Chapter 10
Magnetohydrodynamics

Magnetohydrodynamics (MHD for the experts) is often impressive for its complexity. However, it is only the dynamics of electrically conducting fluids. It is indeed complicated because of a new vector field that enters the game, namely the magnetic field. The dynamics is different because of a new force: the Laplace force. Since conducting fluids support electric currents that may generate magnetic fields, we easily imagine that the evolution of both velocity and magnetic fields may be quite complex. In this chapter we wish to remain introductive and therefore we shall focus only on the very basis of magnetohydrodynamics.

10.1 Approximations Leading to Magnetohydrodynamics

Magnetohydrodynamics deals with the motions of a conducting fluid where there is no free charge. This implies that some approximations are met.

The first one is that the fluid motion is not relativistic, namely that the fluid velocity is much less than that of light c, i.e.

$$V/c \ll 1 \tag{10.1}$$

Two other conditions come from the absence of free charges. If our medium is a fully ionized plasma made of electrons (of charge $-e$) and ions (of charge Ze), then a first condition which should be met by a flow characterized by a length scale L is

$$L \gg \lambda_D$$

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where λ_D is the Debye length. This latter length is the mean distance beyond which the charge of an ion is screened by electrons. It depends on the temperature T and the electron density n_e of the medium, namely

$$\lambda_D = \sqrt{\frac{\varepsilon_0 kT}{n_e e^2}}$$

where ε_0 is the vacuum permittivity and k Boltzmann constant. Using international units this length reads $\lambda_D = 2.8 \times 10^{-12}\sqrt{AT/\rho Z}$ metres, where A is the mass number of the ions and ρ is the mass density. This scale may invalidate the use of the MHD equations if the fluid is too hot or too dilute.

Charge separation may also occur if the time frequency of the fluid motion is too large, i.e. larger than the plasma frequency ω_P. Thus we should also demand that the time scales of the fluid flow $T = L/V$ verifies

$$T \gg \omega_P^{-1} = \sqrt{\frac{m_e \varepsilon_0}{n_e e^2}} = 7.2 \times 10^{-16} \sqrt{\frac{A}{\rho Z}} \text{ seconds}$$

The two foregoing constraints are important in fluids with very low densities. A typical example is the solar wind. In this flow the density is very low and a more refined approach from plasma physics is often needed.

Finally, we shall suppose that the electrical conductivity is isotropic, namely that Ohm's law reads

$$\mathbf{j} = \sigma \mathbf{E} \tag{10.2}$$

where $\mathbf{j}$ is the current density, σ is the electrical conductivity and $\mathbf{E}$ is the electric field. A common source of anisotropy of the electrical conductivity is the magnetic field which induces cyclotronic motion of the electrons. Here again, dilute plasmas are more likely to be prone to such anisotropy. Conductivity is indeed much higher in the direction parallel to the magnetic field than orthogonally to it. In dense plasma, however, collision frequency is much higher than the cyclotron one ($\omega_{cyclo} = eB/m_e$). The mean charge motion is thus hardly influenced by the magnetic field and thus conductivity is a scalar.[1]

[1]Let us recall that electrons moving in a magnetic field, without shocks, follow helicoidal trajectories around field lines. Their rotation frequency is the cyclotron frequency.

10.2 The Flow Equations

As stated in the introduction the main peculiarity of the dynamics of a conducting fluid is the action of the Laplace force[2]

$$\mathbf{F}_L = \mathbf{j} \times \mathbf{B}$$

where **B** is the local magnetic field. The momentum equation is therefore

$$\rho \frac{D\mathbf{v}}{Dt} = -\nabla P + \mu \Delta \mathbf{v} + \mathbf{j} \times \mathbf{B} \tag{10.3}$$

for an incompressible fluid. Two other equations are needed to complete the momentum equation: one should give **j** and the other **B**. They will be derived from Maxwell equations and Ohm's law.

10.2.1 j *and* B *Equations*

Let us first recall the Maxwell equations for a medium whose dielectric properties are similar to those of the vacuum. Hence

$$\begin{cases} \nabla \cdot \mathbf{E} = \nabla \cdot \mathbf{B} = 0 \\ \dfrac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E} \\ \nabla \times \mathbf{B} = \mu_0 \left(\mathbf{j} + \varepsilon_0 \dfrac{\partial \mathbf{E}}{\partial t} \right) \end{cases} \tag{10.4}$$

Following a fluid particle, Ohm's law reads

$$\mathbf{j}' = \sigma \mathbf{E}' \tag{10.5}$$

where $\mathbf{j}'$ and $\mathbf{E}'$ are the current and electric field measured in a frame attached to the fluid particle. The motion of this fluid element is supposed to be non-relativistic.

[2]This force is called the Lorentz (1853–1928) force in the Anglo-Saxon world while this is Laplace force in the French literature. Laplace (1749–1827) actually gave the first analytic expression of the force that Biot & Savart measured for the action of a magnetic field on a wire carrying an electric current. It is therefore close to the force that we encounter in MHD. Lorentz force was derived for the charged particles and leads of course to the same expression for the action of a magnetic fields on an electrically conducting fluid.

Thus, the electric field viewed by the particle, $\mathbf{E}'$, is related to the one measured in the laboratory by

$$\mathbf{E} = \mathbf{E}' - \mathbf{v} \times \mathbf{B} \tag{10.6}$$

Since the fluid does not contain free charges, $\mathbf{j} = \mathbf{j}'$ so that (10.5) now reads

$$\mathbf{j} = \sigma(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \tag{10.7}$$

Finally, the fourth Maxwell equation can be simplified using (10.1). This inequality allows us to estimate the order of magnitude of the displacement term $\varepsilon_0\mu_0\partial\mathbf{E}/dt$ compared to other terms:

$$\varepsilon_0\mu_0\frac{\partial\mathbf{E}}{\partial t} = \frac{1}{c^2}\frac{\partial\mathbf{E}}{\partial t} \sim \frac{E}{c^2T} \sim \frac{LB}{c^2T^2} \sim \frac{V^2}{c^2}\nabla\times\mathbf{B} \ll \nabla\times\mathbf{B}$$

where we noted that $E \sim VB$. The displacement field is therefore very small and will be neglected. The magnetic field thus verifies:

$$\begin{cases} \nabla\cdot\mathbf{B} = 0 \\ \nabla\times\mathbf{B} = \mu_0\mathbf{j} \\ \dfrac{\partial\mathbf{B}}{\partial t} = -\nabla\times\mathbf{E} \\ \mathbf{j} = \sigma(\mathbf{E} + \mathbf{v}\times\mathbf{B}) \end{cases} \tag{10.8}$$

The third equation is called the *induction equation*. It is usually written without the electric field, namely as:

$$\frac{\partial\mathbf{B}}{\partial t} = \nabla\times(\mathbf{v}\times\mathbf{B}) - \nabla\times(\eta\nabla\times\mathbf{B}) \tag{10.9}$$

where $\eta = 1/(\mu_0\sigma)$ is the *magnetic diffusivity*. This quantity is expressed in m^2/s just as the kinematic viscosity. Thus, a new number arises, namely *the magnetic Prandtl number*, which is defined as

$$\mathcal{P}_m = \frac{\nu}{\eta}$$

When η is a constant, the induction equation reads:

$$\frac{\partial\mathbf{B}}{\partial t} = \nabla\times(\mathbf{v}\times\mathbf{B}) + \eta\Delta\mathbf{B} \tag{10.10}$$

We observe that this is a linear equation for **B**. Using non-dimensional variables, we rewrite this equation as

$$\frac{\partial \mathbf{B}}{\partial \tau} = \nabla \times (\mathbf{u} \times \mathbf{B}) + \frac{1}{\mathrm{Re}_m} \Delta \mathbf{B} \tag{10.11}$$

where $\mathrm{Re}_m = \frac{VL}{\eta}$ is the *magnetic Reynolds number*.

10.2.2 Boundary Conditions on the Magnetic Field

The partial differential equations that we derived for the magnetic field need to be completed by boundary conditions. Basically, the magnetic field must be continuous. However, in many situations it is desirable to idealize the medium which bounds the flow. This is often a way to avoid the computation of the magnetic field outside the fluid domain. Just like temperature, two ideal cases are used: the perfect conductor or the perfect insulator.

10.2.2.1 Boundary Conditions at an Electrical Insulator

If the fluid is bounded by an electrical insulator, the current outside the domain is vanishing, namely **j** is zero. Hence, outside the fluid domain

$$\nabla \times \mathbf{B} = \mathbf{0} \qquad \Longleftrightarrow \qquad \mathbf{B} = \nabla \Phi$$

Since **B** is continuous at the surface, **B** must match a potential field. On the other hand, no current crosses the boundary so

$$\mathbf{j} \cdot \mathbf{n} = 0 \qquad \Longleftrightarrow \qquad \mathbf{n} \cdot \nabla \times \mathbf{B} = 0$$

This equation shows that if the field is continuous, this is not the case for all its derivatives. The normal component of the curl is the only combination of the derivatives that is continuous.

10.2.2.2 Boundary Conditions at a Perfect Electrical Conductor

These boundary conditions are definitely more delicate to establish. To be as clear as possible, we shall consider the case of a fluid meeting a (solid) conductor whose conductivity will be increased up to infinity. We then focus on the field inside the wall.

Let us assume that the magnetic field includes a time-variation like $e^{i\omega t}$ (this might just be the Fourier component of a more complex time dependence). In the solid, we assume that $\mathbf{v} = \mathbf{0}$. Hence, $\mathbf{B}$ verifies

$$i\omega\mathbf{B} = \eta_s \Delta\mathbf{B}$$

where η_s is the diffusivity inside the solid. We shall let this quantity vanish. If we assume the surface separating the fluid and the solid to be the plane $z = 0$ (solid $z \geq 0$, fluid $z < 0$), then we might also assume that the variations of $\mathbf{B}$ along z are much faster that along the other directions. In other words, we assume the existence of a boundary layer. In this layer, $\mathbf{B}$ verifies

$$i\omega\mathbf{B} = \eta_s \frac{\partial^2\mathbf{B}}{\partial z^2}$$

whose solution is

$$\mathbf{B} = \mathbf{B}_0(x, y)e^{-(1+i)z/\delta} \tag{10.12}$$

Here, $\mathbf{B}_0(x, y)$ is the field at $z = 0$ and δ is the boundary layer thickness (the skin depth in electromagnetism). Let us underline the similarity of (10.12) with the Ekman layer: the field shows an oscillatory damping (but without changing direction). The thickness of the layer reads:

$$\delta = \sqrt{\frac{2\eta_s}{\omega}} = \sqrt{\frac{2}{\omega\sigma_s\mu_0}} \tag{10.13}$$

The foregoing expression shows that the magnetic field does not penetrate into the solid if the product $\omega\sigma_s$ goes to infinity.

Now, if we use the flux conservation in the solid, namely that $\nabla \cdot \mathbf{B} = 0$, the normal component of $\mathbf{B}$ may be expressed with the divergence of the tangent field, i.e.

$$B_{0z} = (1 + i)\delta\nabla_h \cdot \mathbf{B}_0(x, y)$$

Again, this equation is very similar to that of the Ekman pumping (8.53). It shows that when the thickness of the layer goes to zero, this component vanishes. We thus find that at the boundary of a perfect conductor

$$\mathbf{B} \cdot \mathbf{n} = 0 \tag{10.14}$$

Namely, that the field does not penetrate into a perfect conductor. We may also observe that in the solid the field is proportional to $e^{-z/\delta}$ with $\delta \to 0$. Thus, the field is zero inside the perfect conductor. However, the tangential component of the field may remain finite at the surface, thus having a jump at the surface (note that the normal component is continuous though). The gradient of the field, and thus the current density diverges near the surface. Indeed,

$$\mathbf{j} = \frac{1}{\mu_0}\nabla \times \mathbf{B} = \frac{(1+i)}{\mu_0\delta}\left|\begin{array}{l} B_{0y} \\ -B_{0x} \\ 0 \end{array}\right. e^{-(1+i)z/\delta}$$

Thus, there is always some current at the surface of the conductor. One usually introduces a *surface current density* $\mathbf{j}_S$ defined by

$$\mathbf{j}_S = \int_0^{+\infty} \mathbf{j} dz = \mathbf{n} \times \mathbf{B}/\mu_0$$

which is finite and measures the jump in the tangential components of the magnetic field when one crosses the bounding surface.

Now, let us focus on the electric field in the solid. We have

$$\mathbf{E} = \frac{(1+i)}{\sigma_s\mu_0\delta}\mathbf{e}_z \times \mathbf{B}_0 e^{-(1+i)z/\delta} = \mathcal{O}(\sqrt{\frac{\omega}{\sigma_s}})$$

It shows that this field disappears as expected when the conductivity is infinite. Thus at the interface with a perfect conductor we should write

$$\mathbf{E} \times \mathbf{n} = \mathbf{0} \tag{10.15}$$

namely, the tangential electric field vanishes. Using Ohm's law and the fact that **B** et **j** are both tangential to the bounding surface, this condition also reads

$$\mathbf{j} \times \mathbf{n} = \mathbf{0} \qquad \text{or} \qquad (\nabla \times \mathbf{B}) \times \mathbf{n} = \mathbf{0} \tag{10.16}$$

10.2.3 The Energy Equation with a Magnetic Field

10.2.3.1 The Maxwell Tensor

We shall first note that the Laplace force $\mathbf{j} \times \mathbf{B}$ is also the divergence of a tensorial field. Indeed,

$$(\mathbf{j} \times \mathbf{B})_i = \partial_j \Sigma_{ij}$$

where

$$\Sigma_{ij} = \frac{1}{\mu_0}(B_i B_j - \frac{1}{2}B^2\delta_{ij})$$

$[\Sigma]$ is the magnetic stress tensor or Maxwell tensor.

10.2.3.2 Joule Heating

To derive the equation governing the local evolution of internal energy we need to start with the energy balance that lead us to (1.29) and to introduce the magnetic terms.

For that, we consider some volume independent of time because the magnetic field is not attached to the fluid. The energy balance in this volume reads

$$\frac{d}{dt}\int_{(V)}\left[\rho(\frac{1}{2}v^2+e)+\frac{B^2}{2\mu_0}\right]dV = -\int_{(S)}\rho(\frac{1}{2}v^2+e)\mathbf{v}\cdot d\mathbf{S} - \int_{(S)}\mathbf{E}\times\mathbf{B}/\mu_0\cdot d\mathbf{S}$$
$$+\int_{(V)}\mathbf{f}\cdot\mathbf{v}dV + \int_{(S)}v_i\sigma_{ij}dS_j$$
$$-\int_{(S)}\mathbf{F}\cdot d\mathbf{S} + \int_{(V)}\mathcal{Q}dV$$

In this expression the new terms are the magnetic energy density that completes internal and kinetic energies, and the Poynting flux $\mathbf{E}\times\mathbf{B}/\mu_0$ that represents the surface flux of electromagnetic energy through the boundary. We may be surprised of the absence of the Laplace force. This is natural: the Laplace force does not modify the energy content of the volume. It just permits exchanges between the kinetic and magnetic energies reservoirs. To simplify the foregoing energy balance we need using the magnetic field (10.9). After a scalar product by $\mathbf{B}/\mu_0$ and using (12.40), we obtain

$$\frac{\partial}{\partial t}\left(\frac{B^2}{2\mu_0}\right) = \nabla\cdot[(\mathbf{v}\times\mathbf{B}-\eta\nabla\times\mathbf{B})\times\mathbf{B}]/\mu_0 - \mathbf{v}\cdot(\mathbf{j}\times\mathbf{B}) - \eta(\nabla\times\mathbf{B})^2/\mu_0 \qquad (10.17)$$

This expression shows that the power of the Laplace force $\mathbf{v}\cdot(\mathbf{j}\times\mathbf{B})$ extracts energy from the magnetic reservoir (and so fills that of kinetic energy).

Now, combining the magnetic energy equation with that of kinetic energy (1.28) (completed with the Laplace force work), we finally get:

$$\rho\frac{De}{Dt} = \nabla\cdot(\chi\nabla T) - P\nabla\cdot\mathbf{v} + \frac{\mu}{2}(\nabla:\mathbf{v})^2 + \zeta(\nabla\cdot\mathbf{v})^2 + \eta(\nabla\times\mathbf{B})^2/\mu_0 \qquad (10.18)$$

This equation shows that the magnetic field is at the origin of a new source of internal energy (and entropy) through the term $\eta(\nabla\times\mathbf{B})^2/\mu_0$ which represents the Joule heating (see exercises).

10.3 Some Properties of MHD Flows

10.3.1 The Frozen Field Theorem

When the diffusion time of the magnetic field L^2/η is large compared to the advection time L/V, the magnetic field is just like "frozen" in the fluid. The field lines are attached to the fluid particles. Then, we can show the following theorem:

When the magnetic Reynolds number increases to infinity, the magnetic field flux through a surface attached to fluid particles is constant.

To prove this theorem, we need to show that the integral $\int \mathbf{B} \cdot d\mathbf{S}$ is constant when S is attached to fluid particles. If we use the vector potential $\mathbf{A}$ of the magnetic field, the demonstration is quite similar to that of Kelvin's theorem that we studied in Chap. 3. Indeed, setting $\eta = 0$, we have

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}) \qquad \Longleftrightarrow \qquad \frac{\partial \mathbf{A}}{\partial t} = \mathbf{v} \times \nabla \times \mathbf{A} + \nabla Q \tag{10.19}$$

where Q is an arbitrary function.

Let us call $\phi(t)$ the magnetic flux through the surface $S(t)$ which leans on the contour $C(t)$ carried by the fluid. We write

$$\phi(t) = \int_{S(t)} \mathbf{B} \cdot d\mathbf{S} = \oint_{C(t)} \mathbf{A} \cdot d\mathbf{l} \tag{10.20}$$

Just like in Kelvin's theorem demonstration, we use relation (1.13) and find

$$\frac{d\phi}{dt} = \oint_{C(t)} \left(\frac{\partial A_i}{\partial t} - (\mathbf{v} \times \mathbf{B})_i \right) dl_i = \oint_{C(t)} \partial_i Q dl_i = 0$$

where we used (10.19). The flux of $\mathbf{B}$ through an open surface given by a contour attached to fluid particles is therefore a constant.

10.3.2 Magnetic Pressure and Magnetic Tension

The Laplace force may also be written as

$$\mathbf{j} \times \mathbf{B} = \frac{1}{\mu_0} (\nabla \times \mathbf{B}) \times \mathbf{B} = \frac{1}{\mu_0} (\mathbf{B} \cdot \nabla)\mathbf{B} - \nabla \left(\frac{B^2}{2\mu_0} \right)$$

thanks to (12.43). In this new expression we note that part of the force derives from a scalar potential which is known as the *magnetic pressure*. We set

$$P_m = \frac{B^2}{2\mu_0} \tag{10.21}$$

The magnetic pressure is therefore identical to the magnetic energy density.[3]

The remaining term, $(\mathbf{B} \cdot \nabla)\mathbf{B}$ can be split, just as $(\mathbf{v} \cdot \nabla)\mathbf{v}$ in Euler's equation (see Chap. 3, 3.12), as

$$(\mathbf{B} \cdot \nabla)\mathbf{B} = B\frac{\partial B}{\partial s}\mathbf{e}_s + \frac{B^2\mathbf{n}}{R_s} \tag{10.22}$$

The first term is called the *magnetic tension* since it is parallel to the field line while the second term is the *curvature force* since it grows as the radius of curvature R_s decreases. Note that the tension term is equal to the longitudinal component of the magnetic pressure gradient. This is a consequence of the fact that the Laplace force has no component along **B**.

10.3.3 Force-Free Fields

When the current density is parallel to the magnetic field, the Laplace force vanishes. Such situations are thought to exist (approximately) in regions where the magnetic pressure is strong enough to control the distribution of matter, and therefore that of currents. The most famous example is the atmosphere of the Sun: there the magnetic field is dominating and shapes the distribution of matter. The most spectacular illustration of this situation is given by solar prominences (see Fig. 10.1). The structure of these magnetic features is often approximated using a force-free field. Below, we shall also use a force-free field to get a simple example of a dynamo.

If the Laplace force is vanishing, then the magnetic field verifies the following extra-equation

$$\nabla \times \mathbf{B} = K(\mathbf{r})\mathbf{B} \tag{10.23}$$

The function $K(\mathbf{r})$ is unknown, but since $\nabla \cdot \mathbf{B} = 0$, it must verify:

$$\mathbf{B} \cdot \nabla K = 0$$

which means that it is constant along the field lines.

[3]The reader may verify that an energy volumic density is dimensionally identical to a pressure.

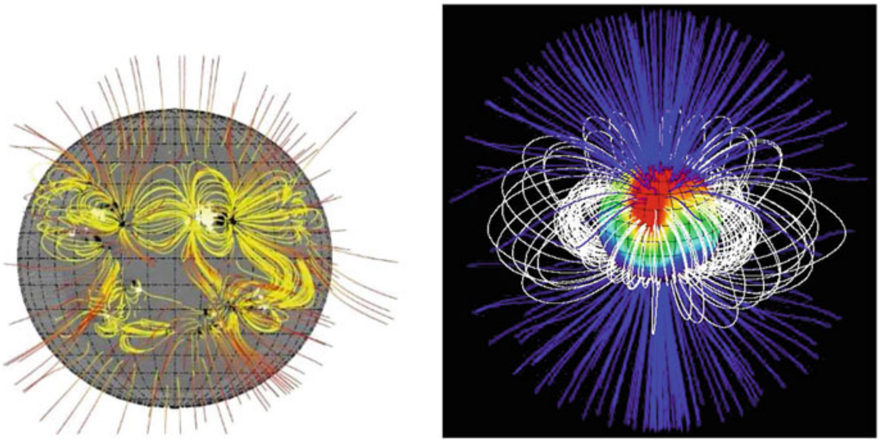

Fig. 10.1 *Left* a modelling of the solar coronal magnetic fields using force-free fields by Tadesse *et al.* (2013). *Right* extrapolation of the magnetic field of the star V374 Pegasi from spectropolarimetric observations. This star is a red dwarf of 0.28 solar mass with a radius about one third that of the Sun. Its magnetic field is quite strong (0.2 T) and generated by a turbulent dynamo triggered by the thermal convection that transport heat from the central regions to the surface (see Morin *et al.*, 2008). (Picture by M. Jardine & J.-F. Donati)

One example of a force-free field may be obtained if we assume K to be constant. Then, $-\Delta\mathbf{B} = K\nabla\times\mathbf{B} = K^2\mathbf{B}$, which means that $\mathbf{B}$ verifies Helmholtz equation, namely:

$$(\Delta + K^2)\mathbf{B} = \mathbf{0}$$

Since we took the curl of (10.23), the solution of this equation are too general, but among them, there are some where the curl of $\mathbf{B}$ is parallel to $\mathbf{B}$. A simple example is given by

$$\mathbf{B} = B_0 \left| \begin{array}{l} \cos Kz \\ \sin Kz \\ 0 \end{array} \right.$$

which is indeed a force-free field.

There are other solutions in cylindrical or spherical geometry, but they are more complex (see Moffatt 1978 for instance).

10.3.4 The Equipartition Solutions and Elsässer Variables

When the fluid is incompressible and when diffusive effects are neglected ($\nu = \eta = 0$), there exists a simple steady solution to the equations of MHD. The equations for a steady flow indeed read:

$$\rho \mathbf{v} \cdot \nabla \mathbf{v} = -\nabla(P + P_m) + \frac{1}{\mu_0}\mathbf{B} \cdot \nabla \mathbf{B}$$

$$\nabla \times (\mathbf{v} \times \mathbf{B}) = \mathbf{0}$$

$$\nabla \cdot \mathbf{v} = \nabla \cdot \mathbf{B} = 0$$

These equations are satisfied if

$$\mathbf{v} = \pm \frac{\mathbf{B}}{\sqrt{\rho\mu_0}} \qquad \text{and} \qquad P = \mathrm{Cst} - P_m \tag{10.24}$$

This is the *equipartition solution* because

$$\frac{1}{2}\rho \mathbf{v}^2 = \frac{\mathbf{B}^2}{2\mu_0}$$

The kinetic energy density equals the magnetic energy density. This solution shows that the quantity $B/\sqrt{\rho\mu_0}$ is a velocity. This is the *Alfvén speed*. The equipartition solution is a solution of the nonlinear equations. It slowly fades with time when diffusion is included (over a typical diffusion time $\min(L^2/\nu, L^2/\eta)$). Chandrasekhar (1961) has shown that this solution is linearly stable.

We shall come back later on the physical meaning of the Alfvén speed. We may however notice that it naturally introduces new variables, called the *Elsässer variables*, which are defined as:

$$\mathbf{z}^{\pm} = \mathbf{v} \pm \frac{\mathbf{B}}{\sqrt{\rho\mu_0}} \tag{10.25}$$

Combining the momentum and induction equations (and taking into account diffusion terms), we can derive this other form of the MHD equations, namely

$$\begin{cases} \dfrac{\partial \mathbf{z}^+}{\partial t} + \mathbf{z}^- \cdot \nabla \mathbf{z}^+ = -\nabla \pi + \nu_+ \Delta \mathbf{z}^+ + \nu_- \Delta \mathbf{z}^- \\[2ex] \dfrac{\partial \mathbf{z}^-}{\partial t} + \mathbf{z}^+ \cdot \nabla \mathbf{z}^- = -\nabla \pi + \nu_- \Delta \mathbf{z}^+ + \nu_+ \Delta \mathbf{z}^- \end{cases} \tag{10.26}$$

where we set $\pi = (P + P_m)/\rho$ and $\nu_\pm = \frac{1}{2}(\nu \pm \eta)$. The equipartition solutions are simply $\mathbf{z}^{\pm} = \mathbf{0}$.

10.4 The Waves

10.4.1 Alfvén Waves

Let us consider an incompressible fluid bathed by a uniform magnetic field $\mathbf{B}$. The fluid is in equilibrium. Small amplitude perturbations are denoted $\mathbf{b}$ for the magnetic field, $\mathbf{v}$ for the velocity and p for the pressure. We neglect diffusion. These perturbations are governed by the following equations:

$$\begin{cases} \rho_0 \dfrac{\partial \mathbf{v}}{\partial t} = -\nabla p + \dfrac{1}{\mu_0}(\nabla \times \mathbf{b}) \times \mathbf{B} \\ \\ \dfrac{\partial \mathbf{b}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}) \\ \\ \nabla \cdot \mathbf{v} = \nabla \cdot \mathbf{b} = \mathbf{0} \end{cases} \tag{10.27}$$

We first derive the dispersion relation of the associated freely propagating waves. We set

$$\mathbf{v} \propto \exp(i\omega t + i\mathbf{k}\cdot\mathbf{x}), \qquad \mathbf{b} \propto \exp(i\omega t + i\mathbf{k}\cdot\mathbf{x})$$

The system (10.27) turns into

$$\begin{cases} \rho_0 \omega \mathbf{v} = -p\mathbf{k} + (\mathbf{k} \times \mathbf{b}) \times \mathbf{B}/\mu_0 \\ \\ \omega \mathbf{b} = \mathbf{k} \times (\mathbf{v} \times \mathbf{B}) \\ \\ \mathbf{v} \cdot \mathbf{k} = \mathbf{b} \cdot \mathbf{k} = 0 \end{cases} \tag{10.28}$$

which leads to

$$\begin{cases} \mathbf{b} = \frac{\mathbf{k}\cdot\mathbf{B}}{\omega}\mathbf{v} \\ \\ \left(\omega^2 - \frac{(\mathbf{B}\cdot\mathbf{k})^2}{\rho_0\mu_0}\right) \mathbf{v} \times \mathbf{k} = \mathbf{0} \end{cases} \tag{10.29}$$

so that the dispersion relation is

$$\omega^2 = \frac{(\mathbf{B}\cdot\mathbf{k})^2}{\rho_0\mu_0} \tag{10.30}$$

We now introduce the Alfvén speed $V_A = B/\sqrt{\rho_0\mu_0}$ and θ the angle between the wave vector $\mathbf{k}$ and $\mathbf{B}$. We get

$$\frac{\omega}{k} = V_A \cos\theta \tag{10.31}$$

The Alfvén speed is therefore the maximum velocity of the waves. These waves are the *Alfvén waves*. Just like inertial waves or gravity waves, these waves do not propagate isotropically. They cannot propagate in a direction perpendicular to the field, and are the fastest in the direction parallel to the field lines. These waves may be thought as the ones propagating along a string, where the magnetic field lines play the role of the string. Their group velocity,

$$\mathbf{v}_g = \nabla_k\omega(\mathbf{k}) = \frac{\mathbf{B}}{\sqrt{\rho_0\mu_0}} = V_A\frac{\mathbf{B}}{B}$$

shows that the energy only propagates along the field lines at the Alfvén speed. Unlike inertial and gravity waves, the phase and group velocities are not orthogonal.

10.4.2 Magnetosonic Waves

When the compressibility of the fluid cannot be neglected, Alfvén waves are coupled with acoustic waves and form the set of *magnetosonic waves* which we now study.

To analyse their properties, we start from (10.27) but now taking into account the density perturbations. Still considering infinitesimal amplitudes, mass conservation now implies:

$$\frac{\partial\rho}{\partial t} + \rho_0\nabla\cdot\mathbf{v} = 0$$

Just like for acoustic waves, the density perturbation is related to the pressure one by $p = c_s^2\rho$, where c_s is the sound speed (see 5.16). As usual, we decompose the disturbances on the plane waves and get the following relations between the amplitudes:

$$\begin{cases} \mathbf{b}\cdot\mathbf{k} = 0 \\ \omega\rho + \rho_0\mathbf{v}\cdot\mathbf{k} = 0 \\ \omega\mathbf{b} = \mathbf{k}\times(\mathbf{v}\times\mathbf{B}) \\ \rho\omega\mathbf{v} = -p\mathbf{k} + (\mathbf{k}\times\mathbf{b})\times\mathbf{B}/\mu_0 \\ p = c_s^2\rho \end{cases} \tag{10.32}$$

Eliminating the pressure, the magnetic field and the density, we are left with an equation where there is only the velocity amplitude:

$$\rho_0\mu_0\omega^2\mathbf{v} = (\rho_0\mu_0c_s^2 + B^2)(\mathbf{v}\cdot\mathbf{k})\mathbf{k} + (\mathbf{k}\cdot\mathbf{B})^2\mathbf{v} - (\mathbf{v}\cdot\mathbf{B})(\mathbf{k}\cdot\mathbf{B})\mathbf{k} - (\mathbf{k}\cdot\mathbf{B})(\mathbf{k}\cdot\mathbf{v})\mathbf{B}$$

The dispersion relation is conveniently derived if we write this equation in a matrix form (like 5.8). Then, we set to zero the determinant. Very generally, we may fix the direction of some vectors. For instance, we may choose $\mathbf{B} = B\mathbf{e}_z$ and, using cylindrical coordinates, set $\mathbf{k} = k_s\mathbf{e}_s + k_z\mathbf{e}_z$. As before, θ is the angle between $\mathbf{B}$ and $\mathbf{k}$. We also set $B^2 = \rho_0\mu_0 V_A^2$. Thus, the foregoing equation reads

$$\Omega^2\mathbf{v} = [(c_s^2 + V_A^2)\mathbf{k}\cdot\mathbf{v} - V_A^2 k_z v_z]\mathbf{k} - V_A^2(\mathbf{k}\cdot\mathbf{v})k_z\mathbf{e}_z$$

where $\Omega^2 = \omega^2 - k_z^2 V_A^2$. More explicitly,

$$\begin{cases} \Omega^2 v_s = (V_B^2 k_s^2)v_s + (c_s^2 k_s k_z)v_z \\ \Omega^2 v_\varphi = (V_B^2 k_\varphi k_s)v_s + (c_s^2 k_\varphi k_z)v_z \\ \Omega^2 v_z = (c_s^2 k_s k_z)v_s + (c_s^2 - V_A^2)k_z^2 v_z \end{cases} \tag{10.33}$$

where $V_B^2 = V_A^2 + c_s^2$. Zeroing the determinant of this system leads to the dispersion relation:

$$(\omega^4 - \omega^2 k^2(V_A^2 + c_s^2) + k^2 c_s^2 k_z^2 V_A^2)(\omega^2 - k_z^2 V_A^2) = 0 \tag{10.34}$$

This new relation contains three types of waves that are specified by their phase velocity ω/k. The first one is the pure Alfvén wave:

$$\frac{\omega}{k} = V_A \cos\theta$$

For this wave, compressibility does not play any role. Indeed, taking the last two equations of (10.33), we find that $k_s v_s = k_z v_z = 0$ and v_φ undetermined. This is always a transverse wave, namely $\mathbf{k}\cdot\mathbf{v}=0$, and the density fluctuation is always zero. The velocity field is perpendicular to the plane formed by the wave vector and the magnetic field.

The two other waves have the following phase velocity:

$$\begin{cases} V_f = \sqrt{\dfrac{V_A^2 + c_s^2 + \sqrt{(V_A^2 + c_s^2)^2 - 4\cos^2\theta\, V_A^2 c_s^2}}{2}} \\ V_s = \sqrt{\dfrac{V_A^2 + c_s^2 - \sqrt{(V_A^2 + c_s^2)^2 - 4\cos^2\theta\, V_A^2 c_s^2}}{2}} \end{cases} \tag{10.35}$$

They correspond, respectively, to *the fast magnetosonic wave* and *the slow magnetosonic wave*. In general these waves are neither transverse nor longitudinal.

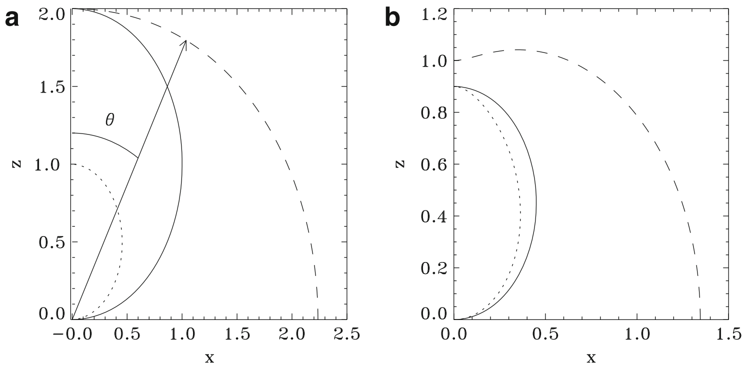

Fig. 10.2 Phase velocity of the three magnetosonic waves in polar coordinates. The *solid line* denotes the Alfvén wave, the *dotted line* is for the slow magnetosonic wave and the *dashed line* the fast magnetosonic wave. *Left* The case where the medium is such that $V_A > c_s$. *Right* when $V_A < c_s$ (here $V_A = 0.9c_s$)

Two cases deserve some attention: the case when the wave propagates either perpendicularly ($\theta = \pi/2$) to the magnetic field or along it ($\theta = 0$).

- If $\theta = 0$, then $V_f = c_s$ and $v_s = v_\varphi = 0$. The wave is longitudinal. This is just a plain acoustic wave. There is no magnetic field perturbation. The slow wave verifies $V_s = V_A$: this is a pure Alfvén wave.
- If $\theta = \pi/2$, we find only the fast magnetosonic wave, propagating with the phase velocity $V_f = \sqrt{V_A^2 + c_s^2}$. It is longitudinal. In fact, this is an acoustic wave which propagates in a fluid whose pressure is increased by the magnetic pressure.

We show in Fig. 10.2 the phase velocity of all these waves for every angle θ. This diagram is sometimes called *Friedrich diagram*.

10.5 The Dynamo Problem

One of the fascinating properties of conducting fluids is their ability to generate magnetic fields by the famous *dynamo effect*. Thanks to this effect, planets like the Earth or Jupiter or stars like the Sun own a magnetic field.

The dynamo problem is also one of the most complex in Fluid Mechanics, because no simple solution of this problem exists.

10.5.1 The Kinematic Dynamo

To make a first step into the dynamo problem, it is convenient to start with the one of the kinematic dynamo. A kinematic dynamo is a velocity field that amplifies the magnetic field without being perturbed by the Lorentz force. Such a velocity field is prescribed and whether the solution of the induction equations grows or not the velocity field is considered as a dynamo or not. Thus we give **v** and solve:

$$\begin{cases} \dfrac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}) + \eta \Delta \mathbf{B} \\ \\ \nabla \cdot \mathbf{B} = 0 \end{cases} \tag{10.36}$$

with boundary conditions. Note that this problem is linear for **B**. We may set

$$\mathbf{B}(\mathbf{r}, t) = \mathbf{B}(\mathbf{r}) e^{\lambda t}$$

and the velocity field is a kinematic dynamo if and only if there exist a critical diffusivity $\eta_{\rm crit}$ such that if $\eta < \eta_{\rm crit}$ then $\mathrm{Re}(\lambda) > 0$. Introducing a length scale L and a velocity scale V, we may associate with $\eta_{\rm crit}$ a critical magnetic Reynolds number beyond which the magnetic field is amplified.

The reader may have guessed that finding a kinematic dynamo is much more difficult than determining the stability of a flow. Indeed, in this new problem, a very large number of "parameters" control the stability of the magnetic field. These are all the values of the function $\mathbf{v}(\mathbf{r})$ in the fluid's domain. In fact, a kinematic dynamo is to be found in a function space. In addition, we shall see below that, when a dynamo exists, it cannot be a simple velocity field.

Two kinds of kinematic dynamos are usually distinguished: the *fast* dynamos and the *slow* dynamos. If the time scale which controls the growth of the magnetic field is the diffusive one, namely L^2/η then the dynamo is said to be slow. If, on the other hand, this time scale is the advective one, i.e. L/V, then the dynamo is said to be fast.

Presently, nobody knows a criterion on the velocity field that tells whether a dynamo is slow or fast. We only know that some characteristics of the flow are favourable for a fast dynamo. For instance, if the trajectory of the Lagrangian particles are chaotic or if the velocity field owns shear discontinuities, the amplification of the magnetic field may be fast.

10.5.2 The Amplification of the Magnetic Field

As a first step, we shall examine the ways magnetic fields can be amplified by a flow. This is actually the role of the term $\nabla \times (\mathbf{v} \times \mathbf{B})$ in the induction equation.

To show this, let us consider the case where the velocity field is zero and show that necessarily the magnetic field disappears. **B** verifies

$$\begin{cases} \dfrac{\partial \mathbf{B}}{\partial t} = \eta \Delta \mathbf{B} \\ \\ \nabla \cdot \mathbf{B} = 0 \\ \text{Boundary conditions} \end{cases} \tag{10.37}$$

We scalarly multiply the first equation by **B** and integrate it over the whole space. This gives us the evolution of the magnetic energy (up to the μ_0 factor):

$$\frac{d}{dt}\int \frac{1}{2}B^2 dV = \eta \int \mathbf{B}\cdot\Delta\mathbf{B}dV \tag{10.38}$$

with (12.40) we find

$$\nabla\cdot(\mathbf{B}\times(\nabla\times\mathbf{B})) = (\nabla\times\mathbf{B})^2 - \mathbf{B}\cdot\nabla\times\nabla\times\mathbf{B}$$

thus

$$\int_{(V)} \mathbf{B}\cdot\Delta\mathbf{B}dV = -\int_{(V)} \mathbf{B}\cdot\nabla\times\nabla\times\mathbf{B}dV = \int_{(S)} (\mathbf{B}\times\nabla\times\mathbf{B})\cdot d\mathbf{S} - \int_{(V)} (\nabla\times\mathbf{B})^2 dV$$

Now we assume that the conducting fluid does not fill the whole space, so that $\nabla\times\mathbf{B}$ is zero on some sufficiently large surface S. The magnetic energy verifies:

$$\frac{dE_M}{dt} = -\eta\int_{(V)} (\nabla\times\mathbf{B})^2 dV \tag{10.39}$$

which shows that it decreases with time. If there is some amplification, it must come from the $\nabla\times(\mathbf{v}\times\mathbf{B})$-term, which we now study.

Using the equality (12.41), the induction equation may be rewritten as:

$$\frac{D\mathbf{B}}{Dt} = \overbrace{(\mathbf{B}\cdot\nabla)\mathbf{v}}^{\text{(I)}} - \overbrace{\mathbf{B}\nabla\cdot\mathbf{v}}^{\text{(II)}} + \eta\Delta\mathbf{B} \tag{10.40}$$

Two terms are potentially able to amplify the magnetic field. The role of the second one (II) is easy to understand: when the (compressible) fluid flow is convergent ($\nabla\cdot\mathbf{v} < 0$) the field lines are gathered, the flux density (namely, the field) increases (see Fig. 10.3a). This term disappears when the compressibility of the fluid vanishes. It remains however the first term (I) which is also able to increase the magnetic field. We see that this phenomenon occurs when the velocity gradient is parallel to the magnetic field. In this case the component of the magnetic field along the velocity vector grows. This is illustrated in Fig. 10.3b. There, we see that if the magnetic field

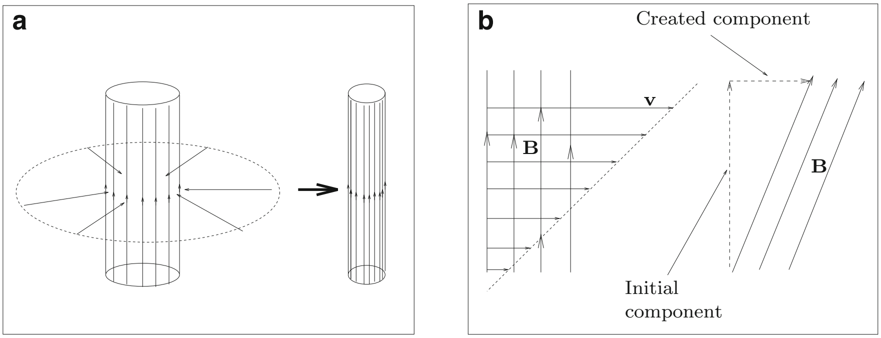

Fig. 10.3 Amplification of **B** by (**a**) a converging flow: the flow convergence gathers the *field lines* thus increasing **B** but does not change its direction. In (**b**) **B** is amplified through the raise of a new component

owns a single component and the velocity field is in the direction orthogonal to it, with some shear, then the magnetic field gets a new component parallel to **v** while the initial component is not affected. Hence, the magnetic energy increases locally. One may note the analogy with vorticity (see Chap. 3).

10.5.3 Some Anti-Dynamo Theorem

The dynamo problem is a difficult one because there is no simple velocity field that is able to amplify a magnetic field. Generally speaking, a dynamo has a low degree of symmetry. We show below that no purely axisymmetric dynamo exists. This result is the first antidynamo theorem. It was first demonstrated by Cowling (1933) and may be stated as follows:

An axisymmetric magnetic field cannot be sustained by an axisymmetric velocity field.

We demonstrate this theorem by showing that if both **B** and **v** are axisymmetric, then **B** necessarily decays. For this, we first write **B** in cylindrical coordinates:

$$\mathbf{B}(s,z,t) = \left| \begin{array}{l} \dfrac{1}{s}\dfrac{\partial A}{\partial z} \\ B \\ -\dfrac{1}{s}\dfrac{\partial A}{\partial s} \end{array} \right. \tag{10.41}$$

Here B is the toroidal component of the field while A is related to the toroidal component of the vector potential by $A = -sA_\varphi$. A controls the meridional field (also called the poloidal component). We now derive equations governing the evolution of A and B (details are given in appendix).

Using the components along $\mathbf{e}_s$ and $\mathbf{e}_z$ of the induction equation, we find the equation for A:

$$\frac{\partial A}{\partial t} + \mathbf{v} \cdot \nabla A = \eta \left(\Delta - \frac{2}{r} \frac{\partial}{\partial r} \right) A \tag{10.42}$$

This equation shows that A is simply advected and diffused. As time passes, it evolves towards a constant, which means that the meridional magnetic fields go to zero.

When A is a constant, then B verifies

$$\frac{\partial B}{\partial t} + s\mathbf{v} \cdot \nabla (B/s) = \eta(\Delta - 1/s^2)B \tag{10.43}$$

where we assumed here that $\nabla \cdot \mathbf{v} = 0$. This is a similar equation as the one for A and therefore B also converges to a constant, which is necessarily zero (why?).

Thus, no axisymmetric velocity field can sustain an axisymmetric magnetic field.

10.5.3.1 Other Cases

The foregoing theorem is one case among a larger set of theorems which state cases where a magnetic field cannot be (re)generated. Here are some examples:

1. No magnetic field independent of one space coordinate can be sustained by a velocity field of the same type.
2. A divergence-free velocity field without any radial component (namely always tangent to a sphere) cannot sustain a magnetic field.
3. A two-dimensional flow cannot sustain a magnetic field.
4. A purely radial flow cannot sustain a magnetic field.

10.5.3.2 Conclusions

All these theorems show that a two-dimensional velocity field cannot sustain a two-dimensional magnetic field, whatever the surface we work on (plane, cylinder or sphere). A magnetic field can be generated only in three dimensions.

10.5.4 An Example: The Ponomarenko Dynamo

According to the foregoing discussion, a simple example of a dynamo flow is not easy to find. The Ponomarenko dynamo is one of them and was found not so long ago. The flow has the following form:

$$\mathbf{v} = \omega s \mathbf{e}_\varphi + U \mathbf{e}_z \quad s < a$$
$$\mathbf{v} = \mathbf{0} \quad s > a$$

in a fluid that fills the whole space. This is an axisymmetric flow and therefore only non-axisymmetric magnetic fields can be amplified. We are thus looking for solutions of the form:

$$\mathbf{B} = \mathbf{B}_0(s) e^{i(m\varphi + kz) + \lambda t}$$

We leave the resolution of this problem to the reader as an exercise. The result is the following: if the product ωU is large enough, then there exist unstable magnetic modes, for which $Re(\lambda) > 0$ and $m \neq 0$ of course. This flow can thus amplify magnetic fields. Two ingredients are indeed very favourable to this property: first, it is a helical flow. Helicity

$$H = \mathbf{v} \cdot \nabla \times \mathbf{v} = 2\omega U$$

is non-zero and we note that a critical value of it determines the dynamo action. We shall see below that this is indeed an important quantity for dynamos. Second, the flow owns a very steep (actually infinite) velocity gradient at $r = a$. This is a very useful feature for a dynamo because we noticed that magnetic field amplification only depends on the velocity gradients. This infinite gradient implies that all the scales of magnetic field are amplified, and in fact lead to a fast dynamo (see Gilbert 1988).

10.5.5 The Turbulent Dynamo

We learnt that dynamos are necessarily flows of low symmetry. Hence, it is no surprise that turbulent flows are very good candidates to be dynamos. Actually, natural dynamos in stars or planets are all turbulent flows. Of course, using turbulence to make magnetic fields is not an obvious way since we have no general theory of turbulent flows as we saw in the previous chapter. Nevertheless, the analysis of equations reveals some general laws that are helpful to understand what we see on the Sun or planets (including the Earth) as far as magnetic field are concerned (see Fig. 10.4 for an illustration of the solar magnetic cycle).

To study the role of turbulence in the generation of magnetic field it is useful to split the fields into their (ensemble) average and fluctuating parts. Thus

$$\mathbf{B} = \langle \mathbf{B} \rangle + \mathbf{B}' \qquad \text{and} \qquad \mathbf{v} = \langle \mathbf{v} \rangle + \mathbf{v}'$$

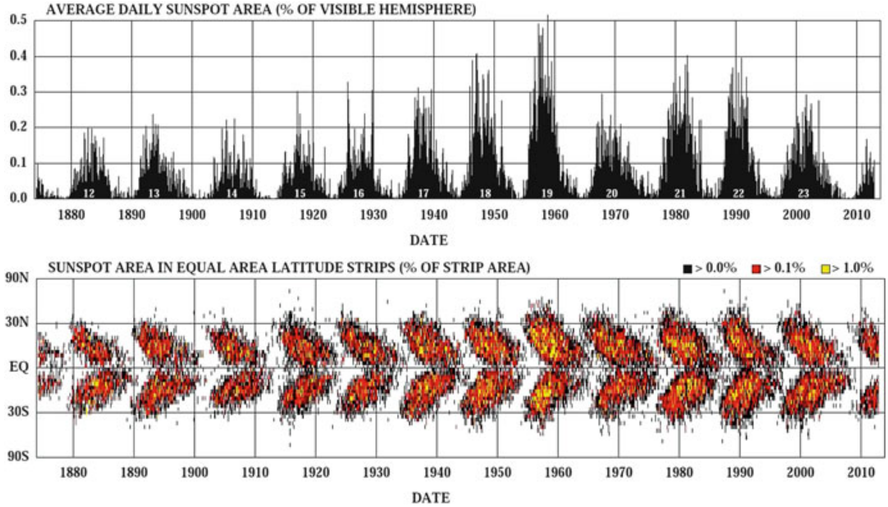

Fig. 10.4 The magnetic cycle of the Sun. *Up* The surface covered by sunspots as a function of time. *Bottom* The distribution of sunspots as a function of latitude and time (color indicates the importance of the spotted area). This diagram shows that spots appear at a latitude that decreases with time. When the equator is reached, a new cycle starts with the emergence of new spots around latitudes $\pm 30\,^{\circ}$(source: Dr. David Hathaway, NASA)

Reporting this decomposition into the induction equation and taking the average, we get

$$\frac{\partial\langle\mathbf{B}\rangle}{\partial t} = \nabla \times (\langle\mathbf{v}\rangle \times \langle\mathbf{B}\rangle) + \nabla \times \langle\mathbf{v}' \times \mathbf{B}'\rangle + \eta\Delta\langle\mathbf{B}\rangle$$

Correlations between velocity and magnetic field fluctuations appear. They generate a mean electric field:

$$\mathcal{E}_i = \langle\mathbf{v}' \times \mathbf{B}'\rangle_i$$

To go forward, we need to model this correlation. When the mean magnetic field is not too strong this may be done rather precisely. Indeed, the magnetic field fluctuations verify:

$$\frac{\partial\mathbf{B}'}{\partial t} = \nabla \times \big(\langle\mathbf{v}\rangle \times \mathbf{B}' + \mathbf{v}' \times \langle\mathbf{B}\rangle\big) + \nabla \times (\mathbf{v}' \times \mathbf{B}' - \langle\mathbf{v}' \times \mathbf{B}'\rangle) + \eta\Delta\mathbf{B}'$$

If the velocity fluctuations are independent of the mean magnetic field, which is expected if this mean field is weak enough, then the magnetic field fluctuations depend linearly on $\langle\mathbf{B}\rangle$, as well as $\langle\mathbf{v}' \times \mathbf{B}'\rangle$. Thus, we may write:

$$\langle \mathbf{v}' \times \mathbf{B}' \rangle_i = a_{ij} \langle B_j \rangle + b_{ijk} \partial_j \langle B_k \rangle + \cdots \tag{10.44}$$

where the tensors $[a]$ and $[b]$ are functions of the turbulent velocity field (not perturbed by the mean magnetic field). We may observe that they are two pseudo-tensors that are not invariant by parity transformations. For an isotropic turbulence we can write

$$a_{ij} = \alpha \delta_{ij} \qquad \text{and} \qquad b_{ijk} = \beta \epsilon_{ijk} \; .$$

β is a true scalar but α is a pseudo-scalar which vanishes if turbulence is parity-invariant. β has the dimension of a diffusivity and is consequently interpreted as the turbulent diffusivity of the magnetic field. α has the dimension of a velocity and gives birth to the now famous *alpha effect*, which we now discuss.

10.5.6 The Alpha Effect

The alpha effect is important in natural dynamo because this is an efficient way to generate magnetic fields. To get a more precise idea of the way it works, we take the example of a mean force-free magnetic field with an alpha effect. We assume that the mean velocity field is zero. Thus, the mean magnetic field verifies:

$$\frac{\partial \mathbf{B}}{\partial t} = \alpha \nabla \times \mathbf{B} + \eta_{turb} \Delta \mathbf{B} \tag{10.45}$$

where we assumed that α and β are constants. We also defined $\eta_{turb} = \eta + \beta$. Let us now determine the condition under which the mean magnetic field is amplified by turbulence. We set

$$\mathbf{B} = \mathbf{B}_0(\mathbf{r}) e^{\lambda t}$$

where $\nabla \times \mathbf{B}_0(\mathbf{r}) = K \mathbf{B}_0(r)$. Using (10.45), we find the following dispersion relation of the Fourier modes:

$$\lambda = \alpha K - \eta_{turb} K^2 \tag{10.46}$$

It shows that if $\alpha > \eta_{turb} K$, the field is amplified.

Another way to see the amplification effect of the new term $\alpha \nabla \times \mathbf{B}$ is to reconsider Cowling's antidynamo theorem. Equation (10.42) now reads:

$$\frac{\partial A}{\partial t} + \mathbf{v} \cdot \nabla A = \alpha s B + \eta_{turb} \left(\Delta - \frac{2}{s} \frac{\partial}{\partial s} \right) A \tag{10.47}$$

There we see that we no longer have the simple advection-diffusion of the potential A that leads to the disappearance of the field. The toroidal component of the field B comes into play and allows the regeneration of A, which in turn regenerates B.

10.6 Exercises

1. Show that the electric current that would result form the presence of free charges in the non-relativistic flow of a conducting fluid is always negligible compared to the induced current. Show that it implies that the force exerted on the fluid by the electrostatic forces is always very small compared to the Laplace force.
2. Show that the power dissipated by Joule effect in an electric circuit (RI^2 where R is the resistance of the circuit and I the intensity of the current that circulates) has the same origin as the magnetic dissipation that appears in (10.18).
3. Study the dispersion relation of a plane wave in an homogeneous isotropic turbulence where the alpha effect is present and where the mean velocity is zero.
4. *The magnetorotational instability*: This instability is much studied in Astrophysics since it is thought to be the main source of turbulence in accretion discs (see Chap. 6). Here we propose a simplified study of this instability.

   We start with the system made of a differentially rotating incompressible fluid contained between two infinitely long cylinders (see Sect. 6.2.1). The fluid is now bathed by a uniform magnetic field parallel to the rotation axis $\mathbf{e}_z$. Let $\mathbf{U} = U(s)\mathbf{e}_\varphi = s\Omega(s)\mathbf{e}_\varphi$ be the basic differential rotation and $\mathbf{u}$ the velocity perturbation, $B_0\mathbf{e}_z$ the imposed magnetic field and $\delta\mathbf{B} = B_0\mathbf{b}$ its perturbation. We assume that all perturbations are axisymmetric, of vanishing amplitude and proportional to $\exp(i\omega t)$.

   (a) Show that

$$i\omega\mathbf{u} = 2\Omega u_\varphi \mathbf{e}_s - \frac{\kappa^2}{2\Omega} u_s \mathbf{e}_\varphi - \nabla p/\rho + v_a^2 \nabla\times\mathbf{b}\times\mathbf{e}_z \tag{10.48}$$

$$i\omega\mathbf{b} = \nabla\times(\mathbf{u}\times\mathbf{e}_z) + (\mathbf{b}\cdot\nabla)\mathbf{U} - (\mathbf{U}\cdot\nabla)\mathbf{b} \tag{10.49}$$

   where magnetic diffusion is neglected. κ is the epicyclic frequency as given by (6.10). What is the expression of v_a^2 ?

   (b) Show that

$$(\mathbf{b}\cdot\nabla)\mathbf{U} - (\mathbf{U}\cdot\nabla)\mathbf{b} = s\frac{d\Omega}{ds} b_s \mathbf{e}_\varphi$$

   (c) We now assume that disturbances just depend on z and are proportional to $\exp(ikz)$. Show that

$$\nabla\times(\mathbf{u}\times\mathbf{e}_z) = ik\mathbf{u} \qquad \text{and} \qquad u_z = b_z = p = 0$$

   (d) and deduce that

$$\begin{cases} i\omega u_s - 2\Omega u_\varphi = v_a^2 ikb_s \\ i\omega u_\varphi + \frac{\kappa^2}{2\Omega}u_s = v_a^2 ikb_\varphi \\ i\omega b_s = iku_s \\ i\omega b_\varphi = iku_\varphi + s\frac{d\Omega}{ds}b_s \end{cases} \tag{10.50}$$

(e) From the foregoing relations show that the dispersion relation reads

$$\omega^4 - (\kappa^2 + 2v_a^2k^2)\omega^2 + v_a^2k^2\left(v_a^2k^2 + s\frac{d\Omega^2}{ds}\right) = 0 \tag{10.51}$$

(f) Show that at least one root of this equation may lead to an instability. Derive the following condition for instability:

$$s\frac{d\Omega^2}{ds} < -v_a^2k^2 \tag{10.52}$$

What is the general condition on the flow that can be deduced?

(g) We set $y = v_a^2k^2$. Show that the growth rate of the instability is maximum when

$$y = -s\frac{d\Omega^2}{ds}\left(\frac{1}{4} + \frac{\kappa^2}{16\Omega^2}\right)$$

(h) Show that the maximal growth rate is given by

$$\omega_{\max} = \frac{s}{2}\left|\frac{d\Omega}{ds}\right| \tag{10.53}$$

(i) Show that the keplerian flow $\Omega \propto s^{-3/2}$ of an accretion disc of thickness H can be unstable if the background magnetic field is less than a limiting value. Give the expression of this upper limit.

Appendix: Equations of the Axisymmetric Field

We start from (10.41) and write the diffusion term and the curl of the electric field $\mathbf{E} = \mathbf{v} \times \mathbf{B}$:

$$\Delta\mathbf{B} \left| \begin{array}{l} (\Delta - s^{-2})\left(\frac{1}{s}\frac{\partial A}{\partial z}\right) \\ \\ (\Delta - s^{-2})B \\ \\ -\Delta\left(\frac{1}{s}\frac{\partial A}{\partial s}\right) \end{array} \right. \qquad \nabla \times \mathbf{E} \left| \begin{array}{l} -\frac{\partial E_\varphi}{\partial z} \\ \\ \frac{\partial E_s}{\partial z} - \frac{\partial E_z}{\partial s} \\ \\ \frac{1}{s}\frac{\partial sE_\varphi}{\partial s} \end{array} \right.$$

The $\mathbf{e}_s$ and $\mathbf{e}_z$-components of the induction (10.10) lead to:

$$\frac{\partial A}{\partial t} = -sE_\varphi + \eta s(\Delta - s^{-2})A/s + f(s)$$

$$-\frac{\partial}{\partial t}\frac{\partial A}{\partial s} = \frac{\partial sE_\varphi}{\partial s} - \eta s\Delta\left(\frac{1}{s}\frac{\partial A}{\partial s}\right)$$

We take the s-derivative of the first equation and add it to the second equation. We find:

$$0 = f'(s) + \eta\left(\frac{\partial}{\partial s}s(\Delta - s^{-2})A/s - s\Delta\left(\frac{1}{s}\frac{\partial A}{\partial s}\right)\right)$$

but

$$\frac{\partial}{\partial s}s(\Delta - s^{-2})A/s - s\Delta\left(\frac{1}{s}\frac{\partial A}{\partial s}\right) = \frac{\partial^2}{\partial s^2}\left(s\frac{\partial}{\partial s}\frac{A}{s}\right) - \frac{\partial}{\partial s}\left(\frac{A}{s^2}\right) - \frac{\partial}{\partial s}\left[s\frac{\partial}{\partial s}\left(\frac{1}{s}\frac{\partial A}{\partial s}\right)\right]$$

which shows that the terms in parenthesis cancel and that $f'(s) = 0$. Noting that

$$sE_\varphi = v_s\frac{\partial A}{\partial s} + v_z\frac{\partial A}{\partial z}$$

we find (10.42) up to a constant.

To derive the equation for B we take the φ-component of the induction equation; hence

$$\frac{\partial B}{\partial t} = \frac{\partial E_s}{\partial z} - \frac{\partial E_z}{\partial s} + \eta(\Delta - s^{-2})B$$

setting $v_\varphi = s\omega$, then

$$E_s = -Bv_z - \omega\frac{\partial A}{\partial s} \qquad \text{and} \qquad E_z = Bv_s - \omega\frac{\partial A}{\partial z}$$

Considering the case where $A \to \text{Cst}$, the equation for B now reads

$$\frac{\partial B}{\partial t} + \mathbf{v}\cdot\nabla B = -B\left(\frac{\partial v_z}{\partial z} + \frac{\partial v_s}{\partial s}\right) + \eta(\Delta - s^{-2})B$$

using $\nabla\cdot\mathbf{v} = 0$, we rearrange the terms so that

$$\frac{\partial B}{\partial t} + sv_s\frac{\partial}{\partial s}\left(\frac{B}{s}\right) + sv_z\frac{\partial}{\partial z}\left(\frac{B}{s}\right) = \eta(\Delta - s^{-2})B$$

from which we find (10.43).

Further Reading

A classical reference to the subject of fluid dynamos is the book of K. Moffatt *Magnetic field generation in fluids* (1978) unfortunately out of print. The lectures of A. Pouquet and P. Roberts in Les Houches volume *Astrophysical Fluid Dynamics* (1992) give another introduction to MHD turbulence and dynamos, but see also *An Introduction to Magnetohydrodynamics* by Davidson (2001). One may also consult *Lectures on Solar and Planetary Dynamos* (Proctor & Gilbert Edts, 1994), or *Principles of Magnetohydrodynamics: With Applications to Laboratory and Astrophysical Plasmas* by Goedbloed & Poedts (2004).

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