Chapter 4
Flows of Incompressible Viscous Fluids

As it was shown in Sect. 3.2.5, the density variations in a fluid flow decrease with the square of the Mach number (the ratio of the fluid velocity to the sound speed). Hence, for many fluid flows, and especially for those of liquids, incompressibility is an excellent approximation. Moreover, it simplifies very much the equations of motion. This simplification provides us with the easiest context to study the effects of viscosity that we have neglected until now.

Thus, in this chapter we study the effects of viscosity using solely incompressible fluids. We first discuss the laws of similarity, which appear thanks to viscosity, then we deal with two limits: that of flows with a strong viscous force and that of flows with a slight viscous effect. Next, we review some classical solutions of Navier equation, and we end the chapter with a short study of forces exerted on solids by viscous fluid flows.

4.1 Some General Properties

4.1.1 The Equations of Motion

We have seen in the first chapter that the flow of an incompressible fluid with constant viscosity is governed by only two equations: the equation of continuity and the Navier–Stokes equation, namely

$$\begin{cases} \nabla \cdot \mathbf{v} = 0 \\ \rho \dfrac{D\mathbf{v}}{Dt} = -\nabla P + \mu \Delta \mathbf{v} \end{cases} \tag{4.1}$$

M. Rieutord, *Fluid Dynamics*, Graduate Texts in Physics,
DOI 10.1007/978-3-319-09351-2_4

Indeed, the third equation, the energy equation, uncouples completely from the two others. An important consequence of this uncoupling is that the pressure does not have a dynamic role. It doesn't drive the flow but is driven by the flow. This property follows from the fact that the velocity field is entirely determined by

$$\begin{cases} \nabla \cdot \mathbf{v} = 0 \\ \nabla \times \left(\dfrac{D\mathbf{v}}{Dt} - \nu \Delta \mathbf{v} \right) = \mathbf{0} \end{cases} \tag{4.2}$$

and the boundary conditions

$$\mathbf{v} = \mathbf{v}_s \quad \text{on} \quad (S)$$

where S is the boundary (supposedly solid with a velocity $\mathbf{v}_s$) which delimits the fluid. The pressure is thus entirely determined, up to a constant, by

$$\nabla P = \mu \Delta \mathbf{v} - \rho \frac{D\mathbf{v}}{Dt}$$

Another important property of this system of equations is that, if the viscosity of the fluid is large enough, the solution is unique. "Large enough" means larger than some critical value below which the system has several solutions. Physically, this shows up with the raise of an instability in the original solution. We shall discuss this point later on, but presently we just note that this phenomenon is a consequence of the nonlinear character of the equations.

4.1.2 Law of Similarity

A fluid flow always involves the *dynamic time scale*, which is the typical time that it takes for a fluid particle to cover the distance L, namely

$$T_d = \frac{L}{V} \tag{4.3}$$

where V is the typical velocity of the fluid. If the fluid is viscous then another time scale comes about; this is

$$T_\nu = \frac{L^2}{\nu} \tag{4.4}$$

also called *the viscous diffusion time*. The origin of this definition is the following: if we consider a very slow flow, the quadratic term $\mathbf{v} \cdot \nabla \mathbf{v}$ in the Navier–Stokes equation is very small compared to other terms. Neglecting this term and taking the curl of the momentum equation, we get the linearized equation of the vorticity $\boldsymbol{\omega} = \nabla \times \mathbf{v}$, namely

$$\frac{\partial \boldsymbol{\omega}}{\partial t} = \nu \Delta \boldsymbol{\omega} \tag{4.5}$$

This is the diffusion equation (see Sect. 12.6.4 for a presentation of its basic properties). Schematically, if $\boldsymbol{\omega}$ varies on a length scale L, then $\Delta\boldsymbol{\omega} \sim \boldsymbol{\omega}/L^2$ and $\partial\boldsymbol{\omega}/\partial t \sim \nu\boldsymbol{\omega}/L^2$ which shows that $\boldsymbol{\omega}$ evolves on a time scale of order L^2/ν.

The dynamic and viscous time scales are compared through the non-dimensional ratio

$$Re = \frac{T_\nu}{T_d} = \frac{VL}{\nu} \tag{4.6}$$

also called the *Reynolds number*. It characterizes the ratio between two transport velocities: the macroscopic (dynamic) transport and the microscopic (diffusive) transport. This non-dimensional number is the only parameter intervening in the equations of motion of an incompressible viscous fluid. Indeed, if we make the following change in the variables:

$$\mathbf{v} = V\mathbf{u}\,, \qquad P = \rho V^2 p\,, \qquad \mathbf{r} = L\mathbf{x} \qquad \text{and} \qquad t = \frac{L}{V}\tau$$

then, $\mathbf{u}$, p, $\mathbf{x}$, τ represent respectively the non-dimensional velocity, pressure, spatial coordinates and time. The equations of motion read

$$\begin{cases} \nabla \cdot \mathbf{u} = 0 \\ \dfrac{\partial \mathbf{u}}{\partial \tau} + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla p + \dfrac{1}{\mathrm{Re}} \Delta \mathbf{u} \end{cases} \tag{4.7}$$

Save for the parameters that may be added in the boundary conditions, the solution $\mathbf{u}$ depends on just one quantity, which is the Reynolds number. All the flows having the same Reynolds number are identical up to a constant scale factor: they are said to be *similar*. This conclusion is true only if the solution is unique, that is to say, if the viscosity is large enough or if the Reynolds number is small enough.

Let us consider a simple example of the use of the similarity between flows. A solid represented by a cube of 1 cm side moves in air at a speed of 1 cm/s. The air flow is exactly the same as one around a cube of 1 m side moving at 0.1 mm/s. A practical application of the similarity relation is the use of reduced models to study some complex flows.

4.1.3 Discussion

System (4.7) gives us the first example of the flow equations written with non-dimensional variables. The use of non-dimensional variables is the rule in Fluid Mechanics. Thus doing, we are able to compare the various scales that intervene in a fluid flow. The foregoing example is very simple, but as we progress, we shall see that many non-dimensional numbers come into play. These numbers are crucial to compare the flows to each others and eventually evaluate the difficulties to compute them.

Finally, let us observe that perfect fluids correspond to the limit of infinite Reynolds numbers. However, this limit is *singular* because, as viscosity vanishes, second order derivatives disappear from the equations thus making some boundary conditions unmatched. This singularity is at the origin of the boundary layers, which appear when the Reynolds number is very large (see Sect. 4.3).

We shall further explore the dynamics of viscous fluids with the help of two limiting cases: the one of very viscous fluids and the one of nearly inviscid fluids. In other words, we shall study the two limits: the very small and the very large Reynolds numbers. We begin with the first case, which is the easiest one.

4.2 Creeping Flows

Creeping flows are all the flows for which the inertia of the fluid is negligible. Their Reynolds number is therefore very small compared to unity.

Examples of such flows come from the very viscous fluids (magma, for instance) or from the flows with very small scales (lubrication, microfluidic,...).

4.2.1 Stokes' Equation

We consider the momentum equation in (4.7) and multiply it by the Reynolds number while carrying out the substitution $p \to p/\mathrm{Re}$. We get

$$\mathrm{Re}\frac{\partial \mathbf{u}}{\partial \tau} + \mathrm{Re}\ \mathbf{u}\cdot\nabla\mathbf{u} = -\nabla p + \Delta\mathbf{u}$$

Setting $\mathrm{Re} = 0$, we get *Stokes' Equation* :

$$-\nabla p + \Delta\mathbf{u} = \mathbf{0} \tag{4.8}$$

We may observe that, by taking the limit Re=0, we eliminated the time derivative of the velocity. Does this mean that all flows with very small Reynolds number are stationary? Not quite, of course! It means that if the appropriate time scale is L/V

then the temporal variations are truly negligible and the flow is steady. But we can easily envision a flow where there is a time dependent forcing. In this case, the time scale of this forcing is a new independent parameter which controls the amplitude of the term $\partial \mathbf{v}/\partial t$.

Stokes equation can take two other equivalent forms for an incompressible fluid:

$$\nabla p + \nabla \times \boldsymbol{\omega} = \mathbf{0} \qquad \text{or} \qquad \Delta \boldsymbol{\omega} = \mathbf{0} \tag{4.9}$$

where $\boldsymbol{\omega} = \nabla \times \mathbf{v}$ is the vorticity. We also note, by taking the divergence of (4.8), that the pressure verifies Laplace's equation:

$$\Delta p = 0$$

The essential property of these equations is their linear character. The solutions thus own all the properties associated with linearity. For instance, an interesting property is *the reversibility* : if $\mathbf{v}$ is a solution then $-\mathbf{v}$ is also a solution. Any fluid particle goes back to its initial position if the forcing is reversed (the nonlinear terms break this symmetry).

Another important consequence of the linearity of Stokes' equation is the uniqueness of the solution for a given set of boundary conditions. Below, we demonstrate this property by showing that the solutions obey a variational principle. The unicity of the solutions also resolves the problem of stability: the solutions are always stable.

Finally, note a third possible form of Stokes' equation:

$$\mathbf{Div}[\sigma] = \mathbf{0} \qquad \text{or} \qquad \partial_j \sigma_{ij} = 0 \tag{4.10}$$

where $[\sigma]$ is the stress tensor. This form is more general than the previous ones since it does not make use of the explicit form of the stress tensor.

4.2.2 Variational Principle ♠

Equation (4.10) can be obtained with the help of a variational principle, such as the least action principle. This means that the solutions of (4.10) render extremum a functional of the velocity field defined on the space occupied by the fluid. This functional is just the viscous dissipation.

In order to show this result, we shall consider a Newtonian fluid inside a given volume, limited by a surface (S), where the velocity is given as on a solid wall. On this surface the variations of the velocity $\delta\mathbf{v}$ vanish. The dissipation in this volume is given by:

$$D = \int_{(V)} \left(\frac{\mu}{2} c_{ij} c_{ij} + \zeta (\nabla \cdot \mathbf{v})^2 \right) dV$$

where we do not assume incompressibility. A variation of D associated with the variations of $\mathbf{v}$ is easily obtained by a functional derivation of the integral with respect to the velocity field:

$$\delta D = \int_{(V)} \left\{ \mu c_{ij} \delta c_{ij} + 2\zeta (\nabla \cdot \mathbf{v})(\nabla \cdot \delta \mathbf{v}) \right\} dV \tag{4.11}$$

According to the rheological law of Newtonian fluids $\mu c_{ij} = \sigma_{ij}^v - \zeta \nabla \cdot \mathbf{v}\, \delta_{ij}$ where $[\sigma^v]$ is the viscous stress tensor. Moreover, $\delta c_{ii} = 0$; hence, the foregoing expression may be rewritten as

$$\delta D = \int_{(V)} \left\{ \sigma_{ij} \delta c_{ij} + 2\zeta (\nabla \cdot \mathbf{v}) \nabla \cdot \delta \mathbf{v} \right\} dV$$

The expression (1.40) of c_{ij} and the symmetry of the viscous stress tensor allows us to simplify the preceding expression. Thus

$$\delta D = 2 \int_{(V)} \sigma_{ij} \partial_i \delta v_j \, dV$$

Using the equation of motion (4.10) together with the divergence theorem, we finally obtain

$$\delta D = 2 \int_{(S)} \sigma_{ij} \delta v_j \, dS_i = 0 \tag{4.12}$$

This last integral is zero because of the boundary conditions imposed on $\delta \mathbf{v}$. The dissipation is therefore at an extremum for the velocity field verifying Stokes' equation. We now show that this extremum is a minimum. For this, we observe that the dissipation is a linear function of the squared gradient of velocity. Symbolically, we can write that

$$D(v) = \mathcal{L}\left[(\partial v)^2\right]$$

where $\mathcal{L}$ is a linear operator. We have shown that

$$\delta D = 2\mathcal{L}\left((\partial v)(\partial \delta v)\right) = 0$$

Now we make the difference between the dissipation associated with the field v solution of the equations and that associated with the field $v + \delta v$ where δv is a variation. We have:

$$\begin{aligned} D(v + \delta v) - D(v) &= \mathcal{L}\left([\partial(v + \delta v)]^2\right) - \mathcal{L}\left((\partial v)^2\right) \\ &= \mathcal{L}\left((\partial v)^2 + (\partial \delta v)^2 + 2(\partial v)(\partial \delta v)\right) - \mathcal{L}\left((\partial v)^2\right) \\ &= \mathcal{L}\left((\partial \delta v)^2\right) = D(\delta v) \geq 0 \end{aligned}$$

This result shows that whatever the variation of v made around the true solution, the dissipation increases. This quantity is therefore a minimum for the true solution.

Let us now show that this implies the uniqueness of the solution. For that, it is convenient to take two solutions and show that they are in fact identical. Let v_1 and v_2 be two such solutions; their dissipation being at the minimum is therefore identical. From the foregoing results, we necessarily have

$$\mathcal{L}\left([\partial(v_1 - v_2)]^2\right) = 0$$

The operator $\mathcal{L}$ being only an integration, we infer from the preceding equation that the integrant is zero everywhere within the fluid $\partial(v_1 - v_2) = 0$. In fact $\partial(v_1 - v_2)$ symbolizes all the components of the shear tensor and the divergence; we therefore have:

$$c_{ij}(\mathbf{v}_1 - \mathbf{v}_2) = 0 \qquad \text{and} \qquad \nabla \cdot (\mathbf{v}_1 - \mathbf{v}_2) = 0$$

From these two equations we derive a third one, namely

$$s_{ij}(\mathbf{v}_1 - \mathbf{v}_2) = 0$$

which means that the symmetric part of the velocity gradient tensor is zero. We have seen in Chap. 1 that this implies that $\mathbf{v}_1 - \mathbf{v}_2$ is the combination of a solid body rotation and a translation. But $\mathbf{v}_1$ and $\mathbf{v}_2$ satisfy the same boundary conditions, thus, in general, the rotation and the translation are both zero. Thus, $\mathbf{v}_1$ and $\mathbf{v}_2$ are identical.

The preceding results are not valid when the Reynolds number is large. In this case the solution is not unique and does not produce a minimum of dissipation.

4.2.3 Flow Around a Sphere

As a first example we shall consider the flow of a viscous fluid around a sphere moving slowly. We assume that the fluid fills the whole space and that the flow is steady. The Reynolds number, based on the velocity of the sphere and its diameter, is very small compared to unity so that we can use Stokes' equation. Returning to the dimensional variables, we have

$$\begin{cases} \nabla \cdot \mathbf{v} = 0 \\ \mathbf{0} = -\nabla P + \mu \Delta \mathbf{v} \end{cases} \tag{4.13}$$

Using a reference frame whose origin is at the centre of the sphere, the boundary conditions are

$$\mathbf{v} = \mathbf{0} \quad \text{at} \quad r = R \quad \text{and} \quad \mathbf{v} \to U\mathbf{e}_z \quad \text{when} \quad r \to \infty$$

Note that $-Ue_z$ is the velocity of the sphere in a rest frame. We use the spherical coordinates. Since the flow is axisymmetric around the z-axis, the velocity and the pressure only depend on r and θ. Moreover, $\mathbf{v}$ has no component along $\mathbf{e}_\varphi$.

In order to solve system (4.13) we expand the functions on the basis of Legendre's polynomials in the following manner:

$$\begin{cases} v_r &= \sum_\ell u_\ell(r) P_\ell(\cos\theta) \\ v_\theta &= \sum_\ell v_\ell(r) \frac{dP_\ell}{d\theta} \\ P &= \sum_\ell p_\ell(r) P_\ell(\cos\theta) \end{cases} \tag{4.14}$$

Legendre's polynomials satisfy the differential equation

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dP_\ell}{d\theta}\right) + l(l+1)P_\ell = 0$$

Using the equation of continuity, we derive the relation between $u_\ell(r)$ and $v_\ell(r)$

$$\ell(\ell+1)v_\ell(r) = \frac{1}{r}\frac{dr^2 u_\ell}{dr}$$

We also find that

$$\nabla \times \mathbf{v} = -\sum_\ell \frac{\Delta_\ell(ru_\ell)}{\ell(\ell+1)} P_\ell(\cos\theta)\mathbf{e}_\varphi,$$

and

$$\Delta\mathbf{v} = -\nabla\times(\nabla\times\mathbf{v}) = \frac{\Delta_\ell(ru_\ell)}{r}P_\ell(\cos\theta)\mathbf{e}_r + \frac{1}{r}\frac{d}{dr}\left(\frac{r\Delta_\ell(ru_\ell)}{\ell(\ell+1)}\right)\frac{dP_\ell}{d\theta}\mathbf{e}_\theta$$

Δ_ℓ being the operator

$$\Delta_\ell = \frac{1}{r}\frac{d^2}{dr^2}r - \frac{\ell(\ell+1)}{r^2}$$

As Legendre's polynomials form an orthogonal basis as well as their derivative, we easily find that

$$\begin{cases} \dfrac{dp_\ell}{dr} = \mu\Delta_\ell(ru_\ell)/r \\[2ex] p_\ell(r) = \mu\dfrac{d}{dr}\left(\dfrac{r\Delta_\ell(ru_\ell)}{\ell(\ell+1)}\right) \end{cases} \tag{4.15}$$

which yields

$$\Delta_\ell \Delta_\ell (r u_\ell) = 0 \tag{4.16}$$

The general solution of this equation is in the form of the powers of r; namely

$$u_\ell(r) = A r^{\ell-1} + B r^{\ell+1} + C r^{-\ell} + D r^{-\ell-2}$$

In order to find these four constants, we need four boundary conditions; $v_r(R) = v_\theta(R) = 0$ imply

$$u_\ell(R) = v_\ell(R) = 0, \qquad \forall \ell$$

while at infinity we have

$$\mathbf{v} \to U \mathbf{e}_z = U(\cos\theta \mathbf{e}_r - \sin\theta \mathbf{e}_\theta), \qquad \text{as} \quad r \to \infty$$

One may verify that this last condition leads to

$$\lim_{r\to\infty} u_1(r) = U \qquad \text{and} \qquad \lim_{r\to\infty} u_{\ell\neq 1}(r) = 0$$

The boundary conditions implies that all the coefficients, except those of u_1, are zero. More explicitly, we have

$$u_1(r) = A + B r^2 + \frac{C}{r} + \frac{D}{r^3} \tag{4.17}$$

Using the conditions at infinity we find $A = U$ and $B = 0$, while with those on the sphere it turns out that

$$C = -3UR/2, \qquad D = UR^3/2$$

Since $P_1(\cos\theta) = \cos\theta$, we finally obtain

$$\begin{cases} v_r = U \cos\theta \left(1 - \frac{3R}{2r} + \frac{R^3}{2r^3}\right) \\ v_\theta = -U \sin\theta \left(1 - \frac{3R}{4r} - \frac{R^3}{4r^3}\right) \\ p = -\frac{3\mu U R}{2r^2} \cos\theta \end{cases} \tag{4.18}$$

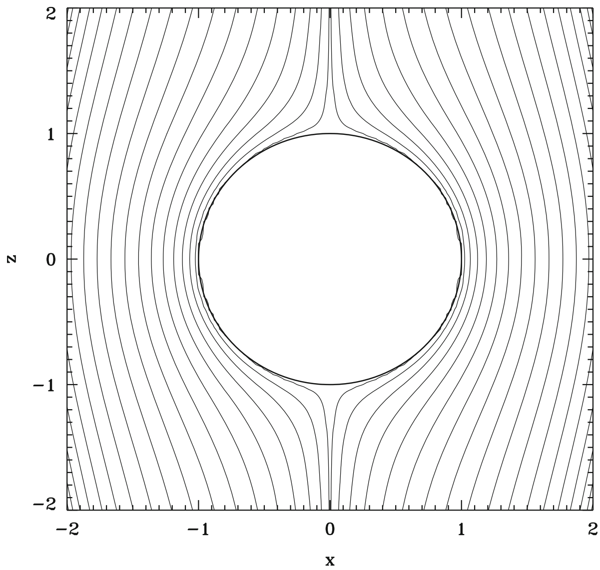

Fig. 4.1 Streamlines in the meridional plane of the Stokes' flow around a sphere in uniform motion in a viscous fluid

The velocity field can be expressed with a stream function ψ describing the streamlines in the meridional plane as shown in Fig. 4.1. The expression of the stream function is

$$\psi = \frac{1}{2}Ur^2\sin^2\theta\left(1 - \frac{3R}{2r} + \frac{R^3}{2r^3}\right)$$

since

$$v_r = \frac{1}{r^2\sin\theta}\frac{\partial\psi}{\partial\theta} \qquad \text{and} \qquad v_\theta = -\frac{1}{r\sin\theta}\frac{\partial\psi}{\partial r}$$

From the expression of the velocity, we also infer the expression of the vorticity field:

$$\nabla\times\mathbf{v} = -\frac{3UR}{2r^2}\cos\theta\,\mathbf{e}_\varphi, \tag{4.19}$$

which will be used later on to compute the drag force exerted on the sphere.

4.2.4 Oseen's Equation

Stokes' equation has been derived for vanishing Reynolds numbers. However, in any experiment, even if the Reynolds number is very small, it is finite. This makes Stokes' equation invalid far from the solid. This strange property comes from the nature of the solutions of Stokes equation, which is a kind of Laplace equation. The solutions of Stokes's equation that vanish at infinity are power laws of the distance (like (4.17) for instance). For these solutions, the length scale characterizing the velocity variations grows with r. Indeed, a typical length scale of the velocity field V is $L = (d \ln V/dr)^{-1}$, thus if $V \sim 1/r^n$ then $L \sim r/n$. The consequence of this growing scale is that nonlinear terms decrease more slowly than viscous ones as they contain only first order derivatives. The distance by which nonlinear terms overtake the linear one can be guessed from an order of magnitude estimate

$$(\mathbf{u} \cdot \nabla)\mathbf{u} \sim \frac{1}{\mathrm{Re}} \Delta \mathbf{u} \Longrightarrow \frac{1}{L} \sim \frac{1}{L^2 \mathrm{Re}} \Longrightarrow L \sim \mathrm{Re}^{-1}$$

where the Reynolds number is computed from the dimensions of the object. The foregoing result shows that this critical length goes to infinity as the Reynolds number vanishes.

Hence, the computation of flows extending to distance larger than Re^{-1} must take into account the corrections imposed by nonlinear terms. For instance, if we wish to compute the flow around a solid body moving at constant speed in a very viscous fluid, we may set $\mathbf{u} = \mathbf{U}_\infty + \delta\mathbf{u}$ (U_∞ is the fluid velocity at infinity in a frame attached to the solid) and first solve Stokes equation. However, at distances larger than Re^{-1}, corrections from nonlinear terms are important. These are taken into account by keeping the leading order of these terms. Hence, in these regions, one has to solve

$$(\mathbf{U}_\infty \cdot \nabla)\delta\mathbf{u} = -\nabla p + \frac{1}{\mathrm{Re}} \Delta \delta\mathbf{u} \tag{4.20}$$

which is *Oseen's equation*. Although this equation seems more complete than Stokes one, it is not valid close to the solid as the flow is not close to a uniform velocity field. Thus, Oseen's equation is useful to complement Stokes' equation when the fluid's domain is larger than Re^{-1}; in this case the solution of both equations must be matched together, which may be delicate. Of course, if the domain is not that large, Stokes' equation is sufficient.

4.2.5 The Lubrication Layer

We end this section with the study of another type of flows at very small Reynolds number, namely the case of the lubrication layer, which was analysed for the first

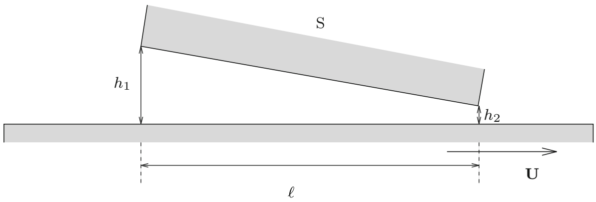

Fig. 4.2 Schematic view of a lubrication layer. In the reference frame of the solid S the ground moves to the right at speed U entraining the fluid to flow in the same direction

time by Reynolds in 1886. The flow in a lubrication layer has numerous applications, especially in *tribology* (the study of friction). The lubricating effect of a thin fluid layer between two solids is shown by an experiment of everyday life: that of a sheet of paper, which glides practically without friction on a smooth floor. A thin layer of air forms between the floor and the paper, making an air cushion, which reduces drastically the friction. We may also observe that in the same conditions of incidence and velocity, but far above the floor, the sheet of paper has not a sufficient lift to compensate its weight, and falls.

In order to understand the fundamentals of lubrication, we shall consider the simple system illustrated in Fig. 4.2. A solid with a length ℓ glides with a velocity U above a fixed solid plane. An incompressible viscous fluid flows between the two solids, forming the lubrication layer. The Reynolds number of this flow, based on the thickness of the fluid film, is supposedly very small:

$$\frac{Uh}{\nu} \ll 1$$

We assume the contact surface of the solid S to be plane and slightly inclined as in Fig. 4.2. As shown, the thickness h varies linearly with the abscissa; we thus set

$$h(x) = h_1 + (h_2 - h_1)x/\ell \qquad \text{and} \qquad \alpha = (h_1 - h_2)/\ell \ll 1 \; .$$

In order to analyse the flow, we use a frame attached to the moving solid. The boundary conditions are therefore

$$\mathbf{v} = U\mathbf{e}_x \quad \text{at} \quad z = 0 \quad \text{and} \quad \mathbf{v} = \mathbf{0} \quad \text{at} \quad z = h(x)$$

The flow is stationary and with a very small Reynolds number. It therefore satisfies Stokes' equation (4.13). Using the x-component of the momentum equation, we get

$$-\frac{\partial P}{\partial x} + \mu \frac{\partial^2 v_x}{\partial z^2} = 0, \tag{4.21}$$

where we neglected the x-dependence of the velocity field. Terms coming from this dependence are $\mathcal{O}(\alpha)$ or smaller, thus (4.21) is just the zeroth order in α. At this same order we see that $G_p = \partial P/\partial x$ is independent of x. Solving for the z-dependence gives the following form of v_x:

$$v_x = U\frac{h-z}{h} + \frac{G_p}{2\mu}(z-h)z \tag{4.22}$$

This solution is the superimposition of two exact solutions of the Navier–Stokes equation: Couette's flow

$$\mathbf{u} = U(h-z)/h\ \mathbf{e}_x$$

and Poiseuille's flow

$$\mathbf{u} = G_p z(z-h)/2\mu\ \mathbf{e}_x$$

that we shall discuss in Sect. 4.4.1.

In the expression of the velocity, G_p is an unknown. However, it may be related to the volume flux Q which is the same at any x since the fluid is incompressible. Neglecting the third dimension, we have

$$Q = \int_0^h v_x dz = \frac{Uh}{2} - \frac{G_p h^3}{12\mu}$$

which leads to

$$G_p = \frac{dP}{dx} = \frac{12\mu}{h^3}\left(\frac{Uh}{2} - Q\right) \tag{4.23}$$

If the solid S is completely immersed in the same fluid (as the sheet of paper), the pressure on the two ends is identical. Rewriting (4.23) as

$$\frac{G_p}{\mu} = \frac{1}{\mu}\frac{dP}{dx} = \frac{1}{\mu}\frac{dh}{dx}\frac{dP}{dh} = -\frac{\alpha}{\mu}\frac{dP}{dh} = \frac{6U}{h^2} - \frac{12Q}{h^3}$$

and integrating between h_1 and h_2, we can express the volume flux as a function of the parameters of the problem. We find

$$Q = U\frac{h_1 h_2}{h_1 + h_2} \tag{4.24}$$

We can then give the expression of the pressure field in the domain $[0, \ell]$:

$$P(x) - P_o = \frac{6\mu U(h_1 - h(x))(h(x) - h_2)}{\alpha(h_1 + h_2)h(x)^2} \tag{4.25}$$

where P_o is the pressure at the ends of the solid. Using this expression, in which we insert $h(x) = h_1 - \alpha x$, we can observe that

$$\delta P = P(x) - P_o \propto x(\ell - x)/(h_1 - \alpha x)^2,$$

showing that the pressure reaches a maximum in the neighborhood of $\ell/2$. The maximum value of the pressure may be expressed as a function of the parameters of the problem, namely

$$\delta P_{max} = \frac{6\mu U\alpha\ell}{(h_1 + h_2)^3} \tag{4.26}$$

This result shows that the pressure strongly increases when the thickness of the fluid layer vanishes. Furthermore, the total pressure force can be derived from $F_l = \int_0^\ell \delta P dx$. After little algebra,[1] we find

$$F_p = \frac{6\mu U}{\alpha^2}\left\{\ln\left(\frac{h_1}{h_2}\right) - 2\frac{h_1 - h_2}{h_1 + h_2}\right\} \tag{4.27}$$

It is interesting to compare this lift force to the total shear stress exerted upon the moving solid, which is just the drag force. By using a similar calculation, we find

$$F_d = \int_0^\ell \mu\frac{\partial v_x}{\partial z}dx = \frac{2\mu U}{\alpha}\left\{3\frac{h_1 - h_2}{h_1 + h_2} - \ln\left(\frac{h_1}{h_2}\right)\right\} \tag{4.28}$$

so that

$$\frac{F_{\rm drag}}{F_{\rm lift}} = \frac{\alpha}{3}\frac{\left\{3\frac{h_1 - h_2}{h_1 + h_2} - \ln\left(\frac{h_1}{h_2}\right)\right\}}{\left\{\ln\left(\frac{h_1}{h_2}\right) - 2\frac{h_1 - h_2}{h_1 + h_2}\right\}} \tag{4.29}$$

Since $\alpha \ll 1$, the horizontal force is very much smaller than the vertical one. This is why a large mass can be moved effortlessly with bearings. An example is given in exercise.

[1] The reader may note that after an integration by part

$$\int_0^\ell \frac{x(\ell - x)}{(h_1 - \alpha x)^2}dx = -\frac{1}{\alpha}\int_0^\ell \frac{\ell - 2x}{h_1 - \alpha x}dx$$

while

$$\int_0^\ell \frac{\ell - 2x}{h_1 - \alpha x}dx = \int_0^\ell \left\{\frac{\ell - 2h_1/\alpha}{h_1 - \alpha x} + \frac{2}{\alpha}\right\} dx$$

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4.3 Boundary Layer Theory

In the foregoing section we considered flows with very low Reynolds numbers, fully dominated by the viscous force. Now, we shall examine the opposite limit, that of laminar flows at large Reynolds numbers.

4.3.1 Perfect Fluids and Viscous Fluids

Contrary to the low Reynolds number flows, the present ones are not always stable. Beyond some critical Reynolds number, several solutions are possible. However, this critical value may be large compared to unity. Thus it is often the case that the flow is stable even if $\mathrm{Re} \gg 1$. This is the typical situation that we shall investigate now.

A convenient example to bear in mind is the one of a flow around a solid body. Let us think of a car or an airplane moving at constant speed. In the frame attached to the solid, the fluid shows a uniform velocity field in the far distance of the body. We assume that the Reynolds number, based on the typical size of the object, is large compared to unity. Such a set-up was already discussed in the case of perfect fluids (see Sect. 3.3.7). There we argued that the flow was irrotational, so that there exist ϕ such that $\mathbf{v} = \boldsymbol{\nabla}\phi$. It is interesting to note that such kind of solution is almost acceptable for a viscous fluid. Indeed, for an incompressible fluid ($\boldsymbol{\nabla} \cdot \mathbf{v} = 0$), the viscous force associated with a potential flow is zero, since:

$$\Delta \boldsymbol{\nabla}\phi = \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{v}) - \boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \boldsymbol{\nabla}\phi = \mathbf{0}$$

However, this solution is not fully acceptable since it does not meet the no-slip boundary conditions on the solid. The fluid sticks to the solid and we can surmise that close enough to it, the viscous force dominates over the other forces, which is clearly not possible if the flow remains irrotational.

To specify what is meant by "close enough", we have to go back to the momentum equation (4.7). If the viscous force is important in some region of the flow, then, in this place

$$\Delta \mathbf{u} \gtrsim \mathcal{O}(\mathrm{Re})$$

since other terms are supposedly of order unity. Such an inequality can be realized in only two ways: either $\mathbf{u}$ is very large compared to unity or its spatial variations are very rapid. The first possibility can be eliminated thereof since close to the solid the velocity cannot grow much as it vanishes on the boundary. The second possibility is

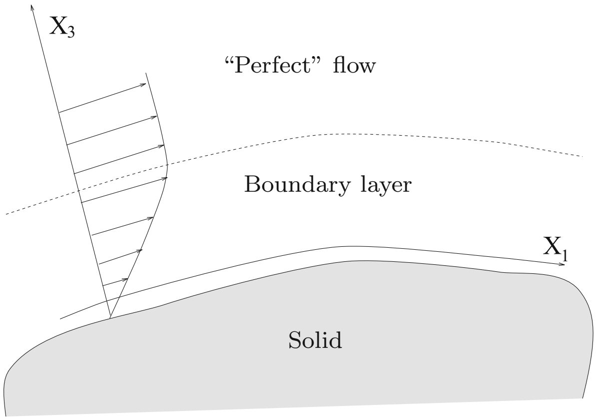

Fig. 4.3 Geometry of a boundary layer flow

therefore the right one. The function $\mathbf{u}$ varies very rapidly in the vicinity of the solid. If ε is the characteristic scale of the velocity field, the viscous force is dominating if

$$\frac{1}{\mathrm{Re}}\Delta\mathbf{u} \sim \frac{1}{\mathrm{Re}}\frac{u}{\varepsilon^2} \sim \mathcal{O}(1) \qquad \Longleftrightarrow \qquad \varepsilon = \mathcal{O}(1/\sqrt{\mathrm{Re}})$$

since we assumed that $\mathbf{u} \sim \mathcal{O}(1)$.[2]

We thus find that around a solid body in a large Reynolds number flow, there is a very thin layer of thickness $\varepsilon = 1/\sqrt{\mathrm{Re}} \ll 1$ where the viscous force may control the flow. This is the *boundary layer*. At a distance of a few times ε, the viscous force is usually negligible and the fluid behaves as if perfect. This example shows us that high Reynolds number flows may be split into regions with very different dynamics: (*i*) the boundary layers controlled by viscosity (or other diffusion processes in more general situations) and (*ii*) the "remaining" where Euler's equation is sufficient to describe the flow. This is schematically illustrated in Fig. 4.3.

Before closing this heuristic introduction to boundary layers, let us mention that regions where viscous force is important are not systematically attached to boundaries. It turns out that in some cases strong shear layers occur in the middle of the fluid. They are no longer boundary layers but *detached (shear) layers*. Let us also underline that the technique of splitting the fluid domain into various subdomains is not specific to fluid mechanics but is in fact a way of obtaining an approximate solution of (partial) differential equations that are too difficult to be solved analytically.[3]

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[2]Here is a very simple example of such a property: $\sin(x/\varepsilon)$ is $\mathcal{O}(1)$ but $\frac{d^2}{dx^2}\sin(x/\varepsilon)$ is $\mathcal{O}(1/\varepsilon^2)$.

[3]An example where we determine the solution of a differential equation using boundary layer theory is given in Sect. 12.4.

4.3.2 Method of Resolution

The preceding discussion has shown that a small parameter $\varepsilon \ll 1$ is naturally introduced in this problem. A way of taking advantage of this peculiarity is to expand the solutions into powers of ε and write:

$$\mathbf{u} = \mathbf{u}_0 + \varepsilon\,\mathbf{u}_1 + \varepsilon^2\,\mathbf{u}_2 + \cdots \tag{4.30}$$

Furthermore, we can take advantage of the partition of the flow into the boundary layer and the inviscid outer flow. In the boundary layer viscosity is important and the full equation need to be solved. However, the shape of the flow is rather simple as it is parallel to the boundaries. On the other hand, the outer flow does not need viscosity, which may be neglected at zeroth order. Hence, in both domains the solution can be simplified. The strategy is therefore obvious: in each domain solutions are expanded according to (4.30). Each order is solved in each domains and the solutions are matched together. This technique is called asymptotic matching. The final result is an asymptotic solution valid up to some higher order correction in ε^n. We shall now detail all these steps.

4.3.3 Flow Outside the Boundary Layer

Outside the boundary layer, the derivatives are all of order unity. Thus using the expansion (4.30) and identifying each order in ε, we get

$$\begin{cases} \mathbf{u}_0 \cdot \nabla \mathbf{u}_0 = -\nabla p_0 \\ \nabla \cdot \mathbf{u}_0 = 0 \\ \mathbf{u}_0 \cdot \mathbf{n} = 0 \quad \text{on } S \end{cases} \tag{4.31}$$

at zeroth order, and

$$\begin{cases} \mathbf{u}_0 \cdot \nabla \mathbf{u}_1 + \mathbf{u}_1 \cdot \nabla \mathbf{u}_0 = -\nabla p_1 \\ \nabla \cdot \mathbf{u}_1 = 0 \end{cases} \tag{4.32}$$

at first order. We do not write the boundary conditions yet; they need further discussion and will be introduced at the end of the next subsection. Note that at second or higher orders, viscous terms need to be taken into account, even if we are outside the boundary layer.

In the simple case where the zeroth order velocity field is irrotational, equation may be simplified

$$\begin{cases} \Delta \Phi_0 = 0 \\ \mathbf{n} \cdot \nabla \Phi_0 = 0 \;\; \text{on S} \end{cases} \tag{4.33}$$

4.3.4 Flow Inside the Boundary Layer

The foregoing equations give solutions that are not valid close to the boundaries of the fluid. There, the viscous force is important. Viscous terms make the equations of higher order, but, as noted above, the geometry of the flow is simpler, as almost parallel to the boundary. To take advantage of this property, it is natural to introduce three curvilinear coordinates (x_1, x_2, x_3) where $x_3 = \text{Cst}$ is the equation of the boundary. In order to simplify the following discussion, we shall assume that the bounding surface of the fluid is just the $z = 0$-plane. Plain cartesian coordinates are thus sufficient.

As in the "inviscid domain", we expand the unknowns in powers of the small parameter ε. We use the tilde to denote boundary layer quantities; hence, the boundary layer flow is expressed:

$$\tilde{u} = \tilde{u}_0 + \varepsilon\,\tilde{u}_1 + \varepsilon^2\,\tilde{u}_2 + \cdots$$

Boundary layer quantities are characterized by their rapid variations in the direction perpendicular to the boundary. Partial derivatives should therefore be ordered as

$$\partial/\partial x, \quad \partial/\partial y \ll \partial/\partial z \tag{4.34}$$

This inequality can be made more quantitative since we know the thickness of the boundary layers, namely ε. Thus, a typical boundary layer function $\tilde{f}$ reads

$$\tilde{f} \equiv \tilde{f}\left(x, y, \frac{z}{\varepsilon}\right)$$

One usually introduces the *stretched coordinate* $\tilde{z} = z/\varepsilon$, so that $\tilde{f} \equiv \tilde{f}(x, y, \tilde{z})$. With this new coordinate, the inviscid region, which is at $z = \mathcal{O}(1)$, is now rejected at infinity, since for a fixed z, $\tilde{z} \to \infty$ as $\varepsilon \to 0$.

The thickness of the boundary layer, and thus the stretched coordinate, is such that $\frac{\partial \tilde{f}}{\partial \tilde{z}} = \mathcal{O}(1)$; hence, normal variations of the boundary layer functions, namely $\partial_z \tilde{f}$, are all of order ε^{-1}. Besides, the horizontal variations of the fields (velocity and pressure) are controlled by those in the perfect domains. Indeed, the solutions in the boundary layer match those of the perfect domain at each point on the bounding surface, thus horizontal variations in the boundary layer are the same as those just outside of it. In the perfect domain, all the scales are of order unity, therefore horizontal gradient in the boundary layer are also of order unity; thus inequalities (4.34) mean

$$\partial_x \tilde{f} \equiv \mathcal{O}(1), \qquad \partial_y \tilde{f} \equiv \mathcal{O}(1), \qquad \text{and} \qquad \partial_z \tilde{f} \equiv \mathcal{O}(\varepsilon^{-1})$$

Let us now consider the equation of mass conservation. Using (4.30) together with $\nabla \cdot \mathbf{v} = 0$, we find that the lowest order is $\mathcal{O}(\varepsilon^{-1})$. It yields

$$\frac{\partial \tilde{u}_{0,z}}{\partial \tilde{z}} = 0 \,.$$

This equation implies that

$$\tilde{u}_{0,z} = 0 \tag{4.35}$$

since the velocity is zero on $\tilde{z} = 0$. This result shows an important property of boundary layers: the component of the velocity that is perpendicular to the layer is much smaller than the one parallel to it; it is at least of the next order in ε.

The following order of the equation of continuity reads

$$\frac{\partial \tilde{u}_{0,x}}{\partial x} + \frac{\partial \tilde{u}_{1,z}}{\partial \tilde{z}} = 0 \tag{4.36}$$

The Navier–Stokes equation develops in the same way and the first terms, of zeroth order, yield the two equations

$$\begin{cases} \tilde{u}_{0,x}\dfrac{\partial \tilde{u}_{0,x}}{\partial x} + \tilde{u}_{1,z}\dfrac{\partial \tilde{u}_{0,x}}{\partial \tilde{z}} = -\dfrac{\partial \tilde{p}_0}{\partial x} + \dfrac{\partial^2 \tilde{u}_{0,x}}{\partial \tilde{z}^2} \\[2ex] 0 = -\dfrac{\partial \tilde{p}_0}{\partial \tilde{z}} \end{cases} \tag{4.37}$$

(4.36) and (4.37) are known as *Prandtl's equations* of the boundary layer. These equations show that the pressure does not depend on the coordinate $\tilde{z}$; in other words, it is determined, like $\tilde{u}_{0,z}$, by the flow outside the boundary layer. This implies that in (4.37) $\frac{\partial \tilde{p}_0}{\partial x}$ is given by the "perfect fluid flow", so that

$$\frac{\partial \tilde{p}_0}{\partial x} = \frac{\partial p_0}{\partial x}$$

Prandtl's equations can be rearranged in the following way. We derive $\tilde{u}_{1,z}$ from (4.37), and we substitute it into (4.36). Thus

$$\tilde{u}_{1,z} = \frac{-\tilde{u}_{0,x}\frac{\partial \tilde{u}_{0,x}}{\partial x} - \frac{\partial \tilde{p}_0}{\partial x} + \frac{\partial^2 \tilde{u}_{0,x}}{\partial \tilde{z}^2}}{\frac{\partial \tilde{u}_{0,x}}{\partial \tilde{z}}} \tag{4.38}$$

and

$$\frac{\partial \tilde{u}_{0,x}}{\partial x} + \frac{\partial}{\partial \tilde{z}}\left(\frac{-\tilde{u}_{0,x}\frac{\partial \tilde{u}_{0,x}}{\partial x} - \frac{\partial \tilde{p}_0}{\partial x} + \frac{\partial^2 \tilde{u}_{0,x}}{\partial \tilde{z}^2}}{\frac{\partial \tilde{u}_{0,x}}{\partial \tilde{z}}}\right) = 0 \tag{4.39}$$

This last expression shows that the boundary layer equations are nonlinear and of the third order. Three boundary conditions are thus necessary to determine the solution. These are:

- $\tilde{u}_{0,x} = 0$ at $\tilde{z} = 0$,
- $\tilde{u}_{0,x} \longrightarrow u_{0,x}$ when $\tilde{z} \longrightarrow +\infty$,
- $\tilde{u}_{1,z} = 0$ at $\tilde{z} = 0$.

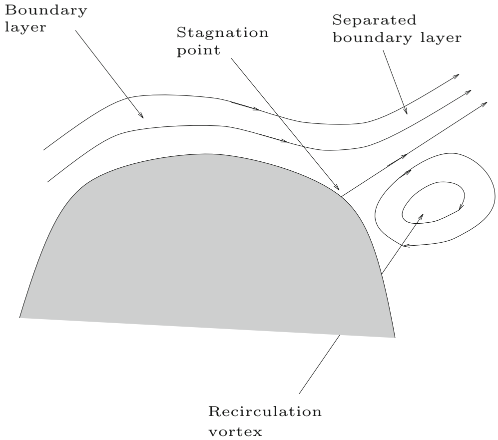
We observe that the limit value of $\tilde{u}_{1,z}$ when $\tilde{z} \to +\infty$ is not specified. It cannot be since $\tilde{u}_{1,z}$ obeys a first order equation (4.36). In general, $\lim_{\tilde{z}\to+\infty} \tilde{u}_{1,z} \neq 0$ so that *there exist a mass flux between the boundary layer and the perfect fluid domain*. The value of $\tilde{u}_{1,z}$ plays in this way the role of the boundary value for the first order terms in the perfect domain. Thus, system (4.32) is completed by the boundary condition

$$u_{1,z}(z=0) = \lim_{\tilde{z}\to+\infty} \tilde{u}_{1,z} \tag{4.40}$$

4.3.5 Separation of the Boundary Layer

We may observe that (4.38), which gives $\tilde{u}_{1,z}$, becomes singular if, at some point on the boundary layer,

$$\frac{\partial \tilde{u}_{0,x}}{\partial \tilde{z}} = 0 \tag{4.41}$$



When approaching such a point, the vertical variations of $\tilde{u}_{0,x}$ are on an increasingly larger scale, in other words the boundary layer becomes thicker up to the point of being "infinite". One says that there is a *separation of the boundary layer*. Let us note that the boundary layer becomes infinitely thick with respect to the coordinate $\tilde{z}$. It does not mean that the boundary layer is overrunning all the fluid domain, but simply that its true scale is no longer ε but a much greater scale. For example, if the thickness is $\varepsilon^{\frac{1}{2}}$ and that it is developed in spite of everything in powers of ε, the thickness will be $\mathcal{O}(1/\varepsilon^{\frac{1}{2}})$ in the coordinate $\tilde{z}$ and therefore infinite when ε is vanishing. Thus, at the point of separation, $\tilde{u}_{1,z}$ diverges but in the neighbourhood of such a point $\tilde{u}_{1,z}$ is no longer of order unity and the expansion in powers of ε is no longer valid.

Equation (4.41) determines the position of the point of separation when solving Prandtl's equation. In fact, it is not necessary to solve this equation in order to know the position of this point. Indeed, if $\frac{\partial \tilde{u}_{0,x}}{\partial \tilde{z}} = 0$, then, using boundary conditions, $\tilde{u}_{0,x} = 0$. As the tangential velocity in the boundary layer is also the tangential velocity of the perfect fluid on the solid, we find that the separation of the boundary layer occurs close to a (downstream) stagnation point of the perfect fluid flow (see figure).

4.3.6 *Example of the Laminar Boundary Layer: Blasius' Equation*

We shall now illustrate the foregoing general theory with a very classical example which is Blasius flow. This is the boundary layer flow generated by a thin horizontal plate parallel to the flow at infinity (see Fig. 4.4). Far from the plate the pressure is

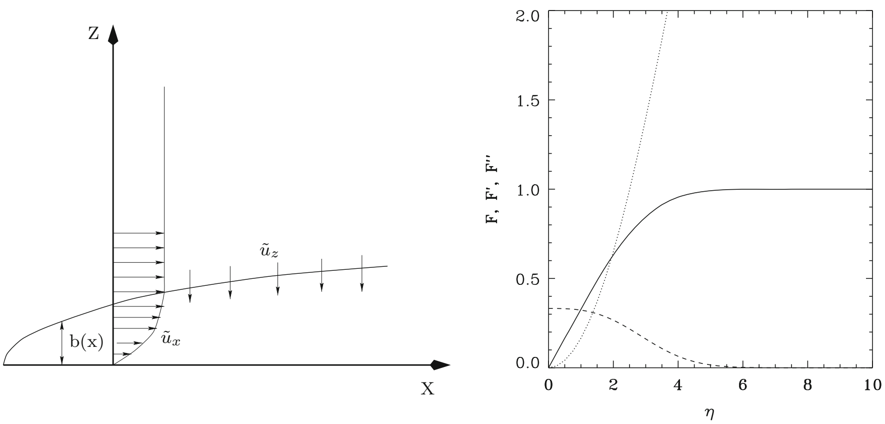


Fig. 4.4 *Left*: Shape of the boundary layer on a plate in a uniform flow. *Right*: View of the functions F (*dotted line*), f (*solid line*) and f' (*dashed line*)

uniform. Equations (4.36) and (4.37) thus reduce to

$$\begin{cases} \tilde{u}_{0,x}\dfrac{\partial \tilde{u}_{0,x}}{\partial x} + \tilde{u}_{1,z}\dfrac{\partial \tilde{u}_{0,x}}{\partial \tilde{z}} = \dfrac{\partial^2 \tilde{u}_{0,x}}{\partial \tilde{z}^2} \\ \dfrac{\partial \tilde{u}_{0,x}}{\partial x} + \dfrac{\partial \tilde{u}_{1,z}}{\partial \tilde{z}} = 0 \end{cases} \tag{4.42}$$

Despite their apparent complexity, these equations admit analytical solutions when one imposes self-similarity. Indeed, we may set

$$\tilde{u}_{0,x} = Uf\left(\frac{\tilde{z}}{b(x)}\right) \qquad \text{and} \qquad \tilde{u}_{1,z} = V(x)g\left(\frac{\tilde{z}}{b(x)}\right) \tag{4.43}$$

where U is the velocity at infinity. Such a velocity profile is said to be *self-similar*, because for all x, its shape is identical and given by f. Then, we introduce the similarity variable $\eta = \tilde{z}/b(x)$ and rewrite the equations. The equation of mass conservation (4.42b) gives

$$V(x)g'(\eta) = U\eta f'(\eta)b'(x) \tag{4.44}$$

The primed functions designate the derivatives. This equation shows that if self-similar solutions exist then $V(x)/b'(x)$ is a constant. Dimensionally, this constant is a velocity that can be set to U without loss of generality.

Turning to the momentum equation, we find

$$f'' = -Ub(x)b'(x)\eta ff' + b(x)V(x)gf' \tag{4.45}$$

As before, this equation admits self-similar solutions if $b(x)b'(x)$ and $b(x)V(x)$ are constants. We therefore set

$$b(x) = \sqrt{Lx} \qquad \Longrightarrow \qquad bb' = L/2 \qquad \text{and} \qquad b(x)V(x) = UL/2$$

$b(x)$ is the thickness of the boundary layer. The preceding expressions show that this quantity grows like the square root of the distance to the leading edge of the plate. We shall return further on to the physical interpretations of this result.

Finally, (4.45) is rewritten as

$$f'' = \frac{UL}{2}\left(gf' - \eta ff'\right)$$

where U and L are dimensionless constants, which represent a velocity and a length scale respectively. We use them as the velocity scale and length scale, which is equivalent to setting $U = L = 1$. By using (4.44), which we now write $g'(\eta) = \eta f'(\eta)$, we thus deduce a first form of Blasius' equation :

$$\left(\frac{f''}{f'}\right)' = -\frac{1}{2}f \quad \Longleftrightarrow \quad f'' = -\frac{1}{2}f'\int_0^\eta f(y)dy \tag{4.46}$$

Another classical form of Blasius' equation may be found by introducing a stream function like $F = \displaystyle\int f d\eta$. Equation (4.46) yields then

$$2F''' + FF'' = 0 \tag{4.47}$$

This equation is completed by three boundary conditions:

- $F'(0) = 0$, ($v_x = 0$ on the plate),
- $F'(\tilde{z} \to +\infty) = 1$, (velocity is constant at infinity)
- $F(0) = 0$, ($v_z = 0$ on the plate).

The functions f or F need to be determined numerically[4] and are shown in Fig. 4.4.

The solution of Blasius' equation allows us to show two general phenomena of boundary layers: The flow in a boundary layer vanishes exponentially in the outer region and there is a flux of matter between the boundary layer and the rest of the fluid. This is the so-called boundary layer pumping. We can demonstrate this last point by recapitulating the asymptotic form of f' for the large values of η. In this case, $f' \sim \exp(-\eta^2/4)$ and we get the component of the velocity v_z by way of the equation of mass conservation $g' = \eta f'$, which we integrate taking into account that $\lim_{\eta\to\infty} g = 0$. Hence,

$$g \sim -2e^{-\eta^2/4}$$

[4] We can get an idea of the shape of the function $f(\eta)$ by considering the asymptotic limits $\eta \sim 0$ and $\eta \to \infty$.

Near the origin, (4.46) and the boundary conditions impose that $f(0) = f''(0) = f'''(0) = 0$; hence, a Taylor expansion yields

$$f(\eta) \approx a\eta - \frac{a^2}{48}\eta^4 + \mathcal{O}(\eta^7)$$

where $a = f'(0) \simeq 0.332058$ (this value is determined by the boundary condition $f(\infty) = 1$). This expression shows that the profile of the velocity is almost linear just before reaching the asymptotic value where $f(\eta) \simeq 1$. In this region ($\eta \gg 1$), the function f verifies approximately $f'' = -\eta f'/2$ whose solution for f' is Gauss function and thus for f the error function:

$$f'(\eta) = Ae^{-\eta^2/4} \qquad \Longrightarrow \qquad f(\eta) \sim \mathrm{erf}(\eta) \qquad \eta \to \infty$$

so that v_z is negative far from the boundary; everything happens as if the boundary layer "breathes" the exterior fluid.

4.4 Some Classic Examples

We continue our tour of flows with incompressible viscous fluids by a short review of the very classic examples, which are either very simple solutions of the Navier–Stokes equation or just very common flows.

4.4.1 Poiseuille's Flow

4.4.1.1 Stationary Regime

One of the simplest cases of steady flows is that of a viscous fluid in a very long cylindrical pipe. In this case the velocity has just one component that is parallel to the pipe axis and which we identify to the z-axis. We also assume that the flow is axisymmetric. These two symmetries imply that the velocity field may be written as $\mathbf{v} = v(r, z)\mathbf{e}_z$. Using mass conservation, we find that $\partial v/\partial z = 0$ so that $\mathbf{v} = v(r)\mathbf{e}_z$. This velocity field belongs to the class of *plane-parallel shear flows*: it has just one component, which varies in a direction perpendicular to it. As a consequence, the velocity gradient is orthogonal to the velocity itself and thus the nonlinear term $(\mathbf{v} \cdot \boldsymbol{\nabla})\mathbf{v}$ is zero. The momentum equation reads

$$-\boldsymbol{\nabla} p + \mu \Delta \mathbf{v} = \mathbf{0}$$

and we find Stokes equation again. We note, however, that in this case the Reynolds number is not necessarily small. If we project this equation along $\mathbf{e}_r$ and $\mathbf{e}_z$ we find that:

$$\frac{\partial p}{\partial r} = 0 \qquad \text{and} \qquad \frac{\partial p}{\partial z} = \frac{\mu}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v}{\partial r}\right)$$

The pressure is therefore independent of r and if we differentiate the second equation with respect to z, we see that the pressure gradient is necessarily constant. We call this gradient G_p and integrate the equation of v, which leads to

$$v(r) = -\frac{G_p}{4\mu}(R^2 - r^2) \tag{4.48}$$

where the constants of integration have been chosen so that $v(0)$ is finite and $v(R) = 0$. Solution (4.48) is *Poiseuille's flow*.[5] The velocity profile is parabolic. We see in this expression that the flow is in the opposite direction to the pressure gradient. The fluid flows from the high pressures toward the low pressures.

Such a velocity profile may also be found for the laminar flow of a viscous fluid between two infinite flat plates staying at a distance d from each other. If a pressure gradient G_p is set up along, say, the x-axis, then

$$v_x(z) = -\frac{G_p}{\mu} z(d - z)$$

which is also a parabolic profile with a maximum velocity of $z = d/2$.

4.4.1.2 Transients to a Poiseuille's Flow

We shall now briefly examine the way the Poiseuille flow sets up. For this, we consider two situations. The steady flow inside a pipe but close to the inlet, and the transient flow occurring in an infinitely long pipe when a pressure gradient is abruptly set up.

- When a viscous fluid enters a pipe, the Poiseuille flow is not immediately set up, especially if the Reynolds number is large. Indeed, at large Reynolds numbers, a boundary layer appears. Such a layer is very similar to the one described by Blasius' equation. It thickens as the square root of the distance to the entrance as illustrated in Fig. 4.5. When the thickness of the layer has reached

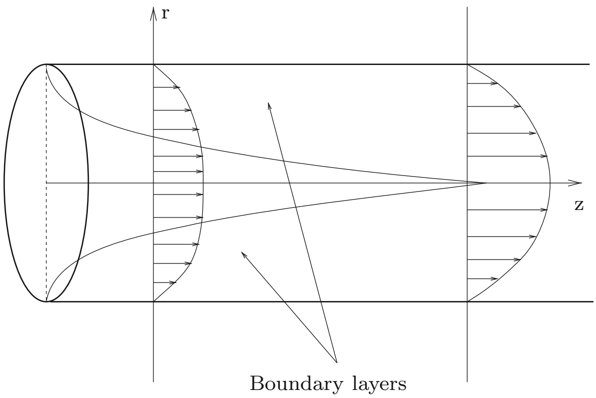


Fig. 4.5 Boundary layers at the inlet of a cylindrical pipe

[5]Sometimes called the Hagen–Poiseuille flow. Hagen studied it in 1839 and Poiseuille in 1840.

the radius of the pipe, Poiseuille flow is almost established. Using the results of Blasius boundary layer, we can estimate the distance from the entrance at which Poiseuille flow appears. The thickness of the layer is given by

$$e = \frac{D}{\sqrt{\mathrm{Re}}}\sqrt{z}$$

In this expression, D is the diameter of the pipe. From the boundary layer theory, we know that the boundary layer thickness scales like $D/\sqrt{\mathrm{Re}}$, so that using the result of the Blasius flow we find the above expression (which only gives an order of magnitude). We thus see that Poiseuille flow appears at a distance from the entrance which is typically $\mathrm{Re}/2$ times the diameter.

– We now consider the case of an infinite pipe in which a pressure gradient is suddenly set up. Such a situation occurs when one rapidly opens a tap or a sluice gate. In this case, the velocity field evolves according to

$$\frac{\partial v}{\partial \tau} = -G_p + \mathrm{Re}^{-1}\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v}{\partial r}\right)$$

If we solve this equation numerically, we find a result similar to that of Fig. 4.6. In this figure we clearly see the boundary layers at early times and their progressive diffusion towards the interior up until the formation of the parabolic profile.

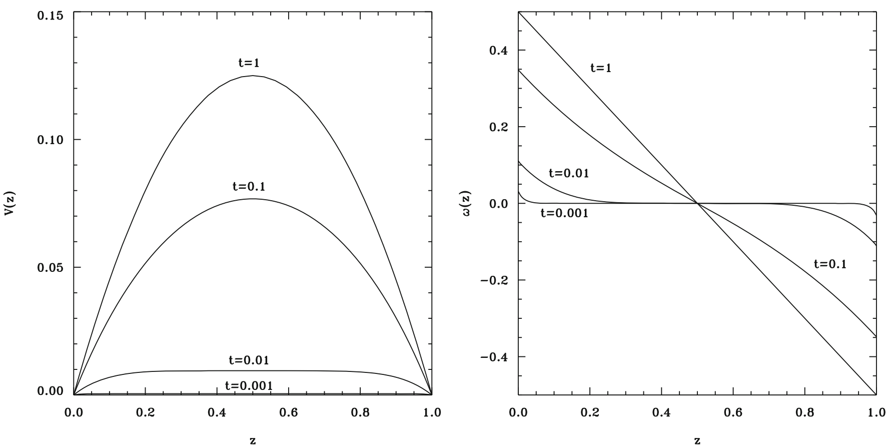


Fig. 4.6 Time evolution of the velocity and vorticity during a transient leading to the Poiseuille flow between two plates. The equation $\frac{\partial v}{\partial \tau} = -\frac{\partial p}{\partial x} + \frac{\partial^2 v}{\partial z^2}$ with $\partial p/\partial x = -1$ has been integrated numerically, giving the velocity while vorticity is $\omega_y = \partial v/\partial z$

4.4.1.3 Generation of Vorticity

The set-up of Poiseuille flow in the foregoing examples comes from a single phenomenon: the diffusion of vorticity from the walls. The case of the time evolution is very clear: just after the pressure gradient is set up, at $t = 0.001$, vorticity is zero everywhere except near the bounding planes. As time passes, it diffuses slowly to the interior until the steady state is reached. This shows the key role played by the no-slip boundary conditions in the generation of vorticity. These conditions prevent the flow from remaining irrotational.

In the other example, vorticity also diffuses from the walls, but it is simultaneously advected by the main stream. These two effects combine, and give birth to the square root law that we met in analysing the Blasius layer. Indeed, the diffusion of a quantity proceeds with the square root of time (see the discussion of the diffusion equation in the maths complements) : if δ is the distance to the wall at which the vorticity takes a given value, then $\delta \propto \sqrt{t}$. But t is such that $z = Vt$; thus $\delta \propto \sqrt{z/V}$. We thus find again the square root law of Blasius boundary layer. It is a consequence of advection and diffusion acting simultaneously.

4.4.2 Head Loss in a Pipe

When we studied the motion of perfect fluids, we introduced the notion of head losses which we connected to energy dissipation. We are now in a position to estimate these losses in some simple cases.

4.4.2.1 Regular Head Losses

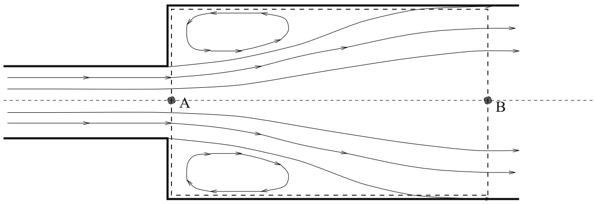
We begin with the case of the Poiseuille's flow of an incompressible fluid. We can easily calculate the head loss between two points separated by a distance L and belonging to the same streamline. Since the kinetic energy is constant along each streamline (this is mandatory because of mass conservation and incompressibility), the loss of mechanical energy $P + \frac{1}{2}\rho v^2$ comes from the pressure gradient G_p. Thus, over a distance L, the loss is $G_p L$; the head therefore decreases linearly in the downstream direction. One says that the head loss is *regular*.

More quantitatively, if the volume flux in the pipe is Q and the volumic mechanical energy is $E_m(x)$, x being the coordinate along the streamline, the power dissipated between the two points is just:

$$D = (E_m(x) - E_m(x + L))Q$$

Since $E_m = \frac{1}{2}\rho v^2 + P$ and since the velocity does not depend on x, we find that $D = LG_p Q$. We may verify that this expression also comes out of a direct calculation of the viscous dissipation, and that D does not depend on the velocity profile inside the pipe, but just on the pressure gradient.

4.4.2.2 Singular Head Losses



Let us now imagine the case of a pipe flow at the place where the pipe's cross section abruptly increases, just as shown in the above figure. Such a change in the pipe, provokes the separation of streamlines from the wall and gives birth to a jet. Further downstream, this jet reconnects to the wall of the pipe. In between, we find a region of "dead water" where recirculation vortices stand. The flow is quite complicated there, but an evaluation of the losses and gains of momentum, between the upstream and downstream sides, allows us to find out the head loss due to this singularity of the cross section. Such kinds of head losses are called *singular*. Other examples of singularities are pipe junctions, pipe bends, etc.

To understand the effect of the abrupt change of pipe section, we consider a fixed control surface (shown with dashed lines in the figure). The difference between the in and out momentum flux is compensated by the pressure difference in A and B. In B the pressure is uniform in the cross section since streamlines are all parallel to the pipe boundaries (see Sect. 3.2.2), however in A this is less obvious. In fact it is almost uniform there also. The reason is similar as for B: in a cross section of the jet, pressure does not vary because of its almost parallel streamlines; it is equal to the one just outside it. Thus, provided the pressure is constant in the dead water region (this is approximate of course), we may assume that the pressure is constant all over the section in A. Thus we write

$$(P_A - P_B)S_B = \rho V_B^2 S_B - \rho V_A^2 S_A$$

But since the volume flux is conserved, $V_A S_A = V_B S_B$, and thus we get

$$P_A - P_B = \rho V_B (V_B - V_A) \tag{4.49}$$

This result is sometimes called *Bélanger's Theorem*. We show now that some energy is lost in the crossing of the enlargement; for this we calculate the difference

$$\Delta = (P_A + \frac{1}{2}\rho V_A^2) - (P_B + \frac{1}{2}\rho V_B^2) \ .$$

From the preceding formula we get

$$\Delta = \frac{1}{2}\rho(V_A - V_B)^2 > 0 \qquad (4.50)$$

This difference is thus always positive: there is always a head loss due to the sudden change of cross section.

We examined here the case of an abrupt enlargement of the cross section; in the opposite case of a cross section narrowing abruptly, some head loss also exists but not as large. This is because no jet forms, so that recirculating vortices are much smaller.

4.4.3 Flows Around Solids

Flows around solids constitute a wide class of flows with numerous applications. They are sometimes called *external flows* to underline the differences with pipe flows which are therefore *internal flows*. We shall describe these kinds of flows with the vorticity field, considering examples with increasingly high Reynolds numbers.

When the Reynolds number is small compared to unity, vorticity fills the whole space, although decreasing like $1/r^2$ as shown by (4.19) in the case of the sphere. When the Reynolds number is large compared to unity, we have seen that it is confined inside the boundary layer. However, this confinement is not complete: the boundary layer always separate somewhere on the downstream side of the solid and forms *the wake*. Thus, far from the solid, the vorticity may be found in the wake only.

When we discussed Stokes' equation in Sect. 4.2.1, we noticed the symmetry of the solutions between upstream and downstream sides. We observed that the nonlinear terms break this symmetry. Here, we see that this symmetry breaking actually occurs through the raise of the wake on the downstream side.

The shape of a wake much depends on the Reynolds number. In Fig. 4.7, we see that the wake consists of two recirculating vortices. When $\mathrm{Re} \gtrsim 50$, these vortices separate from the solid and form a *vortex streak* also called von Kármán streak (see Fig. 4.8). Finally, if $\mathrm{Re} \gtrsim 1000$, the wake becomes turbulent (e.g. Fig. 4.9).

Fig. 4.7 A glimpse at unsteady recirculating vortices behind a cylinder at Re$\sim$ 330 (photo of the author)

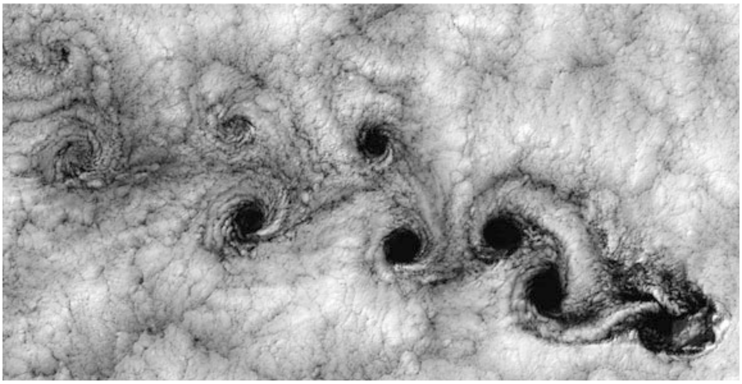


Fig. 4.8 Vortex street in the wake of Juan Fernandez islands imprinted in the clouds (Landsat 7 image, NASA)

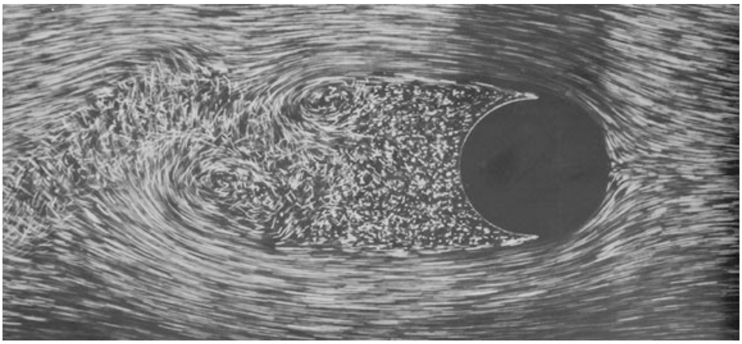


Fig. 4.9 Turbulent wake behind a cylinder at Re=2000 (ONERA photograph in Werlé and Gallon 1972).

The computation of such flows is solely possible with a direct numerical resolution of the Navier–Stokes equation (together with mass conservation) and only when the Reynolds number is not too large (presently less than a few thousands). Such flows are extremely complex.

4.5 Forces Exerted on a Solid

When the flows are known, the stresses they exert on the boundaries can be computed. We therefore continue our investigations on viscous fluids by a look at forces that they may apply on solid bodies.

4.5.1 General Expression of the Total Force

The total force exerted on a solid by a viscous fluid flowing around it is the sum of all the stresses applied to its surface, namely

$$\mathbf{F} = \int_{(S)} [\sigma] d\mathbf{S}$$

where the surface element $d\mathbf{S}$ is directed outside the solid.

To illustrate this formula we take the simple example of Poiseuille flow for which we have an explicit expression of $\mathbf{v}$. In this case, the expression of $[\sigma]$ is

$$[\sigma] = \begin{pmatrix} -p & 0 & \mu\frac{\partial v_z}{\partial r} \\ 0 & -p & 0 \\ \mu\frac{\partial v_z}{\partial r} & 0 & -p \end{pmatrix}$$

now $d\mathbf{S} = -2\pi R dz \mathbf{e}_r$ since the surface must be oriented towards the fluid which exerts the stress. The resultant force is therefore

$$\mathbf{F} = -\int \left| \begin{matrix} -p \\ 0 \\ \mu\left(\frac{\partial v_r}{\partial z}\right)_R \end{matrix} \right. dS = -\mu 2\pi R L \left(\frac{\partial v_z}{\partial r}\right)_{r=R} \mathbf{e}_z$$

where L is the length of the tube and R its radius. But $\left(\frac{\partial v_z}{\partial r}\right)_R = G_p R/2\mu$ so that the fluid entrains the tube with a force $\mathbf{F} = -\pi R^2 G_p L \mathbf{e}_z$ in the same direction as the flow. We observe that viscosity has disappeared from the expression of the force which means that this force can be obtained without knowing the details of the flow, simply by an integral balance (see exercises).

4.5.2 Coefficient of Drag and Lift

The expression of the force exerted on a solid is rarely accessible by direct calculation, in particular when the Reynolds number is large compared to unity. It is then necessary to resort to numerical calculation and/or to modelling (if there is turbulence). However, even if we totally ignore the form of the flow it is always possible to connect this force, may be just dimensionally, to the fundamental quantities of the flow. This is why one introduces non-dimensional coefficients which concentrate our ignorance about the flow.

When the Reynolds number is large, the pressure field due to the inertia of the fluid is the main source of stresses exerted on a solid moving in a fluid. When we studied the motion of perfect fluids, we saw that pressure at an upstream stagnation point was $\frac{1}{2}\rho v^2$ (also called dynamic pressure); multiplying this pressure by a surface typical of the solid, we get an order of magnitude for the force. This may be expressed in the following way:

$$\mathbf{F} = \frac{1}{2}\rho V^2 \left| \begin{array}{l} S_x C_x \\ S_y C_y \\ S_z C_z \end{array} \right.$$

where ρ is the density of the fluid, V the velocity of the solid, and S_x, S_y and S_z are the surfaces projected on planes perpendicular to each axis (see Fig. 4.10).

If the solid moves along Ox, C_x is called *drag coefficient*, C_z *lift coefficient*, while C_y, rarely utilized, could be called *coefficient of lateral lift*.

These coefficients depend on the shape of the body and on the Reynolds number: a well-shaped body has a smaller C_x than an ill-shaped one. When the Reynolds number is very large ($\gtrsim 10^6$), these coefficients are almost constant.

The dependence on the shape of a body is most easily shown with the case of a wing. In this example, the coefficients much depend on the incidence of the wing: for small values the lift increases with the sine of this angle. But if this angle exceeds some critical value, the lift drops abruptly: the boundary layer separates from the wing near the leading edge. Streamlines are no longer curved and the resulting pressure drop, which is responsible of the lift, disappears. In addition, the drag strongly raises: this is known as *the wing stall.* This situation is illustrated in Fig. 4.11.

4.5.3 Example: Stokes' Force

To conclude this section, we examine the case of the sphere in uniform motion in a viscous fluid when the Reynolds number is very small. The solution that we obtained in Sect. 4.2.3 allows us to calculate the expression of the resultant force. This force

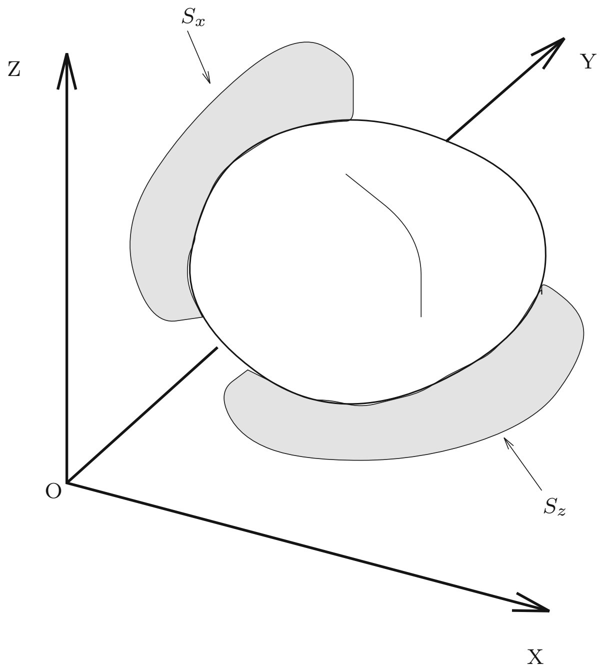


Fig. 4.10 Schematic views of the various projections of a solid on a plane

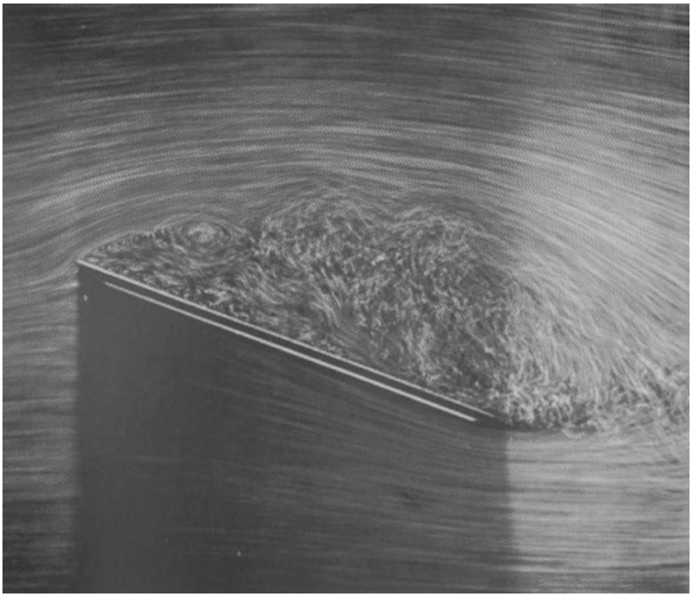


Fig. 4.11 Flow around an inclined plate at stall (© ONERA photograph, Werlé 1974)

expresses as

$$\mathbf{F} = \int_{(S)} [\sigma]\mathbf{e}_r dS = \int_{4\pi} (\sigma_{rr}\cos\theta - \sigma_{r\theta}\sin\theta) R^2 \sin\theta d\theta d\phi \mathbf{e}_z$$

where we have used the symmetry of the flow around Oz. The boundary conditions on the sphere allow us to write:

$$\sigma_{rr}(R) = -p(R) \qquad \text{and} \qquad \sigma_{r\theta}(R) = \mu \left(\frac{\partial v_\theta}{\partial r}\right)_{r=R}$$

By using the solution (4.18) and after evaluation of the integrals we get

$$F_z = 6\pi\mu R U \tag{4.51}$$

which is the expression of *Stokes' force*. This formula may be used in various ways, but an interesting one is the measurement of the dynamic viscosity of Newtonian fluids. Indeed, if we let a ball falling in a viscous fluid, provided its radius is small enough, its velocity quickly reaches a constant value. This value results from the balance between the weight (minus the buoyancy force) and the viscous friction (Stokes force). The result is that dynamic viscosity is given by

$$\mu = (m - m_f)g/(6\pi R V)$$

where V is the velocity of the falling ball, m is its mass and m_f the mass of the displaced fluid. g is the local gravity. Since Stokes' formula is valid only at very low Reynolds number, it is necessary to check that this condition is verified once the viscosity is determined. For instance, a small glass ball, weighing 0.02 g, left in glycerin, falls with a constant speed of 1 cm/s. This corresponds to a Reynolds number $\sim$0.04. However, if the same experiment is made with water, we would expect, from Stokes formula, a final velocity of 10 m/s and a Reynolds number of 20,000, which is certainly not consistent with the use of Stokes equation.

From Stokes formula, we may also compute the drag coefficient of a sphere at low Reynolds numbers. We find

$$C_x = \frac{6\pi\mu R U}{\frac{1}{2}\rho U^2 \pi R^2} = \frac{24}{\mathrm{Re}}$$

using a Reynolds number based on the diameter of the sphere.

Thus, the C_x coefficient decreases like the inverse of the Reynolds number. At infinite Reynolds number it is zero which is reminiscent of d'Alembert's paradox, but in this limit, again, Stokes' formula is not valid!

The variations of the sphere's C_x with Reynolds number has been well studied experimentally. Figure 4.12 reproduces the curve derived from experiments like in Fig. 4.13. We see the decrease in 1/Re for the small numbers, then a plateau and

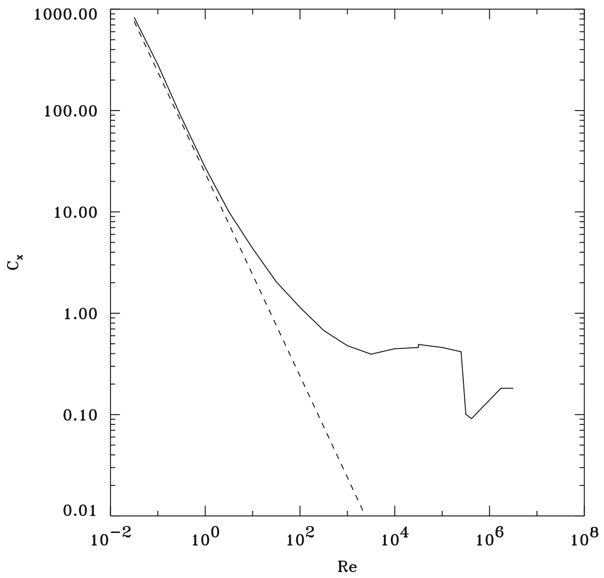


Fig. 4.12 Variation of the C_x coefficient of a sphere with the Reynolds number (*solid line*). The *dashed line* shows the law $C_x = 24/\mathrm{Re}$ valid at small Reynolds numbers

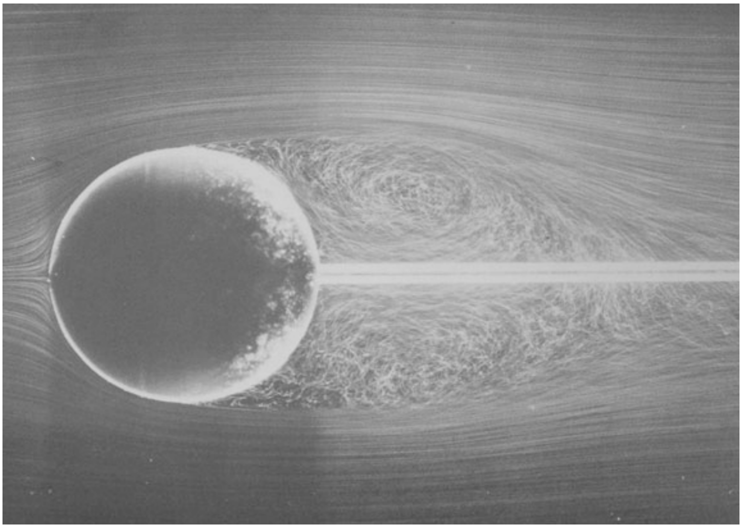


Fig. 4.13 View of the turbulent wake of a sphere at Re = 15,000 (© ONERA photograph, H. Werlé)

an abrupt jump which correspond to the transition of the boundary layers towards turbulence. It is usually admitted that beyond this value C_x remains constant.

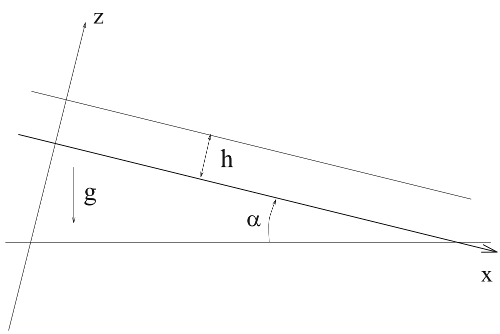
4.6 Exercises

1. Find the expression of the force exerted by a Poiseuille flow in a cylindrical pipe using an integral balance. L is the length of the pipe, R its radius and G_p the applied constant pressure gradient.
2. *Lubrication layer:* show that if the solid inclination is very small, i.e. that $h_1 = h_2(1+\varepsilon)$, then

$$F_{\text{drag}} \simeq \frac{h_2}{3\ell} F_{\text{lift}}$$

Compute the force needed to push a solid of 10^3 kg at constant speed on an oil film 1 mm thick if the length of the contact surface is 1 m (the contact surfaces are assumed to be perfect planes).

3. *Flow of a viscous fluid on a slope*



The fluid layer, as shown in the above figure, meets free-slip boundary conditions on the top plane and no-slip ones on the bottom plane. The planes make an angle α with the horizontal. The fluid is incompressible and of kinematic viscosity ν. The flow is steady.

(a) Determine the velocity profile assuming that $\mathbf{v} = V(z)\mathbf{e}_x$.
(b) Find the volume flux through a cross section S.

4. *The Taylor–Couette flow or cylindrical Couette flow*
We consider a viscous fluid contained between two rotating cylinders of radii R_1 and R_2, respectively. Their angular velocity is Ω_1 and Ω_2.

(a) We look for a solution like $\mathbf{v} = v(s)\mathbf{e}_\varphi$. What are the symmetries of this solution? Show that $v(s)$ is the solution of a linear equation. Solve it and give the solution verifying the boundary conditions.
(b) What is the torque exerted by the interior cylinder on the outer one.
(c) How can we measure the viscosity of a fluid with such a device?

5. *Falkner–Skan equation*: We take the Prandtl equations of the laminar boundary layer (4.37), but we look for solutions more general than the Blasius ones. We set $\tilde{u}_{0,x} = U(x)f(\eta)$ and $\tilde{u}_{1,z} = V(x)g(\eta)$, where η is still the self-similarity variable and $U(x)$ the x-component of the velocity at infinity.

(a) Give the expression of $\partial p_0/\partial x$ as a function of $U(x)$.
(b) Show that the existence of such solutions implies that $U(x)$, $V(x)$ and $b(x)$ verify:

$$2Ubb' = c_1, \qquad U'b^2 = c_2 \qquad \text{and} \qquad Vb = c_3$$

where c_1, c_2, c_3 are constants.
(c) Derive the general form of $U(x)$ and $b(x)$.
(d) Show that if one chooses $c_1/2 + c_2 = 1$ (why is that always possible?), then $F = \int f d\eta$ verifies Falkner–Skan equation:

$$F''' + FF'' - \frac{2m}{m+1}(F'^2 - 1) = 0 \tag{4.52}$$

where m is a constant to be related to c_1, c_2, c_3.
(e) For which value of m is the boundary layer thickness constant? How can we find the Blasius equation again?

Further Reading

The matter of this chapter belongs to the very base of Fluid Mechanics and therefore may be found in all the books aimed at introducing Fluid Mechanics; for instance, Batchelor (1967), Faber (1995), Guyon et al. (2001), Landau and Lifchitz (1971–1989), Paterson (1983), Ryhming (1985, 1991).

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