Chapter 8
Rotating Fluids

8.1 Introduction

The most spectacular effect of rotation on a fluid flow is certainly the huge hurricanes surging up in the Earth's atmosphere when the waters of the ocean are warm enough. These huge flows, so typical in pictures of the Earth, would not exist if the Earth were not rotating. They owe their existence to the Coriolis acceleration.

In this chapter we wish to introduce the reader to the fundamentals of fluid dynamics in a rotating frame. Rotating fluids are indeed those fluids whose motion is essentially a solid body rotation supplemented by a small velocity field. Thus, even if hurricanes generate terrific winds, let say with speeds of 60 m/s, this is still small compared to the Earth rotation velocity (460 m/s). Such a velocity field is thus conveniently analysed in a rotating frame. As we shall see, all the novelties come from the Coriolis force, which deeply modifies the dynamics, imposing the quasi-bidimensionality of steady flows, generating new sorts of waves, new boundary layers, etc.

8.1.1 The Equation of Motion

The basic change in the equations governing a fluid flow in a rotating frame comes from the existence of inertial forces associated with the Coriolis and centrifugal accelerations. Thus, the equation of momentum is the only one to be modified. Its expression is easily derived from Newton equation, which controls the motion of a point mass particle. Let $\mathbf{r}$ be the position of the particle; it evolves according to

$$\rho \frac{d^2\mathbf{r}}{dt^2} = \mathbf{f}$$

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in a galilean frame. When the frame rotates at an angular velocity $\mathbf{\Omega}$ the same equation reads

$$\rho\left(\frac{d^2\mathbf{r}}{dt^2} + 2\mathbf{\Omega} \times \frac{d\mathbf{r}}{dt} + \mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{r})\right) = \mathbf{f}$$

This equation also gives the trajectory of a fluid particle according to the Lagrangian approach. Going back to the eulerian formalism, the preceding equation is translated into

$$\rho\left(\frac{D\mathbf{v}}{Dt} + 2\mathbf{\Omega} \times \mathbf{v} + \mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{r})\right) = \mathbf{f} \tag{8.1}$$

which gives the evolution of the velocity field. This expression could have been derived directly from the one we met in the first chapter; however, this derivation is lengthy and left to the reader as an exercise.

8.1.2 New Numbers

The importance of rotation may be appreciated if we use the right non-dimensional numbers. For this, we first introduce a length scale L, a velocity scale V and a time scale that we relate to rotation. This time scale is $(2\Omega)^{-1}$. 2Ω is known as *the Coriolis frequency*. In order to concentrate on the effects of rotation, we shall consider a simple fluid like the incompressible viscous fluid.

The momentum and continuity equation read:

$$\rho\left(\frac{D\mathbf{v}}{Dt} + 2\mathbf{\Omega} \times \mathbf{v} + \mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{r})\right) = -\nabla P + \mu\Delta\mathbf{v} \tag{8.2}$$

$$\nabla \cdot \mathbf{v} = 0 \tag{8.3}$$

where we left aside an eventual gravity force. If we observe that

$$\mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{r}) = -\nabla[\frac{1}{2}(\mathbf{\Omega} \times \mathbf{r})^2]$$

namely, the fact that the centrifugal acceleration may be derived from a potential, then, we can rewrite the momentum equation as:

$$\frac{D\mathbf{v}}{Dt} + 2\mathbf{\Omega} \times \mathbf{v} = -\nabla\Pi + \nu\Delta\mathbf{v} \tag{8.4}$$

where $\Pi = P/\rho - \frac{1}{2}(\mathbf{\Omega} \times \mathbf{r})^2$ is called the *reduced pressure*. We are now in a position to use non-dimensional quantities and we find:

$$\frac{\partial \mathbf{u}}{\partial \tau} + \mathbf{e}_z \times \mathbf{u} + \mathrm{Ro}\, \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla p + \mathrm{E}\Delta \mathbf{u} \tag{8.5}$$

where we set $\mathbf{\Omega} = \Omega \mathbf{e}_z$ and $p = \Pi/(2\Omega LV)$. Two numbers appeared:

$$\mathrm{E} = \frac{\nu}{2\Omega L^2} \qquad \text{and} \qquad \mathrm{Ro} = \frac{V}{2\Omega L} \tag{8.6}$$

which are respectively the *Ekman number* and the *Rossby number*. We note that the Ekman number measures the ratio of the viscous force to the Coriolis one, while the Rossby number shows the importance of the nonlinear advection terms with respect to the Coriolis acceleration.

When a fluid flow, in some inertial frame, is essentially a solid body rotation, we should write $\mathbf{V} = \mathbf{\Omega} \times \mathbf{r} + \mathbf{v}$ where $||\mathbf{v}|| \ll ||\mathbf{\Omega} \times \mathbf{r}||$. Since $||\mathbf{v}||$ is just the magnitude of the flow in the rotating frame, we see that flows dominated by rotation are such that their Rossby number is very small compared to unity.

We may observe that the Rossby and Ekman numbers decrease when the scale of the flow increases. Rotation is therefore expected to be important in the large scales. Let us consider two examples: a wind of 20 m/s in the Earth atmosphere is dominated by the Earth rotation when it affects a scale larger than 140 km. For these scales, the Rossby number is less than unity. An ocean current, like the Gulf Stream, is even more affected by rotation since its speed is much lower, typically 1 m/s. For this value, rotation is important for all scales larger than 7 km. This shows that an oceanic current, spanning thousands of kilometers, is very much dominated by the effects of rotation.

Now, if we turn to the Ekman number, it is usually extremely small. For instance, a water flow with a scale of 7 km, has an Ekman number around 10^{-10}. This implies, as we shall see, the existence of very thin boundary layers.

8.2 The Geostrophic Flow

8.2.1 Definition

The geostrophic flow is a steady flow where the viscous force and the nonlinear terms play a negligible part. The momentum equation is therefore reduced to

$$\rho 2\mathbf{\Omega} \times \mathbf{v} = -\nabla P \tag{8.7}$$

This is called *the geostrophic balance*. The pressure gradient balances the Coriolis force.

8.2.2 The Taylor–Proudman Theorem

The geostrophic flow has one remarkable property: it is independent of the coordinate parallel to the rotation axis. Indeed, let us take the curl of (8.7); we find

$$\nabla \times (2\boldsymbol{\Omega} \times \mathbf{v}) = \mathbf{0} \quad \Longleftrightarrow \quad (\boldsymbol{\Omega} \cdot \nabla)\mathbf{v} = \mathbf{0} \quad \Longleftrightarrow \quad \frac{\partial \mathbf{v}}{\partial z} = \mathbf{0} \tag{8.8}$$

where we used (12.41). The velocity field therefore only depends on the coordinates in the plane orthogonal to $\boldsymbol{\Omega}$. This result is known as *Taylor–Proudman Theorem*.

8.2.3 The Expression of the Geostrophic Flow

The geostrophic (8.7) can easily be solved. One finds:

$$\mathbf{v} = \frac{1}{2\Omega\rho}\mathbf{e}_z \times \nabla P + F(x, y)\mathbf{e}_z \tag{8.9}$$

In this expression $F(x, y)$ is an arbitrary function to be determined with the boundary conditions. This solution shows that the pressure also depends solely on the plane coordinates. The pressure plays the role of a stream function since isobars are also streamlines.

To further illustrate the properties of geostrophic flows, let us consider the case where the rotating fluid is bounded by a surface defined by:

$$\begin{cases} z - f(x, y) = 0 & \text{if} \quad z \geq 0 \\ z + g(x, y) = 0 & \text{if} \quad z \leq 0 \end{cases} \tag{8.10}$$

The outgoing (unnormalized) normal vector is

$$\begin{cases} \mathbf{n} = \mathbf{n}_{sup} = \nabla(z - f(x, y)) = \mathbf{e}_z - \nabla f \\ \mathbf{n} = \mathbf{n}_{inf} = -\nabla(z + g(x, y)) = -\mathbf{e}_z - \nabla g \end{cases} \tag{8.11}$$

from which we derive the equality:

$$\mathbf{n}_{sup} - \mathbf{n}_{inf} + \nabla(f - g) = 2\mathbf{e}_z$$

However, on the bounding surface $\mathbf{n}_{sup} \cdot \mathbf{v} = 0$ or $\mathbf{n}_{inf} \cdot \mathbf{v} = 0$, but since $\mathbf{v}$ does not depend on z, the foregoing equality may be used everywhere. Thus, taking the scalar product with $\mathbf{v}$, we find

$$2v_z = 2F(x, y) = \mathbf{v} \cdot \nabla(f - g)$$

which we report in (8.9). This yields

$$\mathbf{v} = \frac{1}{4\Omega\rho}\left[\mathbf{n}_{sup} - \mathbf{n}_{inf} + \nabla(f-g)\right] \times \nabla P + \frac{1}{2}\mathbf{v}\cdot\nabla(f-g)\mathbf{e}_z$$

This new expression may be simplified if we note that

$$\mathbf{v}\cdot\nabla(f-g) = (\mathbf{e}_z \times \nabla P)\cdot\nabla(f-g)/(2\Omega\rho);$$

since $\nabla(f-g)\times\nabla P$ is parallel to $\mathbf{e}_z$. It turns out that

$$\mathbf{v} = \frac{1}{4\Omega\rho}(\mathbf{n}_{sup} - \mathbf{n}_{inf}) \times \nabla P \tag{8.12}$$

This expression may further be arranged as follows. If we take the scalar product of (8.12) with $\mathbf{n}_{inf}$, we find

$$(\mathbf{n}_{sup} \times \mathbf{n}_{inf})\cdot\nabla P = 0$$

but $\mathbf{n}_{sup} \times \mathbf{n}_{inf} = \nabla(f+g)\times\mathbf{e}_z + \nabla f \times \nabla g$, so that

$$(\nabla h \times \mathbf{e}_z)\cdot\nabla P = 0 \quad\Longleftrightarrow\quad \nabla P \times \nabla h = \mathbf{0}$$

We introduced $h = f + g$ and observed that $\nabla f \times \nabla g$ and $\nabla h \times \nabla P$ are along $\mathbf{e}_z$. One should note that $h(x, y)$ is just the height of the container at (x, y). The foregoing relation shows that the pressure only depends on h. Noting that $\mathbf{n}_{sup} + \mathbf{n}_{inf} = -\nabla h$, (8.12) may be rewritten in its final form:

$$\mathbf{v} = \frac{1}{2\Omega\rho}\left(\frac{dP}{dh}\right)\mathbf{n}_{inf} \times \mathbf{n}_{sup} \tag{8.13}$$

This solution is valid only if the normal vectors are continuous in the x, y-plane. It may be observed that $\mathbf{v}$ is parallel to the curves of constant height since $\mathbf{v}\cdot\nabla h = 0$ because $\mathbf{n}_{inf} \times \mathbf{n}_{sup} = \mathbf{e}_z \times \nabla h$. These curves are also called *geostrophic contours*. Since they are streamlines they must be closed.

Another property of the geostrophic flow is that it possesses circulation around the rotation axis. Indeed, along a geostrophic contour

$$\oint_{(C)} \mathbf{v}\cdot\mathbf{dl} = \frac{1}{2\Omega\rho}\left(\frac{dP}{dh}\right)\oint_{(C)} ||\mathbf{n}_{inf} \times \mathbf{n}_{sup}||dl \neq 0 \tag{8.14}$$

Thus, in general, the geostrophic flow owns angular momentum.

8.2.4 *Examples*

8.2.4.1 The Geostrophic Flow in a Sphere

To give a simple illustration of the foregoing results, we now take the case of a geostrophic flow in a spherical container, which is typical of planetary or stellar situations. In this case, geostrophic contours are just circles of constant latitude and the velocity is constant on cylinders centered on the rotation axis.

Expliciting the results of the previous section, we note that for a sphere of radius R, the equations of the boundary are such that

$$f = g = \sqrt{R^2 - x^2 - y^2}$$

Letting $s^2 = x^2 + y^2$, the direction of the velocity is

$$\mathbf{n}_{inf} \times \mathbf{n}_{sup} = -\frac{2s}{\sqrt{R^2 - s^2}}\mathbf{e}_\varphi$$

which confirms that the velocity is purely azimuthal. If we now observe that $h = 2\sqrt{R^2 - s^2}$, the solution (8.13) gives

$$\mathbf{v} = \frac{1}{2\Omega\rho}\frac{\partial P}{\partial s}\mathbf{e}_\varphi \tag{8.15}$$

This relation could have been derived directly from (8.7), of course.

Solution (8.13) is more interesting when one deals with a more complicated geometry, like a spheroid for instance. One just needs to derive h and normal vectors from the shape of the surface boundary.

Let us note that if the sphere is truncated, like in Fig. 8.1b, some geostrophic contours are no longer closed. This ruins the existence of the geostrophic solution which disappears. As shown by Greenspan (1969), no steady state is possible, and the geostrophic flow is replaced by a set of Rossby waves, which form a subset of inertial modes (see below for their detailed presentation).

8.2.4.2 The Vortex of an Emptying Reservoir

When a reservoir like a bath tube is emptied, a strong vortex is often observed above the exit. The question of whether the rotation of this vortex is controlled by the Coriolis force due to the Earth rotation is often raised. Should the vortex rotate in opposite directions when one makes the experiment in the northern or southern hemisphere?

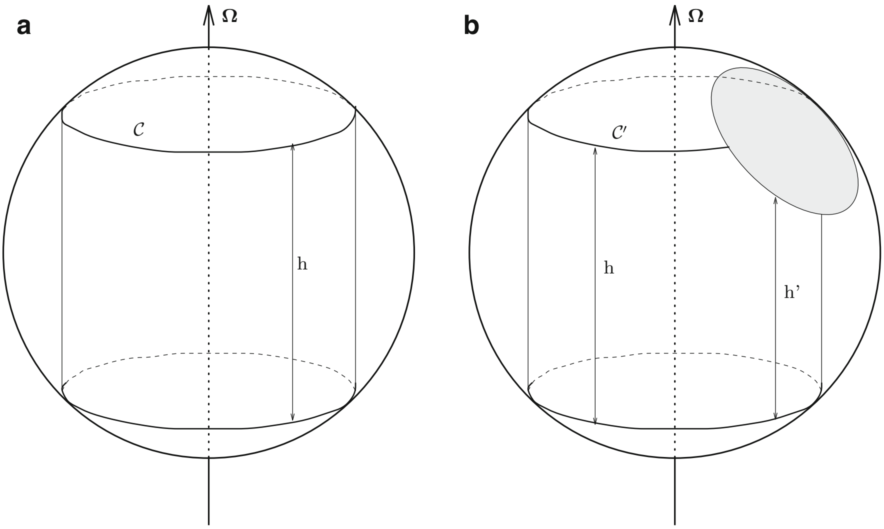

Fig. 8.1 (**a**) Geostrophic contours on a sphere. (**b**) A truncated sphere: some geostrophic contours are not closed (like $\mathcal{C}'$)

The answer is negative because the Coriolis force imposed by the Earth rotation is much too weak to be effective compared to other forces. To make things clear, it is useful to appreciate the orders of magnitude associated with such a flow. First, we may observe that the scale at which the Rossby number passes below unity for a flow whose typical velocity is 10 cm/s, is 700 m. Thus, unless the bath tube is of the size of a lake, the Rossby number will be very large, letting the $(\mathbf{v}\cdot\nabla)\mathbf{v}$ term a thousand time greater than the Coriolis acceleration.

Another way of understanding this question is to suppose that the flow is geostrophic (it is indeed almost steady, and viscous effects are small). In such a case, the amplitude of the fluid velocity would be $V \sim |\nabla P|/2\Omega$. We may estimate the pressure gradient by noting that on the bottom of the bath tube the pressure varies between $\rho g h + P_{atm}$ and P_{atm}, far from the exit and at the exit. Taking h=10 cm, we find a fluid velocity of 20,000 km/s which is absurd.

So what's going on in reality? The key point is to be found in the initial conditions. In general, the fluid is not strictly at rest when one empties a bath tube. With respect to the exit, the water owns some residual angular momentum. When the emptying is started, conservation of this momentum implies an amplification of the rotation near the exit. Actually, the convergence of the streamlines on exit strongly amplifies the vorticity. Thus, a flow which was not perceptible to the eye before the reservoir is emptying, shows up neatly when the exit is open. The low pressure at the vortex centre makes this structure clearly visible.

8.3 Waves in Rotating Fluids

We continue our exploration of the properties of rotating fluids by focusing on the waves that are specific to them.

8.3.1 Inertial Waves

Inertial waves owe their existence to the Coriolis force which is their restoring force. We recall that the existence of the Coriolis force is the consequence of the conservation of angular momentum. If this force were absent, the free motion in a rotating frame would not conserve the angular momentum.

To fully appreciate its effects, it is useful to consider the motion of a particle which is solely driven by this force. Its velocity verifies

$$\frac{d\mathbf{v}}{dt} + 2\mathbf{\Omega} \times \mathbf{v} = \mathbf{0}$$

This equation is easily solved and yields:

$$v_x = v_0 \cos(2\Omega t) \qquad \text{and} \qquad v_y = v_0 \sin(2\Omega t)$$

if we choose that $v_x = v_0$ and $v_y = 0$ at $t = 0$. A further integration gives the trajectory:

$$x = x_0 + \frac{v_0}{2\Omega}\sin(2\Omega t) \qquad \text{and} \qquad y = y_0 - \frac{v_0}{2\Omega}\cos(2\Omega t)$$

This shows that particles have a circular motion. The Coriolis force brings the particles back to their initial position after making a circular trajectory with a radius $v_0/2\Omega$.

Let us now focus on the dispersion relation of these waves. We take (8.5) and set $\mathrm{E} = \mathrm{Ro} = 0$. As needed, we assume that the pressure and velocity perturbations are plane waves, namely:

$$(p, \mathbf{v}) = (p, \mathbf{v})_0 e^{i(\omega t - \mathbf{k}\cdot\mathbf{x})}$$

Incompressibility implies that

$$\mathbf{k} \cdot \mathbf{v} = 0 \tag{8.16}$$

which shows that these waves are transversal. The equation of momentum, $i\omega\mathbf{v} + 2\Omega\mathbf{e}_z \times \mathbf{u} = i\mathbf{k}P$, leads to

$$\begin{cases} 2\Omega \mathbf{e}_z \cdot (\mathbf{u} \times \mathbf{k}) = ik^2 P \\ i\omega v_z = ik_z P \\ i\omega \mathbf{k} \times \mathbf{v} = 2\Omega k_z \mathbf{v} \end{cases} \tag{8.17}$$

from which we derive the following dispersion relation:

$$\omega^2 = (2\Omega)^2 \frac{k_z^2}{k^2} \tag{8.18}$$

This relation shows that the frequency of inertial waves is bounded by 2Ω, since $k_z \leq k$. Inertial waves are thus long period waves whose shortest period is half the rotation period.

The dispersion relation also shows that these waves propagate in a very anisotropic way. This is shown by the phase velocity:

$$\mathbf{v}_\phi = \frac{\omega}{k}\mathbf{e}_k = 2\Omega \frac{k_z}{k^3}\mathbf{k} \tag{8.19}$$

This expression shows that no propagation is possible if it is restricted to a plane perpendicular to the rotation axis rotation. Propagation preferentially occurs along the rotation axis.

Now let us consider the group velocity. We find

$$\mathbf{v}_g = \nabla_k \omega(\mathbf{k}) = 2\Omega \frac{\mathbf{k} \times (\mathbf{e}_z \times \mathbf{k})}{k^3} \tag{8.20}$$

This expression shows that $\mathbf{v}_g \cdot \mathbf{k} = 0$: like for internal gravity waves, energy propagates perpendicularly to the phase!

8.3.2 Inertial Modes

If the fluid domain is bounded, the equations of motion need to be completed by the boundary conditions $\mathbf{u} \cdot \mathbf{n} = 0$. The inertial modes are the oscillation modes of a rotating inviscid fluid contained in a reservoir. Setting ω as the mode frequency, the associated flow verifies

$$\begin{cases} i\omega \mathbf{u} + \mathbf{e}_z \times \mathbf{u} = -\nabla P \\ \nabla \cdot \mathbf{u} = 0 \\ \mathbf{u} \cdot \mathbf{n} = 0 \qquad \text{on} \quad S \end{cases} \tag{8.21}$$

which we wrote with non-dimensional variables following (8.5). Now, let ω_n and ω_m be two distinct eigenfrequencies, then

$$\int_{(V)} \mathbf{u}_n \cdot \mathbf{u}_m^* \, dV = 0 \tag{8.22}$$

i.e. inertial modes are orthogonal with respect to this scalar product. This property is a consequence of (8.21) when it is written for two different modes of frequency ω_n and ω_m. Indeed,

$$\begin{cases} i\omega_n \mathbf{u}_n + \mathbf{e}_z \times \mathbf{u}_n = -\nabla P_n \\ -i\omega_m \mathbf{u}_m^* + \mathbf{e}_z \times \mathbf{u}_m^* = -\nabla P_m^* \end{cases} \tag{8.23}$$

Taking the scalar product of the first equation with the complex conjugate $\mathbf{u}_m^*$ and the second equation with $\mathbf{u}_n$, adding the results and integrating over the whole fluid volume leads to

$$i(\omega_n - \omega_m) \int_{(V)} \mathbf{u}_n \cdot \mathbf{u}_m^* dV + \int_{(V)} \left[\mathbf{u}_m^* \cdot (\mathbf{e}_z \times \mathbf{u}_n) + \mathbf{u}_n \cdot (\mathbf{e}_z \times \mathbf{u}_m^*) \right] dV = 0$$

where we used the boundary conditions to eliminate the pressure term. The last two terms are of opposite sign so that we are left with (8.22) since $\omega_m \neq \omega_n$.

Another important property of inertial modes is that, like their wavy counterpart, their frequency is less that 2Ω or, for the scaled ω, less than unity. This result comes from the momentum equation when projected on $\mathbf{u}^*$ and integrated over the fluid's volume. It turns out that

$$\omega = \frac{\int_{(V)} \mathrm{Im}[(\mathbf{u}^* \times \mathbf{u}) \cdot \mathbf{e}_z] dV}{\int_{(V)} |\mathbf{u}|^2 dV}$$

Schwarz inequality, $|\mathrm{Im}[(\mathbf{u}^* \times \mathbf{u}) \cdot \mathbf{e}_z]| \leq |\mathbf{u}^* \times \mathbf{u}| \leq |\mathbf{u}|^2$, leads to

$$|\omega| \leq \frac{\int_{(V)} |\mathrm{Im}[(\mathbf{u}^* \times \mathbf{u}) \cdot \mathbf{e}_z]| dV}{\int_{(V)} |\mathbf{u}|^2 dV} \leq 1 \tag{8.24}$$

For some simple containers, the spectrum, namely all the possible values of ω, can be computed. In this case, the eigenvalues are dense in the interval [0,1]. This means that for any real number in this interval, we may find a frequency ω as close to this number as we wish (see the box "The inertial modes in the sphere" for a detailed example).

8.3.3 The Poincaré Equation

If we now take the divergence of (8.21), the system can be reduced to a single equation on the pressure, namely

$$\Delta P - \frac{1}{\omega^2}\frac{\partial^2 P}{\partial z^2} = 0 \tag{8.25}$$

This equation is known as *Poincaré's equation* since the work of Élie Cartan (1922). This equation is completed by the boundary condition $\mathbf{u}\cdot\mathbf{n} = 0$, which can be reexpressed with the pressure as:

$$-\omega^2\mathbf{n}\cdot\nabla P + (\mathbf{n}\cdot\mathbf{e}_z)(\mathbf{e}_z\cdot\nabla P) + i\omega(\mathbf{e}_z\times\mathbf{n})\cdot\nabla P = 0 \tag{8.26}$$

The Poincaré equation completed with the foregoing boundary condition is peculiar as it constitutes a mathematically ill-posed problem. Indeed, since $\omega < 1$, this equation is of hyperbolic type, like the wave equation. However, unlike the wave equation, boundary conditions are imposed to the solutions of the Poincaré equation. This makes an ill-posed problem in the sense of Hadamard. In the general case, solutions own many singularities which endow inertial modes with very unusual properties as illustrated in Fig. 8.2.

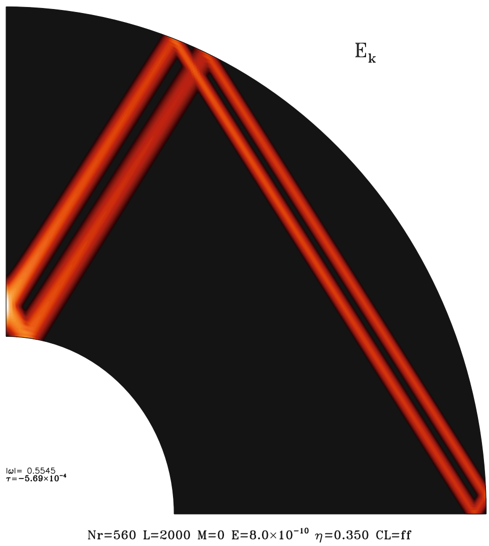

Fig. 8.2 A singular inertial mode: this figure shows the kinetic energy of an inertial mode inside a spherical shell. This meridian cut shows that the mode is concentrated along a periodic path of the characteristics of the Poincaré equation (this periodic path is called an attractor). When viscosity decreases (here the Ekman number is 8×10^{-10}), the mode gets more focused around the attractor and becomes singular at a vanishing viscosity (see Rieutord et al. 2001, for more details)

The inertial modes in the sphere

Let us write the Poincaré equation (8.25) as

$$\frac{\partial^2 P}{\partial x^2}+\frac{\partial^2 P}{\partial y^2}-\left(\frac{1-\omega^2}{\omega^2}\right)\frac{\partial^2 P}{\partial z^2}=0 \tag{8.27}$$

The boundary condition $\mathbf{v}\cdot\mathbf{n}=0$ yields

$$s\frac{\partial P}{\partial s}+\frac{1}{i\omega}\frac{\partial P}{\partial\varphi}+\left(1-\frac{1}{\omega^2}\right)z\frac{\partial P}{\partial z}=0 \tag{8.28}$$

when cylindrical coordinates are used. It is derived by using

$$\mathbf{n}=\frac{1}{\sqrt{r^2+z^2}}(r\mathbf{e}_r+z\mathbf{e}_z)$$

$$v_s=-\frac{1}{1-\omega^2}\left(i\omega\frac{\partial P}{\partial s}+\frac{1}{s}\frac{\partial P}{\partial\varphi}\right) \tag{8.29}$$

$$v_z=-\frac{1}{i\omega}\frac{\partial P}{\partial z} \tag{8.30}$$

The dispersion relation of inertial modes in a full sphere has been first obtained by Bryan (1889), who proposed to change the z-coordinate into

$$z'=-\frac{i\omega}{\sqrt{1-\omega^2}}z$$

so that Poincaré equation turns into Laplace's one. With this new system of coordinates, the bounding sphere of radius R becomes:

$$\frac{x^2+y^2}{R^2}-\frac{z'^2}{B^2}=1 \tag{8.31}$$

with $B^2=\frac{\omega^2}{1-\omega^2}R^2$. This is the equation of a one sheet axisymmetric hyperboloid. To solve Laplace equation, we need to use a coordinate system which is appropriate to this new geometry. These coordinates are those of the oblate ellipsoid Angot (1949,1972).
This coordinate system uses the following surfaces:

$$\frac{x^2+y^2}{a^2\cos^2\chi}-\frac{z'^2}{a^2\sin^2\chi}=1$$

$$\frac{x^2+y^2}{a^2\cosh^2\xi}+\frac{z'^2}{a^2\sinh^2\xi}=1$$

where we identify

$$a^2=\frac{R^2}{1-\omega^2},\qquad \sin^2\chi=\omega^2$$

The ellipsoidal coordinates ξ,χ,φ are related to the cartesian ones by

$$\begin{cases} x=a\cosh\xi\cos\chi\cos\varphi\\ y=a\cosh\xi\cos\chi\sin\varphi\\ z'=a\sinh\xi\sin\chi\end{cases} \tag{8.32}$$

and the solutions of the Laplace equation are of the form:

$$P(\xi,\chi,\varphi)=\sum_{l,m}A_{l,m}P_l^m(\sin\chi)\,\tilde{P}_l^m(i\sinh\xi)e^{im\varphi}$$

where the P_l^m are the Legendre polynomials. Noting that

$$z=\frac{\sqrt{1-\omega^2}}{\omega}a\,i\sinh\xi\sin\chi$$

we can set $\mu=i\sinh\xi$ and $\eta=a\sin\chi$; then

$$\begin{cases} s=\sqrt{x^2+y^2}=\sqrt{(a^2-\eta^2)(1-\mu^2)}\\ z=\dfrac{\mu\eta}{\sqrt{a^2-1}}\end{cases} \tag{8.33}$$

and $a=1/(1-\omega^2)$ if we set $R=1$. The solution is therefore:

$$P=\sum_{l,m}A_{l,m}P_l^m\left(\frac{\eta}{a}\right)P_l^m(\mu)e^{im\varphi}$$

$$\Longleftrightarrow\ P=\sum_{l,m}A_{l,m}P_l^m(\eta\sqrt{1-\omega^2})P_l^m(\mu)e^{im\varphi}$$

The inertial modes in the sphere

The boundary conditions (8.28) need to be rewritten with the variables (η, μ). We find

$$\begin{cases} \frac{\partial\mu}{\partial s} = \frac{s\mu}{\Delta}, & \frac{\partial\mu}{\partial z} = \frac{(a^2-1)(1-\mu^2)z}{\mu\Delta} \\ \frac{\partial\eta}{\partial s} = -\frac{s\eta}{\Delta}, & \frac{\partial\eta}{\partial z} = \frac{\mu(\eta^2-a^2)\sqrt{a^2-1}}{\Delta} \end{cases} \tag{8.34}$$

with $\Delta = \eta^2 - a^2\mu^2$. Using these relations on the sphere, at $\eta = \frac{\omega}{\sqrt{1-\omega^2}}$, we finally obtain the dispersion relation:

$$(1-\omega^2)\frac{dP_l^m}{d\omega} = mP_l^m \tag{8.35}$$

which permits the computation of the frequencies of inertial modes in the sphere. If we consider the case of axisymmetric modes, $m = 0$, this relation is now simply $\frac{dP_l^m}{d\omega} = 0$ or $\omega = \pm 1$. The eigenfrequencies are thus the roots of the Legendre polynomial $P_l^1(\omega) = \sqrt{1-\omega^2}\frac{dP_l^0}{d\omega}$. All these roots are between -1 and 1 (which meets the constraint (8.24)) and when $\ell \longrightarrow +\infty$, these root form a dense set in this interval.

8.3.4 *Rossby Waves*

Rossby waves constitute a wave category which is very important in planetary atmospheres. They are often called planetary waves.[1]

Let us restart the derivation of the dispersion relation for inertial waves but with the assumption that the fluid is contained in a very thin layer like the Earth atmosphere. We set the z-axis along the vertical, the x-axis towards East (parallel to the Earth rotation) and the y-axis to the North. We look for a purely two-dimensional wave solution, where vertical motions are negligible compared to the horizontal ones. The dispersion relation of such waves cannot be derived from the one of inertial waves since we now impose the condition $v_z = 0$. The simplification by v_z, which is needed to derive (8.18) is no longer possible. Thus, the derivation needs to be started *ab initio*. The equations of the flow are:

$$\begin{cases} i\omega\mathbf{v} + 2\mathbf{\Omega}(y)\times\mathbf{v} = -\nabla P \\ \nabla\cdot\mathbf{v} = 0 \end{cases} \tag{8.36}$$

When writing this system, we explicitly mention the dependence $\mathbf{\Omega} \equiv \mathbf{\Omega}(y)$ since the local rotation vector depends on the latitude. We underline that since we restrict the motions to the horizontal ones, the horizontal part of $\mathbf{\Omega}$ does not play any role; we just need to consider the component of $\mathbf{\Omega}$ along the z-axis. We thus write:

$$i\omega\mathbf{v} + 2\Omega(y)\mathbf{e}_z\times\mathbf{v} = -\nabla P$$

[1] See Longuet-Higgins (1964).

where $\Omega(y) = \Omega \sin \lambda(y)$, λ being the latitude. We have

$$\begin{cases} i\omega v_x - 2\Omega(y)v_y = -\dfrac{\partial P}{\partial x} \\[2ex] i\omega v_y + 2\Omega(y)v_x = -\dfrac{\partial P}{\partial y} \\[2ex] \dfrac{\partial v_x}{\partial x} + \dfrac{\partial v_y}{\partial y} = 0 \end{cases} \tag{8.37}$$

Eliminating the pressure by taking the curl, we find the vertical component ζ of the vorticity:

$$i\omega\zeta = 2v_y \frac{d\Omega}{dy} \tag{8.38}$$

This equation shows that the relation between Ω and the latitude is essential. Now a standard approximation, called *the β-plane approximation*, consists in setting $\frac{d\Omega}{dy}$ to a constant value. In atmospheric sciences, β is called the gradient of the planetary vorticity ($2d\Omega/dy$). With this assumption, we easily find the dispersion relation of Rossby waves:

$$\omega = -\frac{2k_x}{k_x^2 + k_y^2}\left(\frac{d\Omega}{dy}\right) \tag{8.39}$$

This relation shows that $\omega k_x < 0$ since $\frac{d\Omega}{dy} > 0$. Rossby waves thus propagate to the $x < 0$, that is to say to the West, opposite to the Earth rotation. They are *retrograde* waves. The group velocity

$$\mathbf{v}_g = 2\frac{d\Omega}{dy}\left((k_y^2 - k_x^2)\mathbf{e}_x + 2k_xk_y\mathbf{e}_y\right)/k^4$$

shows that energy has no preferred direction.

The form of the dispersion relation of Rossby waves shows why we could not have derived it from the one of inertial waves: the variation of Ω is crucial. In particular, we note that if the velocity field of the perturbation is a plane wave, this is not the case for the pressure fluctuation because $\frac{\partial P}{\partial x} \neq ik_xP$. In fact, Rossby waves are rather a class of inertial modes which meet some constraints like bidimensionality. This is why, when a container does not admit closed geostrophic contours, the geostrophic flow is replaced by an infinite sum of Rossby waves, namely by inertial modes which are quasi 2D and of very low frequency.[2]

[2] A detailed discussion of this question may be found in Greenspan (1969).

8.3.4.1 Planetary Modes

Let us now generalize the foregoing results by considering the Rossby waves over the whole sphere. We still consider that they propagate in a very thin fluid layer covering a sphere, but we abandon the β-plane approximation. Such modes are usually called *planetary modes*.

Because the velocity field is two-dimensional, we may use a stream function to describe the flow. We thus introduce $\chi(\theta, \varphi)$, which is such that

$$\mathbf{v} = \boldsymbol{\nabla} \times (\chi \mathbf{e}_r)$$

We derive the equation for χ by applying the $\mathbf{e}_r \cdot \boldsymbol{\nabla}\times$ operator to (8.21). We get

$$i\omega \mathbf{e}_r \cdot \boldsymbol{\nabla} \times \boldsymbol{\nabla} \times (\chi \mathbf{e}_r) + \mathbf{e}_r \cdot \boldsymbol{\nabla} \times (\mathbf{e}_z \times \mathbf{u}) = 0$$

which leads to

$$i\omega \Delta\chi + \frac{\partial \chi}{\partial \varphi} = 0$$

on a sphere of unit radius. It is natural to decompose the stream function χ on the set of spherical harmonics, which is a complete base for the functions defined on the sphere. Thus, setting

$$\chi = \sum_{\ell,m} \chi_m^\ell Y_\ell^m$$

we find that an eigenmode is represented by a single spherical harmonic to which corresponds the eigenfrequency

$$\omega_{\ell m} = \frac{m}{\ell(\ell+1)} \tag{8.40}$$

We derived this dispersion relation using the spherical harmonics differential equation $\Delta Y_\ell^m = -\ell(\ell+1) Y_\ell^m$ (see 12.31).

The expression of $\omega_{\ell m}$ shows that the (angular) phase velocity $-\omega/m = -1/\ell(\ell+1)$ is always negative.[3] Thus, like Rossby waves, planetary modes propagate to the West. Figure 8.3 illustrates the wind pattern generated by these waves in the Earth atmosphere.

[3] We recall that we set χ proportional to $e^{i(\omega t + m\varphi)}$.

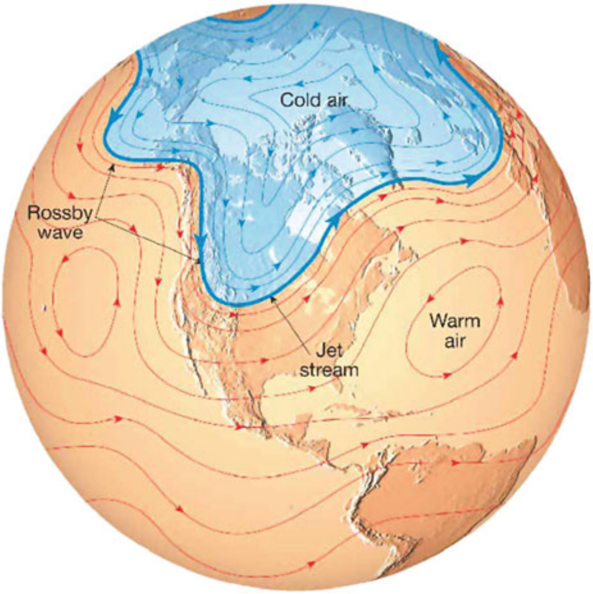

Fig. 8.3 Rossby waves in the Earth atmosphere: in shaping the interface between cold air and warm air, Rossby waves have a determining influence on the weather at mid-latitudes. Credit: City University of New York

8.4 The Effects of Viscosity

Until now we neglected the viscosity. We showed, while introducing the rotation time scale, that viscous terms are controlled by the Ekman number, which value is usually very small. Hence, the effects of viscosity are important only in places where the gradient of velocity is strong, namely in boundary (or shear) layers.

As above, we shall consider the limit of vanishing Rossby numbers so as to (again) neglect the nonlinear terms. Then, boundary layers usually result from a balance between viscous terms and the Coriolis term. They are called *Ekman layers*. The boundary layer flow can be formally solved, as we shall see below, because the equations governing the flow are linear, unlike the general boundary layers that we studied in Chap. 4 (like the Blasius flow for instance).

8.4.1 The Method

In order to simplify the discussion, we shall concentrate on the steady case only, namely on the geostrophic flow (an example of the unsteady case can be found in Rieutord 2001). The idea of the method is to divide the solution into small subsolutions which are easy to derive. For this, we first expand the solution into powers of the small parameter $\sqrt{\mathrm{E}}$, which is the thickness of the layer (just like $1/\sqrt{\mathrm{Re}}$ in Chap. 4). We thus write

$$\mathbf{u} = \mathbf{u}_0 + \sqrt{\mathrm{E}}\,\mathbf{u}_1 + \mathrm{E}\,\mathbf{u}_2 + \dots, \qquad p = p_0 + \sqrt{\mathrm{E}}\,p_1 + \dots \tag{8.41}$$

Then, each order $\mathbf{u}_n$ is split into two parts: the boundary layer part $\tilde{u}_n$ and the interior part $\overline{u}_n$. The derivation of each of these terms is much simpler than the full solution. Summing them together allows us to obtain a solution valid up to the chosen order (usually one or two).

8.4.2 The Boundary Layer Solution

The boundary layer solution is simpler than the general one because the flow is along the boundary and the velocity variations are dominated by the gradients along the normal to the wall (see Chap. 4).

Let $\mathbf{n}$ be the outer normal of the wall, and let us rewrite (8.5) with $\mathrm{Ro} = \frac{\partial}{\partial\tau} = 0$. We find

$$\mathbf{e}_z \times \mathbf{u} = -\boldsymbol{\nabla} p + \mathrm{E}\Delta\mathbf{u} \tag{8.42}$$

We now make the decomposition

$$\mathbf{u} = \overline{u}_0 + \tilde{u}_0, \qquad p = \overline{p}_0 + \tilde{p}_0$$

There, $\overline{u}_0$ is nothing but the geostrophic solution. $\tilde{u}_0$ and $\tilde{p}_0$ are the corrections to add to the geostrophic solution so that the boundary conditions are met. Since $\mathbf{e}_z \times \overline{u}_0 = -\boldsymbol{\nabla}\overline{p}_0$ then

$$\mathbf{e}_z \times \tilde{u}_0 = -\boldsymbol{\nabla}\tilde{p}_0 + \mathrm{E}\Delta\tilde{u}_0 \tag{8.43}$$

where we neglected $\mathrm{E}\Delta\overline{u}_0$ since it is $\mathcal{O}(\mathrm{E})$ while other terms are of order unity.

Let ξ be the coordinate along the normal of the wall directed towards the container's interior. Projected along $\mathbf{n}$ (8.43) yields

$$\frac{\partial\tilde{p}_0}{\partial\xi} = \mathbf{n}\cdot(\mathbf{e}_z \times \tilde{u}_0)$$

Since $\tilde{p}_0$ is a boundary layer quantity, its variation along ξ is very fast. If $\sqrt{\mathrm{E}}$ is the thickness of the layer as shown below, then $\partial\tilde{p}_0/\partial\xi$ is $\mathcal{O}(1/\sqrt{\mathrm{E}})$ but $\mathbf{n}\cdot(\mathbf{e}_z \times \tilde{u}_0)$ is $\mathcal{O}(1)$. This implies that the normal derivative of $\tilde{p}_0$ is zero. Hence, $\tilde{p}_0$ is a constant across the boundary layer. We find here again the result derived from the Prandtl equations, which control a general boundary layer (see 4.37b). Since the value of $\tilde{p}_0$ is zero outside the boundary layer, $\tilde{p}_0$ is vanishing everywhere. The pressure correction in the boundary layer is therefore of the next order, that is $\mathcal{O}(\sqrt{\mathrm{E}})$. We thus have to write

$$p = \overline{p}_0 + \sqrt{\mathrm{E}}\,(\overline{p}_1 + \tilde{p}_1)$$

Keeping only the $\mathcal{O}(1)$ terms in (8.43), we have

$$\mathbf{e}_z \times \tilde{u}_0 = \frac{\partial \tilde{p}_1}{\partial \zeta}\mathbf{n} + \frac{\partial^2 \tilde{u}_0}{\partial \zeta^2} \tag{8.44}$$

where we introduced the stretched coordinate ζ such that $\xi = \sqrt{\mathrm{E}}\,\zeta$. Taking the cross product of this equation with $\mathbf{n}$ and observing that $\tilde{u}_0 \cdot \mathbf{n} = 0$, we find that

$$-(\mathbf{n}\cdot\mathbf{e}_z)\tilde{u}_0 = \frac{\partial^2 \mathbf{n}\times\tilde{u}_0}{\partial \zeta^2} \tag{8.45}$$

On the other hand $\tilde{u}_0 = (\tilde{u}_0\cdot\mathbf{n})\mathbf{n} + (\mathbf{n}\times\tilde{u}_0)\times\mathbf{n} = (\mathbf{n}\times\tilde{u}_0)\times\mathbf{n}$ since we are dealing with boundary layer quantities. (8.44) may be rewritten as

$$(\mathbf{n}\cdot\mathbf{e}_z)(\mathbf{n}\times\tilde{u}_0) - \mathbf{e}_z\cdot(\mathbf{n}\times\tilde{u}_0)\mathbf{n} = \frac{\partial \tilde{p}_1}{\partial \zeta}\mathbf{n} + \frac{\partial^2 \tilde{u}_0}{\partial \zeta^2}$$

$$\Longrightarrow \qquad (\mathbf{n}\cdot\mathbf{e}_z)(\mathbf{n}\times\tilde{u}_0) = \frac{\partial^2 \tilde{u}_0}{\partial \zeta^2} \tag{8.46}$$

where we identified the vectors belonging to the tangent plane. Multiplying (8.46) by i and adding it to (8.45), we deduce that

$$\frac{\partial^2}{\partial \zeta^2}(\mathbf{n}\times\tilde{u}_0 + i\,\tilde{u}_0) = i\,(\mathbf{n}\cdot\mathbf{e}_z)(\mathbf{n}\times\tilde{u}_0 + i\,\tilde{u}_0) \tag{8.47}$$

This equation is easily solved. We find

$$(\mathbf{n}\times\tilde{u}_0 + i\,\tilde{u}_0) = (\mathbf{n}\times\tilde{u}_0 + i\,\tilde{u}_0)_{\zeta=0}\exp\left(-\zeta\sqrt{i\,(\mathbf{n}\cdot\mathbf{e}_z)}\right) \tag{8.48}$$

The integration constant $(\mathbf{n}\times\tilde{u} + i\,\tilde{u})_{\zeta=0}$ is given by the flow outside the boundary layer. For instance, if the boundary conditions are $\mathbf{u} = \mathbf{0}$ on the wall, then, the solution must be such that $\tilde{u}_0 + \overline{u}_0 = \mathbf{0}$ on the wall. Hence $(\mathbf{n}\times\tilde{u}_0 + i\,\tilde{u}_0)_{\zeta=0} = -(\mathbf{n}\times\overline{u}_0 + i\,\overline{u}_0)_{\mathrm{wall}}$, so that

$$(\mathbf{n}\times\tilde{u}_0 + i\,\tilde{u}_0) = -(\mathbf{n}\times\overline{u}_0 + i\,\overline{u}_0)_{\mathrm{wall}}\exp\left(-\zeta\sqrt{i\,(\mathbf{n}\cdot\mathbf{e}_z)}\right) \tag{8.49}$$

The solution (8.48) calls for some comments: First let us note that the velocity has a changing orientation within the boundary layer. Indeed, let us assume, for instance, that $\mathbf{n} = \mathbf{e}_z$ and $\overline{u}_0 = U\mathbf{e}_x$ on the wall. Then, (8.48) changes into

$$\begin{cases} \tilde{u}_x = -Ue^{-\zeta'}\cos\zeta' \\ \tilde{u}_y = Ue^{-\zeta'}\sin\zeta' \end{cases} \tag{8.50}$$

where we set $\zeta' = \zeta/\sqrt{2}$. The "complete" solution reads

$$\mathbf{u} = \overline{u}_0 + \tilde{u}_0 = U\left((1-\cos\zeta' e^{-\zeta'})\mathbf{e}_x + \sin\zeta' e^{-\zeta'}\ \mathbf{e}_y\right) \tag{8.51}$$

To illustrate the shape of the velocity field, we draw the velocity vector as a function of the "depth" ζ'. This yields a spiral known as the *Ekman spiral* (see Fig. 8.4).

A second comment about (8.48) concerns the thickness of the Ekman layer which is

$$e = L\sqrt{\frac{2\mathrm{E}}{|\mathbf{n}\cdot\mathbf{e}_z|}}$$

where L is the length scale. This expression shows that if the wall is parallel to the rotation axis, the thickness of the Ekman layer is infinite. In fact, in this very case, the analysis that led to (8.48) is no longer valid. This difficulty arises for instance when one deals with the geostrophic flow inside a sphere. At the equator, the boundary layer is singular: this is *the equatorial singularity*. It may be shown that for latitudes within an equatorial band of latitudinal extension $\mathcal{O}(\mathrm{E}^{1/5})$, the thickness of the layer is $\mathcal{O}(\mathrm{E}^{2/5})$. Thus, for a development in powers of $\mathrm{E}^{1/2}$, the new thickness of the layer, scaling like $\mathrm{E}^{2/5}$ appears to be of infinite size since $\lim_{\mathrm{E}\longrightarrow 0}\mathrm{E}^{2/5-1/2} = \infty$. More details may be found in the original paper of Roberts and Stewartson (1963).

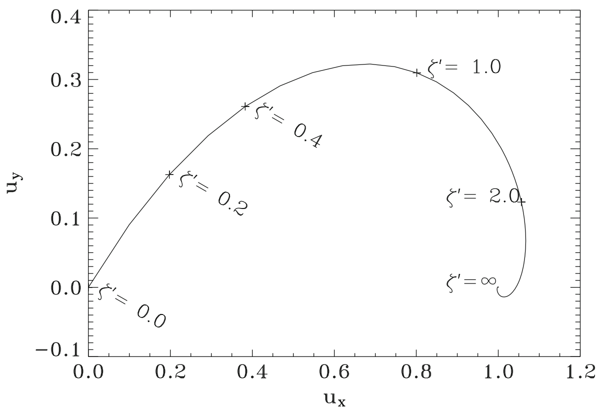

Fig. 8.4 The Ekman spiral: on the boundary $\mathbf{u}(\zeta' = 0) = \mathbf{0}$, while outside the boundary layer $\mathbf{u}(\zeta' \to \infty) = \mathbf{e}_x$

8.4.3 Ekman Pumping and Ekman Circulation

When we derived (8.48), we just solved the momentum equation, leaving aside mass conservation. It may well be that $\nabla \cdot \tilde{u}_0 \neq 0$. Fortunately, we need not throw away our solution (8.48). Let us be more explicit. If x and y are the coordinates in the tangent plane, the projection of (8.48) on this basis gives

$$\begin{cases} \tilde{u}_x^0 = -(\overline{u}_x^0 \cos\zeta' + \overline{u}_y^0 \sin\zeta')e^{-\zeta'} \\ \tilde{u}_y^0 = (\overline{u}_x^0 \sin\zeta' - \overline{u}_y^0 \cos\zeta')e^{-\zeta'} \end{cases} \tag{8.52}$$

where $\overline{u}_x^0$ and $\overline{u}_y^0$ are the x and y-components of the geostrophic flow taken on the wall. We now compute the divergence of $\tilde{u}_0$ noting that $\nabla \cdot \overline{u}_0 = 0$. We have:

$$\frac{\partial \tilde{u}_x^0}{\partial x} + \frac{\partial \tilde{u}_y^0}{\partial y} = \left(\frac{\partial \overline{u}_x^0}{\partial y} - \frac{\partial \overline{u}_y^0}{\partial x}\right) \sin\zeta' e^{-\zeta'} = -(\mathbf{n} \cdot \nabla \times \overline{u}_0) \sin\zeta' e^{-\zeta'}$$

This derivation is purely formal because we did not take into account the curvilinear nature of the coordinates; however, it keeps the dominant terms. This expression shows that the divergence is actually proportional to the normal component of the vorticity of the geostrophic flow.

This divergence is generally non-zero and is compensated by a flow along $\mathbf{n}$. Let us denote this flow $\tilde{u}'$. It verifies

$$\frac{\partial \tilde{u}_x^0}{\partial x} + \frac{\partial \tilde{u}_y^0}{\partial y} + \frac{\partial \tilde{u}'}{\partial \xi} = 0$$

Setting $R(x, y) = \mathbf{n} \cdot \nabla \times \mathbf{u}_{geo}$, then

$$\frac{\partial \tilde{u}'}{\partial \xi} = R(x, y) \sin\zeta' e^{-\zeta'}$$

which is easily integrated, remembering that $\xi = \sqrt{2\mathrm{E}}\,\zeta'$; it turns out

$$\tilde{u}' = -\sqrt{\mathrm{E}}\,R(x, y)\,e^{-\zeta'} \cos\left(\zeta' - \frac{\pi}{4}\right) \tag{8.53}$$

The important point shown by this expression is the fact that this new component of the boundary layer flow is of a higher order in powers of $\sqrt{\mathrm{E}}$, so that the foregoing results are still valid, fortunately! We thus write

$$\tilde{u}' = \sqrt{\mathrm{E}}\,\tilde{u}_1$$

This new component of the boundary layer flow is very important for the large-scale dynamics. We indeed observe that it is non-zero on the boundary at $\zeta = 0$. This means that in order for the boundary conditions to be verified at first order, the boundary layer flow needs to be completed by an interior one of the same order. Let us call this new interior flow $\overline{u}_1$. As $\overline{u}_0$ it verifies the geostrophic (8.7) but it meets a different boundary condition. Indeed, we now demand that

$$(\overline{u}_1 + \tilde{u}_1) \cdot \mathbf{n} = 0 \tag{8.54}$$

on the boundary.

The new component of the boundary layer flow, $\tilde{u}_1$, is called the *Ekman pumping*. This "pumping" is similar to the one we met with the Blasius flow. It may just be either positive of negative, meaning that the layer either pumps in or out the matter, depending on the sign of the local vorticity. This pumping forces the component $\overline{u}_1$ of the interior flow. This new component is known as *the Ekman circulation*. We shall see below that despite its small amplitude, Ekman circulation is crucial to the large-scale dynamics.

We have now all the pieces to write down the steady solution complete at first order. With obvious notations, we may write it:

$$\mathbf{u} = \mathbf{u}_{\mathrm{geo}} + \tilde{u}_{\mathrm{geo}} + \sqrt{\mathrm{E}}(\tilde{u}_{\mathrm{pump}} + \mathbf{u}_{\mathrm{circ}}) + \mathcal{O}(\mathrm{E}) \tag{8.55}$$

8.4.4 An Example: The Spin-Up Flow

The spin-up flow is the large-scale flow that arises within a rotating fluid when an exterior stress increases the angular velocity. For instance, when a liquid in some container rotates as a solid body, like the container, at an angular velocity $\mathbf{\Omega}$, a sudden change of the angular velocity of the container, by $\Delta\mathbf{\Omega}$, will generate a fluid flow, that will lead to the new solid body rotation at $\mathbf{\Omega} + \Delta\mathbf{\Omega}$. This transient flow may be split in several steps one of which is quasi-steady and called the spin-up (or spin-down) flow.

8.4.4.1 Spin-Up Driven by a Solid Plane

The simplest set-up to study a spin-up flow is to consider a viscous incompressible fluid in the neighbourhood of a solid plane staying at $z = 0$. The plane rotates uniformly at $\mathbf{\Omega} = \Omega\mathbf{e}_z$. The viscous fluid is in the half-space $z > 0$. The rotation of the plane is increased instantaneously by $\Delta\Omega\mathbf{e}_z$. After a transient of a few rotation periods, Ekman layers have formed and a quasi-steady flow takes place.

To study this flow we use a frame attached to the bounding plane. Far from this plane the fluid rotates at the angular velocity $-\Delta\Omega\mathbf{e}_z$. In this region, viewed from the plane, there is a geostrophic flow $\mathbf{v}_{\text{geo}} = -\Delta\Omega\mathbf{e}_z \times \mathbf{r}$. This is our basic geostrophic solution that needs to be completed by Ekman layers. Inserting this solution into (8.49), we get the needed boundary layer corrections so that $\mathbf{v} = \mathbf{0}$ at $z = 0$. Using cylindrical coordinates (s, φ, z), we get

$$\begin{cases} u_s = \Delta\Omega s \sin\zeta' e^{-\zeta'} \\ u_\varphi = \Delta\Omega s(\cos\zeta' e^{-\zeta'} - 1) \end{cases} \tag{8.56}$$

This solution shows that the spin-up flow is diverging in the boundary layer ($u_s > 0$), which shows that this boundary layer "sucks" the outer fluid. Since the boundary is plane we can use (8.53). Noting that $\mathbf{n} = \mathbf{e}_z$ and

$$R(x, y) = \mathbf{e}_z \cdot \nabla \times (-2\Delta\Omega\mathbf{e}_z \times \mathbf{r}) = -2\Delta\Omega$$

we deduce that

$$\tilde{u}_z = 2\Delta\Omega\sqrt{E}\cos\left(\zeta' - \pi/4\right) e^{-\zeta'}$$

This component of the boundary layer flow induces a pumping of the outer fluid into the boundary layer because $\tilde{u}_z(0) \neq 0$. Thus, in order that the boundary condition $u_z = 0$ be verified, the outer solution needs to be completed by an $\mathcal{O}(\sqrt{E})$ solution of the inviscid equations such that

$$\overline{u}(\zeta' = 0) = -\tilde{u}_z(\zeta' = 0)$$

In Fig. 8.5, we show schematically the radial and vertical components of the boundary layer flow. A solid body rotation should be added in thought.

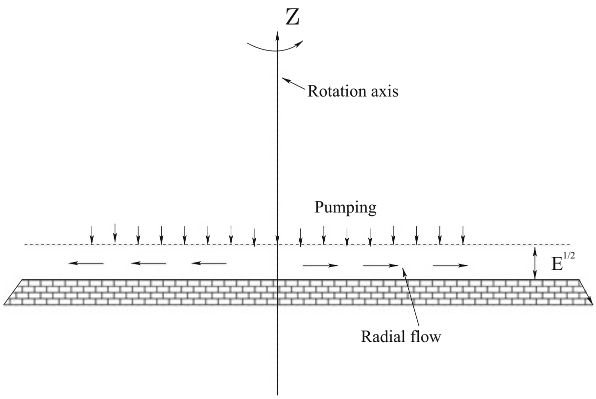

Fig. 8.5 Meridian view of a spin-up flow in the neighbourhood of a rotating plane. The radial component of the flow only exists in the boundary layer. To insure mass conservation the boundary layer absorb some mass from the interior. In a spin-dow flow all the flows would be reversed

The radial flow in the boundary layer is easily observable in a glass of water when we try to dissolve some sugar by stirring the water with a tea-spoon. When we stop stirring, we observe that the sugar gathers on the rotation axis of the water on the bottom of the glass. This is the signature of the radial component of the boundary layer flow, which is converging in the spin-down case, gathering the sugar at the centre.

8.4.4.2 Spin-Up Within a Sphere

As a second example, we now consider a viscous incompressible fluid inside a sphere whose angular velocity increases very slowly with time. In this ideal case the spin-up flow is steady. Let $\dot{\Omega}$ be the acceleration of the rotation, the natural scaling of the velocity field is

$$\mathbf{v} = \frac{\dot{\Omega} R}{2\Omega}\mathbf{u}$$

If we choose $(2\Omega)^{-1}$ as the time scale and the radius of the sphere R as the length scale, the momentum equation written in a frame corotating with the sphere reads

$$\frac{\partial \mathbf{u}}{\partial \tau} + \mathrm{Ro}(\mathbf{u}\cdot\nabla)\mathbf{u} + \mathbf{e}_z \times \mathbf{u} + \mathbf{e}_z \times \mathbf{r} = -\nabla p + E\Delta\mathbf{u}$$

The acceleration term $\dot{\boldsymbol{\Omega}} \times \mathbf{r}$ that yields the term $\mathbf{e}_z \times \mathbf{r}$ is sometimes called *the Euler force*. The Rossby number is assumed to be vanishingly small since we focus on very small accelerations. The nonlinear terms are therefore neglected and since we look for steady solutions, we'll have to solve

$$\begin{cases} \mathbf{e}_z \times \mathbf{u} + \mathbf{e}_z \times \mathbf{r} = -\nabla p + E\Delta\mathbf{u} \\ \nabla\cdot\mathbf{u} = 0 \\ \mathbf{u} = \mathbf{0} \qquad \text{on} \qquad r = 1 \end{cases} \tag{8.57}$$

To solve this system, it is convenient to split the solution in the following way:

$$\mathbf{u} = 2z\mathbf{e}_z - s\mathbf{e}_s + \mathbf{u}_{\mathrm{geo}}(s) + \tilde{u}$$

where we used the cylindrical coordinates (s, φ, z). The $2z\mathbf{e}_z - s\mathbf{e}_s$ terms represent a particular solution of vanishing divergence, that cancels the forcing term $\mathbf{e}_z \times \mathbf{r}$. But this particular solution does not meet the boundary condition $\mathbf{n}\cdot\mathbf{u} = 0$. Unfortunately the geostrophic solution which is parallel to $\mathbf{e}_\varphi$ cannot help. The mass flux of this particular solution on the bounding sphere needs thus to be compensated by the boundary layer mass flux. The particular solution therefore represents the

Ekman circulation part of the solution. Since this circulation is $E^{1/2}$ times smaller than the geostrophic part, we conclude that $\mathbf{u}_{\text{geo}}$ is $\mathcal{O}(E^{-1/2})$. It means that that $\mathbf{u}_{\text{geo}}$ diverges at zero viscosity, but this is not surprising since the sphere cannot entrains an inviscid fluid!

Let us come back to the resolution of our problem. We note that the Ekman pumping on the wall is such that

$$\tilde{u}_r + \mathbf{e}_r \cdot (2z\mathbf{e}_z - s\mathbf{e}_s) = 0 \qquad \Longrightarrow \qquad \tilde{u}_r = 3\sin^2\theta - 2$$

where θ is the polar angle of the spherical coordinates. On the other hand (8.15) shows that $\mathbf{u}_{\text{geo}}(s) = U(s)\mathbf{e}_\varphi$ and induces a boundary flow given by (8.49)

$$\begin{cases} \tilde{u}_\theta = -U(\sin\theta)\sin\alpha\, e^{-\alpha} \\ \tilde{u}_\varphi = -U(\sin\theta)\cos\alpha\, e^{-\alpha} \end{cases} \tag{8.58}$$

where

$$\alpha = \zeta\sqrt{\frac{\cos\theta}{2}}$$

Here, we'll assume that $\cos\theta > 0$ thus restricting our discussion to the Northern hemisphere. We note that on the bounding sphere $r = 1$ and $s = \sin\theta$. Finally, the geostrophic flow with its boundary layer correction reads

$$\begin{cases} u_\theta = -U(\sin\theta)\sin\alpha\, e^{-\alpha} \\ u_\varphi = U(s) - U(\sin\theta)\cos\alpha\, e^{-\alpha} \end{cases} \tag{8.59}$$

Mass conservation gives the relation between pumping and the foregoing flow. At the leading order we have

$$\frac{\partial\tilde{u}}{\partial r} + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\,\tilde{u}_\theta) = 0 \qquad \text{at} \quad r = 1$$

Using the boundary layer variable $\zeta = (1-r)/\sqrt{E}$ and the previous expression of $\tilde{u}_\theta$, we get

$$\frac{\partial\tilde{u}}{\partial\zeta} = -\frac{\sqrt{E}}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\, U(\sin\theta)\sin\alpha\, e^{-\alpha}\right)$$

This equation is integrated between 0 and $+\infty$ and leads to:

$$\tilde{u}_r(\zeta = 0) = \frac{\sqrt{E}}{\sin\theta}\frac{\partial}{\partial\theta}\left(\frac{\sin\theta\, U(\sin\theta)}{\sqrt{2\cos\theta}}\right)$$

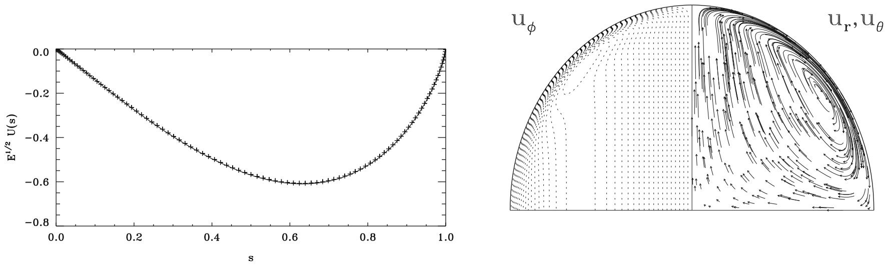

Fig. 8.6 *Left* the analytic solution (8.60) of the geostrophic flow associated with the spin-up flow in a sphere (*solid line*) and the numerical solution (pluses). The Ekman number is 10^{-7}. The difference between the two curves is less than a percent outside the Ekman layer. *Right* a meridian view of the velocity field. The Ekman number is $\mathrm{E} = 4 \times 10^{-4}$ large enough to make the boundary layer flow clearly visible

Since we know the expression of $\tilde{u}_r(\zeta = 0)$, we get the differential equation for $U(\sin\theta)$. We finally get

$$U(\sin\theta) = -\sqrt{\frac{2}{E}} \sin\theta \left(1 - \sin^2\theta\right)^{3/4}$$

valid on the sphere $r = 1$. The geostrophic flow therefore reads

$$\mathbf{u}_{\mathrm{geo}}(s) = -\sqrt{\frac{2}{E}} s \left(1 - s^2\right)^{3/4} \mathbf{e}_\varphi \tag{8.60}$$

using the cylindrical coordinate $s = r\sin\theta$.

We have plotted this solution together with the full numerical solution of (8.57) in Fig. 8.6 (left) when $\mathrm{E} = 10^{-7}$. The difference between the analytic and numerical solution is not noticeable. That would not be the case if we used $\mathrm{E} = 4 \times 10^{-4}$ as in Fig. 8.6 (right) to better show the meridian flow. In fact, at $\mathrm{E} = 4 \times 10^{-4}$ the boundary layer theory is not performing well (although E is small).

8.4.4.3 Conclusion: The Spin-Up Time

We have summarized in Fig. 8.7 all the components of a spin-up flow in a spherical container, including the boundary layer singularity.

One remarkable property of the spin-up flow is that the Ekman circulation controls the time scale of the spin-up. Indeed, this circulation insure the transport of angular momentum from the walls to the interior. We may thus evaluate the time scale of the process of synchronization between the fluid and the container. This is typically the turnover time scale of the Ekman circulation. If L is the characteristic size of the container, the amplitude of this circulation is $\Omega L\sqrt{\mathrm{E}}$. It leads to the spin-up time scale:

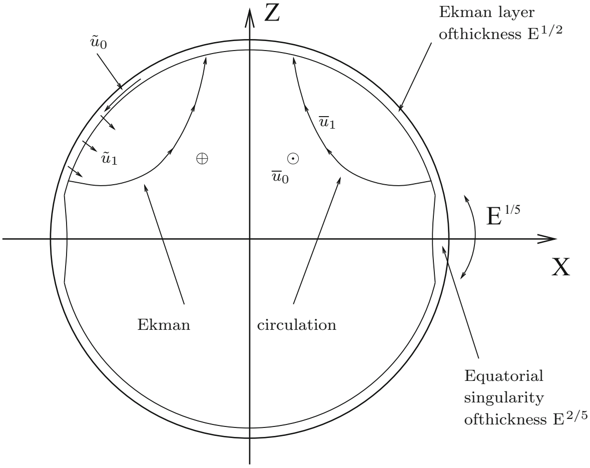

Fig. 8.7 Schematic view of a spin-up flow within a sphere. $\overline{u}_0$ is the geostrophic azimuthal flow; $\tilde{u}_0$ is the meridional flow within the Ekman layer and $\tilde{u}_1$ is the Ekman pumping forcing the Ekman circulation $\overline{u}_1$

$$t_{\text{spin-up}} = \frac{L}{\Omega L\sqrt{\mathrm{E}}} = \frac{\Omega^{-1}}{\sqrt{\mathrm{E}}} = \frac{L}{\sqrt{\Omega\nu}} \tag{8.61}$$

This expression shows that the spin-up of the fluid is realized on a time scale much shorter than the one imposed by viscous diffusion. Indeed,

$$T_{visc} = \frac{L^2}{\nu} = \frac{\Omega^{-1}}{\mathrm{E}} \gg \frac{\Omega^{-1}}{\sqrt{\mathrm{E}}}$$

since $\mathrm{E} \ll 1$.

This new time-scale may be revealed by a simple experiment using a glass of water. If we make rotating the water within the glass, we can measure the time by which the water has ceased to rotate after our forcing has been stopped. Using a glass of water of 7 cm in diameter, rotating the water at one round per second, we find that the fluid flow has almost vanished 2.5 min after. Computing the diffusion time scale, we find $T_{visc} = 0.035^2/10^{-6} \approx 20$ min, which is much larger. This spin-down flow has a time scale, which we evaluate from (8.61), of 20 s, which is much closer to our observation. Within such an experiment, nonlinear effects are quite strong since the rotation ends at zero; however, orders of magnitude are correct, especially if we take a mean rotation of half a round per second.

8.5 Hurricanes

8.5.1 A Qualitative Presentation

In the introduction to this chapter, we mentioned one of the most violent phenomena in the terrestrial atmosphere, namely the hurricanes. We are now ready to explore their dynamics in some details.

First, let us observe that a low pressure region in the Earth atmosphere cannot be filled up by the geostrophic flow that it triggers: the winds are orthogonal to the pressure gradient. Thus, only non-geostrophic effects may fill up a low pressure region. Because of their weakness, the lifetime of such a pressure field is quite long.

In the case of a hurricane, the low pressure field has an especially long life time due to the existence of an energy source: the tropical ocean.

The dynamics of a hurricane can be understood with a simple model. One can then derive the value of the central depression as a function of the temperature of the ocean and of the upper atmosphere. However, before getting into these details, we shall first give a qualitative description of the hurricanes.

Let us consider the air near the surface of a tropical ocean: the percentage of water vapour may be quite high there, due to the important evaporation. Such a mixture is unstable to convection (see Chap. 7): a rising fluid element will face an adiabatic expansion which triggers the condensation of water vapour, releasing latent heat, which amplifies the rise. This process is at the origin of cloud formation and is called *wet convection.*

The low pressure created by the rising elements forces a geostrophic wind, which contributes to make the sea more rough. The fraction of water vapour within the air thus increases. The boundary layer has a radial drift which tries to fill up the depression. Within the centre of the depression air is forced to rise, releasing more and more latent heat and thus making the pressure even lower. Thus, the phenomenon amplifies. However, we may observe that there is a maximum value to the fraction of water vapour in the air: this is the saturation.

Thus we understand why hurricanes appear only in the tropics: the hotter the air, the larger the mass fraction of water vapour. At lower temperatures, the water vapour content is not enough to maintain the winds. It is also clear that above a continent a hurricane dies. Finally, computing the resulting Coriolis force on the ascending air column, we find that the depression should drift to the West as usually observed. This tendency is sometimes counteracted by an anticyclonic air mass.

8.5.2 The Steady State: A Carnot Engine

A hurricane is actually a true Carnot engine, running with a heat source, the ocean, and a cold source, the upper atmosphere. The thermal energy of the ocean is partly converted into mechanical energy: the wind.

In Fig. 8.8 we show the Carnot cycle followed by a fluid element. From A to B the fluid is heated at constant temperature: its water vapour mass fraction increases. From B to C, it follows an adiabatic expansion but during the rise of a fluid parcel water droplets form: it rains! From C to D, the fluid radiates its heat into space and cools down. Next, from D to A the model assumes that the fluid supports an adiabatic compression which is never realized actually. There is no streamline between A and D but this is no problem if the fluid elements have the same entropy at these two points. This is usually assumed.

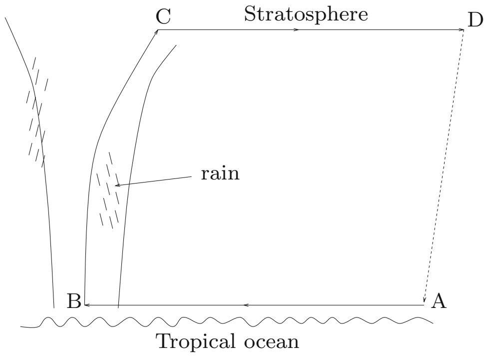

Fig. 8.8 A meridional view of a hurricane

Let us now be a little more quantitative. In a steady regime, along a streamline, we have

$$d(\frac{1}{2}v^2 + gz) + \frac{dP}{\rho} - \mathbf{f} \cdot d\mathbf{l} = 0 \tag{8.62}$$

where $\mathbf{f}$ represents the forces due to viscosity. We now integrate this relation along the cycle that we just described. It turns out that

$$\oint \mathbf{f} \cdot d\mathbf{l} = \oint \frac{dP}{\rho}$$

For a mixture of air and water vapour, assumed to be an ideal gas, we have

$$dh = c_p dT = Tds + \frac{dP}{\rho} - d(L_v q) \tag{8.63}$$

where L_v is the latent heat of vapourization and q is the mass fraction of water. Therefore along the cycle $\oint dP/\rho = -\oint Tds$, and

$$\oint \mathbf{f} \cdot d\mathbf{l} = -\oint Tds$$

which shows that the entropy production is due to the friction (viscous dissipation). Now, if T_{sc} and T_{sf} are the temperature of the hot and cold sources respectively, s_A and s_B the entropy at A and B, then

$$-\oint \mathbf{f} \cdot d\mathbf{l} = \oint Tds = (T_{sc} - T_{sf})(s_B - s_A) \tag{8.64}$$

since $s_C = s_B$ and $s_D = s_A$. Besides, from (8.63) and $\rho = P/(R_* T_{sc})$, we have

$$s_B - s_A = R_* \ln(P_A/P_B) + \frac{L_v}{T_{sc}}(q_B - q_A) \tag{8.65}$$

Now we have to evaluate the power dissipated by friction. The main contribution to $\oint \mathbf{f} \cdot d\mathbf{l}$ comes from the leg AB which is in the atmospheric boundary layer. The flow there follows a spiral, namely the azimuthal geostrophic flow combined with a radial drift of the (turbulent) Ekman layer. Using (8.62) between A and B, we see that the work of friction forces comes from the pressure (just like for the Poiseuille flow). It turns out that

$$\oint_{AB} \mathbf{f} \cdot d\mathbf{l} = -\oint_{AB} \frac{dP}{\rho} = R_* T_{sc} \ln(P_A/P_B) \tag{8.66}$$

since the temperature is constant on AB and $P = R_* \rho T$. Finally, combining (8.64), (8.65) and (8.66), we get

$$R_* T_{sf} \ln(P_A/P_B) = \varepsilon L_V (q_B - q_A) \tag{8.67}$$

where we introduced $\varepsilon = (T_{sc} - T_{sf})/T_{sc}$ which is nothing but the efficiency of the Carnot cycle. Equation (8.67) shows that the depression of the hurricane will be all the stronger that q_B be the larger. However, the highest quantity of water vapour in the air is reached when the air is saturated. Setting q_B to this maximum value, we obtain the minimum central pressure, that is the strongest hurricane. If T_{sc} is expressed in Celsius degrees, a very good approximation of q at saturation is given by

$$q_{sat} = \frac{380.2}{P} \exp\left[\frac{17.67\, T_{sc}}{243.5 + T_{sc}}\right]$$

where P is expressed in Pascals.

Let us take the case of a hurricane blowing in the Northern Atlantic ocean. A typical temperature in the Caribbean sea is 28 °C, while that of the stratosphere is $T_{sf} = -60$ °C; using L_V= 2.3 10^6 J/kg and assuming that outside the hurricane the partial pressure of water vapour is 75 %, we find that the ratio P_A/P_B verifies

$$\ln(P_A/P_B) = 0.256(P_A/P_B - 0.75)$$

which solution is $P_A/P_B \simeq 1.09$. The strongest hurricane has a central pressure about 930 hPa. For comparison, Emily (Fig. 8.9) had a central pressure at 960 hPa. However, some hurricanes in the Eastern Pacific ocean have reached pressures as low as 870 hPa.

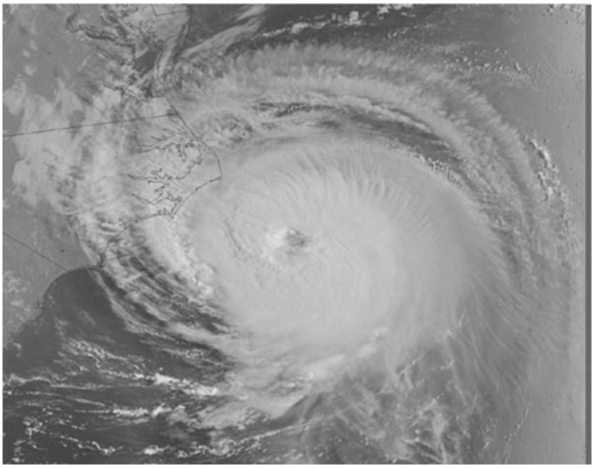

Fig. 8.9 The hurricane Emily near the coast of North Carolina (USA) in September 1993

8.5.3 The Birth of Hurricanes

The foregoing very simplified model allows us to understand the way hurricanes work. However, it does not teach us how such vortices arise. Indeed, the conditions for their existence are realized most of the time in tropical oceans. Thus, we would expect that they would be always present. However, this is by far not the case: Hurricanes are rather rare features in the atmosphere. According to recent studies, this scarcity seems to be the consequence of the finite-amplitude nature of the instability that leads to a hurricane. At the origin of the phenomenon, we mentioned the wet convection. This convection is usually giving birth to gentle clouds like cumulus, which extend over a fraction of the troposphere. When a hurricane sets in, wet convection is able to connect the ocean (the heat source) with the stratosphere (the cold source), otherwise the Carnot engine does not work. This is like if a cumulus extends over the whole troposphere. Only, a small fraction of tropical storms reach such an amplitude and turn into a hurricane.

In fact, many sides of the hurricanes dynamics remain obscure because of their complexity. For instance, only very few hurricanes reach the strongest state. Likely, the storm sweeping the ocean, generate an upwelling of cold water, which cools the surface water and decreases the water vapour content of the air near the surface. Hence, a good model needs to take into account the dynamic coupling between the ocean and the atmosphere, and this is not an easy matter.

8.6 Exercises

1. Show by a direct transformation (8.1). This demonstration may be done in three steps: (i) splitting the velocity field into a solid rotation and a remaining flow, (ii) noting that this flow should be expressed using the unit vectors of the rotating frame and (iii) observing that the time dependence of the velocity changes.
2. Show that even with viscosity, the frequency of inertial is such that $|\omega| \leq 1$.
3. Show that the stream function of axisymmetric inertial modes obeys a hyperbolic equation similar to the Poincaré one.
4. Explain why hurricanes do not appear on the equator.

Further Reading

There is only one monograph dealing entirely with rotating fluids: *The theory of rotating fluids* by Greenspan (1969), unfortunately out of print. However, some insights may be found in *Geophysical fluid dynamics* by Pedlosky (1979) but in the context of the ocean and atmosphere dynamics. A recent review of spin-up flows is given by Duck and Foster (2001). A more detailed presentation of hurricanes may be found in Emanuel (1991).

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