Chapter 12
Complements of Mathematics

12.1 A Short Introduction to Tensors

The idea of tensors arose when physicists started dealing with forces inside elastic solids (tensions lead to tensors). Mathematically speaking, tensors are multilinear forms. For instance, the scalar product between two vectors is a bilinear form:

$$\delta : (\mathbf{a}, \mathbf{b}) \longrightarrow \mathbf{a} \cdot \mathbf{b} = \sum_{i=1}^{3} a_i b_i$$

This is an application which associates a scalar with two vectors (hence bi-) and it is linear with respect to each vector. The foregoing expression could be rewritten using the components of the unit tensor, namely δ_{ij}, like:

$$\delta : (\mathbf{a}, \mathbf{b}) \longrightarrow \delta(\mathbf{a}, \mathbf{b}) = \sum_{i=1}^{3} \sum_{j=1}^{3} a_i \delta_{ij} b_j$$

This is the same as above since $\delta_{ij} = 0$ when $i \neq j$ and $\delta_{ii} = 1$. Very often the $\sum$ symbol is dropped, and Einstein notations are used: repeated indices meaning summation. Using these notation $\delta(\mathbf{a}, \mathbf{b}) = a_i \delta_{ij} b_j$. However, $\delta_{ij} b_j = b_i$ thus $\delta(\mathbf{a}, \mathbf{b}) = a_i b_i$.

To simplify the presentation of these notions, we stay in the metric space $\mathbb{R}^3$ with an orthonormal basis. This means that the notion of variance, namely contravariant and covariant tensors, is not useful at all. So the place of indices (up or down) is not meaningful here. We just use the Einstein convention of implicit summation on repeated indices. Thus,

$$a_i b_{ij} \qquad \text{means} \qquad \sum_{i=1}^{3} a_i b_{ij}$$

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12.1.1 Definitions

A multilinear form f may be simply defined by

$$f(\mathbf{X}^1, \mathbf{X}^2, \ldots, \mathbf{X}^n) = \sum_{i_1, i_2, \ldots, i_n} f_{i_1, i_2, \ldots, i_n} X^1_{i_1} X^2_{i_2} \ldots X^n_{i_n}$$

where the summation runs from 1 to 3 in a three-dimensional space. The set

$$\{ f_{i_1, i_2, \ldots, i_n} / i_1 = 1, 2, 3, i_2 = 1, 2, 3, \ldots i_n = 1, 2, 3 \}$$

is the set of the 3^n components of the n^{th}-order tensor $[f]$. Very often, one calls, abusively, $f_{i_1, i_2, \ldots, i_n}$ a tensor, while this is only one of its components.

With the foregoing definition, we observe that *a zeroth order tensor is a scalar and a first order tensor is a vector*.

12.1.1.1 Notations

The cartesian coordinates are often denoted (x_1, x_2, x_3). The vectors are denoted $\mathbf{v}$ and their components v_i. Tensors of order greater or equal to 2 are denoted $[a]$ and their components $a_{ij\ldots}$.

The gradient of a scalar function f is ∇f and its components are:

$$(\nabla f)_i \equiv \frac{\partial f}{\partial x_i} \equiv \partial_i f$$

12.1.1.2 Contraction

The contraction of two indices is the summation over these two indices. For instance, the contraction of index i and j of the second order tensor a_{ij} gives a_{ii} which is the *trace* of the tensor: a_{ii} =Tr[a]. We may note that $\nabla \cdot \mathbf{v} = \partial_i v_i$ is also the trace of the velocity gradient tensor.

12.1.1.3 The Tensorial Product of Two Vectors

From vectors one can construct tensors of higher order. This operation is called the *tensorial product* and usually denoted $\otimes$. For example, from the velocity vector $\mathbf{v}$, we can build the Reynolds stress tensor $\rho[\mathbf{v} \otimes \mathbf{v}]$ whose components are $\rho v_i v_j$.

12.1.2 ϵ_{ijk}

To ease operations using the cross product, one uses the set of numbers ϵ_{ijk} which are defined as follows:

$$\begin{array}{ll} \epsilon_{ijk} = 1 & \text{if } ijk \text{ is an even permutation of } 123 \\ \epsilon_{ijk} = -1 & \text{if } ijk \text{ is an odd permutation of } 123 \\ \epsilon_{ijk} = 0 & \text{if two indices are identical} \end{array}$$

This is not a tensor but a pseudo-tensor often called the *completely antisymmetric pseudo-tensor*. It may indeed also be defined from a determinant:

$$\begin{aligned} \epsilon : \quad \mathbb{R}^3 \quad &\longrightarrow \mathbb{R} \\ (\mathbf{a}, \mathbf{b}, \mathbf{c}) &\longmapsto \epsilon_{ijk} a_i b_j c_k = Det(\mathbf{a}, \mathbf{b}, \mathbf{c}) \end{aligned}$$

ϵ is a pseudo-tensor because the sign of the determinant depends on the orientation of the basis. This is also the reason why the determinant is often called a pseudo-scalar, and the cross product a pseudo-vector.

In the following we give some useful relations for the manipulation of ϵ_{ijk}:

$$\begin{aligned} \epsilon_{ijk}\epsilon_{lmn} &= \delta_{il}\delta_{jm}\delta_{kn} + \delta_{kl}\delta_{im}\delta_{jn} + \delta_{jl}\delta_{km}\delta_{in} - \delta_{kl}\delta_{jm}\delta_{in} - \delta_{jl}\delta_{im}\delta_{kn} - \delta_{il}\delta_{km}\delta_{jn} \\ &= \begin{vmatrix} \delta_{il} & \delta_{jl} & \delta_{kl} \\ \delta_{im} & \delta_{jm} & \delta_{km} \\ \delta_{in} & \delta_{jn} & \delta_{kn} \end{vmatrix} \end{aligned} \tag{12.1}$$

$$\epsilon_{ijk}\epsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl} \tag{12.2}$$

$$\epsilon_{ikl}\epsilon_{jkl} = 2\delta_{ij} \tag{12.3}$$

$$\epsilon_{ijk}\epsilon_{ijk} = 6 \tag{12.4}$$

Here are also the relation between some expressions using vectors and the same expressions using tensorial notation:

$$\mathbf{a} \cdot \mathbf{b} = a_i b_i$$

$$(\mathbf{a} \times \mathbf{b})_i = \epsilon_{ijk} a_j b_k$$

$$det(\mathbf{a}, \mathbf{b}, \mathbf{c}) = \epsilon_{ijk} a_j b_k c_i$$

If [A] is a symmetric tensor, then $A_{ij} = A_{ji}$ and

$$\epsilon_{ijk} A_{jk} = 0 \tag{12.5}$$

because terms cancel two by two. A more detailed and more formal presentation may be found in textbooks of Hladik (1993), Bass (1978) or Lebedev et al. (2010).

12.2 The Divergence Theorem

12.2.1 Statement and Demonstration

If $T_{i_1 i_2 \cdots i_n}$ is a tensorial field of order n, smooth enough so that derivatives are defined, then

$$\int_{(S)} T_{i_1 i_2 \cdots i_k \cdots i_n} dS_{i_k} = \int_{(V)} \partial_{i_k} T_{i_1 i_2 \cdots i_k \cdots i_n} dV \tag{12.6}$$

where the first integration is taken over the surface S surrounding the volume V. This relation is true in a space of dimension $n \geq 1$, but we shall give its demonstration only in the two-dimensional space (a plane). Its generalization to a higher dimension is straightforward (Fig. 12.1).

We first write the volume integral under the form:

$$\int_{(V)} (\partial_x A_x + \partial_y A_y) dxdy = \int_{y_{min}}^{y_{max}} (A_x(x_+) - A_x(x_-)) dy + \int_{x_{min}}^{x_{max}} (A_y(y_+) - A_y(y_-)) dx$$

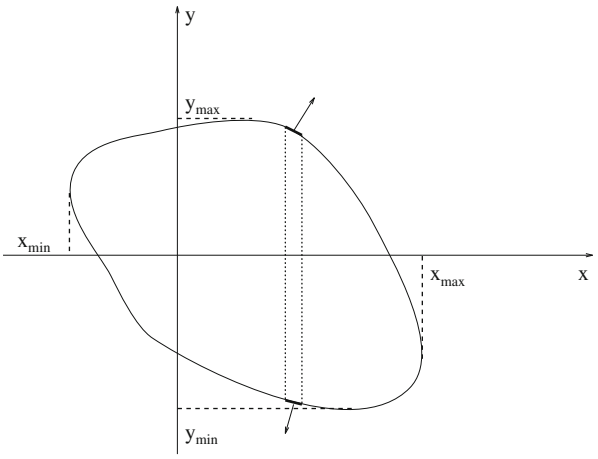

Fig. 12.1 The various quantities needed in the demonstration of the divergence theorem

where $x_{\pm}(y)$ and $y_{\pm}(x)$ are the curves defining the boundary of the "volume". Let us concentrate on one of the terms, for instance $A_y(y_+)dx$. This term also reads

$$A_y(y_+)\mathbf{e}_y \cdot \mathbf{n}dl$$

where $\mathbf{n}$ is the outward normal unit vector and dl is the "surface" element. Thus,

$$A_y(y_+)dx = (A_y(y_+)\mathbf{e}_y) \cdot d\mathbf{S}$$

but y_+ defines the upper part of the surface while $y_-(x)$ defines the lower part. Together, when x runs from x_{min} to x_{max}, y_- and y_+ describe the whole surface. We may therefore write

$$\int_{x_{min}}^{x_{max}} (A_y(y_+) - A_y(y_-))dx = \int_{(S)} (A_y\mathbf{e}_y) \cdot d\mathbf{S}$$

where we noted that $-A_y(y_-)dx = -(A_y(y_-)\mathbf{e}_y) \cdot (-\mathbf{n})dl = (A_y\mathbf{e}_y) \cdot d\mathbf{S}$.

Then, we follow the same reasoning for the other half of the integral finally get $\int_{(S)} A_i dS_i$. QED

12.2.2 Corollary

Similarly we may show the following relation:

$$\int_{(S)} \epsilon_{jik_m} T_{i_1 i_2 \cdots i_k \cdots i_n} dS_m = \int_{(V)} \epsilon_{jik_m} \partial_m T_{i_1 i_2 \cdots i_k \cdots i_n} dV \tag{12.7}$$

12.2.3 A Few Consequences

The general result (12.6) has some interesting specific cases. In the usual three-dimensional space, we find again some classical formulae if we set the tensor $[T]$ to a scalar, a vector or a second order tensor, namely:

$$\int_{(S)} f d\mathbf{S} = \int_{(V)} \boldsymbol{\nabla} f \; dV \tag{12.8}$$

$$\int_{(S)} \mathbf{A} \cdot d\mathbf{S} = \int_{(V)} \boldsymbol{\nabla} \cdot \mathbf{A} \; dV$$

$$\int_{(S)} [a] d\mathbf{S} = \int_{(V)} \mathbf{Div}[a] \; dV$$

The corollary (12.7) allows us to write:

$$\int_{(S)} \mathbf{A} \wedge d\mathbf{S} = -\int_{(V)} \boldsymbol{\nabla} \times \mathbf{A}\, dV \tag{12.9}$$

In a two-dimensional space, volumes are surfaces and surfaces are contours, hence:

$$\oint_{(C)} f\, dl\, \mathbf{n} = \int_{(S)} \boldsymbol{\nabla} f\ dS$$

$$\oint_{(C)} \mathbf{A} \cdot \mathbf{n}\, dl = \int_{(S)} \boldsymbol{\nabla} \cdot \mathbf{A}\ dS$$

$$\oint_{(C)} \mathbf{A} \times \mathbf{n}\, dl = -\int_{(S)} \boldsymbol{\nabla} \times \mathbf{A}\ dS$$

where $\mathbf{n}$ is the unit outward normal vector to the contour C.

12.3 Radius of Curvature

12.3.1 For a Plane Curve

The radius of curvature of a plane curve defined parametrically by the functions $x(t)$ and $y(t)$ is the radius of the osculating circle at the given point. It may be computed as follows.

Let M be a point of the curve and $M + dM$ another point infinitely close to M and on the same curve. If these two points also belong to the same osculating circle, then their distance is $Rd\theta$ where $d\theta$ is the angle between the vectors that are tangent to the curve in M and $M + dM$. Thus we have:

$$Rd\theta = \sqrt{\dot{x}^2 + \dot{y}^2}dt \tag{12.10}$$

where the dot means the derivative with respect to t. In addition

$$d\theta \simeq \sin d\theta = -\frac{(\mathbf{T} \times (\mathbf{T} + d\mathbf{T})) \cdot \mathbf{e}_z}{\|\mathbf{T}\|\ \|\mathbf{T} + d\mathbf{T}\|}$$

$$\iff \quad d\theta \simeq -\frac{(\mathbf{T} \times d\mathbf{T}) \cdot \mathbf{e}_z}{\|\mathbf{T}\|^2}$$

where $\mathbf{T}$ is the tangent vector. Noting that

$$\mathbf{T}\begin{vmatrix}\dot{x}\\ \dot{y}\end{vmatrix}, \quad d\mathbf{T}\begin{vmatrix}\ddot{x}dt\\ \ddot{y}dt\end{vmatrix} \qquad \text{and} \qquad d\theta = -\frac{\dot{x}\ddot{y} - \dot{y}\ddot{x}}{\dot{x}^2 + \dot{y}^2}dt$$

(12.10) leads to the result:

$$\frac{1}{R} = \frac{\ddot{x}\dot{y} - \ddot{y}\dot{x}}{(\dot{x}^2 + \dot{y}^2)^{3/2}} \tag{12.11}$$

For a curve defined by the equation $y = y(x)$, then

$$\frac{1}{R} = -\frac{\ddot{y}}{(1 + \dot{y}^2)^{3/2}} \tag{12.12}$$

12.3.2 For a Curve in Space

Now, if the curve is defined in a three-dimensional space by its parametric equation $[x(s), y(s), z(s)]$, then we define the tangent vector as:

$$\mathbf{e}_s = \frac{1}{\sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}}\begin{vmatrix}\dot{x}(s)\\ \dot{y}(s)\\ \dot{z}(s)\end{vmatrix}$$

The curvature radius R is defined as:

$$\frac{\partial \mathbf{e}_s}{\partial s} = \frac{\mathbf{n}}{R} \tag{12.13}$$

where $\mathbf{n}$ is a unit normal vector to the curve. The radius R is a positive number.

Finally, one then introduce the *binormal vector* $\mathbf{e}_b$, which defines, with $\mathbf{e}_s$ and $\mathbf{n}$, the Frenet frame. It is defined as

$$\frac{\partial \mathbf{n}}{\partial s} = \frac{\mathbf{e}_b}{T}$$

The length T is called the *torsion*.

12.4 The Boundary Layer Theory Viewed from Differential Equation Theory

We introduced in Chap. 4 the idea of boundary layers, basing our approach on physical considerations to find out the various possible balances of forces. We stressed the physical existence of this layer when we presented its destabilization that induces rapid variation of the drag coefficient C_x. However, the notion of boundary layer has deep roots in the theory of differential equations. Here, we shall briefly sketch out these ideas, but we strongly encourage the reader to have a look to the book of Bender and Orszag where many aspects of this technique are detailed.

One of the various ways of solving differential equations is to use the theory of singular perturbations. To illustrate this theory we readily take an example, namely the differential equation:

$$\varepsilon \frac{d^2y}{dx^2} + (1+\varepsilon)\frac{dy}{dx} + y = 0 \tag{12.14}$$

where ε is a small parameter. If we consider the terms factoring ε as a perturbation of an original equation where $\varepsilon = 0$, then this perturbation is said to be singular because the order of the differential equation is not the same when $\varepsilon = 0$ and $\varepsilon \neq 0$. This is said to be singular because the change of order leads to the appearance of a singularity when ε tends to zero.

Let us now study in more detail the example given by (12.14), which we complete by boundary conditions so as to fully determine the solution. We choose

$$y(0) = 0 \qquad \text{and} \qquad y(1) = 1 \tag{12.15}$$

(12.14) is a second order differential equation with constant coefficients. Its solutions are (in the general case) a linear combination of exponentials. Thus, we find:

$$y(x) = \frac{e^{-x} - e^{-x/\varepsilon}}{e^{-1} - e^{-1/\varepsilon}} \tag{12.16}$$

This solution is already quite interesting as it reveals the singularity which arises when ε goes to zero. The solution is discontinuous at the origin. In addition, when $\varepsilon \ll 1$, the solution varies very rapidly near this point (see Fig. 12.2). We see the presence of a boundary layer!

We shall now retrieve the solution (12.16) through a perturbative approach similar to the one we used for the boundary layer analysis. We need four steps:

1. First step: we determine the *outer solution* (outside the boundary layer). We set $\varepsilon = 0$ and we solve. However, we face a first difficulty: We have two boundary

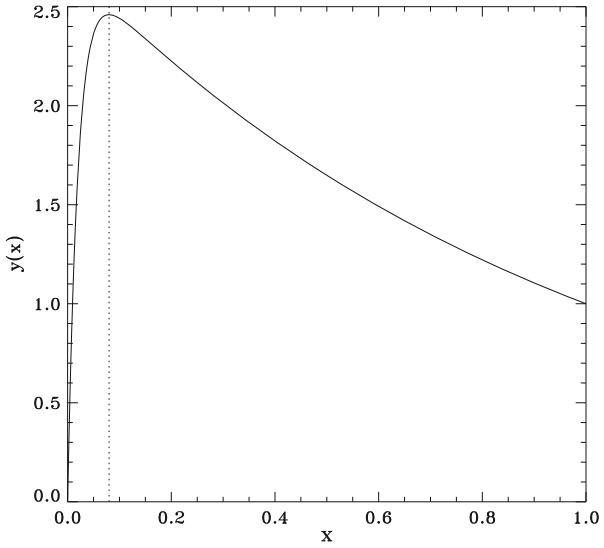

Fig. 12.2 A plot of the function $y(x) = (e^{-x} - e^{-x/\varepsilon})/(e^{-1} - e^{-1/\varepsilon})$ for $\varepsilon = 0.02$. The *vertical dotted line* shows the separation between the inner and outer regions

conditions for a first order differential equation. We have to dismiss one of the boundary conditions. In this case, the choice is quite simple since one of the boundary conditions is in the boundary layer. Thus we take the other one, namely $y(1) = 1$. Of course, this choice is not always so simple, and some trial and error is sometimes required.
We thus solve:

$$y' + y = 0 \qquad \text{with} \qquad y(1) = 1$$

The solution is $y_e(x) = e^{1-x}$. Let us point out that this is the solution we would have obtained if we had let $\varepsilon \to 0$ in the solution (12.16) for $x \neq 0$.

2. The second step is to determine the solution in the boundary layer. For this, we use the stretched coordinate $\zeta = x/\varepsilon$ (the boundary layer coordinate) and we rewrite the differential equation using ζ:

$$y'' + (1 + \varepsilon)y' + \varepsilon y = 0$$

Again, we let $\varepsilon \to 0$ and find:

$$y'' + y' = 0$$

This new differential equation is of second order and needs two boundary conditions. One of them is obvious: at $x = 0$, $y = 0$. We do not specify

the second one at the moment and leave the constant undetermined. Thus, the boundary layer solution is

$$y_{bl} = C(1 - e^{-\zeta}) \tag{12.17}$$

where C is an arbitrary constant. We note that this solution can be derived from the exact solution (12.16) by making the change of variable $x \to \zeta$ and letting $\varepsilon \to 0$.

3. The third step is the asymptotic matching between the two previous solutions. We need to find an interval where both solutions are valid. In the present case, this is where:

$$\varepsilon \ll x \ll 1 \tag{12.18}$$

In this interval, the two functions y_e and y_{bl} must tend to the same limit. Hence

$$\lim_{\zeta \to \infty} y_{bl} = \lim_{x \to 0} y_e = e$$

This fixes the constant $C = e$.

4. The last step is to gather all the foregoing "pieces" and construct the function that solves the problem in the whole interval. This function is identical to y_e when $x \gg \varepsilon$ and equals y_{bl} when $x \leq \varepsilon$. It is obtained by the combination:

$$y_{unif} = y_e + y_{bl} - e = e^{1-x} - e^{1-\zeta} \tag{12.19}$$

The solution $y_{unif}(x)$ is said to be the *uniform approximation at order n* if, in the given interval,

$$y_{unif}(x) - y(x) = \mathcal{O}(\varepsilon^{n+1}) \qquad \text{when} \quad \varepsilon \to 0$$

We may observe that in our example $|y_{unif}(x) - y(x)|$ is $\mathcal{O}(e^{-1/\varepsilon})$ which is less than any power of ε. Our approximation is therefore valid at "infinite" order, however it always differs from the exact solution by an undetermined $\mathcal{O}(e^{-1/\varepsilon})$ quantity. This last point underlines the fact that *the boundary layer theory is a theory of singular perturbations and not of regular perturbations.*

12.5 The Sturm–Liouville Problem

The Sturm–Liouville problem is usually enounced in the following way:

Let three functions $p(x), q(x)$ and $w(x)$, continuous and defined on the interval $[a, b] \subset \mathbb{R}$, such that $p(x) > 0$, $q(x) \geq 0$ and $w(x) > 0$ in the whole interval. The Sturm–Liouville problem is the following boundary value problem:

$$(py')' - qy + \lambda wy = 0 \tag{12.20}$$

where the solution $y(x)$ verifies the boundary conditions:

$$y(a) = 0 \quad \text{or} \quad y'(a) = 0, \qquad y(b) = 0 \quad \text{or} \quad y'(b) = 0 \tag{12.21}$$

It may be shown that the solutions of this problem form a denumerable suite of eigenfunctions y_n associated with real eigenvalues λ_n.

Two eigenfunctions associated with two distinct eigenvalues are orthogonal with respect to the scalar product:

$$\langle f|g\rangle = \int_a^b f(x)g(x)w(x)dx$$

Indeed, let y_m and y_n be two solutions associated respectively with λ_m and λ_n, then

$$(py'_m)' - qy_m + \lambda_m w y_m = 0$$

$$(py'_n)' - qy_n + \lambda_n w y_n = 0$$

Multiplying the first equation by y_n and the second by y_m, integrating and subtracting the two equations, one finds:

$$(\lambda_m - \lambda_n)\langle y_m|y_n\rangle = \left[y_m(wy'_n) - y_m(wy'_n)\right]_a^b = 0$$

where we used the boundary conditions at a and b. Since $\lambda_m \neq \lambda_n$, $\langle y_m|y_n\rangle = 0$.

This property of the Sturm–Liouville problem is at the origin of *spectral methods* for solving numerically differential equations. The unknown solutions are expanded on the basis formed by a set of functions solution of a Sturm–Liouville problem.

When one deals with a problem of stability, the sign of the real part of the eigenvalues is important. In a Sturm–Liouville problem, it can be determined quite easily. Multiplying (12.20) by y and integrating over $[a, b]$, one gets:

$$\lambda = \frac{\int_a^b (py'^2 + qy^2)dx}{\int_a^b y^2 w dx}$$

showing that all the eigenvalues are positive. A slightly more general result can be obtained if we note that $w(x) < 0$ in the whole interval, then all the eigenvalues are negative. A more complicated case is when w changes sign in the interval. It can be shown then that the eigenvalue spectrum spans from $-\infty$ to $+\infty$.

A complete study of the Sturm–Liouville problem may be found in the book of Ince (1956).

12.6 Second Order Partial Differential Equations

12.6.1 The Different Types

Second order partial differential equations are classified in four categories: they are either of *hyperbolic, parabolic, elliptic or mixed* type. They are categorized according to a property of the coefficients of the second order derivatives. The most general form of this kind of equation is:

$$A(x,y)\frac{\partial^2 f}{\partial x^2} + B(x,y)\frac{\partial^2 f}{\partial x \partial y} + C(x,y)\frac{\partial^2 f}{\partial y^2} + \cdots = 0$$

where the dots are for the first and zeroth order terms. The function $D(x,y) = B^2 - 4AC$ determines the type of the equation. If in the whole domain where f is defined,

- $D(x,y) > 0$, the equation is hyperbolic
- $D(x,y) = 0$ the equation is parabolic
- $D(x,y) < 0$ the equation is elliptic

If $D(x,y)$ changes sign in the domain, the equation is said to be of mixed type.

To be more familiar with the basic properties of these types of equations, we shall focus on three very classical examples: the wave equation, the heat equation and the Laplace equation. All these equations are met in Fluid Mechanics.

As a first step, we need to introduce the notion of characteristics.

12.6.2 An Introduction to Characteristics

Let $f(x,y)$ be a function defined on a plane (or in a domain of a plane) and verifying

$$A(x,y)\frac{\partial f}{\partial x} + B(x,y)\frac{\partial f}{\partial y} = 0 \tag{12.22}$$

The characteristic curves are the curves where f is constant, namely where:

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = 0 \tag{12.23}$$

Using (12.22) and (12.23), we derive the equation of characteristics:

$$\frac{dy}{dx} = \frac{B}{A}$$

The solution of this equation together with the boundary conditions permits, in some cases, a complete determination of f. Let us take an example of this favourable case. We assume that $f(x,0) = \cos x$, $A(x,y) = y$ and $B = 1$. The characteristics are such that:

$$y\frac{dy}{dx} = 1 \implies y = \pm\sqrt{2(x+K)}, \quad x \geq -K, \quad K \in \mathbb{R}$$

This is a family of parabola with Ox as their axis. Any point of the plane belongs to a unique characteristics and the value of f there just depends on the constant K defining the characteristics:

$$f(x,y) = f\left(\frac{1}{2}y^2 - x\right)$$

The boundary condition $f(x,0) = \cos x$ determines the function and we obtain the searched solution:

$$f(x,y) = \cos\left(\frac{1}{2}y^2 - x\right)$$

We may observe that the boundary condition on the x-axis, $f(x,0) = \cos x$, has been propagated in the whole plane. If the y-coordinate is replaced by the time, then we have a true propagation. This property is verified by all the equations having characteristics.

12.6.3 A Hyperbolic Equation: The Wave Equation

The equation of a wave propagating at the velocity c is given by:

$$\frac{\partial^2 f}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 f}{\partial t^2} = 0$$

The general solution of this equation is well-known:

$$f(x,t) = \Phi(x - ct) + \Psi(x + ct)$$

where Φ and Ψ are two arbitrary functions to be determined by the initial conditions. This solution is easily obtained after the change of variable: $u = x - ct$ and $v = x + ct$.

The solution is fully determined if we take into account the initial conditions. For instance, we may demand:

$$f(x,0) = \cos x \qquad \text{and} \qquad \left(\frac{\partial f}{\partial t}\right)_0 = 0$$

which leads to

$$f(x,t) = \frac{1}{2}\left[\cos(x-ct) + \cos(x+ct)\right]$$

Initial conditions are necessary to determined completely the solution. With these conditions the problem is well posed. To emphasize the importance of these conditions, let us try the exercise where, instead of imposing the value of the function and its time-derivative at an initial time, we impose the value of the function at two different times. For instance:

$$f(x,0) = I(x) \qquad \text{and} \qquad f(x,T) = F(x)$$

where $I(x)$ and $F(x)$ are given data. Such a problem is mathematically ill-posed. These conditions, which look like boundary conditions are not sufficient to fully determine the solution. It is easy to show that $\Psi(x)$ verifies:

$$\Psi(x) = \Psi(x+2cT) + I(x) - F(x+cT)$$

It means that Ψ is undetermined in the interval $[0, 2cT]$. This kind of problem is faced when one tries to solve Poincaré equation (8.25) in general fluid domains.

12.6.4 A Parabolic Equation: The Diffusion Equation

The diffusion equation has the following general form:

$$\frac{\partial C}{\partial t} = \kappa \Delta C$$

where C is the concentration of a chemical element, for instance, and κ is its diffusion coefficient in the fluid.

To make things as simple as possible, we reduce this problem to one space dimension. $C(x,t)$ is therefore determined by

$$\frac{\partial C}{\partial t} = \kappa \frac{\partial^2 C}{\partial x^2} \tag{12.24}$$

Since this equation is of first order in time we just need one initial condition, namely $C(x,0) = C_0(x)$.

The general method to solve this kind of equation is to use the Laplace transform:

$$\tilde{C}(x,p) = \int_0^\infty C(x,t)e^{-pt}dt$$

The diffusion equation (12.24) changes into:

$$\frac{\partial^2 \tilde{C}}{\partial x^2} - \frac{p}{\kappa}\tilde{C} = -\frac{C_0(x)}{\kappa}$$

We shall solve this equation in the very simple case where $C_0(x) = C_0 \cos kx$. The general solution is then

$$\tilde{C}(x,p) = \frac{C_0 \cos kx}{p + \kappa k^2} + Ae^{-\sqrt{\frac{p}{\kappa}}x} + Be^{\sqrt{\frac{p}{\kappa}}x}$$

If the solution is finite at $\pm\infty$ then $A = B = 0$. We thus obtain the solution $C(x,t)$ by the Mellin–Fourier formula:

$$C(x,t) = \int_{c-i\infty}^{c+i\infty} \frac{T_0 \cos kx}{p + \kappa k^2} e^{pt}dp$$

which leads to:

$$C(x,t) = C_0 \cos kx e^{-\kappa k^2 t}$$

This solution shows us that the initial state described by $C_0(x)$, decreases exponentially on a time scale $1/(\kappa k^2)$ or $\lambda^2/4\pi^2\kappa$ if λ is the wavelength associated with k. This is the same result as the one we found when studying the diffusion of vorticity with (4.5).

Let us now consider the case where the initial conditions do not contain any specific length scale (contrary to the foregoing example). In this case, one looks for a self-similar solution. If $C(x,t)$ is a solution of (12.24), we look for a condition such that $C_1(x,t) = C(Lx,Tt)$ is also a solution. Reporting C_1 into (12.24) and using the fact that C is a solution leads to:

$$\frac{\partial C_1}{\partial t} = \frac{T\kappa}{L^2}\frac{\partial^2 C_1}{\partial x^2}$$

Since L and T are arbitrary, we may choose $L = \sqrt{T}$. C_1 is also a solution for any T. Choosing $T = 1/t$, we find $C_1(x/\sqrt{t},1) \equiv c(x/\sqrt{t})$. The solution only

depends on a single variable $\eta = x/\sqrt{t}$ called the self-similarity variable.[1] $c(\eta)$ is a solution of the differential equation:

$$c''(\eta) = -\eta c'(\eta)/2\kappa$$

whose general solution is

$$c(\eta) = A\,\mathrm{erf}\left(\frac{\eta}{2\sqrt{\kappa}}\right) + B$$

where A et B are constants and erf is the error function.[2] Let us take the example where $c(x,t)$ is such that:

$$x > 0 \quad c(x,0) = 0 \qquad \text{and} \qquad c(0,t) = c_0$$

We find that

$$c(x,t) = c_0\left[1 - \mathrm{erf}\left(\frac{x}{2\sqrt{\kappa t}}\right)\right] \tag{12.25}$$

This example corresponds to the one-dimensional diffusion of a contaminant from a source with a unit concentration (see Fig. 12.3). This solution can of course be determined by the standard method using the Laplace transform. We may note that the iso-concentration lines, where $c(x,t)$ =Cst, move towards $+\infty$ *proportionally to the square root of time*. This law is typical of diffusion phenomena.

12.6.5 An Elliptic Equation: The Laplace Equation

12.6.5.1 Some General Properties

Laplace equation is the equation of potentials, which we find in electrostatics, magnetostatics, irrotational fluids, classic gravitation, etc. We met it several times either in perfect fluids (irrotational flows) or in very viscous ones (creeping flows). Thus, it is useful to know some of its basic properties.

[1] This kind of solution is used to solve the Prandtl equation describing boundary layer flows (see Sect. 4.3.6).

[2] erf is the error function defined by

$$\mathrm{erf}(x) = \frac{1}{\sqrt{\pi}}\int_0^x e^{-u^2}du$$

that is to say as the integral of a Gaussian (erf(∞)=1).

This equation reads:

$$\Delta f = 0 \tag{12.26}$$

It is *a linear equation* and its solutions are the *harmonic functions*. This equation contains no scale: if $f(\mathbf{r})$ is a solution then $f(\mathbf{r}/L)$ is also a solution. Scales only appear through the boundary conditions (the size of the domain) (Fig. 12.3).

If f and g are two harmonic functions, then

$$\nabla \cdot (f \nabla g) = \nabla f \cdot \nabla g$$

Integrating this equation over a volume V, bounded by a surface S characterized by the outward normal vector $\mathbf{n}$, we have

$$\int_{(S)} f \nabla g \cdot d\mathbf{S} = \int_{(V)} \nabla f \cdot \nabla g \, dV \tag{12.27}$$

Let us now assume that f or $\mathbf{n} \cdot \nabla f$ are vanishing on the bounding surface and let us use (12.27), namely

$$\int_{(S)} f \nabla f \cdot d\mathbf{S} = \int_{(V)} (\nabla f)^2 dV \tag{12.28}$$

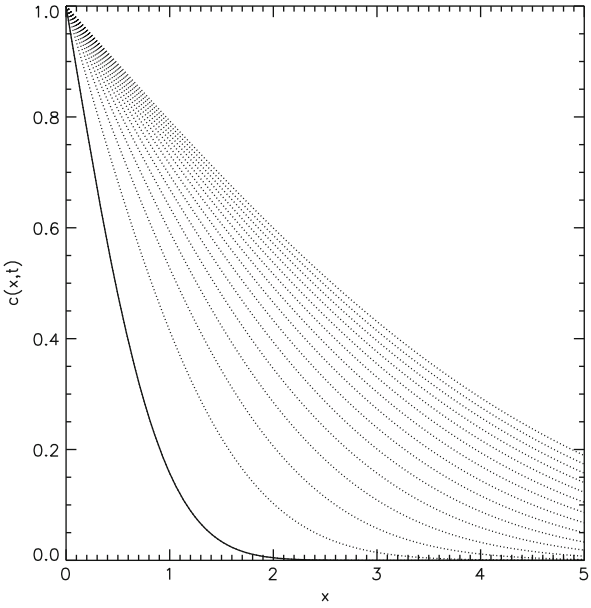

Fig. 12.3 Time evolution of the concentration according to (12.25). The various curves represent the solution at increasing time, the time difference being the same between one curve to the next. Note how they get closer with time because of the $\sqrt{t}$-law

From the boundary conditions met by f, the foregoing equation implies:

$$\boldsymbol{\nabla} f = 0$$

everywhere in V, so that f is a constant in the volume V.

This result leads to the unicity of the solutions of Laplace equation. Indeed, let f_1 and f_2 be two solutions of this equation verifying the same boundary conditions, then $g = f_1 - f_2$ is also a solution of Laplace equation and meets the boundary condition $g = 0$ or $\mathbf{n} \cdot \boldsymbol{\nabla} g = 0$. g is therefore a constant and if on a given point of the surface $g = 0$, this constant is zero. Thus, f_1 et f_2 are different at most by a constant. If f is a potential of a vector field, like the electric field $\mathbf{E} = \boldsymbol{\nabla} f$, we see that the value of the constant has no importance and that the field is unique.

Laplace equation can be solved in several geometries. Here, we shall detail only two cases: the spherical geometry and its two-dimensional counterpart, the plane polar geometry.

12.6.5.2 The Method for Solving Laplace Equation

One classical way to solve Laplace equation is to try to separate the variables: If x and y are the coordinates in a plane, one looks for solutions of the form:

$$V(x, y) = f(x)g(y)$$

We thus form a *Laplace product*. Using this type of solutions leads to

$$\Delta V = 0 \quad \Longrightarrow \quad \frac{1}{f}\frac{\partial^2 f}{\partial x^2} + \frac{1}{g}\frac{\partial^2 g}{\partial y^2} = 0$$

The two parts of the equation depends on two different variables. Their sum can be zero if and only if they are both constants. Let A be this constant, hence:

$$\frac{d^2 f}{dx^2} = -Af \qquad \text{and} \qquad \frac{d^2 g}{dy^2} = Ag$$

If $A > 0$ then

$$V(x, y) = B \cos(\sqrt{A}x + \phi) \operatorname{sh}(\sqrt{A}y + \chi)$$

The four constants A, B, ϕ, χ are arbitrary. Thus we have at our disposal an infinite number of solutions that can be linearly combined. The boundary conditions are then used to fix the values of the constants and determine the unique final solution.

To illustrate this method, we study two examples that are met in Fluids Dynamics.

12.6.5.3 Solutions in Polar Coordinates

In these coordinates the Laplacian reads:

$$\Delta V = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 V}{\partial \theta^2}$$

We thus write Laplace products as $V = F(r)G(\theta)$ and find:

$$\frac{r}{F}\frac{\partial}{\partial r}\left(r\frac{\partial F}{\partial r}\right) = C \qquad \text{and} \qquad \frac{1}{G}\frac{d^2 G}{d\theta^2} = -C$$

We may readily restrict the possible values of the constant C by demanding the periodicity of the solutions with respect to θ, namely $G(\theta) = G(\theta + 2\pi)$. We therefore set $C = n^2$, n being an integer. Hence,

$$G(\theta) = e^{in\theta} \qquad \text{and} \qquad F(r) = r^n$$

Thus, in the present case, we may write the solution of Laplace equation, in the following form:

$$V(r,\theta) = \sum_{n=-\infty}^{n=+\infty} A_n r^n e^{in\theta} \tag{12.29}$$

Here the constant A_n are determined by the boundary conditions.

We may note that $r^n e^{in\theta}$ is just z^n if $z = x + iy$. In other words, the foregoing solution is also the expansion in Laurent series of a function of a complex variable with an essential singularity at the origin. The solution (12.29) may also be expressed as

$$V(r,\theta) \equiv V(z) = \sum_{n=-\infty}^{n=+\infty} A_n z^n$$

In fact $V(z)$ is analytic, i.e. dV/dz exists, except at the origin and any analytic function is a solution of Laplace equation.

We demanded that the solution be periodic in θ but if $V(r,\theta)$ is a potential this condition can be relaxed and replaced by the periodicity of the field ∇V. The solution can include the solutions which follow when $C = 0$. In this case, $F(r) = P \ln r + Q$ and $G(\theta) = R\theta + S$ from which we derive the following linear combination $V = a\theta \ln r + b\theta + c \ln r + d$. If ∇V is periodic then we need to set $a = 0$. With complex number, we may rearrange this expression under the form: $A \ln z + B \ln z^*$ where z^* is the complex conjugate of z.

12.6.5.4 Solutions in Spherical Geometry

In spherical geometry the solution of Laplace equation are obtained in the same way but now the Laplace products are $F(r)G(\theta)H(\varphi)$. The developments are a bit more complicated, but lead to the construction of the *spherical harmonics* which replace the $e^{in\theta}$ in the previous solution. The spherical harmonics $Y_\ell^m(\theta,\varphi)$ are also the eigenfunctions of the horizontal (reduced to θ,ϕ) Laplacian operator. If we assume that the potential can be expanded on the basis of spherical harmonics (which indeed form a complete basis for all functions defined on the sphere), we can write:

$$V(r,\theta,\varphi) = \sum_{\ell=0}^{\ell=\infty} \sum_{m=-\ell}^{m=+\ell} V_m^\ell(r) Y_\ell^m(\theta,\varphi)$$

then, from (12.45) we get:

$$\Delta V = \frac{1}{r^2}\frac{\partial^2}{\partial r^2}(rV) + \frac{\Delta_{\theta\varphi} V}{r^2}$$

and

$$\frac{1}{r^2}\frac{d^2}{dr^2}(rV^\ell) - \frac{\ell(\ell+1)}{r^2}V^\ell = 0 \tag{12.30}$$

since

$$\Delta_{\theta\varphi} Y_\ell^m = -\ell(\ell+1)Y_\ell^m \tag{12.31}$$

The solutions of (12.30) are powers of r, namely r^n with $n = \ell$ or $n = -\ell - 1$. Finally

$$V(r,\theta,\varphi) = \sum_{\ell=0}^{\ell=\infty} \sum_{m=-\ell}^{m=+\ell} (A_{\ell m} r^\ell + B_{\ell m}/r^{\ell+1}) Y_\ell^m$$

These kind of solutions are used when the problem is close to sphericity.

12.7 Exercises

1. From (12.1) show the relations (12.2), (12.3) and (12.4).
2. Show the equivalence between the symmetry of a second order tensor and the relation (12.5).

3. Show that the harmonic oscillator equation

$$\frac{d^2y}{dx^2} + \lambda y = 0$$

with boundary conditions $y(0) = 0$ and $y(1) = 0$, is a Sturm–Liouville problem. Determine the eigenvalue spectrum.

Appendix: Formulae

Gaussian Integrals

We set

$$I_n(\alpha) = \int_0^\infty x^n e^{-\alpha x^2} dx \tag{12.32}$$

where $\alpha > 0$ et $n \neq 0$.

We thus find

$$I_0(\alpha) = \frac{\sqrt{\pi}}{2\alpha^{1/2}}, \qquad I_1(\alpha) = \frac{1}{2\alpha} \tag{12.33}$$

With the following equality

$$I_{n+2}(\alpha) = -\frac{dI_n}{d\alpha} \tag{12.34}$$

we get:

$$I_2(\alpha) = \frac{\sqrt{\pi}}{4\alpha^{3/2}}, \qquad I_3(\alpha) = \frac{1}{2\alpha^2}$$

$$I_4(\alpha) = \frac{3\sqrt{\pi}}{8\alpha^{5/2}}, \qquad I_5(\alpha) = \frac{1}{\alpha^3}$$

$$I_6(\alpha) = \frac{15\sqrt{\pi}}{16\alpha^{7/2}}, \qquad I_7(\alpha) = \frac{3}{\alpha^4}$$

or else

$$I_{2n>0} = \int_0^\infty u^{2n} e^{-\alpha u^2} du = \frac{1 \times 3 \times 5 \times \cdots \times (2n-1)}{2^{n+1}} \sqrt{\frac{\pi}{\alpha^{2n+1}}} \tag{12.35}$$

The symmetry of the gaussian function with respect to the origin implies that

$$\int_{-\infty}^{+\infty} x^{2p+1} e^{-\alpha x^2} dx = 0, \quad \forall p \in \mathbb{N} \tag{12.36}$$

Some Formula of Vectorial Analysis

The following formulae may be demonstrated using tensorial notations:

$$\Delta \mathbf{a} = \boldsymbol{\nabla}\boldsymbol{\nabla} \cdot \mathbf{a} - \boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \mathbf{a} \tag{12.37}$$

$$\boldsymbol{\nabla} \cdot (f\,\mathbf{a}) = \mathbf{a} \cdot \boldsymbol{\nabla} f + f \boldsymbol{\nabla} \cdot \mathbf{a} \tag{12.38}$$

$$\boldsymbol{\nabla} \times (f\,\mathbf{a}) = \boldsymbol{\nabla} f \times \mathbf{a} + f \boldsymbol{\nabla} \times \mathbf{a} \tag{12.39}$$

$$\boldsymbol{\nabla} \cdot (\mathbf{a} \times \mathbf{b}) = \mathbf{b} \cdot \boldsymbol{\nabla} \times \mathbf{a} - \mathbf{a} \cdot \boldsymbol{\nabla} \times \mathbf{b} \tag{12.40}$$

$$\boldsymbol{\nabla} \times (\mathbf{a} \times \mathbf{b}) = \mathbf{a}\boldsymbol{\nabla} \cdot \mathbf{b} - \mathbf{b}\boldsymbol{\nabla} \cdot \mathbf{a} + (\mathbf{b} \cdot \boldsymbol{\nabla})\mathbf{a} - (\mathbf{a} \cdot \boldsymbol{\nabla})\mathbf{b} \tag{12.41}$$

$$\boldsymbol{\nabla}(\mathbf{a} \cdot \mathbf{b}) = \mathbf{a} \times \boldsymbol{\nabla} \times \mathbf{b} + \mathbf{b} \times \boldsymbol{\nabla} \times \mathbf{a} + (\mathbf{a} \cdot \boldsymbol{\nabla})\mathbf{b} + (\mathbf{b} \cdot \boldsymbol{\nabla})\mathbf{a} \tag{12.42}$$

$$\boldsymbol{\nabla}(\mathbf{v} \cdot \mathbf{v}) = 2\mathbf{v} \times \boldsymbol{\nabla} \times \mathbf{v} + 2(\mathbf{v} \cdot \boldsymbol{\nabla})\mathbf{v} \tag{12.43}$$

The Operators in Various Coordinate Systems

Cylindrical Coordinates (s, φ, z)

$$\boldsymbol{\nabla} f = \begin{vmatrix} \dfrac{\partial f}{\partial s} \\ \dfrac{1}{s}\dfrac{\partial f}{\partial \varphi} \\ \dfrac{\partial f}{\partial z} \end{vmatrix} \qquad \boldsymbol{\nabla} \times \mathbf{v} = \begin{vmatrix} \dfrac{1}{s}\dfrac{\partial v_z}{\partial \varphi} - \dfrac{\partial v_\varphi}{\partial z} \\ \dfrac{\partial v_s}{\partial z} - \dfrac{\partial v_z}{\partial s} \\ \dfrac{1}{s}\dfrac{\partial s v_\varphi}{\partial s} - \dfrac{1}{s}\dfrac{\partial v_s}{\partial \varphi} \end{vmatrix} \qquad \Delta \mathbf{v} = \begin{vmatrix} \Delta v_s - \dfrac{v_s}{s^2} - \dfrac{2}{s^2}\dfrac{\partial v_\varphi}{\partial \varphi} \\ \Delta v_\varphi - \dfrac{v_\varphi}{s^2} + \dfrac{2}{s^2}\dfrac{\partial v_s}{\partial \varphi} \\ \Delta v_z \end{vmatrix}$$

$$\Delta f = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial f}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 f}{\partial \varphi^2} + \frac{\partial^2 f}{\partial z^2}, \qquad \boldsymbol{\nabla} \cdot \mathbf{v} = \frac{1}{s}\frac{\partial s v_s}{\partial s} + \frac{1}{s}\frac{\partial v_\varphi}{\partial \varphi} + \frac{\partial v_z}{\partial z}$$

$$(\mathbf{a}\cdot\nabla)\mathbf{b} = \begin{vmatrix} (\mathbf{a}\cdot\nabla)b_s - \dfrac{a_\varphi b_\varphi}{s} \\ (\mathbf{a}\cdot\nabla)b_\varphi + \dfrac{a_\varphi b_s}{s} \\ (\mathbf{a}\cdot\nabla)b_z \end{vmatrix} \qquad (\mathbf{v}\cdot\nabla)\mathbf{v} = \begin{vmatrix} (\mathbf{v}\cdot\nabla)v_s - \dfrac{v_\varphi^2}{s} \\ (\mathbf{v}\cdot\nabla)v_\varphi + \dfrac{v_s v_\varphi}{s} \\ (\mathbf{v}\cdot\nabla)v_z \end{vmatrix}$$

$$\text{Remark} \qquad \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial f}{\partial s}\right) - \frac{f}{s^2} \;=\; \frac{\partial}{\partial s}\left(\frac{1}{s}\frac{\partial s f}{\partial s}\right) \tag{12.44}$$

Spherical Coordinates (r, θ, φ)

$$\nabla f = \begin{vmatrix} \dfrac{\partial f}{\partial r} \\ \dfrac{1}{r}\dfrac{\partial f}{\partial \theta} \\ \dfrac{1}{r\sin\theta}\dfrac{\partial f}{\partial \varphi} \end{vmatrix} \qquad \nabla\times\mathbf{v} = \begin{vmatrix} \dfrac{1}{r\sin\theta}\dfrac{\partial}{\partial\theta}(\sin\theta v_\varphi) - \dfrac{1}{r\sin\theta}\dfrac{\partial v_\theta}{\partial\varphi} \\ \dfrac{1}{r\sin\theta}\dfrac{\partial v_r}{\partial\varphi} - \dfrac{1}{r}\dfrac{\partial r v_\varphi}{\partial r} \\ \dfrac{1}{r}\dfrac{\partial r v_\theta}{\partial r} - \dfrac{1}{r}\dfrac{\partial v_r}{\partial\theta} \end{vmatrix}$$

$$\Delta\mathbf{v} \begin{vmatrix} \Delta v_r - \dfrac{2}{r^2}\left(\dfrac{1}{\sin\theta}\dfrac{\partial(\sin\theta v_\theta)}{\partial\theta} + \dfrac{1}{\sin\theta}\dfrac{\partial v_\varphi}{\partial\varphi} + v_r\right) \\ \Delta v_\theta + \dfrac{1}{r^2}\left(2\dfrac{\partial v_r}{\partial\theta} - \dfrac{v_\theta}{\sin^2\theta} - \dfrac{2\cos\theta}{\sin^2\theta}\dfrac{\partial v_\varphi}{\partial\varphi}\right) \\ \Delta v_\varphi + \dfrac{1}{r^2}\left(\dfrac{2}{\sin\theta}\dfrac{\partial v_r}{\partial\varphi} + \dfrac{2\cos\theta}{\sin^2\theta}\dfrac{\partial v_\theta}{\partial\varphi} - \dfrac{v_\varphi}{\sin^2\theta}\right) \end{vmatrix}$$

where the Laplacian of a scalar field reads

$$\Delta f = \frac{1}{r}\frac{\partial^2 r f}{\partial r^2} + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial\varphi^2} \tag{12.45}$$

and

$$\nabla\cdot\mathbf{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial\sin\theta v_\theta}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial v_\varphi}{\partial\varphi}$$

$$(\mathbf{a}\cdot\nabla)\mathbf{b} = \begin{vmatrix} (\mathbf{a}\cdot\nabla)b_r - \dfrac{a_\theta b_\theta + a_\varphi b_\varphi}{r} \\ (\mathbf{a}\cdot\nabla)b_\theta + \dfrac{a_\theta b_r - a_\varphi b_\varphi \cot\theta}{r} \\ (\mathbf{a}\cdot\nabla)b_\varphi + \dfrac{a_\varphi b_r + a_\varphi b_\theta \cot\theta}{r} \end{vmatrix}$$

The Stress-Tensor of a Newtonian Fluid

Cylindrical Coordinates

$$\begin{aligned}
\sigma_{ss} &= -P + 2\mu\frac{\partial v_s}{\partial s} + (\zeta - 2/3\mu)\nabla\cdot\mathbf{v} \\
\sigma_{\varphi\varphi} &= -P + 2\mu\left(\frac{1}{s}\frac{\partial v_\varphi}{\partial\varphi} + \frac{v_s}{s}\right) + (\zeta - 2/3\mu)\nabla\cdot\mathbf{v} \\
\sigma_{zz} &= -P + 2\mu\frac{\partial v_z}{\partial z} + (\zeta - 2/3\mu)\nabla\cdot\mathbf{v} \\
\sigma_{s\varphi} &= \mu\left(\frac{1}{s}\frac{\partial v_s}{\partial\varphi} + \frac{\partial v_\varphi}{\partial s} - \frac{v_\varphi}{s}\right) \\
\sigma_{sz} &= \mu\left(\frac{\partial v_z}{\partial s} + \frac{\partial v_s}{\partial z}\right) \\
\sigma_{\varphi z} &= \mu\left(\frac{\partial v_\varphi}{\partial z} + \frac{1}{s}\frac{\partial v_z}{\partial\varphi}\right)
\end{aligned}$$

Spherical Coordinates (r, θ, φ)

$$\begin{aligned}
\sigma_{rr} &= -P + 2\mu\frac{\partial v_r}{\partial r} + (\zeta - 2/3\mu)\nabla\cdot\mathbf{v} \\
\sigma_{\theta\theta} &= -P + 2\mu\left(\frac{1}{r}\frac{\partial v_\theta}{\partial\theta} + \frac{v_r}{r}\right) + (\zeta - 2/3\mu)\nabla\cdot\mathbf{v} \\
\sigma_{\varphi\varphi} &= -P + 2\mu\left(\frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial\varphi} + \frac{v_r}{r} + \frac{v_\theta\cot\theta}{r}\right) + (\zeta - 2/3\mu)\nabla\cdot\mathbf{v}
\end{aligned}$$

$$\sigma_{r\theta} = \mu\left(\frac{1}{r}\frac{\partial v_r}{\partial \theta} + r\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right)\right)$$

$$\sigma_{r\varphi} = \mu\left(\frac{1}{r\sin\theta}\frac{\partial v_r}{\partial \varphi} + r\frac{\partial}{\partial r}\left(\frac{v_\varphi}{r}\right)\right)$$

$$\sigma_{\theta\varphi} = \mu\left(\frac{1}{r\sin\theta}\frac{\partial v_\theta}{\partial \varphi} + \sin\theta\frac{\partial}{\partial \theta}\left(\frac{v_\varphi}{r\sin\theta}\right)\right)$$

Further Reading

An introduction to tensors may be found in Lebedev et al. (2010). The theory of singular perturbations, which is very useful to the study of boundary layers, may be found in various books, for instance in Bender and Orszag (1978) or O'Malley (1991). As far as partial differential equations are concerned, there are also a wealth of textbooks, from the classical Courant and Hilbert (1953) or, more recently, Dautray and Lions (1984–1985) or Zwillinger (1992). This last book also deals with all types of ordinary differential equations.

References

Bass, J. (1978). *Cours de Mathématiques*. Paris: Masson.
Bender, C. & Orszag, S. (1978). *Advanced Mathematical Methods for scientists and engineers*. New York: McGraw-Hill.
Courant, R. & Hilbert, D. (1953). *Methods of mathematical physics*. London: Interscience.
Dautray, R. & Lions, J.-L. 1984–1985 *Analyse mathématique et calcul numérique*. Paris: Masson.
Hladik, J. (1993). *Le calcul tensoriel en physique*. Paris: Masson.
Ince, E. L. (1956). *Ordinary differential equations*. New York: Dover.
Lebedev, L., Cloud, M. & Eremeyev, V. (2010). *Tensor analysis with applications in Mechanics*. Singapore: World Scientific.
O'Malley, R. (1991). *Singular Perturbation Methods for Ordinary Differential Equations*. Berlin: Springer.
Zwillinger, D. (1992). *Handbook of differential equations*. New York: Academic Press.