energy levels and line broadening. We make a distinction between two-level based lasers and energy-ladder based lasers.

In a two-level based laser, stimulated transitions occur between two levels of an atomic system. A two-level system of a particular atom or molecule is a subsystem of the energy levels of the atom or the molecule. We characterize the two-level based lasers as: four-level lasers; three-level lasers; two-level lasers; two-band lasers; and quasiband lasers. All presently operating lasers except free-electron lasers can, in principle, be described as two-level based lasers.

In an energy-ladder based laser, stimulated transitions occur between levels of energy-ladder systems. The (yet hypothetical) Bloch laser (Chap. 32) belongs to this type. We will make use (Chap. 19) of the concept of an energy-ladder based laser to illustrate properties of free-electron lasers.

Line broadening can be due to homogeneous or inhomogeneous broadening. We discuss the Lorentzian and Gaussian lineshape functions. We describe the classical oscillator model of an atom and the natural line broadening.

Finally, we introduce low-dimensional active media. In a low-dimensional medium, the free motion of electrons is spatially restricted. There are two-dimensional media, with electrons moving along a plane, or one-dimensional media, with electrons moving along a line. The strongest restriction occurs in a zero-dimensional medium—all three dimensions are restricted (like in an atom). The classification as three-, two-, one-, and zero-dimensional media concerns semiconductor lasers. The most important semiconductor lasers—quantum well lasers—operate with two-dimensional active media.

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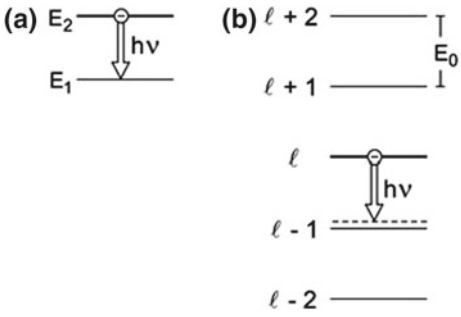

Fig. 4.1 Types of laser.
a Two-level based laser.
b Energy-ladder based laser

4.1 Two-Level Based and Energy-Ladder Based Lasers

A *two-level based laser* contains two-level atomic systems. Coherent radiation is generated by stimulated transitions between the upper and the lower laser levels of two-level systems (Fig. 4.1a). Quantities characterizing a two-level system are:

- E_2 = energy of the upper laser level
- E_1 = energy of the lower laser level
- $E_2 - E_1$ = transition energy
- $\nu_0 = (E_2 - E_1)/h$ = transition frequency = atomic resonance frequency

The laser frequency has a value at or near the transition frequency,

$$\nu_L \sim \nu_0. \tag{4.1}$$

We will characterize the two-level based lasers according to the number of different levels involved in transitions in a laser medium (Sects. 4.2 and 4.3). A two-level system is a subsystem of the energy level system of an atom or a molecule. All presently operating lasers—except free-electron lasers—are two-level based lasers.

An *energy-ladder based laser* contains energy-ladder systems. A free-electron in a spatially periodic field executes oscillations (free-electron oscillations). According to an energy level description, an oscillating free-electron forms an energy-ladder system and occupies one of the levels of the energy-ladder system. Stimulated emission of a photon by the oscillating electron corresponds to a stimulated transition between next-near energy levels of the energy-ladder system (Fig. 4.1b). The energy levels are equidistant,

$$E_l = l\, E_0, \tag{4.2}$$

where l is an integer, Quantities characterizing an energy-ladder system are:

- E_0 = energy distance between next-near levels = transition energy.
- $\nu_0 = E_0/h$ = transition frequency = resonance frequency of the electron oscillation.

The laser frequency ν_L of an energy-ladder based laser is slightly smaller than the resonance frequency,

$$\nu_L < \nu_0. \tag{4.3}$$

We will describe the (yet hypothetical) superlattice Bloch laser (Sect. 32.8) as an energy-ladder based laser. We will furthermore show that the free-electron laser can, in principle, be interpreted as an energy-ladder based laser (Sect. 19.17).

4.2 Four-Level, Three-Level, and Two-Level Lasers

In a four-level laser (Fig. 4.2), a pump excites atoms, molecules, or other atomic systems from the ground state level (level 0) to an excited state level (level 3 = pump level). Relaxation leads to population of the upper laser level (level 2). Stimulated emission by $2 \rightarrow 1$ transitions results in a population of the lower laser level (level 1). Depopulation of the lower level occurs by relaxation to the ground state. We have three relaxation processes, namely $3 \rightarrow 2$, then $2 \rightarrow 1$, and $1 \rightarrow 0$. We assume that the relaxation $3 \rightarrow 2$ is very fast. We ignore other relaxation processes (e.g., $2 \rightarrow 0$). Continuous pumping maintains a *permanent population inversion* ($N_2 > N_1$) if the *relaxation time* τ^*_{rel} of the upper laser level is larger than the relaxation time τ_{rel} of the lower laser level,

$$\tau^*_{\text{rel}} > \tau_{\text{rel}}. \tag{4.4}$$

Without stimulated emission, the population of the upper level is equal to the product of the *pump rate* r and the relaxation time τ^*_{rel},

$$N_2 = r\tau^*_{\text{rel}}. \tag{4.5}$$

Pumping of a four-level laser medium creates two-level atomic systems; a two-level atomic system is either in its ground state (level 1) or in its excited state (level 2).

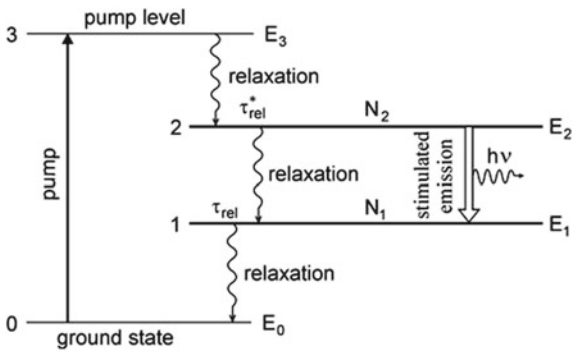

Fig. 4.2 Four-level laser (*principle*)

Table 4.1 Relaxation times of laser levels

Laser	λ	τ^*_{rel}	τ_{rel}
HeNe	633 nm	100 ns	10 ns
CO_2	10.6 μm	5 s	< 5 s
Nd:YAG	1.06 μm	230 μs	$\ll 230$ μs
TiS	790 nm	3.8 μs	10^{-12} s
QCL	5 μm	10^{-11} s	10^{-12} s

The density of two-level atomic systems, $N_2 + N_1$, increases with increasing pump strength. Population inversion in the active medium of a four-level laser medium occurs already at the smallest pump rate.

The maximum efficiency of conversion of a pump quantum (energy $E_3 - E_0$) to a laser light quantum (energy $h\nu$) is the *quantum efficiency*

$$\eta_{\mathrm{q}} = \frac{h\nu}{E_3 - E_0} = \frac{E_2 - E_1}{E_3 - E_0}. \tag{4.6}$$

The quantum efficiency is very small ($\eta_{\mathrm{q}} \ll 1$) if $E_2 - E_1 \ll E_3 - E_0$. Then, a large portion of the pump energy is converted to relaxation energy and therefore to heat (or to radiation produced by spontaneous emission). The quantum efficiency is near unity if the pump level lies only slightly above the upper laser level and if, at the same time, the lower laser level lies only slightly above the ground state level.

Table 4.1 shows values of relaxation times of upper and lower laser levels of a few laser materials. The relaxation times differ by many orders of magnitude. The relaxation $2 \to 1$ can be due to spontaneous emission of radiation or due to nonradiative relaxation. Relaxation of the lower level can also be due to spontaneous emission of radiation or due to nonradiative relaxation, that is, by a radiationless transition. We will specify the relaxation processes later in connection with the discussion of specific lasers. The lasers mentioned in the table can operate as continuous wave lasers ($\tau^*_{\mathrm{rel}} > \tau_{\mathrm{rel}}$).

Optical transitions between two discrete energy levels lead to fluorescence lines and absorption lines that have finite linewidths (Sect. 4.4); the notation *fluorescence* is used for photo luminescence (=optically excited *luminescence*).

Many laser media have levels with energy distributions. A pump band has, in comparison to a single pump level, the advantage that a lamp that emits radiation in a broad spectral range can pump a laser. In the case of pumping with another laser, radiation of different frequencies is suitable for pumping.

Examples of four-level lasers: neodymium YAG laser; titanium–sapphire laser.

We are dealing with a three-level laser if the pump level coincides with the upper laser level (Fig. 4.3a). This type of three-level laser is a special case of the four-level laser. For another type of three-level laser (Fig. 4.3b), the lower laser level is identical with the ground state level. We denote it as ruby laser type. Population inversion requires that more than half of the atoms are in the excited state. Accordingly, the transparency density of a ruby laser type is given by

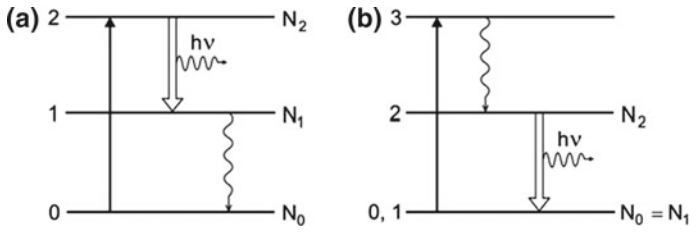

Fig. 4.3 Three-level lasers. **a** Three-level laser with coinciding pump and upper laser level. **b** Ruby laser type

$$N_{\mathrm{tr}} = N_0/2. \tag{4.7}$$

N_0 is the density of impurity ions in a crystal.

In a *two-level laser* only two atomic energy levels play a role.

Example of a two-level laser: the ammonia (NH_3) maser [32]. To obtain population inversion, long-lived excited molecules are permanently injected into a resonator where stimulated emission processes occur. The molecules leave the resonator spatially. The upper laser level is 35 μ eV above the ground state level. Ammonia gas in a box contains NH_3 molecules in the ground state and in the excited state. A hole in a box with NH_3 gas at room temperature is the source of a molecular beam consisting of excited and nonexcited NH_3 molecules. The molecular beam traverses an atomic filter that separates the excited molecules and the nonexcited molecules. The atomic filter consists of an inhomogeneous field (an electric quadrupole field) that exerts forces on the molecules due to their electric dipole moments. The magnitude of the dipole moment of a molecule in the excited state differs from that in the ground state. Therefore, the forces lead to a spatial separation of the molecules. The excited molecules pass the resonator and deliver, via stimulated emission, the excitation energy to the laser field in the resonator. The ammonia laser was the first microwave maser (frequency near 24 GHz).

4.3 Two-Band Laser and Quasiband Laser

A *two-band laser* medium has (besides other energy levels or energy bands) a lower energy band (band 1) and an upper energy band (band 2), separated by an energy gap (Fig. 4.4a). The gap energy is E_{g}. Without pumping, almost all energy levels belonging to the lower band are full and all energy levels of the upper band are empty. We suppose that E_{g} is much larger than kT so that thermal excitation from the lower to the upper band can be ignored.

Pumping—injection of electrons into the upper band and extraction of electrons from the lower band (Fig. 4.4b)—results in a quasithermal population of energy levels in the upper band and in empty levels of the lower band. The electrons in the

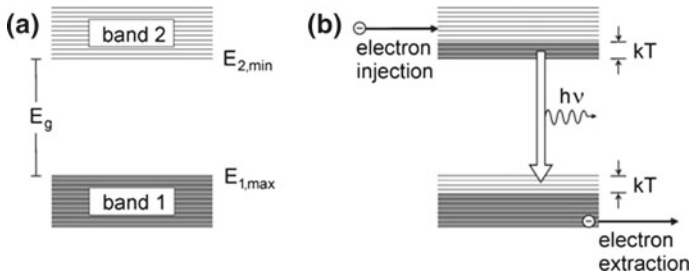

Fig. 4.4 Two-band laser. **a** Energy bands. **b** Laser principle

Table 4.2 Relaxation times of laser levels of two-band laser media and quasiband laser media

Laser	λ (μm)	τ^*_{rel}	τ_{intra} (s)
Semiconductor	0.4–2	1–5 ns	10^{-13}
Fiber	1–2	10^{-2} s	10^{-13}

upper band undergo fast intraband relaxation. The population in the upper band is in a quasithermal equilibrium, which is determined by the lattice temperature of the active medium. The populated levels have energies near the minimum of the upper band. The width of the energy distribution of populated levels is $\sim kT$ (or larger at strong pumping). Energy levels near the energy minimum of the upper band have the largest population. The energy distribution of populated levels in the upper band is governed by Fermi's statistics.

The population in the lower band is also in a quasithermal equilibrium, corresponding to the lattice temperature of the active medium. The empty levels have energies near the maximum of the lower band. The width of the energy distribution of empty levels is $\sim kT$ (or larger at strong pumping). Energy levels near the maximum of the lower band have the lowest population. Fermi's statistics determines the energy distribution of populated levels in the lower band too.

Quasithermal means that the population within an energy band has a thermal distribution according to the lattice temperature of the active medium—but that the population of the upper energy band is, relative to the population in the lower energy band, far out of equilibrium. Stimulated transitions from occupied levels in the upper band to empty levels in the lower band are the source of laser radiation. Establishment of a quasiequilibrium in the upper band and establishment of a quasiequilibrium in the lower band are due to the interaction of electrons with phonons, that is, due to electron–phonon scattering.

Table 4.2 shows relaxation times: the intraband relaxation time τ_{intra} is much smaller than the interband relaxation time τ^*_{rel}. Interband relaxation in an active medium is mainly due to spontaneous emission of radiation.

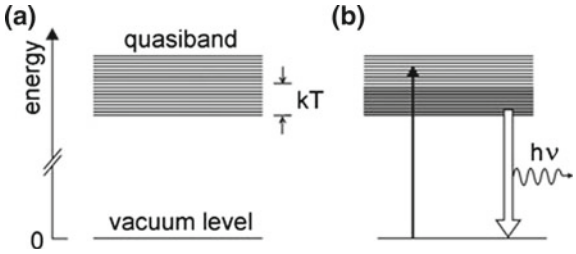

Fig. 4.5 Quasiband laser. **a** Quasiband. **b** Laser principle

The *quasiband laser* represents a model of a glass fiber laser. The active medium (Fig. 4.5a) contains excited-impurity quasiparticles in a *quasiband*. The quasiband lies $\sim 1\,\text{eV}$ above a vacuum level. The width of a quasiband of an impurity-doped glass can have a value of 10–100 meV. Optical pumping via transitions from the vacuum level to the quasiband creates quasiparticles (Fig. 4.5b). Annihilation of quasiparticles via stimulated transitions from the quasiband to the vacuum level is the origin of laser radiation. The quasiparticles in the quasiband have a quasithermal distribution determined by Fermi's statistics; for relaxation times, see Table 4.2. The quasiband model will be described in Chap. 18.

Examples Two-band lasers: all bipolar semiconductor lasers.
Quasiband lasers: erbium-doped fiber laser; other fiber lasers and fiber amplifiers (Sect. 15.7 and Chap. 18).
Two-quasiband lasers (with the active medium having a lower and an upper quasiband): organic and polymer lasers (Sect. 34.4).

4.4 Lineshape: Homogeneous and Inhomogeneous Line Broadening

We use the notation "lineshape" in different ways:

- Lineshape of a luminescence line.
- Lineshape of an absorption line = absorption profile = shape of an absorption coefficient = slope of $\alpha_{\text{abs}}(\nu)$.
- Lineshape of a gain curve = gain profile = shape of a gain coefficient = shape of $\alpha(\nu)$.

We characterize the lineshape of a line that is due to transitions between two levels of an atomic system by:

- ν_0 = center frequency of a line.
- $\Delta\nu_0$ = linewidth = halfwidth = full width at half maximum (FWHM).
- The lineshape function $g(\nu)$ with the normalization $\int_0^\infty g(\nu)\mathrm{d}\nu = 1$ or, alternatively, $\bar{g}(\nu)$ with the normalization $\bar{g}(\nu_0) = 1$.

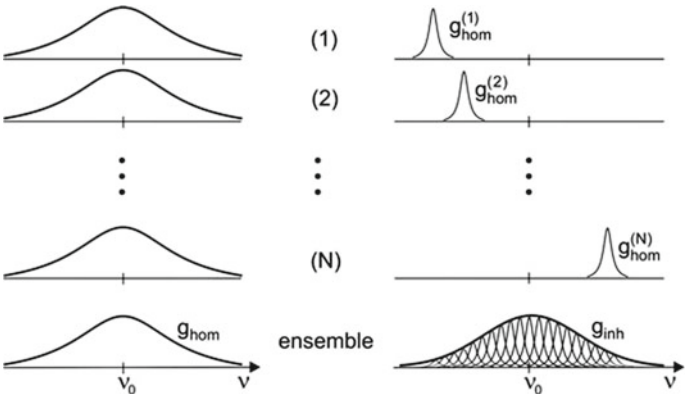

Fig. 4.6 Homogeneous and inhomogeneous line broadening

There are many different mechanisms responsible for lineshapes. Accordingly, there is a large number of different lineshapes. We divide the lineshapes as lineshapes due to homogeneous broadening and lineshapes due to inhomogeneous broadening.

A homogeneous line broadening occurs if all two-level atomic systems have the same lineshape function $g_{\text{hom}}(\nu) = g^{(1)} = g^{(2)} = \ldots = g^{(N)}$, where N is the number of atomic systems (Fig. 4.6, left). The atomic resonance frequency ν_0 is the same for all two-level atomic systems.

In the case that a line is inhomogeneously broadened (Fig. 4.6, right), each two-level atomic system of an ensemble has its own resonance frequency. The linewidth of the ensemble is larger than the transition linewidth of a single two-level atomic system. We can regard an inhomogeneously broadened line (with the lineshape function g_{inh} and the center frequency ν_0) as composed of homogeneously broadened lines with the frequencies $\nu_{0,1}, \nu_{0,2}, \ldots, \nu_{0,N}$ of different two-level systems.

Examples of homogeneous line broadening: collision broadening in gases (Sect. 14.2) and vibronic line broadening of the transition in Ti^{3+}:Al_2O_3 used for operation of the titanium–sapphire laser (Chap. 5; Sects. 7.6 and 15.2; Chap. 17).

Example of inhomogeneous broadening: Doppler broadening of transition lines in gases (Sect. 14.1).

4.5 Lorentz Functions

An important lineshape function is the *Lorentz resonance function* (Fig. 4.7),

$$g_{\text{L}}(\nu) = g_{\text{L,res}}(\nu) = \frac{\Delta\nu_0}{2\pi} \frac{1}{(\nu_0 - \nu)^2 + \Delta\nu_0^2/4}, \tag{4.8}$$

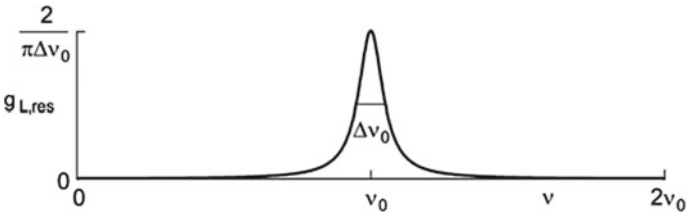

Fig. 4.7 Lorentz resonance function (Lorentzian lineshape function)

where ν_0 is the resonance frequency and $\Delta\nu_0$ the halfwidth (full width at half maximum, FWHM). We suppose that we are dealing with a narrow line. Then, the integral over the Lorentz resonance function, from zero to infinite, is approximately equal to unity,

$$\int_0^\infty g_{\mathrm{L,res}}(\nu)\mathrm{d}\nu = 1. \tag{4.9}$$

The maximum value is equal to

$$g_{\mathrm{L,res}}(\nu_0) \;=\; \frac{2}{\pi\,\Delta\nu_0} \approx \frac{0.64}{\Delta\nu_0}. \tag{4.10}$$

The peak value of the Lorentz resonance curve is equal to the inverse of the halfwidth of the curve (times $2/\pi$); with decreasing halfwidth, the Lorentz resonance curve narrows.

$$g_{\mathrm{L,disp}}(\nu) = \frac{1}{\pi}\,\frac{\nu_0 - \nu}{\left(\nu_0^2 - \nu^2\right)^2 + \Delta\nu_0^2/4} = \frac{\nu_0 - \nu}{\Delta\nu_0/2}\; g_{\mathrm{L,res}}(\nu). \tag{4.11}$$

We write the Lorentz resonance function, normalized to unity at the line center, in dimensionless units:

$$\bar{g}_{\mathrm{L,res}}(\nu/\nu_0) = \frac{(\Delta\nu_0/\nu_0)^2/4}{(1-\nu/\nu_0)^2 + (\Delta\nu_0/\nu_0)^2/4}. \tag{4.12}$$

The corresponding Lorentz dispersion function is equal to

$$\bar{g}_{\mathrm{L,disp}}(\nu/\nu_0) = \frac{(\nu_0 - \nu)(\Delta\nu_0/2)}{(\nu_0 - \nu)^2 + (\Delta\nu_0)^2/4} = \frac{1-\nu/\nu_0}{\Delta\nu_0/2\nu_0}\;\bar{g}_{\mathrm{L,res}}(\nu/\nu_0). \tag{4.13}$$

The Lorentz resonance curve (Fig. 4.8, upper part) is symmetric with respect to the resonance frequency ν_0, while the Lorentz dispersion curve (Fig. 4.8, lower part) is antisymmetric. The Lorentz dispersion curve is zero at $\nu = \nu_0$ and has extrema at the frequencies $\nu_0 \pm \Delta\nu_0/2$. The extrema of $\bar{g}_{\mathrm{L,disp}}$ are equal to $\pm\,0.5$.

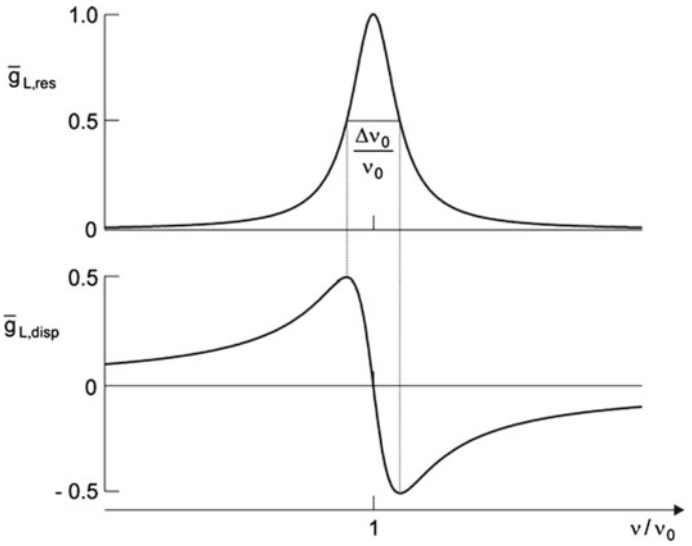

Fig. 4.8 Lorentz functions. **a** Lorentz resonance function. **b** Lorentz dispersion function

The Lorentz resonance function on the ω scale is given by

$$g_{\mathrm{L,res}}(\omega) = \frac{\Delta\omega_0}{2\pi} \frac{1}{(\omega_0 - \omega)^2 + \Delta\omega_0^2/4}. \tag{4.14}$$

We have the relation, because of $g(\omega)\mathrm{d}\omega = g(\nu)\mathrm{d}\nu$, $\omega = 2\pi\nu$ and $\int g(\omega)\mathrm{d}\omega = 1$,

$$g_{\mathrm{L,res}}(\nu) = 2\pi g_{\mathrm{L,res}}(\omega). \tag{4.15}$$

On the ω scale, the Lorentz resonance function, normalized to unity at the line center, has the form

$$\bar{g}_{\mathrm{L,res}}(\omega) = \frac{\pi\Delta\omega_0}{2} g_{\mathrm{L,res}}(\omega) = \frac{\Delta\omega_0^2/4}{(\omega_0 - \omega)^2 + \Delta\omega_0^2/4}. \tag{4.16}$$

The corresponding Lorentz dispersion function is

$$\bar{g}_{\mathrm{L,disp}}(\omega) = \frac{\pi\Delta\omega_0}{2} g_{\mathrm{L,disp}}(\omega) = \frac{(\omega_0 - \omega)\Delta\omega_0/2}{(\omega_0 - \omega)^2 + \Delta\omega_0^2/4}. \tag{4.17}$$

The Lorentz resonance function on the energy scale is given by

$$g_{\mathrm{L,res}}(h\nu) = \frac{\Delta E_0}{2\pi} \frac{1}{(E_{21} - h\nu)^2 + \Delta E_0^2/4}, \tag{4.18}$$

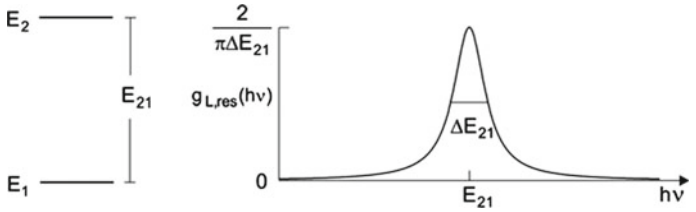

Fig. 4.9 Lorentz resonance function on the energy scale

where we have the quantities:

- $\Delta E_0 = \Delta E_{21} = h\Delta\nu_0$ = halfwidth of the line on the energy scale.
- $E_{21} = E_2 - E_1$ =transition energy.
- $\nu_0 = E_{21}/h = (E_2 - E_1)/h$ =transition frequency.
- $h\nu$ =quantum energy of the photons in a radiation field of frequency ν.

The relation $g_{\mathrm{L,res}}(\nu)\mathrm{d}\nu = g_{\mathrm{L,res}}(h\nu)\mathrm{d}(h\nu)$ leads to

$$g_{\mathrm{L,res}}(h\nu) = \frac{1}{h} g_{\mathrm{L,res}}(\nu). \tag{4.19}$$

Because of line broadening, the photon energy $h\nu$ (Fig. 4.9) does not need to coincide with the transition energy E_{21}.

The Lorentz functions we presented describe narrow resonance lines, $\Delta\omega_0 \ll \omega_0$. Otherwise we have functions that we call *general Lorentz functions*. The general Lorentz resonance function is given by

$$G_{\mathrm{L,res}}(\omega) = \frac{\omega\Delta\omega_0}{(\omega_0^2 - \omega^2)^2 + (\omega\Delta\omega_0)^2}. \tag{4.20}$$

The general Lorentz resonance function normalized to 1 at the line center is equal to

$$\bar{G}_{\mathrm{L,res}}(x) = \frac{a^2 x}{(1 - x^2)^2 + a^2 x^2}, \tag{4.21}$$

where $x = \omega/\omega_0 = \nu/\nu_0$ and $a = \Delta\omega_0/\omega_0 = \Delta\nu_0/\nu_0 = \Delta E_0/E_0$. The Lorentz resonance function increases proportionally to frequency at small frequencies, $\omega \ll \omega_0$, and decreases inversely proportional to the third power of the frequency at large frequencies, $\omega \gg \omega_0$ (Fig. 4.10, upper part); the relative halfwidth of the curves in the figure is $a = 0.03$.

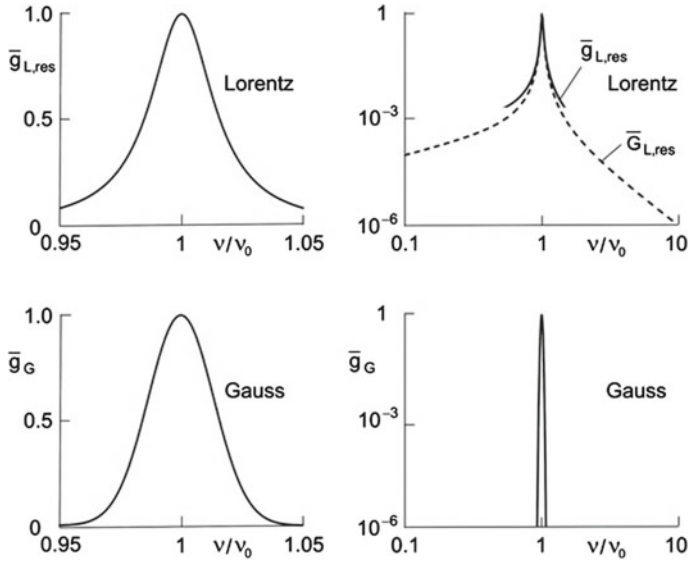

Fig. 4.10 Lorentz resonance function and Gaussian distribution function

4.6 Gaussian Lineshape Function

Line broadening can lead to a *Gaussian lineshape* described by

$$g_G(\nu) = \frac{2}{\Delta\nu_0}\left(\frac{\ln 2}{\pi}\right)^{1/2} \exp\left[-\frac{\ln 2\,(\nu-\nu_0)^2}{\Delta\nu_0^2/4}\right], \tag{4.22}$$

where ν_0 is the center frequency and $\Delta\nu_0$ the half width. The maximum value is

$$g_G(\nu_0) = \frac{2}{\Delta\nu_0}\sqrt{\frac{\ln 2}{\pi}} \approx \frac{0.94}{\Delta\nu_0}. \tag{4.23}$$

The lineshape function is normalized, $\int_0^\infty g_G(\nu)\mathrm{d}\nu = 1$. The Gaussian lineshape function normalized to unity at the line center is

$$\bar{g}_G(\nu) = \exp\left[-\frac{\ln 2\,(\nu-\nu_0)^2}{\Delta\nu_0^2/4}\right]. \tag{4.24}$$

A Gaussian line (Fig. 4.10, lower part) and a Lorentzian line of the same relative halfwidth shows only small differences to at frequencies around the line center. But there are essential differences in the wings. The Gaussian line decreases exponentially and has negligibly small values at frequencies a few halfwidths away from the center frequency; see the double logarithmic plots (Fig. 4.10, right). The Gaussian line has

Table 4.3 Linewidths

Laser	λ	ν (THz)	$\Delta\nu_0$ (GHz)	$\Delta\nu_{\text{nat}}$	τ_{sp}
HeNe	633 nm	474	1.6	1.2 MHz	100 ns
CO_2	10.6 μm	28	(0.07–500)	0.03 Hz	5 s
Nd:YAG	1.06 μm	280	140	1 kHz	230 μs

finite values around the center frequency, while the Lorentzian line (with the same linewidth) extends far into the wings.

Examples of Gaussian lines: Doppler broadened lines (Sect. 14.1) are inhomogeneously broadened; the line of Ti^{3+}:Al_2O_3 used for operation of the titanium–sapphire laser is homogeneously broadened (Sect. 17.4).

4.7 Experimental Linewidths

Table 4.3 shows values of the linewidth $\Delta\nu_0$ of $2 \rightarrow 1$ transition lines together with values of the natural linewidth $\Delta\nu_{\text{nat}}$. The halfwidth $\Delta\nu_{\text{nat}}$ of the upper laser level follows from the relation $\Delta\nu_{\text{nat}} = (2\pi\tau_{\text{sp}})^{-1}$, where τ_{sp} is the lifetime with respect to $2 \rightarrow 1$ spontaneous transitions (Sect. 4.9). Various methods of determination of linewidths are available. We mention a few methods.

- *Helium–neon laser.* The fluorescence line is inhomogeneously broadened (due to Doppler broadening). The linewidth can be calculated by use of the expression of Doppler broadening (Sect. 14.1).
- *CO_2 laser.* The $2 \rightarrow 1$ fluorescence line is Doppler broadened at low gas pressure and collision broadened at high pressure.
- *Nd:YAG laser.* A fluorescence experiment provides the linewidth.

A lower limit of the linewidth of a transition is the natural linewidth. Active media of lasers operated at room temperature show linewidths of the atomic transitions that are always larger than the natural linewidth as a study of the specific lasers shows (see the chapters beginning with Chap. 14).

4.8 Classical Oscillator Model of an Atom

An atom consists of a nucleus and an electron cloud (Fig. 4.11a). In the *classical oscillator model of an atom*, the electron cloud is replaced by an electron located at the center of the electron cloud, that is, at the position of the nucleus. In equilibrium, the electron does not move. When it is brought out of its equilibrium position (Fig. 4.11b), it performs an oscillation with a displacement $x(t)$ and an amplitude x_0. The oscillation of the electron (charge $q = -e$) corresponds to an oscillation of

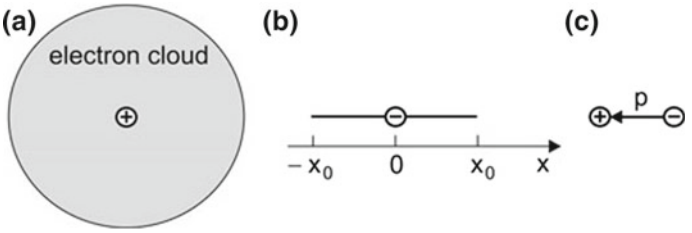

Fig. 4.11 Classical oscillator model of an atom. **a** Electron cloud and nucleus of an atom. **b** Oscillation of an electron around the equilibrium position $x = 0$. **c** Excited atom as an oscillating dipole

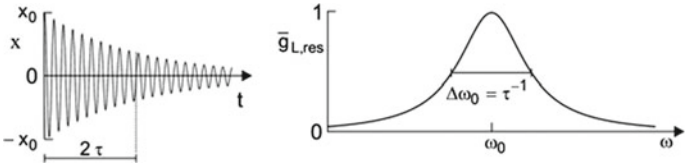

Fig. 4.12 Damped oscillator (*left*) and shape of the fluorescence line (*right*)

a dipole with the dipole moment $p = qx = -ex$ (Fig. 4.11c). An oscillation of a classical electric dipole (= *Hertzian dipole*) gives rise to emission of radiation. Here, we describe the model of the atomic oscillator (= dipole oscillator model = classical oscillator model of an atom = Lorentz model of an atom); later (in Chap. 9), we will make use of the model to derive the gain coefficient of radiation propagating in an active medium.

We consider the atomic oscillation as a damped oscillation. The equation of motion is given by

$$\ddot{x} + \beta\dot{x} + \omega_0^2 x \ = 0, \tag{4.25}$$

where x is the displacement from the equilibrium position ($x = 0$), ω_0 is the resonance frequency, and β the damping constant. We chose the resonance frequency so that $\hbar\omega_0 = E_2 - E_1$, where E_1 is the energy of the ground state, E_2 the energy of the excited state of the two-level atom, and $E_{21} = E_2 - E_1$ is the transition energy.

The solution to the equation of motion has, for $\beta \ll \omega_0$, the form

$$\tilde{x} = x_0 \mathrm{e}^{-\frac{1}{2}\beta t} \mathrm{e}^{-\mathrm{i}\omega_0 t}, \tag{4.26}$$

where x_0 is a displacement at $t = 0$; the displacement is 0 for $t < 0$. The time of decay of the amplitude of the oscillation is 2τ (Fig. 4.12, left), and the time of decay of the energy of the atomic oscillation is the lifetime $\tau = 1/\beta$.

Connected with an oscillation is an electric field

$$\tilde{E} = A\,\mathrm{e}^{-\frac{1}{2}\beta t}\mathrm{e}^{-\mathrm{i}\omega_0 t}. \tag{4.27}$$

The initial value A of the amplitude of the field corresponds to the initial value of the displacement. The field is not monochromatic but has a frequency distribution (Fig. 4.13, right) that follows by Fourier transformation,

$$\tilde{E}(\omega) = \frac{A}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{E}(t)\mathrm{e}^{\mathrm{i}\omega t}\mathrm{d}t, \tag{4.28}$$

which leads, with $\beta = \Delta\omega_0$, to

$$\begin{aligned} \tilde{E}(\omega) &= \frac{A}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp \mathrm{i}[(\omega - \omega_0)t + i(\beta/2)t]\mathrm{d}t \\ &= \frac{-A}{\sqrt{2\pi}}\, \frac{1}{\mathrm{i}(\omega - \omega_0) + \Delta\omega_0/2}. \end{aligned} \tag{4.29}$$

Accordingly, the oscillating electron emits an electromagnetic wave with an intensity distribution that corresponds to a Lorentzian line (Fig. 4.12, right),

$$I(\omega)/I_0 = \bar{g}_{\mathrm{L,res}}(\omega). \tag{4.30}$$

I_0 is the intensity at the line center. The linewidth $\Delta\omega_0$ is equal to τ^{-1}.

In the classical oscillator model description of an atom, the nonoscillating state corresponds to the atomic ground state, and an oscillating state corresponds (independent of the value of the amplitude of the oscillation) to the excited state of the atom.

4.9 Natural Line Broadening

The finite lifetime of the upper level of a two-level atomic system with respect to spontaneous emission of radiation by $2 \rightarrow 1$ transitions leads to a line broadening described by the natural lineshape function

$$g_{\mathrm{nat}}(\nu) = \frac{\Delta\nu_{\mathrm{nat}}}{2\pi}\, \frac{1}{(\nu_0 - \nu)^2 + \Delta\nu_{\mathrm{nat}}^2/4}, \tag{4.31}$$

where ν_0 is the resonance frequency,

$$\Delta\nu_{\mathrm{nat}} = \frac{1}{2\pi\,\tau_{\mathrm{sp}}} \tag{4.32}$$

is the *natural linewidth*, and τ_{sp} the lifetime of the upper level with respect to spontaneous emission of radiation by $2 \rightarrow 1$ transitions. We designate τ_{sp} as the *spontaneous lifetime* of level 2. The maximum value of the lineshape function,

$$g_{nat}(\nu_0) = 4\tau_{sp} = \frac{2}{\pi \Delta \nu_{nat}}, \tag{4.33}$$

is proportional to the spontaneous lifetime, that is, inversely proportional to the natural linewidth.

4.10 Energy Relaxation

We describe energy relaxation in the classical model of an atom. An atomic dipole loses energy due to damping. The amplitude of the oscillation decreases exponentially (Fig. 4.13a). The decay time of the amplitude is $2T_1$ and the decay time of the energy content in the oscillator is equal to the *energy relaxation time*. The linewidth of the frequency distribution of radiation emitted by a dipole oscillator is equal to

$$\Delta\omega_0 = \frac{1}{T_1}. \tag{4.34}$$

The halfwidth of the frequency distribution of radiation emitted by a dipole is equal to the reciprocal of the energy relaxation time.

We will later study an ensemble of dipole oscillators. The decay of the energy contained in an ensemble of dipole oscillators occurs also with the energy relaxation time T_1.

[In an ensemble of electrons, the decay of the energy content is joined with the decay of the *polarization* with the time T_1, which is then called *longitudinal relaxation time* of the polarization; see next section and Sect. 9.9].

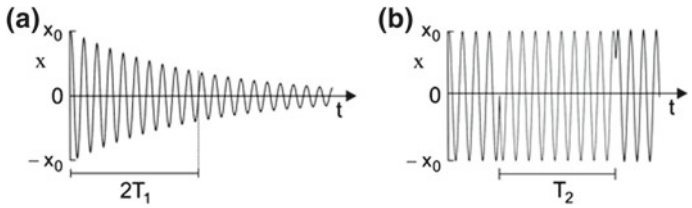

Fig. 4.13 Change of an oscillation. **a** Energy relaxation. **b** Dephasing

4.11 Dephasing

External forces can have the effect that a dipole oscillator randomly changes the phase while the amplitude x_0 and the resonance frequency ω_0 remain unchanged. The average time between two successive changes of the phase is the *dephasing time* T_2 (Fig. 4.13b). Radiation emitted by an oscillator consists of a series of subsequent wave trains $E(t)$ of constant amplitude A. The average duration of a wave train is equal to T_2. A Fourier analysis shows that the power spectrum has a Lorentzian lineshape. The center frequency is an average frequency, the lineshape function is a Lorentz resonance function and the linewidth (halfwidth) is given by

$$\Delta\omega_0 = \frac{1}{T_2/2} = \frac{2}{T_2}. \tag{4.35}$$

Examples Line broadening due to collisions of atoms and molecules in gases (Sect. 14.2); line broadening due to an elastic collision of an ion in a crystal with a phonon (Sect. 15.9).

We will later study an ensemble of dipole oscillators that are prepared in such a way that all elementary atomic oscillators are oscillating with the same phase. The sum of all atomic dipole moments per unit volume is the *dielectric polarization*. Due to randomly occurring dephasing processes of the atomic dipole oscillators, the dielectric polarization decays also with the dephasing time T_2. In connection with the decay of the polarization, T_2 is called *phase relaxation time* or *transverse relaxation time* (Sect. 9.8).

We will (in Chaps. 19 and 32) find dephasing processes characteristic of *monopole oscillators* (next section).

4.12 Dipole Oscillator and Monopole Oscillator

A dipole oscillator (Sect. 4.8) consists of a negative charge and a positive charge oscillating against each other. Interaction of a dipole oscillator with the surrounding can lead to loss of energy and to change of the phase of the oscillation. We are using the dipole oscillator as a classical model of a two-level atomic system (Sect. 4.8).

A *monopole oscillator* consists of an oscillating single charge. Interaction of the charge with the surroundings can change the phase of the oscillation but not the amplitude. An electron traversing a spatially periodic magnetic field represents an example of a monopole oscillator. An electron crossing a spatially periodic electric field is another example.

The monopole oscillation of an electron propagating at a relativistic velocity through a transverse periodic magnetic field is the elementary excitation occurring in a free-electron laser (Chap. 19); interaction of a high frequency electric field with an ensemble of the electron-monopole oscillators leads to gain for the high frequency field in the free-electron laser.

Monopole oscillation of an electron occurs also in the (hypothetical) Bloch laser (Chap. 32). An electron in a periodic potential formed by a semiconductor superlattice executes, under the action of a static electric field, an oscillation (Bloch oscillation); a superlattice consists, in the simplest case, of two different semiconductor layers in turn. The Bloch oscillation is the elementary excitation of a Bloch laser.

A laser based on monopole oscillations as the elementary excitations in an active medium can be regarded as an *inversionless laser*.

An electron that performs a Bloch oscillation can, alternatively, be described as an electron that occupies an energy level of an *energy-ladder system*. The energy ladder consists of equidistant energy levels. In the picture of the energy-ladder system, the Bloch laser is a laser with population inversion in the active medium (Sect. 32.7). Radiation is generated by stimulated transitions between energy levels of the energy-ladder systems.

It is a question whether an electron propagating through a transverse periodic magnetic field can also be characterized by an energy-ladder system. If such a description would be possible, a free-electron laser would be describable as a laser with a population inversion, alternatively to the description as an inversionless laser (Chap. 19).

Interaction of radiation with an active medium based on dipole oscillators leads to a gain coefficient that has the shape of a Lorentz resonance curve (see Chap. 9). Interaction of radiation with a monopole oscillator results in a gain coefficient that has the shape of a Lorentz dispersion curve (Chaps. 19 and 32).

4.13 Three-Dimensional and Low-Dimensional Active Media

We classify media used in lasers as three-dimensional (3D) and low-dimensional media. A low-dimensional active medium is either two-dimensional (2D), one-dimensional (1D), or zero-dimensional (0D). This classification is useful with respect to semiconductor lasers. Semiconductor lasers make use of low-dimensional media. Low-dimensional media are realized by means of semiconductor heterostructures.

The dimensionality concerns solely the question whether electrons *move freely* (between two collisions) in three dimensions, in two dimensions, in one dimension, or cannot move freely at all. In this sense, an atom is a 0D system.

Figure 4.14 gives a survey of active media of different dimensionality:

- *3D active medium.* The electrons move freely in three dimensions. The unit of the density N of electrons is m^{-3}.
  Example electrons in a 3D bipolar semiconductor.
- *2D active medium.* The electron motion is bound to a plane. The unit of the two-dimensional density (area density = sheet density) N^{2D} of electrons is m^{-2}.
  Example electrons in a quantum film (in a quantum well laser).

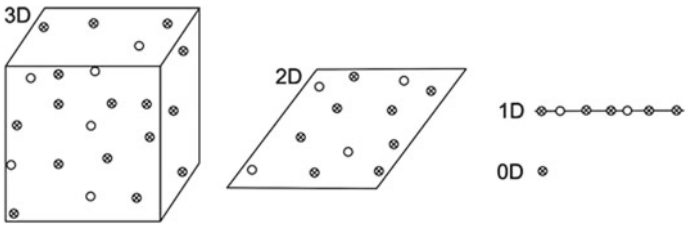

Fig. 4.14 Three-dimensional and low-dimensional active media

- *1D active medium.* The electron motion is bound to a line. The unit of the one-dimensional density (=line density) N^{1D} of electrons is m^{-1}.
  Example Electrons in a quantum wire (in a quantum wire laser).
- *0D active medium.* The electrons are imprisoned. An ensemble of 0D active media forms a 3D active medium.
  Example electrons in a quantum dot (in a quantum dot laser); we can regard a quantum dot as an artificial atom.

Ensembles of atoms, molecules, or ions are three-dimensional media. Each atom (or molecule or ion) is for itself a quantum system. Boltzmann's statistics governs the populations of the energy levels in an ensemble of atoms because the interaction (for instance, by collisions in gases) between the atoms is weak. An electron gas (=ensemble of the electrons) in the upper band in a bipolar semiconductor laser forms a quantum system. The electron gas (=ensemble of the electrons) in the lower band forms another quantum system. The two coexisting electron gases obey, each for itself, Fermi's statistics and have different Fermi energies (called *quasi-Fermi energies*).

References [1–4, 31, 32].

Problems

4.1 Lineshape functions. At which frequency distance from the central line ($\nu_0 = 4 \times 10^{14}$ Hz; $\Delta\nu_0$ =1 GHz) does the lineshape function decrease by a factor of 100 (a) if the line has Lorentzian shape and (b) if the line has Gaussian shape?

4.2 Absolute number of two-level atomic systems. Determine the absolute number N_{tot} of two-level atomic systems for systems of different dimensionality.

(a) Three-dimensional medium with a density $N = 10^{24}\,\text{m}^{-3}$ of two-level systems and a volume of 1 mm × 1 mm × 1 mm.
(b) Two-dimensional medium with an area density $N^{2D} = 10^{16}\,\text{m}^{-2}$ and an area of 1 mm × 1 mm.
(c) One-dimensional medium with a line density $N^{1D} = 10^{7}\,\text{m}^{-1}$ and a length of 1 mm.

4.3 Relate the lineshape function on the frequency scale and the lineshape function on the wave number scale.

4.4 Relate the dimensionless variables of the Lorentz resonance function expressed on the frequency scale and those expressed on the angular frequency scale.

4.5 Area under a Gaussian or Lorentzian curve.

(a) Show that the width of a rectangular curve, which has the same height as a Gaussian curve and encloses the same area, is equal to

$$\Delta\nu_0/(2\sqrt{\ln 2}) \approx 1.06 \times \Delta\nu_0 \approx \Delta\nu_0.$$

(b) Show that the width of a rectangular curve, which has the same height as a Lorentzian curve and encloses the same area, is approximately equal to

$$(\pi/2)\Delta\nu_0 \approx 1.57 \times \Delta\nu_0.$$

4.6 Show that the integral over a narrow Lorentzian curve is approximately unity.

4.7 Derive the maximum value, Eq. (4.10), for the Lorentz resonance function.

4.8 Show that the width of a rectangular shape that has the same height and the same area as a Gaussian $\Delta\nu_0/(2\sqrt{\ln 2}) \approx 1.06 \times \Delta\nu_0$, and, correspondingly, for a Lorentzian shape by $(\pi/2)\Delta\nu_0 \approx 1.57 \times \Delta\nu_0$.