Chapter 3
Flows of Perfect Fluids

3.1 Equations of Motions

In the first chapter we introduced the perfect fluid as a fluid that does not conduct heat and for which the fluid elements interact only through pressure. We then derived the equations of motion of such a fluid:

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot \rho \mathbf{v} = 0 \tag{1.16}$$

$$\rho \frac{D\mathbf{v}}{Dt} = -\boldsymbol{\nabla} P \tag{1.33}$$

$$\frac{Ds}{Dt} = 0 \tag{3.1}$$

These equations express mass, momentum and energy conservation, respectively. The momentum equation is also called Euler's equation and the third equation shows that the motion of fluid particles takes place at constant entropy. In other words a particle of perfect fluid only sustains reversible adiabatic transformations in the course of its motion.[1]

[1] On condition, of course, that the functions are continuous, i.e. that the fluid particles do not cross a shock wave.

M. Rieutord, *Fluid Dynamics*, Graduate Texts in Physics,
DOI 10.1007/978-3-319-09351-2_3

3.1.1 Other Forms of Euler's Equation

Euler's equation (1.33) can be rewritten in several forms. Firstly, using the vector relation $(\mathbf{v} \cdot \nabla)\mathbf{v} = (\nabla \times \mathbf{v}) \times \mathbf{v} + \nabla \frac{1}{2}\mathbf{v}^2$, we obtain Lamb's form:

$$\frac{\partial \mathbf{v}}{\partial t} = \mathbf{v} \times (\nabla \times \mathbf{v}) - \frac{1}{\rho}\nabla P - \nabla \frac{1}{2} v^2 \tag{3.2}$$

But Crocco's form is often more interesting. Let us introduce the enthalpy h, the total derivative of which is connected to that of pressure and entropy by

$$dh = Tds + \frac{1}{\rho} dP$$

This expression relates the differential forms of the three functions (pressure, enthalpy and entropy). It also relates the partial derivatives and therefore the gradients. Thus we can write:

$$\nabla h = T \nabla s + \frac{1}{\rho} \nabla P$$

which leads to Crocco's equation:

$$\frac{\partial \mathbf{v}}{\partial t} = \mathbf{v} \times \nabla \times \mathbf{v} + T \nabla s - \nabla (h + \frac{1}{2} v^2) \tag{3.3}$$

The quantity $h + \frac{1}{2}v^2$ is sometimes called the *total enthalpy*.

3.2 Some Properties of Perfect Fluid Motions

The form of equations (3.1) and (3.3) confers certain conservation properties on the motion of a perfect fluid and we shall study the simplest aspects of these. These properties are summarized by two theorems (Bernoulli and Kelvin) which express the conservation of mechanical energy and of angular momentum.

3.2.1 Bernoulli's Theorem

3.2.1.1 Statement and Proof

Let us consider a steady flow. It is governed by the equations:

$$\nabla \cdot \rho \mathbf{v} = 0 \tag{3.4}$$

$$\mathbf{v} \times (\boldsymbol{\nabla} \times \mathbf{v}) + T\boldsymbol{\nabla} s - \boldsymbol{\nabla}(h + \frac{1}{2}v^2) = \mathbf{0} \tag{3.5}$$

$$\mathbf{v} \cdot \boldsymbol{\nabla} s = 0 \tag{3.6}$$

where we dismissed all the time derivatives as required by steadiness. The last equation shows that entropy is constant along the streamlines. If we now project the momentum equation (3.5) onto the vector $\mathbf{v}$, we obtain

$$\mathbf{v} \cdot \boldsymbol{\nabla} \left(\frac{1}{2}\mathbf{v}^2 + h \right) = 0$$

so that

$$\frac{1}{2}\mathbf{v}^2 + h = \mathrm{Cst} \tag{3.7}$$

along a streamline.

This result constitutes *Bernoulli's Theorem* in its fundamental form. It may be generalized to the case where the fluid flow is driven by a potential force $\mathbf{f} = -\rho\boldsymbol{\nabla}\phi$. In this case

$$\frac{1}{2}\mathbf{v}^2 + h + \phi = \mathrm{Cst} \tag{3.8}$$

along a streamline. This theorem simply expresses the conservation of mechanical energy per unit mass along a streamline. We notice that in this expression, enthalpy plays the role of a potential energy. If the fluid is incompressible (3.8) leads to

$$\frac{1}{2}\rho\mathbf{v}^2 + P + \rho\phi = \mathrm{Cst} \tag{3.9}$$

and pressure plays the role of a potential. The quantity $\frac{1}{2}\rho\mathbf{v}^2$ is called the *dynamic pressure.*

If the fluid is an ideal gas,

$$h = \frac{\gamma}{\gamma - 1}\frac{P}{\rho}$$

and (3.8) now reads

$$\frac{1}{2}\mathbf{v}^2 + \frac{\gamma}{\gamma - 1}\frac{P}{\rho} + \phi = \mathrm{Cst} \tag{3.10}$$

also called *Saint-Venant's relation*.

Finally, it should be noted that the constant in (3.7) or (3.8) is *specific to each streamline* (see exercises).

3.2.2 The Pressure Field

The steady Euler's equation

$$\rho \mathbf{v} \cdot \nabla \mathbf{v} = -\nabla P \tag{3.11}$$

leads to an interesting property of the pressure field associated with steady flows. Let us consider a streamline. We denote by s the curvilinear abscissa of a point on this curve and by $\mathbf{e}_s$ the tangent vector in s. We immediately see that $\mathbf{v} \cdot \nabla \equiv v\partial/\partial s$, therefore

$$(\mathbf{v} \cdot \nabla)\mathbf{v} = v\frac{\partial(v\mathbf{e}_s)}{\partial s} = v\frac{\partial v}{\partial s}\mathbf{e}_s + v^2\frac{\partial \mathbf{e}_s}{\partial s} \; . \tag{3.12}$$

Now

$$\frac{\partial \mathbf{e}_s}{\partial s} = \mathbf{n}/R_s ,$$

where R_s is the radius of curvature of the streamline at s and $\mathbf{n}$ a unit vector perpendicular to $\mathbf{e}_s$ (see Sect. 12.3). If one projects (3.11) on $\mathbf{e}_s$, one obtains

$$\frac{\partial P}{\partial s} = -\rho\frac{\partial}{\partial s}\left(\frac{v^2}{2}\right)$$

which leads to Bernoulli's theorem as we have seen above. However, if we project (3.11) along $\mathbf{n}$, we have

$$\frac{\partial P}{\partial n} = \frac{\rho v^2}{R_s} \tag{3.13}$$

where n is the coordinate along $\mathbf{n}$. This equation expresses the equilibrium that exists between the local centrifugal force $\frac{\rho v^2}{R_s}$ and the normal component of the pressure gradient when the flow is steady. This equation also shows that *the pressure does not vary in the direction perpendicular to a streamline if the streamline is straight (infinite radius of curvature).*

Finally, we note that the relation (3.13) also applies to an unsteady flow because the term $\frac{\partial \mathbf{v}}{\partial t}$ does not have a component along $\mathbf{n}$; in this case it is necessary to replace the streamlines by the trajectories of fluid particles and R_s is the radius of curvature of such a trajectory.

3.2.3 Two Examples Using Bernoulli's Theorem

Waterfalls have been used for a very long time as a source of energy. In this example we calculate the maximum power available from a waterfall of height H having a volume flux q. We assume that water is an incompressible perfect fluid and that the flow is steady. Along a streamline we have, after Bernoulli's theorem:

$$\frac{1}{2}\mathbf{v}^2 + \frac{P}{\rho} + gz = \mathrm{Cst} \tag{3.14}$$

We suppose that the origin of z is at the foot of the waterfall and that the water arrives at the entrance of the fall with a vanishing velocity (originating in a lake, for example).

By applying (3.14) along a streamline lying on the surface of the water, one can obtain the velocity of water at the foot of the waterfall:

$$\frac{1}{2}\mathbf{v}^2 + \frac{P_{atm}}{\rho} + 0 = 0 + \frac{P_{atm}}{\rho} + gH$$

where P_{atm} is the atmospheric pressure. We get

$$v = \sqrt{2gH} \tag{3.15}$$

also called *Torricelli's law*. This relation shows that the velocity at the foot of the waterfall is that of a free particle falling from a height H. The available power here is simply the flux of kinetic energy:

$$P_u = q \times \frac{1}{2}\rho v^2 = q\rho g H$$

For a height H of 10 m and a flow rate q of 10 m^3/s, the available power is around 10^6 W. This is of course a theoretical limit and the study of a realistic case must take losses into account. Nevertheless the performance of hydraulic installations is high (actually higher than 90 %) and the preceding calculation provides a good order of magnitude.

Experts in hydraulics often rewrite (3.14) in the form

$$\frac{\mathbf{v}^2}{2g} + \frac{P}{\rho g} + z = H; \tag{3.16}$$

In this expression where the terms are all homogeneous to a length, permitting an immediate graphical representation (Fig. 3.1), the constant H represents the *hydraulic head* or *load* and

$$h = \frac{P}{\rho g} + z$$

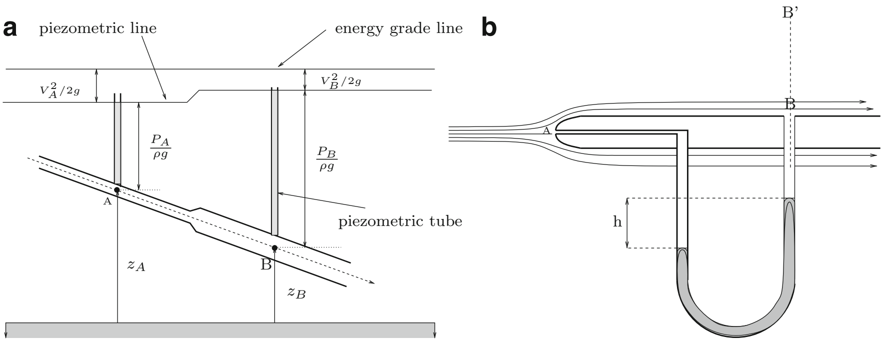

Fig. 3.1 (**a**) A representation of the hydraulic *grade line* in a pipe for a perfect fluid. With a real fluid the energy *grade line* would be inclined towards the downstream side since $H_A > H_B$. (**b**) The Pitot tube

the *piezometric height.*

In a real fluid, the head (or energy) line and the piezometric (or hydraulic grade) line are inclined towards the downstream side and the difference

$$H_A - H_B = \left(\frac{\mathbf{v}_A^2}{2g} + \frac{P_A}{\rho g} + z_A\right) - \left(\frac{\mathbf{v}_B^2}{2g} + \frac{P_B}{\rho g} + z_B\right)$$

on the load line represents the *head loss* between points A and B. The head loss measures the loss of mechanical energy of the flow.

Finally, we note that the power lost (or received) by a flow between two points is proportional to the product of the mass flux and the difference of load ΔH between the two points.

Another simple application of Bernoulli's Theorem is that of an apparatus called the *Pitot tube*,[2] permitting the measurement of velocity within a flow. This apparatus is sketched out in Fig. 3.1b. The principle of the device consists in estimating the difference in pressure between the stagnation point A and a point B along the tube. One admits that the holes for measuring pressure do not disturb the flow, and that the difference of elevation of the measurement points is negligible. If we consider the streamline ending in A, Bernoulli's theorem says that

$$P_A = P_\infty + \frac{1}{2}\rho v^2$$

where P_∞ is the pressure at infinity. The pressure in B is however the same as P_∞. We may see that by considering the streamline that passes through B: noting that the velocity in B is the same as at infinity (the fluid is inviscid), Bernoulli's theorem says

[2]H. Pitot (1695–1771) was a French physicist who invented this device around 1732 in order to measure the velocity of water in a river or the speed of a ship.

that pressure must also be the same as at infinity. Hence $P_B = P_\infty$. However, we may also note that pressure in B is also the same as the pressure along the straight line BB' because all the streamlines are straight lines there (see Sect. 3.2.2). Hence, far enough from the Pitot tube, we find streamlines along which the pressure is uniform (like the velocity) and equal to P_∞. This line of argument is interesting because it applies to real fluids also, and shows that the Pitot tube may measure the velocity even if a slight viscosity of the fluid modifies the flow in the neighbourhood of the solid, as it actually does. So we can write

$$P_A = P_B + \frac{1}{2}\rho v^2 \quad \text{and} \quad v = \sqrt{2(P_A - P_B)/\rho}$$

If the difference in pressure is measured by a U-shaped tube, $P_A - P_B = (\rho_\ell - \rho_f)gh$ where ρ_ℓ and ρ_f are respectively the densities of the liquid and the fluid that one supposes obviously non-mixable (for example, air–water, water–mercury, etc.).

3.2.4 Kelvin's Theorem

3.2.4.1 Statement

Let (C) be a contour moving with the fluid not intersecting any surface of discontinuity: if the fluid is barotropic and subject solely to forces deriving from a potential, then the circulation of the velocity along this curve is constant.

3.2.4.2 Proof

The circulation Γ along a contour (C) moving with the fluid (i.e. made of fluid particles, see Fig. 3.2) is defined as

$$\Gamma(t) = \oint_{C(t)} \mathbf{v}(\mathbf{x}, t) \cdot d\mathbf{l}$$

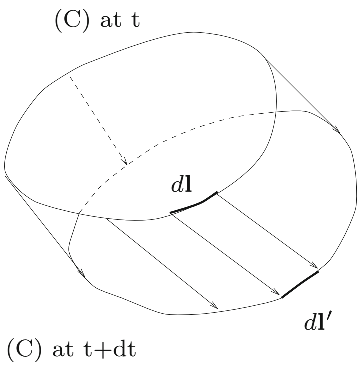

Fig. 3.2 Example of a contour moving with the fluid

We calculate the derivative of this quantity with respect to time with the help of the relation (1.12) :

$$\frac{d\Gamma}{dt} = \oint_{C(t)} \left(\frac{Dv_i}{Dt} dl_i + v_i \partial_j v_i dl_j \right)$$
$$= \oint_{C(t)} \frac{Dv_i}{Dt} dl_i + \oint_{C(t)} \nabla(v^2/2) \cdot d\mathbf{l}$$

whence

$$\frac{d\Gamma}{dt} = \oint_{C(t)} \frac{D\mathbf{v}}{Dt} \cdot d\mathbf{l} \tag{3.17}$$

because the second integral always vanishes. We thus obtain

$$\frac{d\Gamma}{dt} = -\oint_{C(t)} \left(\frac{1}{\rho} \nabla P + \nabla\phi \right) \cdot d\mathbf{l} = -\oint_{C(t)} \left(\frac{1}{\rho} \nabla P \right) \cdot d\mathbf{l}$$

Since the fluid is barotropic $P \equiv P(\rho)$ and

$$\frac{1}{\rho} \nabla P = \nabla \int \frac{dP}{\rho}$$

where $h' = \int \frac{dP}{\rho}$ is a quantity that we can identify as the specific enthalpy if the fluid is isentropic. Finally

$$\frac{d\Gamma}{dt} = -\oint_{C(t)} \nabla h' \cdot d\mathbf{l} = 0$$

and thus

$$\Gamma(t) = \oint_{C(t)} \mathbf{v}(\mathbf{x}, t) \cdot d\mathbf{l} = \mathrm{Cst} \tag{3.18}$$

3.2.4.3 Interpretation

Following Stokes' theorem, this result (3.18) can also be written as

$$\int_{(S(t)} \nabla \times \mathbf{v} \cdot d\mathbf{S} = \mathrm{Cst} \tag{3.19}$$

where $S(t)$ is the surface delineated by the contour $C(t)$. The flux of vorticity across a surface moving with the fluid is constant.

If we consider an infinitesimal cylinder of fluid based on a contour $C(t)$, the angular momentum of this fluid particle is

$$\mathbf{L} = I(\frac{1}{2}\nabla \times \mathbf{v}) \propto mS(\frac{1}{2}\nabla \times \mathbf{v})$$

where we have used the fact that the moment of inertia I is proportional to the base S of the cylinder and that $\frac{1}{2}\nabla \times \mathbf{v}$ is nothing but the local rotation of the fluid element (see Chap. 1). Kelvin's theorem (3.18) implies the constancy of $S\frac{1}{2}\nabla \times \mathbf{v}$ and thus the constancy of the angular momentum $\mathbf{L}$ of the fluid particle of mass m.

Kelvin's theorem shows that in the motion of an inviscid fluid, the angular momentum of the fluid particles is conserved.

3.2.5 Influence of Compressibility

Bernoulli's theorem also allows the determination of the circumstances in which the compressibility of a gas has either a negligible or important role.

To see this, we need considering the flow of an ideal gas and Saint-Venant's relation. We apply it to a streamline that connects points far upstream where the velocity of the fluid is V_∞, the pressure P_∞ and the density ρ_∞, to a stagnation point on a solid surface where the pressure and density are respectively P_m and ρ_m. Then

$$\frac{1}{2}\mathbf{v}_\infty^2 + \frac{\gamma}{\gamma-1}\frac{P_\infty}{\rho_\infty} = \frac{\gamma}{\gamma-1}\frac{P_m}{\rho_m} \tag{3.20}$$

We shall see in Chap. 5 that $\frac{\gamma P}{\rho}$ is simply the square of the local sound speed. The ideal gas flowing as a perfect fluid, fluid elements evolve isentropically and therefore $P \propto \rho^\gamma$. From this relation and (3.20), we deduce the expression of the density at the stagnation point as a function of that far upstream. One obtains

$$\rho_m = \rho_\infty \left(1 + \left(\frac{\gamma-1}{2}\right)\frac{v_\infty^2}{c_\infty^2}\right)^{\frac{1}{\gamma-1}} \tag{3.21}$$

This expression shows that, at low velocity, the changes in density induced by the flow are of the order of v_∞^2/c_∞^2, which is the Mach number of the flow squared. From this particular case, we actually obtain a general result, namely that one can consider a fluid as incompressible as long as its velocity is very small in comparison with the sound speed. For example, the air flow around a car moving at 100 km/h causes variations of density less than a percent, which are therefore negligible in first approximation.

3.3 Irrotational Flows

3.3.1 Definition and Basic Properties

A flow is called *irrotational* if

$$\nabla \times \mathbf{v} = \mathbf{0}$$

or, equivalently, if there exists a function Φ such that

$$\mathbf{v} = \nabla \Phi.$$

This type of flow is also called a *potential flow* and Φ is the *velocity potential.*

Let us consider the case of irrotational flows of perfect fluids, whose motion is driven by a force field derived from a potential ϕ_{ext}. We look for the equations satisfied by the velocity potential Φ. Euler's equation is transformed in the following way:

$$\rho \frac{D\mathbf{v}}{Dt} = -\nabla P - \rho \nabla \phi_{ext}$$

$$\Longleftrightarrow \quad \nabla \left(\frac{\partial \Phi}{\partial t} + \frac{1}{2}\mathbf{v}^2 \right) = -\frac{1}{\rho}\nabla P - \nabla \phi_{ext}$$

We note that in order for this equation to make sense we require that

$$\nabla \times \frac{1}{\rho}\nabla P = \mathbf{0} \Longleftrightarrow \nabla \rho \times \nabla P = \mathbf{0},$$

namely that $P \equiv P(\rho)$, as has been seen in the previous chapter. So we can introduce h' such that $\nabla h' = \frac{1}{\rho}\nabla P$. Hence,

$$\nabla \left(\frac{\partial \Phi}{\partial t} + \frac{1}{2}\mathbf{v}^2 + h' + \phi_{ext} \right) = \mathbf{0}$$

or

$$\frac{\partial \Phi}{\partial t} + \frac{1}{2}\mathbf{v}^2 + h' + \phi_{ext} = \text{Cst} \tag{3.22}$$

We note the similarity of this expression with that obtained for Bernoulli's Theorem, but we must pay attention to the fact that in this new equation the *constant is the same in all the volume occupied by the fluid and thus identical for all streamlines.* Moreover, the expression is also valid for unsteady flows.

To (3.22), we add the equation of continuity

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \nabla \Phi) = 0$$

This last equation takes a special form for incompressible fluids where $\rho = \mathrm{Cst}$, since

$$\Delta \Phi = 0 \tag{3.23}$$

is simply Laplace's equation.

We observe that the potential Φ is defined to within a function of time: since Φ and $\Phi + f(t)$ give the same velocity field.

3.3.2 Role of Topology for an Irrotational Flow

Topology plays a very important role in irrotational flows. Let us first take an illustrative example. We consider a fluid which occupies all space except a cylinder of infinite length with a radius a centered on the axis O_z. The motion of fluid around the cylinder is given by its velocity field

$$\mathbf{v} = \boldsymbol{\Omega} \times \frac{a^2 \mathbf{e}_s}{s} = \frac{\Omega a^2}{s} \mathbf{e}_\varphi$$

which is derived from the potential $\Phi = a^2 \Omega \varphi$ (s, φ, z are the cylindrical coordinates). One will note that this potential possesses a special property: it is not single valued; at a given point, φ can take an infinite number of values like $\varphi + 2n\pi$. The immediate consequence of this property is that the circulation Γ along a closed curve can take many values depending on the chosen curve. In fact, if the curve does not enclose the cylinder $\Gamma = 0$. If, on the other hand, it encloses it n times $\Gamma = 2n\pi\Omega \neq 0$.

This example illustrates the effect of topology on circulation. The space occupied by the fluid here is *doubly connected*: there exist two irreducible paths[3] to connect two points in this space.

Double connectivity implies that the solutions to Laplace's equation are entirely defined only when the circulation around the regions not belonging to the fluid space is given.

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[3]That are paths which cannot be reduced from one to the other by a continuous deformation within the space occupied by the fluid or, equivalently, the surface bounded by the two paths does not belong entirely to this space.

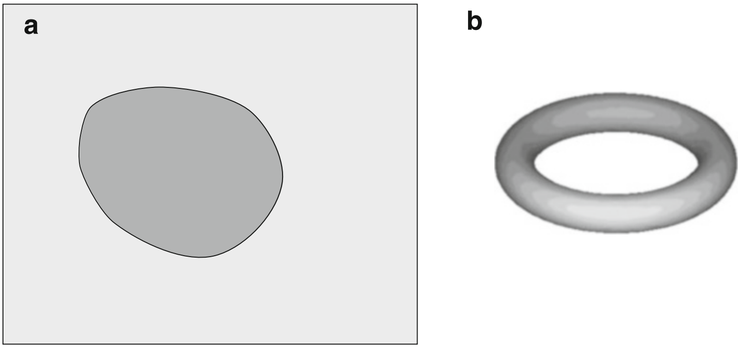


Fig. 3.3 Examples of doubly connected domains: in two-dimensions (**a**) any obstacle creates a doubly connected region; in three-dimensions a toroid (**b**) or an obstacle which is infinite in one dimension implies double connectivity

Two examples of doubly connected spaces are shown in Fig. 3.3. One may note that the presence of an obstacle in a two-dimensional flow renders the space occupied by the fluid doubly connected.

3.3.3 Lagrange's Theorem

If the flow of a barotropic fluid subjected to forces deriving from a potential is irrotational at time t_0 *then it is (irrotational) at all other times.*

In order to prove this theorem we shall suppose the volume occupied by the fluid to be simply connected. According to Kelvin's theorem,

$$\oint_{C(t)} \mathbf{v}\cdot d\mathbf{l} = \mathrm{Cst}$$

at any time. But at t_0

$$\forall C \qquad \oint_C \mathbf{v}\cdot d\mathbf{l} = \oint_C \nabla\Phi\cdot d\mathbf{l} = 0$$

The equality is true for any curve (C) and, from Kelvin's theorem, at all time t. We have therefore

$$\oint_C \mathbf{v}\cdot d\mathbf{l} = \iint_S \nabla\times\mathbf{v}\cdot d\mathbf{S} = 0$$

for any surface S and any time t, thus

$$\Longleftrightarrow \nabla\times\mathbf{v} = \mathbf{0}, \quad \forall t \quad \text{or} \quad \mathbf{v} = \nabla\Phi, \quad \forall t$$

This result is important because it justifies the irrotationality of a large number of flows: in particular if an inviscid fluid is initially at rest and is set in motion by the action of a force deriving from a potential, one can state that the flow will be irrotational because $\mathbf{v} = \mathbf{0}$ is an irrotational flow.

3.3.4 Theorem of Minimum Kinetic Energy

For an incompressible flow of a perfect fluid, the irrotational solution is unique and is that of minimum kinetic energy.

The uniqueness (to within an additive constant) of the solution follows from Laplace's equation satisfied by the potential Φ. The solution is unique when the boundary conditions are specified. As for Lagrange's theorem, we consider only the case where the fluid occupies a simply connected space. If $\mathbf{n}$ is the outward normal at the surface bounding the fluid, the flux of $\mathbf{v}$ across the surface is zero and the potential therefore satisfies

$$\mathbf{n} \cdot \nabla\Phi = 0$$

on it. This boundary condition is called Neumann's boundary condition. Together with Laplace's equation it defines a unique solution for $\mathbf{v}$ (for Φ the solution is defined up to an additive constant). We now show that this solution is that of minimum energy. For this purpose we consider an irrotational flow $\mathbf{v} = \nabla\Phi$ such that $\nabla \cdot \mathbf{v} = 0$ as well as another flow $\mathbf{v}'$ such that $\nabla \cdot \mathbf{v}' = 0$ but which is not necessarily potential. The kinetic energies associated with each of these flows are:

$$E_c = \frac{1}{2}\rho \int_V \mathbf{v}^2 dV \quad \text{and} \quad E'_c = \frac{1}{2}\rho \int_V \mathbf{v}'^2 dV$$

Their difference is

$$E'_c - E_c = \frac{1}{2}\rho \int_V (\mathbf{v}'^2 - \mathbf{v}^2) dV$$

however

$$\mathbf{v}'^2 - \mathbf{v}^2 = (\mathbf{v}' - \mathbf{v})^2 + 2\mathbf{v} \cdot (\mathbf{v}' - \mathbf{v})$$

therefore

$$E'_c - E_c = \frac{1}{2}\rho \int_V (\mathbf{v}' - \mathbf{v})^2 dV + \rho \int_V \mathbf{v} \cdot (\mathbf{v}' - \mathbf{v}) dV$$

but

$$\int_V \mathbf{v}\cdot(\mathbf{v}'-\mathbf{v})dV = \int_V (\mathbf{v}'-\mathbf{v})\cdot\nabla\Phi dV = \int_V \nabla\cdot(\Phi(\mathbf{v}'-\mathbf{v}))dV = \int_{(S)} \Phi(\mathbf{v}'-\mathbf{v})\cdot d\mathbf{S} = 0$$

because $\mathbf{v}$ and $\mathbf{v}'$ both satisfy the boundary condition $\mathbf{v}\cdot\mathbf{n} = \mathbf{v}'\cdot\mathbf{n} = 0$. We find the result

$$E_c' - E_c = \frac{1}{2}\rho\int_V (\mathbf{v}' - \mathbf{v})^2 dV \geq 0$$

This theorem is also due to Kelvin.

3.3.5 Electrostatic Analogy

Laplace's equation is encountered in numerous problems in Physics, in particular in electrostatics where it gives the variations of electrostatic potential in the absence of a charge density. Nevertheless, it is not the electric field that one uses as an analog of the velocity field, but rather a quantity which is proportional to it, like the current density $\mathbf{j}$. Ohm's law states that in a conductive medium, $\mathbf{j} = \sigma\mathbf{E}$, σ being the conductivity assumed constant. In making this analogy we actually substitute the flow of fluid for a flow of charges. The "obstacles" are thus the insulated regions. The situation is easily summed up in the following table:

Fields	$\mathbf{v}$	$\mathbf{j} = \sigma\mathbf{E}$
Equations	$\nabla\times\mathbf{v} = \mathbf{0} \Longleftrightarrow \mathbf{v} = \nabla\Phi$ $\nabla\cdot\mathbf{v} = 0$ $\Delta\Phi = 0$	$\nabla\times\mathbf{E} = \mathbf{0} \Longleftrightarrow \mathbf{j} = \nabla\phi_j$ $\nabla\cdot\mathbf{E} = 0$ $\Delta\phi_j = 0$
Boundary conditions	$\mathbf{v} = \mathbf{0}$ at infinity $\mathbf{v}\cdot\mathbf{n} = 0$ at the surface of the obstacle	$\mathbf{j} = \mathbf{0}$ at infinity $\mathbf{j}\cdot\mathbf{n} = 0$ at the surface of the insulated region

This is the *direct analogy*. We shall later encounter the inverse analogy where the analog of electrostatic potential is the stream function.

3.3.6 Plane Irrotational Flow of an Incompressible Fluid

3.3.6.1 Equation for the Stream Function

We have seen in Sect. 1.3.7 that a two-dimensional flow can be described with the help of a scalar function called the stream function ψ. If the velocity is derived from a potential then ψ also satisfies Laplace's equation. Indeed, $\nabla \times \mathbf{v} = \mathbf{0}$ implies that

$$\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} = 0$$

while $v_x = \partial\psi/\partial y$ and $v_y = -\partial\psi/\partial x$, therefore

$$\Delta\psi = 0 \tag{3.24}$$

It may then be shown (see exercise) that the streamlines ($\psi = \text{Cst}$) are orthogonal to the "equipotentials of velocity" ($\phi = \text{Cst}$).

3.3.6.2 Inverse Analogy

In view of the preceding relation we can make an analogy between the electrostatic potential and the stream function since they both satisfy the same equation. The two functions will be identical if they satisfy the same boundary conditions. For the velocity, these are simply $\psi = \text{Cst}$ along the boundaries and thus for the electrostatic potential we will require that $\phi_e = \text{Cst}$ along the bodies and these will be identified to perfect conductors (this is indeed the inverse of the preceding analogy!).

Fields	$\mathbf{v}$ ψ	$\mathbf{E} \times \mathbf{e}_z$ ϕ_e
Equations	$\nabla \times \mathbf{v} = \mathbf{0}, \mathbf{v} = \nabla \times (\psi\mathbf{e}_z)$ $\Downarrow$ $\Delta\psi = 0$ $\Uparrow$ $\nabla \cdot \mathbf{v} = 0,\ \mathbf{v} = \nabla\Phi$	$\nabla \times (\mathbf{E} \times \mathbf{e}_z) = \mathbf{0}$ $\Uparrow$ $\Delta\phi_e = 0$ $\Uparrow$ $\nabla \cdot \mathbf{E} = 0,\ \mathbf{E} = \nabla\phi_e$
Boundary conditions	$\mathbf{v} = \mathbf{0}$ at infinity $\psi = \text{Cst}$ on an obstacle	$\mathbf{E} = \mathbf{0}$ at infinity $\phi_e = \text{Cst}$ on a conductor

3.3.6.3 The Complex Potential

The existence of two harmonic functions[4] describing the flow allows the study of two-dimensional irrotational incompressible flows in a very thorough manner, thanks to the complex potential. We give here only the broad lines of this approach and refer the reader to the classical works for more details (see for example Batchelor 1967).

We thus introduce the complex function

$$f = \phi + i\psi \tag{3.25}$$

called the *complex potential*. Besides Laplace's equation, this function satisfies

$$\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y} = 0 \tag{3.26}$$

because

$$\begin{cases} \dfrac{\partial \phi}{\partial x} = \dfrac{\partial \psi}{\partial y} = v_x \\ \\ \dfrac{\partial \phi}{\partial y} = -\dfrac{\partial \psi}{\partial x} = v_y \end{cases}$$

Equation (3.26) is also called Cauchy's conditions. It implies that $f \equiv f(x + iy)$, thus f is only a function of the complex variable $z = x + iy$.

We then introduce the *complex velocity* defined by

$$w = \frac{df}{dz} = \frac{\partial f}{\partial x} = v_x - iv_y$$

The interest in introducing the complex potential rests essentially in the ability to use *conformal transformation*. This type of transformation is defined by an analytical function G with non-zero derivative in a domain of the complex plane, which associates with each point z of the first domain a point z' of the image domain, such that

$$z' = G(z)$$

This transformation is called conformal because it conserves angles.

Let us seek the equation for the streamlines (the curves $\psi = \text{Cst}$) in the image plane. $\psi = \text{Cst}$ is the equation of streamlines in the original plane, thus

[4] A harmonic function is a solution of Laplace's equation.

$\psi(G^{-1}(z')) = \text{Cst}$ is the equation in the image plane. $\psi \circ G^{-1}$ is the new stream function. More generally, if $F(z)$ is the complex potential of the flow $F \circ G^{-1}$ is the complex potential in the image plane. We derive from this, the new complex velocity:

$$w' = \frac{dF \circ G^{-1}}{dz'} = \frac{w(z)}{G'(z)} \tag{3.27}$$

In order to illustrate the power of this transformation, we shall use the example of Joukovski's transformation, namely

$$G(z') = z' + R^2/z' \tag{3.28}$$

This function is indeed analytic throughout the plane except at the origin.

We now consider a uniform flow past a flat plate represented by a segment of length $4R$ on the x axis. The velocity is simply $\mathbf{v} = V_0\mathbf{e}_x$ and the associated complex potential is

$$f(z) = V_0 z$$

Let z be the transform of a system of coordinates z' by the conformal transformation G, so that $z = G(z')$. Substituting this in the above equation we have

$$f(G(z')) = V_0 G(z')$$

a new complex potential which is $f \circ G$; but since f is simply the identity (to within a multiplicative constant), G is in the new potential.

In choosing the Joukovsky's transformation for G, we can seek new streamlines and, in particular, the new shape of the obstacle in the (x', y') plane. For that purpose it suffices to take the imaginary part of (3.28)

$$\psi = \text{Im}(z' + R^2/z') = \text{Im}(r'e^{i\theta'} + R^2/r'e^{-i\theta'}) = \frac{r'^2 - R^2}{r'} \sin\theta'$$

which gives the new streamlines. Among them we find those bounding the obstacle: here it consists of the circle $r' = R$, the inverse transform of the line $\text{Im}(z) = 0$.

The inverse of Joukovski's transformation therefore takes us from the (trivial) flow past a flat plate to that past a circle (less obvious). Thus, we determine very easily the flow past more or less complicated forms. For example, starting from the flow past a circle, by shifting a direct Joukovsky transformation we obtain the flow past a wing profile, also called a Joukovsky profile (see Fig. 3.4).

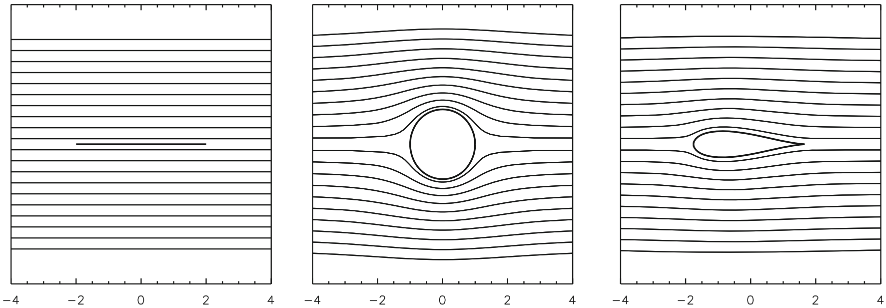


Fig. 3.4 Illustration of possible transformations of a flow past a flat plate. In this example we have first applied an inverse Joukovski transformation which has produced the flow past a circle; then, by application of the slightly shifted Joukovski transformation ($z' = z + c + (1 + c)^2/(z + c)$) one obtains the flow past a wing profile (note that if $c = 0$ the flow past the flat plate is recovered; here $c = -0.17$)

3.3.7 Forces Exerted by a Perfect Fluid

3.3.7.1 d'Alembert's Paradox

Statement:

The steady irrotational flow of an inviscid incompressible fluid around a solid body does not exert any force on it.

Proof:

We assume that the volume occupied by the fluid is simply connected. The solid is supposed to have a constant velocity $\mathbf{V}_s$. The potential satisfies Laplace's equation and the boundary conditions

$$\mathbf{n} \cdot \nabla\phi = \mathbf{n} \cdot \mathbf{V}_s \quad \text{on} \quad S \tag{3.29}$$

$$\phi = \mathcal{O}(1/r^2) \quad \text{if} \quad r \to \infty$$

The second boundary condition results from the properties of the solutions of Laplace's equation (see the mathematical supplement). The force exerted on the solid is just the sum of pressure forces

$$\mathbf{F} = -\int_{(S)} Pd\mathbf{S}$$

Using (3.22) we write

$$P = P_\infty - \frac{1}{2}\rho v^2 - \rho\frac{\partial\phi}{\partial t}$$

where P_∞ is the pressure at infinity assumed constant. We calculate first of all the term $\partial\phi/\partial t$ while remarking that in a region attached to the solid $\phi \equiv \phi(x', y', z')$ where

$$x' = x - V_s t, \quad y' = y, \quad z' = z$$

$$\frac{\partial\phi}{\partial t} = -V_s\frac{\partial\phi}{\partial x} = -\mathbf{V}_s \cdot \nabla\phi$$

thus

$$\mathbf{F} = \frac{1}{2}\rho\int_{(S)} v^2 d\mathbf{S} - \rho\int_{(S)}(\mathbf{V}_s \cdot \mathbf{v})d\mathbf{S} \tag{3.30}$$

Now we examine each component of each of these integrals. In particular,

$$\int_{(S)} v^2 dS_i = \int_{(S\cup S_\infty)} v^2 dS_i$$

where we have introduced a surface S_∞ at infinity which closes the volume of fluid. This is possible and interesting since $\lim_{r\to\infty} v = 0$. We have

$$\begin{aligned}\frac{1}{2}\int_{(S\cup S_\infty)} v^2 dS_i = \int_{(V)}\frac{1}{2}\partial_i v^2 dV = \int_{(V)}(\mathbf{v}\times\nabla\times\mathbf{v} + \mathbf{v}\cdot\nabla\mathbf{v})_i\, dV = \int_{(V)} v_j\partial_j v_i dV\\ = \int_{(V)}\partial_j(v_j v_i)dV = \int_{(S\cup S_\infty)} v_i v_j dS_j\\ = \int_{(S)} v_i v_j dS_j = \int_{(S)} v_i V_{sj} dS_j = V_{sj}\int_{(S)} v_i dS_j\end{aligned}$$

where we used the boundary conditions (3.29). The second integral in (3.30) also reads

$$V_{sj}\int_{(S)} v_j dS_i\ .$$

Finally

$$\begin{aligned} F_i &= -\rho V_{sj}\left(\int_{(S)}(v_j dS_i - v_i dS_j)\right) = -\rho V_{sj}\left(\int_{(V)}(\partial_i v_j - \partial_j v_i)dV\right) \\ &= -\rho V_{sj}\left(\int_{(V)}(\partial_i\partial_j\phi - \partial_j\partial_i\phi)dV\right) = 0 \end{aligned}$$

whence the result.

This shows that a solid body moving in an inviscid fluid is not subjected to any force from the fluid if its motion is uniform. Viscosity is therefore paradoxically an essential element to insure, via the circulation that it induces, the lift of a wing, for example.

3.3.7.2 Case Where the Obstacle is Accelerated

The case of an accelerated body is quite different from the foregoing one and is worth discussing. In a referential attached to the accelerating solid, the flow is now unsteady and subject to an entrainment inertial force but the velocity potential still satisfies $\Delta\Phi = 0$. Therefore the dependence of Φ with respect to time comes from the boundary conditions at infinity where the velocity will be supposedly uniform and of the form $-U(t)\mathbf{e}_z$. One can show from this that the potential of the velocities can be written $\Phi = U(t)f(\mathbf{r})$. The force which is applied to the solid is still the result of the pressure forces, that is

$$\mathbf{F} = -\int_{(S)} Pd\mathbf{S}$$

Noting that the entrainment inertial force ($-\rho\mathbf{a}_e = -\rho\nabla\phi_e$) is derived from a potential, the momentum equation (3.22) reads

$$\frac{\partial\Phi}{\partial t} + \frac{1}{2}\mathbf{v}^2 + \frac{P}{\rho} + \phi_e = \mathrm{Cst}$$

which leads to the following expression for the force exerted on the solid:

$$\mathbf{F} = \int_{(S)}\left(\rho\phi_e + \rho\frac{\partial\Phi}{\partial t}\right)d\mathbf{S} + \int_{(S)}\frac{1}{2}\rho\mathbf{v}^2 d\mathbf{S} \tag{3.31}$$

where we have separated the term of kinetic energy since it is zero as we shall see now. Indeed,

$$\int_{(S)} v^2 d\mathbf{S} = \int_{(S\cup S_\infty)} v^2 d\mathbf{S}$$

because in enclosing the volume by a sphere of infinite radius, the integral remains unchanged since $\mathbf{v} = U(t)\mathbf{e}_z + \mathcal{O}(1/r^3)$. From the calculations of the preceding paragraph, the foregoing integral also reads

$$\int_{(S \cup S_\infty)} \mathbf{v}\,\mathbf{v} \cdot d\mathbf{S}$$

This integral is zero because of the boundary conditions on the solid and because of the form of the velocity at infinity. Finally, the expression for the force is

$$\mathbf{F} = \rho \int_{(S)} \left(\frac{\partial \Phi}{\partial t} + \phi_e \right) d\mathbf{S}$$

so that

$$\mathbf{F} = \rho \dot{U}(t) \int_{(S)} (f + z) d\mathbf{S} \tag{3.32}$$

where we have made use of $\phi_e = \dot{U}(t)z$ assuming a motion along the z-axis.

This integral is non-zero in general. This expression therefore shows that a solid having accelerated motion amidst the fluid, even if inviscid, sustains a force from the fluid. This force is at the origin of all swimming strokes: propulsion in the water is, in fact, efficient only if the solid accelerates with respect to the fluid. For this reason the motion of the fins of a fish is in perpetual acceleration (oscillating motion).

As an example we may calculate the force sustained by a sphere accelerated within of a perfect fluid with constant density. To determine the function f in (3.32) we must solve Laplace's equation in this particular case. In three dimensions and in this geometry we use the expansion of the solution in Legendre's polynomials.

$$\Phi = -U(t) r \cos\theta + \sum_{\ell=0}^{+\infty} \frac{A_\ell(t)}{r^{\ell+1}} P_\ell(\cos\theta) \tag{3.33}$$

where we have taken into account the boundary condition at infinity and the fact that the flow is axisymmetric with respect to the z-axis. The boundary conditions on the sphere, assumed to have radius R, give the functions $A_\ell(t)$. At $r = R$, $\mathbf{v} \cdot \mathbf{e}_r = 0$ so that

$$\left(\frac{\partial \Phi}{\partial r} \right)_{r=R} = 0 = -U(t) \cos\theta + \sum_{\ell=0}^{+\infty} -\frac{(\ell+1) A_\ell(t)}{R^{\ell+2}} P_\ell(\cos\theta)$$

This expression shows[5] that the A_ℓ are all zero except A_1, and

$$A_1(t) = -R^3 U(t)/2 \tag{3.34}$$

Finally, from (3.33)

$$\Phi(r,\theta,t) = U(t)\left(-r - \frac{R^3}{2r^2}\right)\cos\theta$$

and from (3.32)

$$\mathbf{F} = -\rho \int_{(S)} \dot{U}(t) R/2 \cos\theta d\mathbf{S} \quad \Longleftrightarrow \quad \mathbf{F} = -\frac{2\pi}{3} R^3 \rho \dot{U}(t)\mathbf{e}_z$$

The factor $\frac{2\pi}{3}R^3\rho$ is a mass. It is often called the *added mass* because if we exert a force upon the sphere, the latter reacts as if its mass had increased by this quantity (which is equal in this case to half of the mass of the displaced fluid).

3.3.7.3 Drag and Lift of Two-Dimensional Flows

In the foregoing example we assumed that the volume occupied by the fluid was simply connected and therefore its flow was without circulation. In two dimensions, however, the presence of an obstacle makes the fluid "volume" automatically doubly connected and therefore, even if the flow is irrotational, one can have circulation along certain contours.

We shall now consider the same problem as in Sect. 3.3.7.1 but in two dimensions. We assume that the curves surrounding the obstacle possess a circulation Γ. Let us consider a region attached to the solid and assume that the velocity is uniform at infinity:

$$\mathbf{v}_\infty = V_0 \mathbf{e}_x$$

The solution that we are looking for is a solution of Laplace's equation which satisfies $\mathbf{n}\cdot\nabla\Phi = 0$ on the contour of the solid and $\nabla\Phi = V_0\mathbf{e}_x$ at infinity. The general solution of this type of problem is:

$$\Phi = V_0 r\cos\theta + \frac{\Gamma\theta}{2\pi} + \sum_{n=0}^{\infty} A_n \frac{e^{in\theta}}{r^n}$$

[5]We just need to project the equation on Legendre's polynomials and to use their orthogonality with respect to the scalar product $\int_0^\pi P_\ell(\cos\theta)P_k(\cos\theta)d\cos\theta \propto \delta_{\ell k}$.

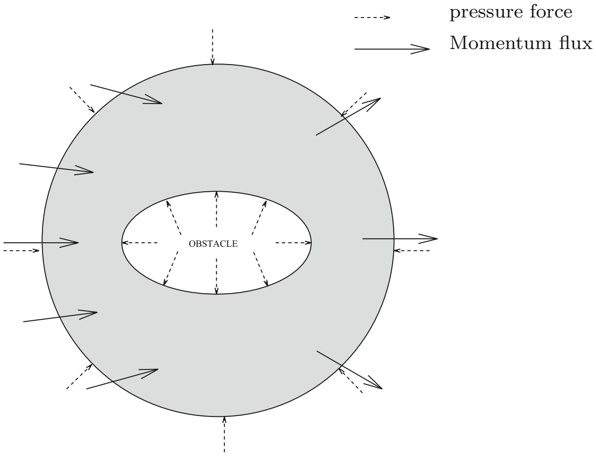


Fig. 3.5 At equilibrium, the sum of the forces and momentum flux is zero, hence $\mathbf{F}_{\text{obst/fluid}} + \mathbf{F}_{\text{press}} + \mathbf{F}_{\text{mom. flux}} = \mathbf{0}$. The force applied to the solid is $-\mathbf{F}_{\text{obst/fluid}} = \mathbf{F}_{\text{press}} + \mathbf{F}_{\text{mom. flux}}$

where the sum represents the multipolar terms that must be added in order to account for the precise shape of the solid.

The associated velocity field is

$$\mathbf{v} = \nabla\Phi = V_0 \cos\theta \mathbf{e}_r + (\frac{\Gamma}{2\pi r} - V_0 \sin\theta)\mathbf{e}_\theta + \cdots$$

If we wish to find the force which is exerted on the solid, a simple method consists in writing the balance of forces and momentum flux that are exerted on a circle surrounding the obstacle at a distance R (see Fig. 3.5). The momentum flux on entry is given by

$$-\int_0^{2\pi} \rho \mathbf{v} v_r R d\theta = -\frac{\Gamma V_0 \rho}{2}\mathbf{e}_y \tag{3.35}$$

while the resultant of the pressure forces is

$$\mathbf{F}_p = -\int_0^{2\pi} P \mathbf{e}_r R d\theta$$

which we calculate using Bernoulli's theorem for an irrotational flow. Equation (3.22) yields

$$P = P_0 - \frac{1}{2}\rho v^2 \qquad \text{and} \qquad v^2 = V_0^2 - \frac{\Gamma}{\pi r} V_0 \sin\theta + \cdots$$

where the dots represent the multipolar terms. The calculation of the integral does not present any difficulty; we find that

$$\mathbf{F}_p = -\frac{\Gamma V_0 \rho}{2}\mathbf{e}_y \tag{3.36}$$

When we let R tend to infinity, the multipolar terms contribution vanishes and only one term remains. Finally, adding (3.35) and (3.36) we find the total force

$$\mathbf{F} = -\Gamma V_0 \rho \mathbf{e}_y = -\rho \mathbf{\Gamma} \times \mathbf{V}_0 \tag{3.37}$$

where $\mathbf{\Gamma} = \Gamma \mathbf{e}_z$. The force just found is called *Magnus' Force*. We see that depending on the sense of the circulation (which is connected to the shape or to the sense of rotation of the body when there is viscosity), the force is directed either upwards or downwards. It is this same force which is responsible for the trajectory of ping-pong balls or tennis balls when they are sliced, and for the lift on wings. Formulae (3.35)–(3.37) are obtained in a two-dimensional space so that the forces are actually forces per unit length. Equation (3.37) leads to the true Magnus force exerted on cylinder of length L by a simple multiplication by L, namely $\mathbf{F} = -\rho L \mathbf{\Gamma} \times \mathbf{V}_0$.

We further note that this force is perpendicular to the motion, consequently there is no resistance to the forward motion *or drag force.*

We could stop here and say that we need the effects of viscosity to calculate the circulation and therefore the lift. Quite surprisingly, this calculation is not necessary for the following reason: when we take into account the effects of viscosity we superimpose upon the irrotational flow the boundary layer corrections which allows the complete solution to verify all the boundary conditions (see next chapter). Actually, we may easily realize (see appendix at the end of this chapter) that the irrotational flow around the profile of a wing has a singularity in the velocity (which becomes infinite) at the trailing edge if the circulation is not adapted. The real flow (with viscosity), which should have for limit this singular irrotational flow, would be very unstable. The problem resolves itself when we observe that for a given circulation, this singularity disappears. This particular value of Γ is that which brings the second stagnation point[6] to the trailing edge (see Fig. 3.6). This condition is usually called *Kutta's Condition*.[7] For a wing profile where the angle of attack is α, we find (see appendix) that:

$$\Gamma = \pi \ell V_0 \sin \alpha \tag{3.38}$$

where ℓ is the wing chord (i.e. it's width).

[6]Point on the solid where the fluid's velocity is zero.

[7]This condition was also found independently by Joukovski in 1906 and is also called sometimes Joukovski's Condition.

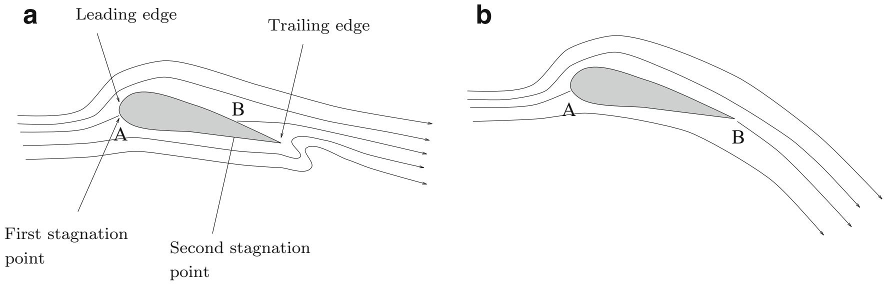


Fig. 3.6 (**a**) A and B are the two stagnation points. On this figure, the circulation is zero and the second stagnation point is located upstream of the trailing edge where the velocity has a singularity. In (**b**) the circulation is such that the trailing edge and the second stagnation point coincide; the velocity is finite everywhere

3.4 Flows with Vorticity

After the irrotational flows, the following step takes us naturally towards flows that own vorticity. These flows are more complex than the preceding ones because the distribution of vorticity is affected by the flow that the vorticity produces. The problem therefore becomes largely nonlinear (we no longer have the equation of the velocity potential $\Delta\Phi = 0$) and consequently only a small number of problems have analytical solutions. We now present the most classic examples.

3.4.1 The Dynamics of Vorticity

In all what follows we call $\boldsymbol{\omega} = \nabla \times \mathbf{v}$ the vorticity. The equation of this quantity is obtained by taking the curl of Euler's equation (1.33) which is made explicit using the following vector equality

$$\nabla \times (\mathbf{v} \cdot \nabla \mathbf{v}) = \nabla \times (\boldsymbol{\omega} \times \mathbf{v}) = (\mathbf{v} \cdot \nabla)\boldsymbol{\omega} - (\boldsymbol{\omega} \cdot \nabla)\mathbf{v} + (\nabla \cdot \mathbf{v})\boldsymbol{\omega}$$

We thus find that the vorticity satisfies:

$$\frac{D\boldsymbol{\omega}}{Dt} = (\boldsymbol{\omega} \cdot \nabla)\mathbf{v} - (\nabla \cdot \mathbf{v})\boldsymbol{\omega} + \frac{1}{\rho^2}\nabla\rho \times \nabla P \tag{3.39}$$

This equation calls for several comments. In the first place, we note that the variations of $\boldsymbol{\omega}$ in a fluid particle result from three different sources:

1. $(\boldsymbol{\omega} \cdot \nabla)\mathbf{v}$ which is a term of *stretching-pivoting*: in order to understand its effect, we take the following simple example where $\boldsymbol{\omega}$ is parallel to $\mathbf{e}_z$ and $\mathbf{v}$ represents a shear along z (see Fig. 3.7). The equation $\frac{D\boldsymbol{\omega}}{Dt} = (\boldsymbol{\omega} \cdot \nabla)\mathbf{v}$ becomes $\frac{D\boldsymbol{\omega}}{Dt} = \omega\frac{\partial \mathbf{v}}{\partial z}$.

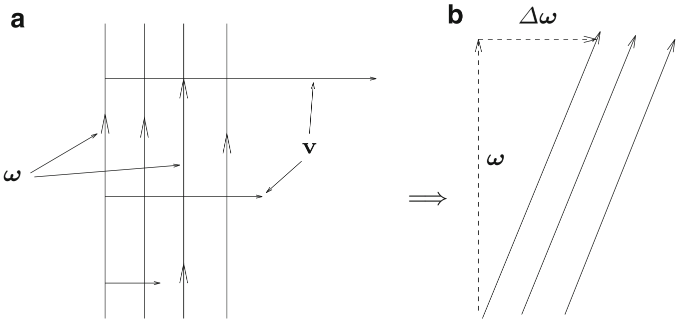


Fig. 3.7 Evolution of the vorticity field subject to a shear flow: from (a) to (b) the vorticity gains a component along the velocity field

It shows that, following each fluid particle, vorticity is created parallel to $\mathbf{v}$ as Fig. 3.7b shows. We will find again such a term when we analyse the evolution of the magnetic field in a fluid with electrical conductivity (Chap. 10).

2. $-(\nabla \cdot \mathbf{v})\,\boldsymbol{\omega}$. We have seen in Chap. 1 the physical meaning of $\nabla \cdot \mathbf{v}$; it represents the volume variations of the fluid elements. This term thus translates the variation in vorticity associated with these variations of volume: if the particle contracts its vorticity increases. Vorticity is created in the same direction and in proportion to the existing one.
3. $\frac{1}{\rho^2}\nabla\rho \times \nabla P$ is the baroclinic torque. This term does not exist (we noted it many times) if $P \equiv P(\rho)$. When it is present, the fluid elements can acquire vorticity, and thus angular momentum, because the pressure force then exerts a torque on them (see Fig. 3.8).

Let us now come back to the barotropic case where $P \equiv P(\rho)$. Equation (3.39) simplifies into

$$\frac{D\boldsymbol{\omega}}{Dt} = (\boldsymbol{\omega} \cdot \nabla)\mathbf{v} - \boldsymbol{\omega}\nabla \cdot \mathbf{v} \tag{3.40}$$

This equation shows that if initially, $\boldsymbol{\omega} = \mathbf{0}$ then $\boldsymbol{\omega}$ remains zero: vorticity cannot be created. This result is, of course, another version of Lagrange's Theorem (see Sect. 3.3.3).

In two dimensions, equation (3.40) takes a very remarkable form if the fluid is incompressible. Indeed, in this case the right-hand side is zero and

$$\frac{D\omega}{Dt} = 0 \tag{3.41}$$

where $\omega = \omega_z$ is the only non-zero component of $\boldsymbol{\omega}$.

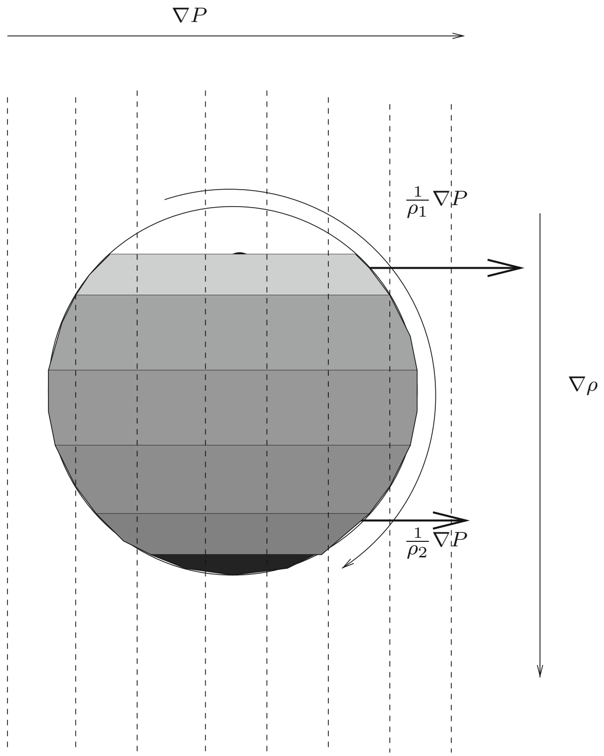


Fig. 3.8 Generation of vorticity by baroclinicity. Density increases towards the *bottom* of the sphere, thus the pressure force per unit mass ($\frac{1}{\rho_1}\boldsymbol{\nabla} P$) is larger than $\frac{1}{\rho_2}\boldsymbol{\nabla} P$. The resulting specific pressure force thus exerts a torque on the fluid element

This equation shows that in this case $\boldsymbol{\omega}$ is a Lagrangian invariant. It implies Kelvin's theorem, but also

$$\frac{D\omega^n}{Dt} = 0 \qquad \Longleftrightarrow \qquad \int_{(S)} \omega^n dS = \text{Cst} \tag{3.42}$$

for all n, S corresponding to a surface advected by the fluid. We shall return to this equation when we study turbulence in two dimensions.

3.4.2 Flow Generated by a Distribution of Vorticity: Analogy with Magnetism

Let's imagine that the distribution of vorticity is given in the space occupied by the fluid. It is then easy to find the distribution of the associated velocity; it is sufficient to solve the equation for **v**

$$\nabla \times \mathbf{v} = \boldsymbol{\omega}$$

where $\boldsymbol{\omega}$ is given. This equation, which is linear, strongly resembles Ampère's equation:

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{j}$$

where **B** is the magnetic field, **j** the volumic current density and μ_0 the permittivity of vacuum. Ampère's equation can be solved quasi-analytically, but for this we must use the vector potential **A** such that $\mathbf{B} = \nabla \times \mathbf{A}$. The transposition of these results to fluid mechanics demands therefore $\nabla \cdot \mathbf{v} = 0$, that is to say, that we need to restrict ourselves to incompressible fluids. In such a case, just as we solve Ampère's equation with

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int_{(V)} \frac{\mathbf{j}(\mathbf{r}')}{\|\mathbf{r} - \mathbf{r}'\|} dx' dy' dz'$$

and

$$\mathbf{B} = -\frac{\mu_0}{4\pi} \int_{(V)} \frac{(\mathbf{r} - \mathbf{r}') \times \mathbf{j}(\mathbf{r}')}{\|\mathbf{r} - \mathbf{r}'\|^3} dx' dy' dz'$$

which is Biot and Savart's law, we have for the velocity field:

$$\mathbf{v}(\mathbf{r}) = -\frac{1}{4\pi} \int_{(V)} \frac{(\mathbf{r} - \mathbf{r}') \times \boldsymbol{\omega}(\mathbf{r}')}{\|\mathbf{r} - \mathbf{r}'\|^3} d^3\mathbf{r}' \tag{3.43}$$

Contrary to the magnetic case, this solution is not the end of the problem because the velocity field thus created modifies the vorticity field by way of the equation (3.40). Problems therefore have simple solutions if the distribution of vorticity is invariant by the advection that it generates. In particular, we can look for a necessary condition for steady flows to be possible. From Euler's equation, assuming that **v** is independent of time, we get

$$\boldsymbol{\omega} \times \mathbf{v} = -\nabla q, \qquad q = \frac{1}{2} v^2 + \int \frac{dP}{\rho} \tag{3.44}$$

where we assumed the fluid to be barotropic. According to this equation

$$\mathbf{v} \cdot \nabla q = \boldsymbol{\omega} \cdot \nabla q = 0 \,,$$

which means that the flow lines and the vorticity lines are on the surfaces $q = \mathrm{Cst}$. If the flow is two-dimensional, the velocity is expressed with a stream function ψ

such that

$$\mathbf{v} = \nabla \times (\psi \mathbf{e}_z), \qquad \boldsymbol{\omega} = \nabla \times \nabla \times (\psi \mathbf{e}_z) = -\Delta \psi \mathbf{e}_z.$$

We can then transform (3.44) into

$$\Delta\psi \nabla\psi = \nabla q \quad \Longrightarrow \quad \nabla\Delta\psi \times \nabla\psi = \mathbf{0}$$

which shows that

$$\Delta\psi = F(\psi) \tag{3.45}$$

where F is an arbitrary function. We are now going to tackle some examples in this category.

3.4.3 *Examples of Vortex Flows*

3.4.3.1 Vortex Sheets

The first example of vortex flows is also the simplest; it concerns the *shear layer* also called the *vortex sheet*: it corresponds to a simple discontinuity in the tangential component of the velocity field, as shown in Fig. 3.9a. It is easy to see that a contour, such as that drawn in Fig. 3.9a, has a circulation; if the length of the longer side is L, the circulation is given by $\Gamma = (V_2 - V_1)L$.

We shall see in Chap. 6 that such a sheet is always unstable. This instability produces individualized vortices such as the vortex ring when the vortex sheet rolls up as indicated in the sketch of Fig. 3.9b under the impulsive motion of the piston.

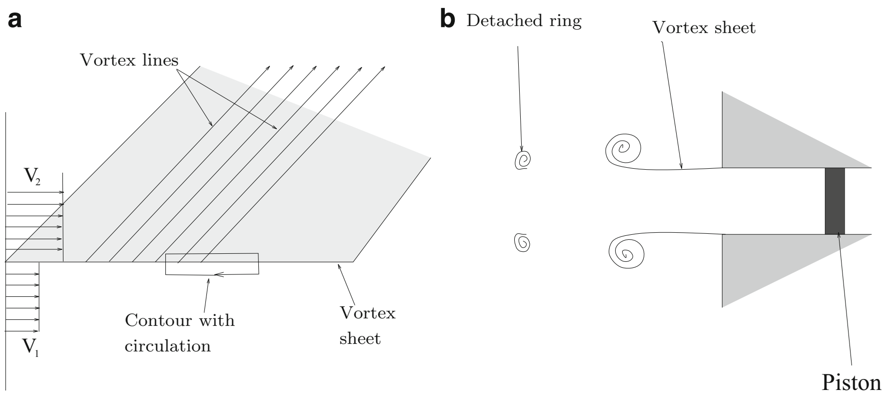


Fig. 3.9 (**a**) Vortex sheet. (**b**) Schematic view of the formation of a vortex ring from a vortex sheet

3.4.3.2 Rankine's Vortex

This is the most simple of the vortex flows. It is made up of a cylindrical kernel in which the vorticity is uniform, and out of which the flow is irrotational. The associated velocity field is then

$$\begin{cases} \boldsymbol{\omega} = \omega \mathbf{e}_z & s \leq a \quad \Longrightarrow \quad \mathbf{v} = \frac{1}{2}\boldsymbol{\omega} \times \mathbf{r} \quad s \leq a \\ \boldsymbol{\omega} = \mathbf{0} & s > a \quad \Longrightarrow \quad \mathbf{v} = \frac{\omega a^2}{2s}\mathbf{e}_\varphi \quad s > a \end{cases} \tag{3.46}$$

where a is the radius of the cylinder and (s, φ, z) are the cylindrical coordinates. We observe that the velocity field is purely azimuthal (only the component along $\mathbf{e}_\varphi$ is non-zero) and therefore the distribution of vorticity does not change with time. The velocity field on the outside of the core has been chosen such that the velocity is continuous at $r = a$.

Rankine's vortex is a very simplified model of the flow generated by a cyclone. We easily show that the pressure passes through a minimum in the centre of such a vortex (see exercises).

3.4.3.3 Hill's Vortex

Another exact solution of Euler's stationary equation consists in distributing the vorticity within a sphere in the following manner:

$$\boldsymbol{\omega} = \frac{\omega r \sin\theta}{a}\mathbf{e}_\varphi \quad \text{if} \quad r \leq a, \qquad \boldsymbol{\omega} = \mathbf{0} \quad \text{if} \quad r > a$$

where (r, θ, φ) are the spherical coordinates. We thus formulate Hill's vortex which moves at constant velocity without being deformed (see Fig. 3.10). We can explain this property by first examining the velocity field of this vortex.

The components v_r and v_θ of the velocity field obey the two following equations:

$$\begin{cases} \dfrac{1}{r}\dfrac{\partial}{\partial r}(r v_\theta) - \dfrac{1}{r}\dfrac{\partial v_r}{\partial \theta} = \dfrac{\omega}{a} r \sin\theta \\ \dfrac{1}{r^2}\dfrac{\partial}{\partial r}(r^2 v_r) + \dfrac{1}{r\sin\theta}\dfrac{\partial \sin\theta v_\theta}{\partial\theta} = 0 \end{cases} \tag{3.47}$$

which express respectively $\nabla \times \mathbf{v} = \boldsymbol{\omega}$ and $\nabla \cdot \mathbf{v} = 0$. We are looking for a solution to this system in the form:

$$v_r = f(r)\cos\theta \quad \text{and} \quad v_\theta = g(r)\sin\theta$$

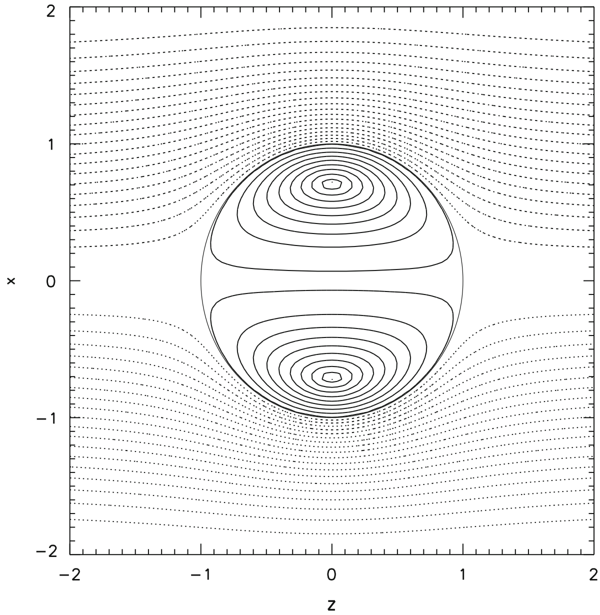


Fig. 3.10 Meridian streamlines associated with Hill's vortex. The *dotted lines* represent the irrotational flow

The equation of continuity yields:

$$g(r) = -\frac{1}{2r}\frac{d}{dr}(r^2 f)$$

The other equation gives the equation verified by f:

$$\frac{d^2}{dr^2}(r^2 f) - 2f = -2\omega r^2/a$$

the solution of which is:

$$f(r) = -\frac{\omega}{5a}r^2 + A + B/r^3$$

The two constants A and B are such that the velocity is regular at the centre of the sphere (so that $B = 0$) and that the radial velocity vanishes at $r = a$. Thus we get:

$$v_r = \frac{\omega}{5a}(a^2 - r^2)\cos\theta \qquad \text{and} \qquad v_\theta = \frac{\omega}{5a}(2r^2 - a^2)\sin\theta \qquad \text{for} \quad r \le a \tag{3.48}$$

We note that on the bounding sphere $v_\theta = \omega a/5 \sin\theta \neq 0$. Outside the sphere the flow is irrotational and the constants of integration must be adjusted such that the velocity field be continuous on the sphere and regular at infinity. The velocity potential being solution of Laplace's equation we find that

$$\Phi(r,\theta) = (A'r + B'/r^2)\cos\theta$$

The boundary conditions $v_r(a) = 0$ and $v_\theta(a) = \omega a/5 \sin\theta$ allow the calculation of A' and B' and we thus infer the velocity field:

$$v_r = \frac{2\omega a}{15}\left(-1+\left(\frac{a}{r}\right)^3\right)\cos\theta \quad \text{and} \quad v_\theta = \frac{2\omega a}{15}\left(1+\frac{1}{2}\left(\frac{a}{r}\right)^3\right)\sin\theta \quad \text{for} \quad r>a \tag{3.49}$$

The remarkable feature in these expressions is the existence of a non-zero velocity at infinity. This velocity represents the velocity of the vortex with respect to the fluid at infinity; it is uniform and along the vortex axis. Its magnitude is:

$$V = \frac{2\omega a}{15} \tag{3.50}$$

The equations for the velocity field also provide the expression for the stream function inside and outside the vortex. For an axisymmetric flow, one notes that:

$$v_r = \frac{1}{r^2 \sin\theta}\frac{\partial\psi}{\partial\theta} \qquad \text{and} \qquad v_\theta = -\frac{1}{r\sin\theta}\frac{\partial\psi}{\partial r}$$

whence, the following two expressions:

$$\psi = \frac{\omega r^2}{10a}(a^2 - r^2)\sin^2\theta \quad \text{if } r \le a \quad \text{and} \quad \psi = \frac{\omega a r^2}{15}\left(-1+\left(\frac{a}{r}\right)^3\right)\sin^2\theta \quad \text{if } r>a$$

These two stream functions give the shape of the streamlines shown in Fig. 3.10.

3.4.3.4 The Vortex Ring

The vortex ring is a spectacular figure of a fluid motion usually known as the smoke ring (see Fig. 3.11). In fact this is a vortex filament that is closed on itself and forms a circular ring, hence the name. Around it, the flow is irrotational and can be calculated with the formula (3.43). The ring being axisymmetric, the velocity is the same at all of its points and thus its motion is a uniform translation. The exact calculation of its velocity can be performed if one assumes a finite interior

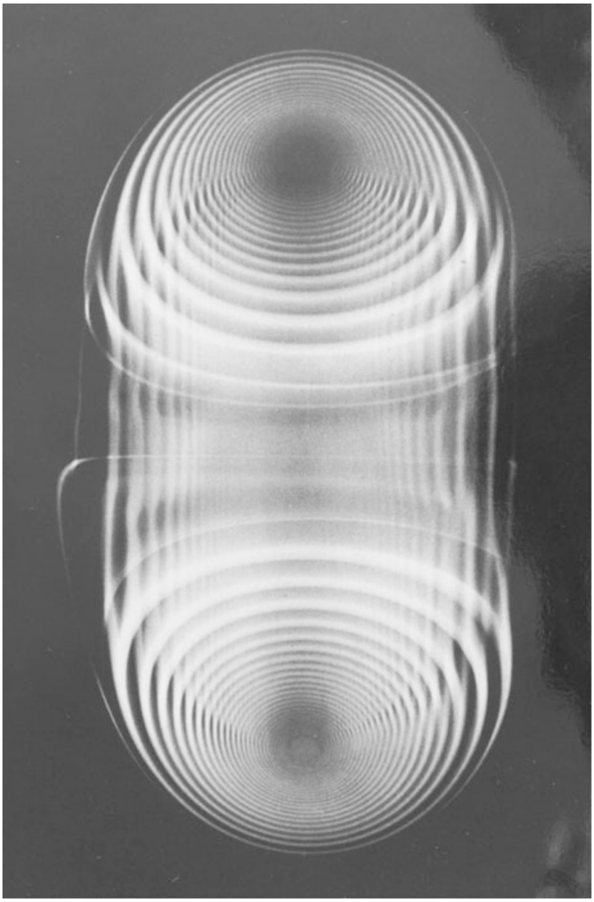


Fig. 3.11 Vortex ring obtained with smoke in the air. The ring structure shows the origin of its formation, namely the roll-up of a vortex sheet; the Reynolds number is 10^4 (from Magarvey and MacLatchy, 1964, © Canadian Science Publishing or its licensors)

radius, but is quite lengthy and we shall limit ourselves to deriving an approximate expression of it. The velocity induced by the filament is, according to (3.43),

$$\mathbf{v} = \frac{1}{4\pi} \int_{(V)} \frac{\mathbf{r} \times \boldsymbol{\omega}}{r^3} dV \tag{3.51}$$

where we have located the origin of the coordinate system on the filament (see Fig. 3.12). The ring is assumed to be a torus of major radius R and minor radius a, with $a \ll R$. One can note that

$$r(\theta) = 2R \sin\theta, \qquad \boldsymbol{\omega}(\theta) = \omega(\cos\theta \mathbf{e}_r + \sin\theta \mathbf{e}_\theta)$$

and

$$dV = \pi a^2 R d\theta'$$

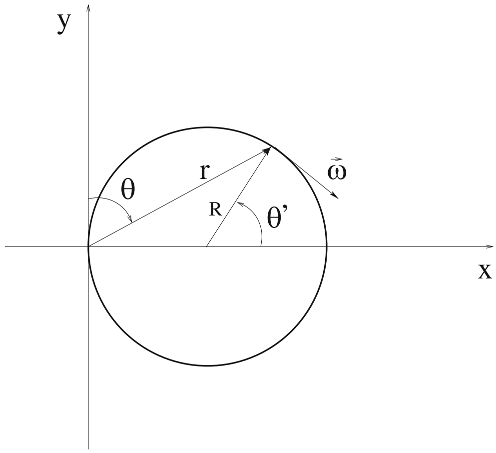


Fig. 3.12 Sketch of the vortex ring. Note that with this representation the equation of the *circle* is $r = 2R\sin\theta$

where θ' is the angle measured from the centre of the torus so that $\theta' = \pi - 2\theta$. Hence,

$$\mathbf{v} = \frac{\omega a^2}{8R}\int_0^{\pi}\frac{d\theta}{\sin\theta}\,\mathbf{e}_z$$

If one recalls that

$$\int\frac{d\theta}{\sin\theta} = \ln\tan\theta/2$$

it appears that the integral diverges at 0 and π. In fact, we have not accounted in this calculation for the fact that the core section is finite and that this effect is important for the points near the origin. An exact integration would involve elliptic integrals which are cumbersome to deal with. We thus simply estimate the order of magnitude of the integral by assuming that the integration domains is $[\varepsilon, \pi - \varepsilon]$ with $\varepsilon \approx a/R$. One finds

$$\mathbf{v} \approx \frac{\Gamma}{4\pi R}\ln(2R/a)\mathbf{e}_z$$

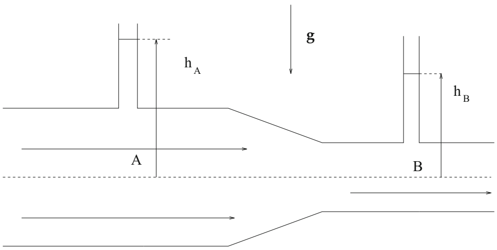
while the exact formula is:

$$\mathbf{v} = \frac{\Gamma}{4\pi R}\left(\ln\frac{8R}{a} - \frac{1}{4}\right)\mathbf{e}_z \tag{3.52}$$

These two expressions have the same asymptotic behavior as $R \to \infty$ or $a \to 0$. Our derivation indicates that the logarithmic singularity is due to the regions that are the closest to the calculation point.

3.5 Problems

1. *Streamlines and velocity equipotentials*
   Show that for an irrotational plane flow of an incompressible fluid, the streamlines are orthogonal to the potential lines.
2. *Flow in a narrowing duct*



   The flow is assumed steady and horizontal between points A and B. Show that along the z-axis, hydrostatic equilibrium is satisfied. Derive from this equilibrium the relation between P_A and h_A. Calculate the difference $h_A - h_B$ in terms of V_A and V_B, assuming an incompressible fluid. What relation holds between V_A and V_B and the cross sections of the pipe S_A and S_B?
3. *Rankine's vortex*
   Let $\mathbf{v}$ be the velocity field of a fluid of constant density ρ:

$$\begin{cases} \mathbf{v} = s\Omega \mathbf{e}_\varphi & s < a \\ \mathbf{v} = \Omega a^2 \mathbf{e}_\varphi / s & s \geq a \end{cases}$$

   where (s, φ, z) are the cylindrical coordinates.

   (a) Show that the flow is irrotational outside the cylinder of radius a.
   (b) Give the expression for the pressure in each of the subdomains. At infinity, $P = P_\infty$.
   (c) What can be said about the quantity $\frac{1}{2}v^2 + P/\rho$ in each of the subdomains? What can be concluded?
   (d) Calculate the minimum pressure at the centre of a storm with winds blowing at a maximum velocity of 50 m/s (180 km/h).
   (e) If the vortex is located over the ocean, find from the previous results the shape of the ocean surface.

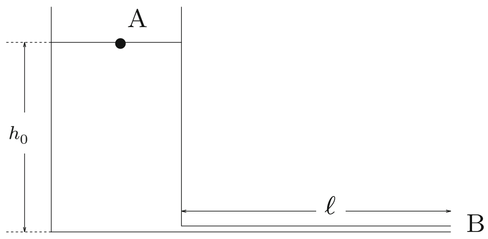
4. *Purge of a tank*

   (a) Lets consider a water tank (assumed to be an inviscid, incompressible fluid), of cross section S with initial level h_0. A valve of cross section s ($s \ll S$) located at the bottom of the tank is open.

      i. Show that the flow is irrotational.
      ii. Assuming quasi-steady flow, derive the differential equation governing $h(t)$, and solve it. Find the time it will take to empty the tank.
      iii. Show "a posteriori" that the time derivatives are indeed negligible.

   (b) One now adds to the reservoir a horizontal pipe of length ℓ, and of very small cross section compared to that of the tank. The tank is filled to level h_0 which is kept constant with time. The fluid is initially at rest. At $t = 0$ the valve at B (see figure) is opened.

      i. Derive the equation of motion of the fluid in the pipe. One assumes that the pressure in the exit jet is equal to the atmospheric pressure; solve the differential equation governing the exit velocity. Let's denote by $v_\infty = \sqrt{2gh_0}$.
      ii. The city water utility pressure is 6 bars; if the length of the connecting pipe from the main pipe to the sink is 10 m, what is the transient time when you open the tap?



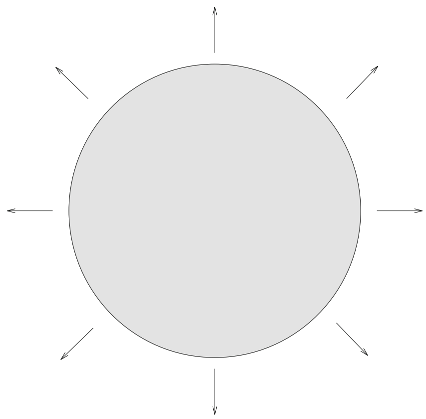
5. A U-tube contains an *incompressible* fluid subject to the gravity field $\mathbf{g} = -g\mathbf{e}_z$. The tube diameter is constant and very small compared to its length. The fluid level at equilibrium is $z = h_0$ and the free surface is at atmospheric pressure. ρ is the fluid density.

   (a) We are interested in the small oscillations of the fluid height about the average value h_0; these oscillations occur for example when the tube is slightly shaken. The fluid is assumed *perfect*. Explain why the fluid motion is necessarily *irrotational*. What can be said about the velocity inside the tube?
   (b) If Φ_A and Φ_B are the values of the velocity potential at the first and second free surfaces of the fluid, L the length of the wetted part of the tube and V the fluid velocity, show that

$$\Phi_A - \Phi_B = LV$$

(c) Derive from this the differential equation governing the time dependent height perturbation δh of the fluid in one of the branches of the tube.

6. *Motion of a liquid near an air bubble*



We assume that the liquid has a radial motion: $\mathbf{v} = v(r,t)\mathbf{e}_r$.

(a) Show that the liquid's flow is irrotational.
(b) Derive the expression for $v(r,t)$ in terms of the bubble radius $R(t)$.
(c) We assume that the air inside the bubble is an ideal gas which follows an isentropic transformation when the bubble radius varies. Neglecting the air flow, give the expression of the pressure inside the bubble in terms of the radius.
(d) Give the evolution equation of $R(t)$ (let P_0 be the value of the pressure at infinity and R_0 the radius of the bubble when $p = P_0$).
(e) If one supposes that the bubble radius oscillates slightly about the equilibrium value R_0, derive the expression for $R(t)$. What is the frequency f of such small oscillations?
(f) Numerical application: calculate f for $R_0 = 1\,\text{mm}$ and $R_0 = 5\,\text{mm}$; we give $\gamma_{\text{air}} = 1.4$, $\rho_{\text{water}} = 10^3\,\text{kg/m}^3$, $P_0 = 10^5\,\text{Pa}$.

7. Show that the potential vorticity of an inviscid compressible fluid, defined by $\boldsymbol{\omega}/\rho$, is governed by the equation

$$\mathcal{H}\left(\frac{\boldsymbol{\omega}}{\rho}\right) = \mathbf{0} \tag{3.53}$$

where $\mathcal{H}$ is the "Helmholtzian" defined by

$$\mathcal{H}(\mathbf{a}) = \frac{\partial \mathbf{a}}{\partial t} + \mathbf{v}\cdot\nabla\mathbf{a} - (\mathbf{a}\cdot\nabla)\mathbf{v} \tag{3.54}$$

Appendix: Flow Past a Plane at Incidence

When we discussed the complex potential, we remarked that the Joukovski transformation can be used to transform a circle into a flat plate. In a previous example, we used the Joukovski transformation to find the flow past a circle from the (trivial) solution of the flow past a flat plate when the velocity is parallel to it.

Now we can do the opposite. Indeed, it is less obvious to find the solution of the flow with circulation Γ past a flat plate at incidence α with respect to the flow at infinity. Conversely, if we consider the circle, we have seen that the velocity potential is:

$$\Phi = VRe(z + \frac{R^2}{z}) = V\left(r\cos\theta + \frac{R^2\cos\theta}{r}\right)$$

It is easy to add circulation to this flow since a potential vortex will still satisfy the boundary conditions; it is also possible to rotate the incoming flow velocity by an angle α with respect to the axes. With these changes, the velocity potential now reads:

$$\Phi = V\left(r\cos(\theta-\alpha) + \frac{\Gamma(\theta-\alpha)}{2\pi V} + \frac{R^2\cos(\theta-\alpha)}{r}\right)$$

This is nothing but the real part of the complex velocity potential:

$$F(z) = Vze^{-i\alpha} + \frac{\Gamma}{2i\pi}\ln z + VR^2e^{i\alpha}/z$$

where we have overlooked the constants. From this expression, one obtains the complex velocity in the image plane:

$$w' = \left(Ve^{-i\alpha} + \frac{\Gamma}{2i\pi z} - \frac{VR^2e^{i\alpha}}{z^2}\right)\left(1 - \frac{R^2}{z^2}\right)^{-1} \tag{3.55}$$

This expression is particular in the sense that it provides the velocity components in the image plane (where the obstacle is a flat plate at incidence) in terms of the coordinates in the initial plane (where the obstacle is a circle). To obtain w' in terms of z', it would be necessary to invert the relation $z' = z + R^2/z$ and to substitute the result into (3.55). But our goal is somewhat different: we only wish to examine the singularities in the flow past the obstacle and find the condition for Γ to eliminate them.

The points with z' on the flat plate correspond to z being on the circle, that is $z = Re^{i\theta}$, $\theta \in [0, 2\pi]$. Along the flat plate w' is given by:

$$
\begin{aligned}
w'(\theta) &= \left(Ve^{-i\alpha} + \frac{\Gamma e^{-i\theta}}{2i\pi R} - Ve^{i(\alpha-2\theta)}\right)\left(1 - e^{-2i\theta}\right)^{-1} \\
&= \left(Ve^{-i(\alpha-\theta)} + \frac{\Gamma}{2i\pi R} - Ve^{i(\alpha-\theta)}\right)\left(e^{i\theta} - e^{-i\theta}\right)^{-1} \\
&= \left(V\sin(\theta-\alpha) - \frac{\Gamma}{4\pi R}\right)\Big/ \sin\theta
\end{aligned}
$$

This expression is singular at $\theta = 0$ and $\theta = \pi$ if Γ is arbitrary. The trailing edge corresponds to $\theta = \pi$; we see that the singularity disappears if the circulation is chosen such that:

$$\Gamma = 4\pi RV\sin\alpha$$

One recovers here the expression (3.38) remembering that the plate length is $4R$. One notices that the flow at the leading edge is also singular, but this singularity can be eliminated by rounding the profile as shown in Fig. 3.4c.

Further Reading

The theory of irrotational flows is often well developed in standard textbooks; one can refer to Batchelor (1967). With regards to the dynamics of vorticity, further developments to the notes presented here will be found in Saffman *Vortex dynamics* (1992) or in Ting and Klein *Viscous vortical flows* (1991). On the properties of the Euler equation, extended material is proposed in Zeytounian *Mécanique des fluides fondamentale* (1991).

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