Chapter 13
The Solutions of Exercises

Chapter 1

1. The solution is $2\boldsymbol{\Omega}$; the velocity field of a solid rotation has a uniform vorticity.
2. One finds that all the components of [s] are zero. The solid body rotation does not introduce any deformation. If $\mathbf{v} = \lambda\mathbf{r}$ then $s_{xx} = s_{yy} = s_{zz} = \lambda$ and $s_{xy} = s_{xz} = s_{yz} = 0$. A solid body rotation can be the velocity field of an incompressible fluid because $Tr([s]) = 0$, but the second velocity field cannot because it is not divergence-free.
3. We just need using the definition of the divergence. For the reciprocal, we observe that

$$\nabla\cdot\mathbf{v} = 0 \iff \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0 \iff v_x = -\int \frac{\partial v_y}{\partial y} dx = -\frac{\partial}{\partial y}\int v_y dx$$

showing that the stream function exists and we just need to choose

$$\psi = -\int v_y dx$$

In polar coordinates, we easily find that

$$v_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta}, \qquad v_\theta = -\frac{\partial \psi}{\partial r}$$

while for an axisymmetric flow in cylindrical coordinates

$$v_r = \frac{1}{r}\frac{\partial \psi}{\partial z}, \qquad v_z = -\frac{1}{r}\frac{\partial \psi}{\partial r}$$

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4. The mass of a volume moving with the fluid is constant. The integral of $\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\mathbf{v})$ is therefore always zero what ever the volume, hence we find the continuity equation again.

5. We have six equations

$$\partial_x v_x = \partial_y v_y = \partial_z v_z = 0$$

$$\partial_x v_y + \partial_y v_x = \partial_y v_z + \partial_z v_y = \partial_z v_x + \partial_x v_z = 0 \quad (13.1)$$

The first three show that

$$v_x \equiv v_x(y,z), \quad v_y \equiv v_y(x,z), \quad v_z \equiv v_z(x,y)$$

The last three lead to

$$\partial_y^2 v_x = \partial_z^2 v_x = 0$$

with similar expressions for v_y and v_z. We derive that

$$\begin{cases} v_x = a_1 yz + b_2 y + c_1 z + d_1 \\ v_y = a_2 xz + b_3 z + c_2 x + d_2 \\ v_z = a_3 xy + b_1 x + c_3 y + d_3 \end{cases} \quad (13.2)$$

We use again relations (13.1) to show that $a_1 = a_2 = a_3 = 0$ and $c_1 = -b_1$, $c_2 = -b_2$, $c_3 = -b3$. It implies that

$$v_i = d_i - \epsilon_{ijk} b_j x_k \qquad \text{or} \qquad \mathbf{v} = \mathbf{d} - \mathbf{b}\times\mathbf{r}$$

which shows that the velocity field is composed of a translation and a solid body rotation.

6. We start from the equation of kinetic energy (1.28) to which we add $\frac{1}{2}v^2$ times the continuity equation. We find

$$\frac{\partial\frac{1}{2}\rho v^2}{\partial t} + \nabla\cdot(\frac{1}{2}\rho v^2\mathbf{v}) = v_i\partial_j\sigma_{ij}$$

We integrate this equation over the volume V. The integration of the second term in the left-hand side gives zero since $\mathbf{v}\cdot d\mathbf{S} = 0$, while the integral of the first term is the time derivative of the kinetic energy. The third integral needs to be transformed in the following way:

$$\int_V v_i\partial_j\sigma_{ij}dV = \int_V \partial_j(v_i\sigma_{ij})dV - \int_V \sigma_{ij}\partial_j v_i dV$$

Here, the first integral may be transformed into an integral over the bounding surface, and therefore vanishes because of boundary conditions. The second integral is just the viscous dissipation within the volume. Thus

$$\frac{dE_c}{dt} = -\int_V \mathcal{D} dV$$

If the fluid is incompressible, using (1.48) gives the expected result.

7. We noticed that a biaxial extension can be derived from a uniaxial extension by just changing the sign of the shear.
So we consider a biaxial extension characterized by a "shear time scale" equal to $-T$. The associated velocity field is $v_x = -x/T, v_y = -y/T, v_z = 2z/T$. Using the definition of the biaxial viscosity, we find

$$\sigma_{xx} - \sigma_{zz} = -\mu_{EB}(-T)/T \;. \tag{13.3}$$

However, in order to use the definition of the uniaxial viscosity (1.79), we have to exchange the axis $x \to y$, $y \to z$, $z \to x$. The velocity field is now $v_x = 2x/T, v_y = -y/T, v_z = -z/T$. From (1.79) we get $\sigma_{xx} - \sigma_{yy} = 2\mu_E(T/2)/T$. Taking into account the exchange of the axis in (13.3) we also get $\sigma_{xx} - \sigma_{yy} = \mu_{EB}(-T)/T$. The result follows.

Chapter 2

1. *About buoyancy*

(a) Let M_g be the mass of the ice and V_{im} the volume below the water level. We also introduce the initial volume of water V_{e_i}, and the final volume of water V_{e_f}, when the ice is melted. Since the container is the same initially and finally, comparing the level of water is equivalent to comparing the volumes $V_{e_i} + V_{im}$ and V_{e_f}. From Archimedes theorem, the equilibrium of the ice cube means that

$$M_g\mathbf{g} - \rho_e V_{im}\mathbf{g} = \mathbf{0} \qquad \Longrightarrow \qquad M_g = \rho_e V_{im}$$

where ρ_e is the density of water. When the ice melts, it transforms into water so that

$$M_g = \rho_e(V_{e_f} - V_{e_i}) = \rho_e V_{im}$$

We therefore derive that $V_{e_i} + V_{im} = V_{e_f}$, showing that the level of water remains identical when the ice melts.

(b) The reasoning is the same as before, but introducing a piece of metal of mass M_m, we now have:

$$(M_g + M_m)\mathbf{g} - \rho_e V_{im}\mathbf{g} = \mathbf{0}$$

After the ice melting $V_F = V_{e_f} + M_m/\rho_m$, where V_F is the final volume of water plus that of the metal, which has to be compared to the initial volume $V_I = V_{e_i} + V_{im}$.

$$V_I = V_{e_i} + (M_g + M_m)/\rho_e$$

from the first equation. But $M_g = \rho_e(V_{e_f} - V_{e_i})$ is still true because the melting ice gives water. Thus

$$V_I = V_{e_f} + M_m/\rho_e > V_{e_f} + M_m/\rho_m = V_F$$

because the density of metal ρ_m is larger than the density of water ρ_e. Hence, the water level decreases.

(c) When the ice is melted, the cork is floating. Let V'_{im} be its volume under the water in the final state. Before the ice melts, the level of water in the glass is given by the volume $V_I = V_{e_i} + (M_g + M_l)/\rho_e$, following the same reasoning as above. M_l is the mass of the cork. Since the cork floats, its equilibrium implies that $V'_{im} = M_l/\rho_e$. It is then easy to check that

$$V_I = V_{e_i} + (M_g + M_l)/\rho_e = V_F = V_{e_f} + V'_{im}$$

since $M_g = \rho_e(V_{e_f} - V_{e_i})$. The level thus remains the same.

(d) This is because the air density decreases with altitude while that of water remains approximately constant.

(e) The balloon moves towards the front of the car. Indeed, when the car starts, the effective gravity has a small component towards the back of the car. The buoyancy force therefore has a small component in the opposite direction, namely towards the front. Do the experiment!

2. (a) The denser goes under the lighter (oil).

(b) $P(z)$ curve is made of two line segments of slope $-\rho_e g$ in the water and $-\rho_h g$ in the oil.

(c) The ball being denser than oil but less dense than water will stay at the interface of the two liquids. Although part of the ball is in the water and the other part in the oil, Archimedes theorem can be used because the pressure field is continuous everywhere over the ball. Thus

$$\rho_{wood} V_{ball} = \rho_{oil} V_{oil} + \rho_{water} V_{water}$$

Of course $V_{ball} = V_{oil} + V_{water}$. It follows that

$$\frac{V_{water}}{V_{ball}} = \frac{\rho_{wood} - \rho_{oil}}{\rho_{water} - \rho_{oil}}$$

Numerically $V_{water}/V_{ball} = 3/4$.

3. Let H_h be the height of the oil free surface, H_{he} be the height of the oil–water interface and H_e the height of the free surface of water in the other branch of the tube. We have $H_h - H_{he} = 2\,\text{cm}$ and $P(H_h) = P(H_e) = P_{atm}$. From Pascal Theorem (2.5),

$$P(H_h) + \rho_{\text{oil}}(H_h - H_{he})g = P(H_e) + \rho_{\text{water}}(H_e - H_{he})g$$

which allows us to express H_e as a function of H_{he}, namely $H_e = 2\,cm\rho_{\text{oil}}/\rho_{\text{water}} + H_{he}$. Now we need another equation. It is given by the fact that the liquids are incompressible, therefore if the level moves of x cm in one arm of the tube it moves of the same quantity in the other arm. Thus $H_e = H_0 + x$ and $H_{he} = H_0 - x$. This leads to $x = \rho_{\text{oil}}/\rho_{\text{water}} = 0.6\,\text{cm}$ and all the other heights, using $H_0 = 10\,\text{cm}$.

4.(a) The wood sphere being in equilibrium, the resultant of forces is vanishing, thus

$$M_b\mathbf{g} - \int_{(S)} P_{\text{water}} d\mathbf{S} + \mathbf{R} = \mathbf{0}$$

where M_b is the mass of the ball and $\mathbf{R}$ is the reaction of the reservoir floor. We therefore need to correctly evaluate the resultant of pressure forces. We note that we cannot use the Archimedes theorem because pressure is not continuous over the whole sphere surface. The integral needs to be computed explicitly. We may note that the contribution of the air pressure is vanishing since it is the same everywhere. We may note in addition that the resultant has just one component along $\mathbf{e}_z$ so that the evaluation of the integral may be done as follows:

$$\mathbf{e}_z \cdot \int_{(S)} P d\mathbf{S} = -\int_0^{\theta_0} P(\theta) 2\pi R \sin\theta \cos\theta R d\theta$$

where θ_0 is such that $\sin\theta_0 = r/R$ and $\cos\theta_0 = -\sqrt{1 - r^2/R^2}$; note the $\cos\theta$ factor due to the projection on $\mathbf{e}_z$. The expression of $P(\theta)$ comes from Pascal's formula : $P(\theta) = g\rho_{\text{water}}[H + R(\cos\theta_0 - \cos\theta)]$. It turns out that

$$\mathbf{e}_z \cdot \int_{(S)} P_{\text{water}} d\mathbf{S} = g\rho_{\text{water}} 2\pi R^2 \left(\frac{1}{2}(H + R\cos\theta_0)\sin^2\theta_0 + \frac{R}{3}(\cos^3\theta_0 - 1)\right)$$

The force exerted by the sphere on the floor of the basin is therefore

$$\mathbf{R} = M_b\mathbf{g}\left\{1 + \frac{\rho_{\text{water}}}{\rho}\left[\frac{3}{4}\left(\frac{H}{R} + \cos\theta_0\right)\sin^2\theta_0 + \frac{1}{2}(\cos^3\theta_0 - 1)\right]\right\}$$

(b) Numerically: $|\mathbf{R}| = 170$ N

(c) The sphere rises up before emerging at the surface. Indeed, this is determined by the vanishing of the resulting force, namely when

$$H \leq H_c = \frac{2R}{3}\frac{1 - \cos^3\theta_0 - 2\rho/\rho_{\text{water}}}{\sin^2\theta_0} - R\cos\theta_0$$

But with $H_c \geq R(1 - \cos\theta)$.

5.(a) The balloon takes off if its buoyancy is larger than its weight. We thus find that $(M_b + M_H)/V_b < \rho_0$.

(b) The cruising altitude of the balloon is reached when its buoyancy compensates its weight or, in other words, when its mean density equals that of the air around, namely $\rho_b = (M_b + M_H)/V_b$. We thus find $z_{\text{flight}} = z_0(1 - \rho_b/\rho_0)^{(\gamma-1)/\gamma}$

6.(a) The hydrogen pressure is identical to that of the outside air.

(b) Air and hydrogen being assumed ideal gases, the equality of pressures and temperatures implies that $V_H = m_a n_H/\rho_a$ where n_H is the mole number of hydrogen, m_a is the mass of an air mole and ρ_a is the air density. Since ρ_a decreases with altitude, η increases with altitude.

(c) We just write that the buoyancy is larger than the weight so that $\rho_a V_H \geq M_b + M_H$. Using the expression of the hydrogen volume and that $M_H = n_H m_H$, we get the condition $n_H \geq M_b/(m_a - m_H)$.

(d) The foregoing condition does not depend explicitly on the altitude so if the amount of hydrogen does not vary, the balloon rises. It will not rise indefinitely because, as we saw before, the volume of hydrogen increases with the altitude. When it exceeds $V_b = m_a n_H/\rho_a$, the balloon loses hydrogen. The altitude of the balloon stabilizes at a value such that $V_b = m_a n_H/\rho_a$ and $n_H = M_b/(m_a - m_H)$. One then finds that the cruising altitude is $z = z_0\left(1 - [\rho_b/\rho_0/(1 - m_H/m_a)]^{(\gamma-1)/\gamma}\right)$, which is quite close to the value of the preceding problem.

7. We just need to compute the surface gravity, to which we withdraw the centrifugal acceleration. We find $g = GM/R^2 = 24.8\,\text{m/s}^2$ and $\Omega^2 R = 2.25\,\text{m/s}^2$. The effective gravity is therefore $g_e = 22.5\,\text{m/s}^2$. The molecular mass of the gas is $M = 0.85 \times 2 + 0,15 \times 4 = 2.3$ g/mole; $c_p = 3.35\ R_*$. Finally, the gradient of temperature is $-g/c_p = -1.9$ K/km.

8.(a) The axial symmetry of the system implies that the resultant of pressure forces has just one component along $\mathbf{e}_z$, namely $F_z = \int_{(S)} P d\mathbf{S}\cdot\mathbf{e}_z$. We note that $dS_z = 2\pi\tan^2\alpha\,(H - z)dz$. Finally, $F_z = \pi\rho g h^2(H - h/3)\tan^2\alpha$.

(b) If $h = H$ then $F_z = 2M_{lg}$. We derive that if the mass of the funnel is smaller than twice the liquid mass, the liquid can move the funnel up and flow away.

9. *A polytropic model of the Sun*:

(a) We note that

$$\frac{dM}{dr} = 4\pi r^2 \rho$$

The equation of state together with the equation of mechanical equilibrium give

$$(1 + 1/n)K\rho^{1/n-1}\frac{d\rho}{dr} = -\frac{GM(r)}{r^2}$$

After multiplication by r^2 and taking the derivative, we introduce θ and get:

$$\frac{1}{r^2}\frac{d}{dr}r^2\frac{d\theta}{dr} + \frac{4\pi G\rho_c^{1-1/n}}{(n+1)K}\theta^n = 0$$

Making the change of variable $r = r_0\xi$ we obtain Emden equation.

(b) We just need to insert the expression of ρ as a function of θ in the equation of state.

(c) First insert the expression of P and ρ in the equation of mechanical equilibrium, then use this equation at the stellar surface. Central pressure is eliminated thanks to the expression of r_0.

(d) Use the expression of r_0.

(e) We find that $r_0 = 8.0\ 10^7$ m. $\rho_c = 1.61\ 10^5\,\text{kg/m}^3$, $p_c = 3.2\ 10^{16}$ Pa.

(f) Helium ions, protons and electrons all contribute to the pressure. But electrons do not contribute to the molecular mass of the gas. Let x_p be the fractional number of protons, x_{He} that of helium ions and x_e that of electrons. The molecular mass of the gas is

$$\mathcal{M} = x_p M_p + x_{\text{He}} M_{\text{He}}$$

because the mass of electrons is negligible. Electrical neutrality imposes:

$$x_e = x_p + 2x_{\text{He}}$$

but we also have

$$x_p + x_e + x_{\text{He}} = 1$$

so that we can deduce

$$2x_p + 3x_{\text{He}} = 1$$

If Y is the mass fraction of helium and X that of protons then $Y = 1 - X$ and

$$Y = \frac{x_{\rm He} M_{\rm He}}{\mathcal{M}} \qquad \text{and} \qquad X = \frac{x_p M_p}{\mathcal{M}}$$

Using this relation and the previous one we get

$$\mathcal{M} = \frac{4}{8 - 5Y} \quad \text{g/mole}$$

If $Y = 0.28$, we obtain $\mathcal{M} = 0.61$ g/mole. This value is smaller than that of protons, in spite of the presence of helium ions. It shows the importance of electrons in a fully ionized plasma. We deduce

$$T_c = 1.43 \times 10^7 \text{ K}$$

The values of the thermodynamic conditions at the Sun's centre are therefore in the right order of magnitude compared to more elaborate models. Polytropic models can thus be used to study some properties of stars without having to deal with the complexity of a realistic equation of state or other specificities (like thermal conductivity).

Chapter 3

1. If the flow is irrotational $\mathbf{v} = \nabla\Phi$, hence $\mathbf{v}$ is perpendicular to surfaces $\Phi = \text{Cst}$. Since streamlines are parallel to $\mathbf{v}$ they are also perpendicular to the equipotentials of Φ.
2. We first project the equation of momentum along Oz, since $\mathbf{v}$ has no component along $\mathbf{e}_z$, we find $\partial_z p = -\rho g \implies P_A = P_{atm} + \rho g h_A$ and $P_B = P_{atm} + \rho g h_B$. Focussing on the streamline going through A and B, Bernoulli's theorem shows that $V_A^2 + P_A/\rho = V_B^2 + P_B/\rho$, hence $h_A - h_B = (V_B^2 - V_A^2)/2g$. We have $V_A S_A = V_B S_B$.
3. (a) One may check that $\nabla \times \mathbf{v} = \mathbf{0}$ or that $\Phi = \Omega a^2 \theta$ is a solution for a velocity potential.
   (b) In the outer domain the flow is irrotational and $p/\rho + \frac{1}{2}v^2 = \text{Cst} \implies p = p_\infty - \rho\Omega^2 a^4/(2r^2)$. In the inner domain, we need Euler equation to derive the pressure field, which turns out to be $p = \frac{1}{2}\rho\Omega^2 r^2 + P_\infty - \rho\Omega^2 a^2$.
   (c) This quantity is constant in the outer domain, but depends on r in the inner domain. There it is constant, only *along* the streamlines. The constant changes from one streamline to the other. On the contrary, in the irrotational region, the constant is the same for all the streamlines.
   (d) The vortex central depression is given by $P_\infty - p(0) = \rho v_{max}^2$ because the velocity is maximum at $r = a$. If v=50 m/s then $P_\infty - p(0) \simeq 3250$ Pa.

(e) If the vortex is in the water, the shape of the air–water interface is given by p(surf)=p_{atm}. The flow being purely horizontal, the vertical variations are given by the hydrostatic balance namely $\partial p/\partial z = -\rho g$. Therefore, the water pressure is given by the previous expressions to which $-\rho g z$ is added. Using p(surf)=p_{atm}, we get the equation of the surface $z_{\text{surf}} = \Omega^2(r^2 - 2a^2)/(2g)$ for the inner domain: this is an axisymmetric paraboloid. For the outer region $z_{\text{surf}} = -\Omega^2 a^4/(2gr^2)$, which is the equation of an axisymmetric hyperboloid.

4. (a) i. The flow is irrotational because at $t = 0$ $\mathbf{v} = \mathbf{0}$, which is irrotational, and because driving forces derive from a potential.

ii. The flow is steady and irrotational. $\frac{1}{2}v^2 + p/\rho + gz$ is constant everywhere in the fluid. Using this expression and its constancy at the reservoir's surface and in the outflow, we find

$$\frac{dh}{dt} = -\frac{s}{S}\sqrt{2gh} \implies t_{\text{purge}} = \frac{S}{s}\sqrt{\frac{2h_0}{g}}$$

where we noticed that the fluid pressure is the same in the two places, and that the reservoir surface velocity is $-dh/dt$.

iii. We need to show that the term $\partial\Phi/\partial t$ is very small compared to the others. The velocity in the reservoir is identical to that of the free surface dh/dt. Since $v_z = \partial\Phi/\partial z$, we determine Φ and $\partial\Phi/\partial t = (s/S)^2 gz \ll gz$. The acceleration term is, as the kinetic energy, very small compared with the potential energy and pressure terms.

(b) i. The equation for the potential (3.22) written between A and B gives

$$\partial_t(\Phi_B - \Phi_A) + (v_B^2 - v_A^2)/2 + (p_B - p_A)/\rho + g(z_B - z_A) = 0.$$

We should note that $\Phi_B - \Phi_A = \oint_A^B \mathbf{v}\cdot d\mathbf{l} \sim v_B l$ Because the fluid velocity is much larger in the tube than in the reservoir, so the part of the path AB which is in the tube dominates the integral. With the assumptions of the text, we find that

$$l\frac{dv}{dt} + v^2/2 = v_\infty^2/2$$

This differential equation is easily solved if we note that $1/(x^2 - 1)$ is the derivative of $argth(x)$. Finally, $v = v_\infty \,\text{th}\,(v_\infty t/(2l))$.

ii. The transient lasts $2l/v_\infty$. We should note that this transient corresponds to the starting motion of the water in the tube, because of its inertia, which grows with the tube length. Numerically we find 0.58 s.

5. (a) Imposing a jolt to the tube is similar to imposing the fluid an inertial force $\mathbf{f} = \rho\mathbf{a}(t)$, where $\mathbf{a}(t)$ is the tube's acceleration. The tube is assumed to be solid, $\mathbf{a}$ is independent of $\mathbf{x}$ and the force is a potential force. We may use

Lagrange theorem. We may observe that the velocity is almost uniform since the fluid is incompressible and the tube radius small compared to its length.

(b) The foregoing remark implies that $\Phi_A - \Phi_B = \oint_B^A \mathbf{v} \cdot d\mathbf{l} = VL$.

(c) We write (3.22) at the two free surfaces of the fluid (at A and B). We find that $\partial_t(\Phi_A - \Phi_B) + g(z_A - z_B) = 0$. Let δh be the level variation at A, then $V = d\delta h/dt$ and $z_A - z_B = 2\delta h$. We find that the surface of the fluid oscillates at a frequency

$$f = \frac{1}{2\pi}\sqrt{\frac{2g}{L}}$$

6.(a) Here too we may verify that $\nabla \times \mathbf{v} = \mathbf{0}$, but it is more efficient to note that the function $\Phi = \int v(r,t)dr$ is a potential for this velocity field.

(b) In the water $\nabla \cdot \mathbf{v} = 0$, consequently $v = C/r^2$. Noting that at $r = R(t)$, $v(r,t) = \dot{R}$, then $v(r,t) = \dot{R}R^2/r^2$.

(c) For an isentropic ideal gas $PV^\gamma = \text{Cst}$, thus $PR^{3\gamma}$=Cst.

(d) The velocity potential is $\Phi(r,t) = -\dot{R}R^2/r$. Equation (3.22) used at $r = R$ gives

$$\ddot{R}R + 3\dot{R}^2/2 = (p - p_0)/\rho = p_0/\rho\left((R_0/R)^{3\gamma} - 1\right) .$$

(e) If the radius of the bubble slightly oscillates around its equilibrium value then, setting $R(t) = R_0(1 + \varepsilon(t))$ with $\varepsilon \ll 1$, we find after linearization, that ε follows the harmonic oscillator equation with a frequency:

$$f = \frac{1}{2\pi R_0}\sqrt{\frac{3\gamma p_0}{\rho}}$$

Numerically, f(1 mm) = 3262 Hz and f(5 mm) = 652 Hz. These frequencies corresponds to sound waves that are readily audible and which give birth to the songs of springs.

7. For a barotropic fluid the equation of vorticity reads

$$\frac{D\boldsymbol{\omega}}{Dt} = (\boldsymbol{\omega} \cdot \nabla)\mathbf{v} - \boldsymbol{\omega}\nabla \cdot \mathbf{v}$$

We divide this equation by ρ and subtract the continuity equation times $\boldsymbol{\omega}/\rho^2$. You should have noted that the continuity equation divided by ρ^2 may also be written:

$$\frac{D(1/\rho)}{Dt} + \frac{\nabla \cdot \mathbf{v}}{\rho} = 0$$

Chapter 4

1. The momentum flux is the same at the inlet and the outlet of the pipe. The resulting force therefore comes from the pressure field at entrance $\pi R^2 p_1 \mathbf{e}_z$ and outlet $(-\pi R^2 p_2 \mathbf{e}_z)$ of the pipe; the sum of it gives $-\pi R^2 G_p L \mathbf{e}_z$.

2. From (4.27) and (4.28), we get

$$F_t = -\frac{\alpha}{3}\frac{3r - 3 - (r+1)\ln r}{2r - 2 - (r+1)\ln r} F_p$$

where $r = h_1/h_2 = 1 + \varepsilon$. With a first order expansion in ε, we find

$$F_t = \simeq \frac{\alpha F_p}{3\varepsilon} = \frac{h_2}{3\ell} F_p$$

The needed force is therefore $3.3\,10^{-4}$ times the weight, so a force similar to that necessary to lift up a weight of 330 g !

3. (a) On the free surface of the fluid the pressure is constant. The z-component of Navier–Stokes equation shows that the pressure does not vary along Ox. The x-component of the Navier–Stokes equation thus gives

$$\nu \frac{\partial^2 V}{\partial z^2} + g \sin\alpha = 0$$

The boundary conditions are $v = 0$ at $z = 0$ and $\partial v/\partial z = 0$ at $z = h$, namely no-slip at the bottom and stress-free at the surface. The solution is easy to derive:

$$V(z) = \frac{g \sin\alpha}{2\nu}(h - z)z$$

(b) The volume flux through a cross section is

$$Q = \int_0^h V(z) dz S/h = \frac{g \sin\alpha}{12\nu} h^2 S$$

4. (a) The flow is axisymmetric (no dependance with respect to θ) and invariant with respect to translation along Oz. We project the Navier–Stokes equation along $\mathbf{e}_\theta$ and verify that the nonlinear terms disappear. The form of the vector Laplacian in cylindrical coordinates leads to the equation $\Delta v(r) - v(r)/r^2 = 0$. Using (12.44), we get

$$\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial r v(r)}{\partial r}\right) = 0$$

which is easily integrated, giving $v(r) = Ar + B/r$. The constants A and B are calculated using the boundary conditions. We find

$$A = \frac{\Omega_1 R_1^2 - \Omega_2 R_2^2}{R_1^2 - R_2^2} \qquad \text{and} \qquad B = \frac{(\Omega_1 - \Omega_2)R_1^2 R_2^2}{R_2^2 - R_1^2}$$

(b) The torque is defined by $\mathbf{C} = \int_{(S)} \mathbf{r} \times [\sigma] d\mathbf{S}$, thus

$$\mathbf{C} = -2\pi R_2 L\mu \left(\frac{\partial}{\partial r}\frac{v}{r}\right)_{r=R_2} \mathbf{e}_z = 4\pi L\mu B R_2^{-2} \mathbf{e}_z$$

$$\mathbf{C} = 4\pi\mu L \frac{(\Omega_1 - \Omega_2)R_1^2}{R_2^2 - R_1^2}$$

where L is the length of the cylinder.

(c) This is possible if one of the cylinders rotates at a given rate and if we measure the torque exerted on the other cylinder (this is the way viscosities are measured with the Couette viscometer).

5.(a) The pressure gradient outside the boundary layer is just $-UU'$.

(b) Mass conservation yields

$$g'(\eta) = \frac{U(x)b'(x)}{V(x)}\eta f'(\eta) - \frac{U'(x)b(x)}{V(x)} f(\eta) \tag{13.4}$$

while momentum conservation implies

$$f''(\eta) = b(x)V(x)gf' + b^2(x)U'(x)(f^2 - 1) - b(x)b'(x)U(x)\eta ff' \tag{13.5}$$

If self-similar solutions exist, then each coefficient depending on x is a constant. This leads to the requested solutions.

(c) Adding the first and second solutions, we observe that $Ub^2 = (c_1 + c_2)x + a$, where a is a new constant. Eliminating b^2 from the second relation, we obtain a differential equation for U whose solution is:

$$U(x) = Ax^{c_2/(c_1+c_2)} \qquad \text{and} \qquad b(x) = \left[\frac{c_1 + c_2}{A}\right]^{1/2} x^{c_1/2(c_1+c_2)}$$

(d) The three constants c_1, c_2, c_3 give the length and velocity scales. One of them is therefore arbitrary and we may require that $c_1/2 + c_2 = 1$ without loss of generality. With this relation (13.4) gives:

$$c_3 g = c_1 \eta f/2 - F$$

Inserting this in (13.5), we obtain Falkner–Skan equation:

$$f'' + FF'' - c_2(f^2 - 1) = 0$$

Then, it turns out that $m = c_2/(2 - c_2)$, so that $U(x) = Ax^m$ and $b(x) \propto x^{(1-m)/2}$

(e) From the form of $b(x)$, the thickness of the layer is constant if $c_1 = 0$ or if $c_2 = 1$ and $m = 1$. Blasius equation is found again when the velocity outside the boundary layer is uniform, namely when $m = 0$. Falkner–Skan equation then reads $F''' + FF'' = 0$. It differs from Blasius equation by a factor 2 because $b(x)$ also misses a factor 2.

Chapter 5

1. The flow associated with the eigenmode of the air column is purely radial, thus depending only on the distance to the centre of the "sphere". The pressure disturbance δp verifies the wave equation (5.17). Denoting the mode frequency ω, we have $(\Delta + \omega^2/c_s^2)\delta p = 0$ but $\delta p \equiv \delta p(r)$, so that

$$\frac{1}{r}\frac{d^2}{dr^2}(r\delta p) + \frac{\omega^2}{c_s^2}\delta p = 0$$

whose solution is $\delta p \propto \frac{\sin kr}{r}$ with $k = \omega/c_s$. At the end of the instrument, at $r = L$, $\delta p = 0$ so that the fundamental mode is such that $kL = \pi$. Let ν be its frequency ($\nu = \omega/2\pi$), then $L = c_s/2\nu$. The bassoon length should therefore be $L = 347/2 * 58.27 = 298$ cm, which is quite close to the real length. One may observe that the dispersion relation of the modes is exactly that of the flute.

2. The mode frequency is proportional to the sound velocity, which is itself proportional to the square root of the temperature. When the temperature increases from 10 to 30 °C (50 to 86 F), the frequency is increased by a factor $\sqrt{303/283} = 1.035$, which a quarter of tone (half a tone between two notes is $2^{1/12} = 1.06$). Thus the temperature variation induces a quarter of tone variation for all the notes of the instrument.

3. The dispersion relation $\omega = \sqrt{gk}$ implies that $\lambda = gT^2/2\pi$, where λ is the wavelength and T the wave period. Numerically, we find $\lambda = 351$ m and $v_\phi = 23$ m/s. In 57 h, these waves cross 4,800 km.

4. These are shallow water waves. Their velocity is $\sqrt{gh} = 221$ m/s. They need 6 h to cross the Atlantic ocean, thus after two crossing they are back, in phase with the tidal potential. There is a resonance. This is the reason why the tides on the Atlantic shores have an important amplitude.

5. We need to use expressions (5.37) and (5.42) with a finite depth. It follows that

$$\omega^2 = \frac{(\rho_e - \rho_a)gk + \gamma k^3}{\rho_e + \rho_a \,\mathrm{th}\,(kH)} \,\mathrm{th}\,(kH)$$

In the shallow water conditions $kH \ll 1$, so that

$$\omega^2 = (1 - \rho_a/\rho_e)gHk^2 + \frac{\gamma H}{\rho_e} k^4$$

6. From the equation of state of an ideal gas:

$$\frac{T_2}{T_1} = \frac{P_2}{P_1}\frac{\rho_1}{\rho_2} = \left(1 + \frac{2\gamma}{\gamma+1}(m-1)\right)\left(\frac{(\gamma-1)m+2}{(\gamma+1)m}\right)$$

where we set $m = M_1^2$. $T_2/T_1 > 1$ is equivalent to

$$(\gamma + 1 + 2\gamma(m-1))\left(\gamma - 1 + \frac{2}{m}\right) > (\gamma+1)^2$$

This inequality may also be written:

$$(\gamma - 1)(\gamma m + 1)(1 - 1/m) > 0$$

which is true when $m > 1$.

7. (a) Starting from the second jump condition (5.68), we eliminate v_2, thanks to the first condition (5.67). We thus derive the expression of v_1 then that of v_2.

(b) Using the definition of Fr_2, (5.67) and (5.69) we can derive the requested expression. Then

$$\mathrm{Fr}_2 \lesseqgtr 1 \Longleftrightarrow 2\mathrm{Fr}_1^{2/3} \lesseqgtr \sqrt{1 + 8\mathrm{Fr}_1^2} - 1$$

Raising this expression to the cubic power, after simplification, we find

$$\sqrt{1 + 8\mathrm{Fr}_1^2} \lesseqgtr 1 + 2\mathrm{Fr}_1^2 \Longleftrightarrow 1 \lesseqgtr \mathrm{Fr}_1^2$$

which is the requested result.

8. We have

$$\frac{d}{d\tau}\int_{-\infty}^{+\infty} \zeta dx = \int_{-\infty}^{+\infty} \frac{\partial \zeta}{\partial \tau} dx = \int_{-\infty}^{+\infty} \left(-\frac{3}{2}\zeta\frac{\partial \zeta}{\partial x} - \frac{1}{6}\frac{\partial^3 \zeta}{\partial x^3}\right) dx$$

Since the function and its second derivative are vanishing in $\pm\infty$, we note that the first integral is independent of time. It reflects mass conservation: the integral

indeed measures the area below the curve $\zeta(x)$. This area is, in two-dimension, the volume and therefore also the mass since ρ =Cst.
Proceeding in the same way for the second integral, it turns out that

$$\frac{d}{d\tau}\int_{-\infty}^{+\infty}\zeta^2 dx = -\left[\frac{\partial\zeta^3}{\partial x}\right]_{-\infty}^{+\infty} - \frac{1}{3}\left[\zeta\frac{\partial^2\zeta}{\partial x^2}\right]_{-\infty}^{+\infty} + \frac{1}{6}\left[\left(\frac{\partial\zeta}{\partial x}\right)^2\right]_{-\infty}^{+\infty} = 0$$

The conserved quantity is here related to the mechanical energy of the system.

Chapter 6

1. The instability can develop only if Jeans length is smaller that the diameter of the sphere. Since the density is constant, $\rho = 3M/4\pi R^3$. Writing $\lambda_J < 2R$ we get

$$R < R_c = \frac{3GM}{\pi^2 c_s^2}$$

where G is the gravitation constant and c_s is the sound velocity. If we observe that GM/R is nothing but the order of magnitude of the escape velocity squared at the surface of the sphere, then the foregoing inequality just means that the molecular velocity (which is similar to the sound velocity), need to be smaller than the escape velocity. The system is gravitationally bound: one may check that the internal energy of the gas is indeed less than the absolute value of the gravitational energy of the sphere. Numerically, we find that $R_c = 1.2$ light-year. An interstellar cloud with diameter of one light-year is stable, while a bigger one with diameter of 10 light-years is unstable (according to this model of course).

2.(a) The specific angular momentum $s^2\Omega(s)$ needs to be a growing function of s (see Sect. 6.2.1).

(b) We write Euler's equation in cylindrical coordinates. The disturbances of the flow $s\Omega(s)$ verify:

$$\begin{cases} (\lambda + im\Omega)u_s - 2\Omega u_\theta = -\frac{\partial p}{\partial s} \\ (\lambda + im\Omega)su_\theta + u_s\frac{\partial}{\partial s}(s^2\Omega) = -imp \\ \frac{\partial}{\partial s}(su_s) + imu_\theta = 0 \end{cases} \tag{13.6}$$

where the third equation is just $\nabla\cdot\mathbf{u} = 0$. We observe that in the first and third domain $\frac{\partial}{\partial s}(s^2\Omega) = 0$, while in the second $\frac{\partial}{\partial s}(s^2\Omega) = 2As$.

(c) The radial component of the velocity and the pressure need to be continuous.

(d) Using the three previous (13.6), we eliminate the pressure and u_θ. Then, we observe that in the equation of u the original flow intervene through $\frac{\partial}{\partial s}\frac{1}{s}\frac{\partial s^2\Omega}{\partial s}$, which is zero in the three domains.

(e) The solutions are :

$$\begin{array}{ll} \text{domain I} & u_{\rm I}(s) = A_1 s^{m-1} \\ \text{domain II} & u_{\rm II}(s) = A_2 s^{m-1} + B_2/s^{m+1} \\ \text{domain III} & u_{\rm III}(s) = B_3/s^{m+1} \end{array}$$

(f) Using the last two equations of (13.6), we get the pressure perturbation:

$$m^2 p = im2asu - (\lambda + im\Omega)s\frac{\partial}{\partial s}(su)$$

With the continuity of pressure and u, we derive the interface conditions:

$$-(\lambda + im\Omega)_\eta \eta^2 u'_{\rm I}(\eta) = im2A\eta u_{\rm II}(\eta) - (\lambda + im\Omega)_\eta \eta^2 u'_{\rm II}(\eta)$$

$$-(\lambda + im\Omega)_1 u'_{\rm III}(1) = im2Au_{\rm II}(1) - (\lambda + im\Omega)_1 u'_{\rm II}(1)$$

$$u_{\rm I}(\eta) = u_{\rm II}(\eta) \qquad \text{and} \qquad u_{\rm II}(1) = u_{\rm III}(1)$$

(g) These four conditions lead to the requested relation. After elimination of A_1 and B_3, we verify that A_2 and B_2 are solutions of

$$\begin{cases} (\lambda + im\Omega_\eta)(A_2 - B_2\eta^{-2m}) = (2iA + \lambda)(A_2 + B_2\eta^{-2m}) \\ (\lambda + im\Omega_1)(A_2 - B_2) = (2iA - \lambda - im\Omega_1)(A_2 + B_2) \end{cases} \tag{13.7}$$

(h) If $m = 1$ we find that $(\lambda + i\Omega_0/2)^2 = -\Omega_0^2/4$ namely that $\lambda = 0$ or $\lambda = -i\Omega_0$. In the two cases the perturbation is neutral, hence the stability. If $m = 2$, we find again that $\lambda = -i\Omega_0$.

3. *Fjørtoft theorem* : Taking the real part of (6.21) we get:

$$\int_a^b \left[(|D\psi|^2 + k^2|\psi|^2) + \frac{k|\psi|^2 U''(-\lambda_I + kU)}{|\lambda + ikU|^2}\right] dz = 0$$

where λ_I is the imaginary part of λ. We consider the case where the instability develops so that $\lambda_R \neq 0$. In this case, because of Rayleigh condition (6.22), we know that

$$\int_a^b \frac{U''|\psi|^2}{|\lambda + ikU|^2} dz = 0$$

The λ_I-term therefore disappears. Hence, we can replace λ_I by any constant we wish. We choose to replace λ_I by kA, which leads to (6.82). We conclude that

$$\int_a^b \frac{k^2|\psi|^2 U''(U - A)}{|\lambda + ikU|^2} dz = -\int_a^b (|D\psi|^2 + k^2|\psi|^2) dz < 0$$

which is possible in $[a, b]$ only if there exists some interval where the quantity

$$U''(U - A)$$

is negative. The constant A is not fixed so we may choose the value of the velocity at the inflexion point. Since U'' and $U - A$ change sign at this point, we see that all the profiles where U'' and $U - A$ are of the same sign can be considered as stable. This stability criterion can be expressed in simpler way: If there exists a constant A such that $U''(U - A) > 0$ in the whole interval $[a, b]$ then $\psi = 0$ and the solution is stable.

Chapter 7

1. Writing $dh = Tds + dP/\rho = c_p dT$, we get a relation on the gradients. Noting that $\nabla P = \rho\mathbf{g}$ we get:

$$\nabla T - \frac{\mathbf{g}}{c_p} = T\nabla s/c_p$$

Since $(\nabla T)_{ad} = \frac{\mathbf{g}}{c_p}$, we derive the requested relation.
Using (7.8) we derive $\nabla T_{pot} = \frac{T_{pot}}{c_p}\nabla s$ and thus (7.9).

2. In the standard model of the stratosphere the temperature gradient is positive or zero. This is certainly larger than the adiabatic, which is negative. The stratosphere is therefore a stable layer as its name reflects.

3. Let A_v be amplitude of the dimensionless velocity $\mathbf{u}$. From the chosen scales we get:

$$\mathrm{Re} = \frac{Vd}{\nu} = \frac{\kappa}{\nu} A_v = \frac{A_v}{\mathcal{P}}$$

From the definition of the stream function, we have

$$u^2 = |D\psi|^2 + k^2|\psi|^2 = 3\pi^2/2|\psi|^2 = A_v^2\ .$$

The amplitude for ψ is given by (7.71) and thus

$$\mathrm{Re} = \frac{6\pi}{\mathcal{P}}\sqrt{\varepsilon} = 6\pi\sqrt{\frac{\mathrm{Ra} - \mathrm{Ra}_c}{\mathrm{Ra}_c\mathcal{P}^2}}$$

4. We linearize the Lorenz equations around zero. The perturbations $\delta X, \delta Y, \delta Z$ verify

$$\begin{cases} \dfrac{d\delta X}{dt} = \mathcal{P}(\delta Y - \delta X) \\ \dfrac{d\delta Y}{dt} = r\delta X - \delta Y \\ \dfrac{d\delta Z}{dt} = -b\delta Z \end{cases} \tag{13.8}$$

We look for a solution proportional to $e^{\lambda t}$, and get λ by setting the determinant of the system to zero. Hence,

$$\lambda^2+(\mathcal{P}+1)\lambda+\mathcal{P}(1-r) = 0 \implies \quad 2\lambda = -(\mathcal{P}+1)\pm\sqrt{(\mathcal{P}+1)^2+4\mathcal{P}(r-1)}$$

When $r > 1$ one of the root has a positive real part showing that the system is unstable.

Chapter 8

1. The demonstration is carried out in three steps:

- We break the velocity field into two parts: the relative velocity $\mathbf{v}_r$ and the background velocity $\boldsymbol{\Omega} \times \mathbf{r}$

$$\mathbf{v} = \mathbf{v}_r + \boldsymbol{\Omega} \times \mathbf{r}$$

  The acceleration reads

$$\frac{D\mathbf{v}}{Dt} = \frac{\partial \mathbf{v}_r}{\partial t} + (\mathbf{v}_r \cdot \boldsymbol{\nabla})\mathbf{v}_r + (\boldsymbol{\Omega} \times \mathbf{r}) \cdot \boldsymbol{\nabla}\mathbf{v}_r + \boldsymbol{\Omega} \times \mathbf{v}_r + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r})$$

  where we used

$$(\mathbf{v}_r \cdot \boldsymbol{\nabla})\boldsymbol{\Omega} \times \mathbf{r} = \boldsymbol{\Omega} \times \mathbf{v}_r \qquad \text{and} \qquad [(\boldsymbol{\Omega} \times \mathbf{r}) \cdot \boldsymbol{\nabla}]\,\boldsymbol{\Omega} \times \mathbf{r} = \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r})$$

- In the second step, we observe that $\mathbf{v}_r$ should be expressed with the vectors attached to the rotating frame, which are time-dependent. In other words, we are not interested in $\mathbf{v}_r$ but in $\mathbf{v}'$ which is such that $\mathbf{v}_r = [\Omega]\mathbf{v}'$. $[\Omega]$ represents the rotation which transform the new coordinates into the old ones. We now assume that $\boldsymbol{\Omega} = \Omega\mathbf{e}_z$. We have

$$\mathbf{r}{=}[\Omega]\mathbf{r}', \quad \nabla{=}[\Omega]\nabla', \quad \mathbf{v}_r\cdot\nabla{=}\mathbf{v}'\cdot\nabla' \qquad \text{and} \quad [\Omega]{=}\begin{bmatrix}\cos\Omega t & -\sin\Omega t\\ \sin\Omega t & \cos\Omega t\end{bmatrix}$$

It is easily verified that:

$$\frac{\partial\mathbf{v}_r}{\partial t} = [\Omega]\frac{\partial\mathbf{v}'}{\partial t} + [\dot{\Omega}]\mathbf{v}' = [\Omega]\left(\frac{\partial\mathbf{v}'}{\partial t} + \boldsymbol{\Omega}\times\mathbf{v}'\right)$$

Applying the rotation to all the vectors of the equation, we find the equation verified by $\mathbf{v}'$, namely

$$\frac{D\mathbf{v}'}{Dt} + 2\boldsymbol{\Omega}\times\mathbf{v}' + (\boldsymbol{\Omega}\times\mathbf{r})\cdot\nabla\mathbf{v}' + \boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\mathbf{r}') = \mathbf{f}'$$

- In the last step, we observe that the new coordinates depend on the old ones and on time. Hence

$$\frac{\partial\mathbf{v}'}{\partial t}(\mathbf{r},t) = \frac{\partial\mathbf{v}'}{\partial t}(\mathbf{r}',t) + \frac{\partial\mathbf{r}'(\mathbf{r},t)}{\partial t}\cdot\nabla'\mathbf{v}'$$

but

$$\mathbf{r}' = [\Omega]^{-1}\mathbf{r} = [-\Omega]\mathbf{r} \qquad \Longrightarrow \qquad \frac{\partial\mathbf{r}'(\mathbf{r},t)}{\partial t} = -\boldsymbol{\Omega}\times\mathbf{r}'$$

which removes the term $(\boldsymbol{\Omega}\times\mathbf{r})\cdot\nabla\mathbf{v}'$ and yields the expected result.

2. When viscosity is taken into account the following term

$$\mathrm{E}\int_{(V)}\mathbf{u}^*\cdot\Delta\mathbf{u}\,dV$$

is important. Introducing $[\sigma^v]$ as the viscous stress tensor, the foregoing integral may also be written

$$\int_{(V)} u_i^*\partial_j\sigma_{ij}^v dV = \int_{(S)} u_i^*\sigma_{ij}^v dS_j - \int_{(V)}\partial_j u_i^*\sigma_{ij}^v dV$$

Because of boundary conditions, the surface integral is always zero (either for no-slip or stress-free conditions). The volume integral is real. Observing that $\sigma_{ij}^v = \mathrm{E}(\partial_j u_i + \partial_i u_j)$, we get

$$\int_{(V)}\partial_j u_i^*\sigma_{ij}^v dV = \frac{E}{2}\int_{(V)}|\partial_j u_i + \partial_i u_j|^2 dV$$

This integral is truly real, therefore the bounds of the eigenspectrum are not modified.

3. The easiest way is to start from the expressions of v_r and v_z given by (8.29) and (8.30), which we combine to eliminate the pressure field. We get

$$\frac{1-\omega^2}{i\omega}\frac{\partial v_s}{\partial z} = i\omega\frac{\partial v_z}{\partial s}$$

but $v_s = \frac{1}{s}\frac{\partial\psi}{\partial z}$, $v_z = -\frac{1}{s}\frac{\partial\psi}{\partial s}$.
Making the substitution, we obtain the equation for ψ :

$$\frac{\partial^2\psi}{\partial s^2} - \frac{1}{s}\frac{\partial\psi}{\partial s} - \left(\frac{1-\omega^2}{\omega^2}\right)\frac{\partial^2\psi}{\partial z^2} = 0$$

which is similar to the Poincaré equation. The only change is the sign of $\frac{1}{s}\frac{\partial\psi}{\partial s}$. The equation is however still of hyperbolic type with the same characteristics.

Chapter 9

1.(a) From the definitions

$$\left\langle \hat{v}_i^*(\mathbf{k})\hat{v}_j(\mathbf{k}')\right\rangle = \left\langle \int v_i(\mathbf{x})e^{i\mathbf{k}\cdot\mathbf{x}}\frac{d^3\mathbf{x}}{(2\pi)^3}\int v_j(\mathbf{x})e^{-i\mathbf{k}'\cdot\mathbf{x}}\frac{d^3\mathbf{x}}{(2\pi)^3}\right\rangle$$

where we noted that v_i is real. Here we transform this expression into a double integral:

$$= \int\int \left\langle v_i(\mathbf{x})v_j(\mathbf{x}')\right\rangle e^{i\mathbf{k}\cdot\mathbf{x}-i\mathbf{k}'\cdot\mathbf{x}'}\frac{d^3\mathbf{x}}{(2\pi)^3}\frac{d^3\mathbf{x}'}{(2\pi)^3}$$

setting $\mathbf{x}' = \mathbf{x} + \mathbf{r}$ and splitting the integrals over $\mathbf{x}$ and $\mathbf{r}$, we get

$$= \underbrace{\int Q_{ij}(\mathbf{r})e^{-i\mathbf{k}'\cdot\mathbf{r}}\frac{d^3\mathbf{r}}{(2\pi)^3}}_{\phi_{ij}(\mathbf{k}')}\underbrace{\int e^{-i(\mathbf{k}'-\mathbf{k})\cdot\mathbf{x}}\frac{d^3\mathbf{x}}{(2\pi)^3}}_{\delta(\mathbf{k}'-\mathbf{k})}$$

hence the result.

(b) We have

$$Z_{ij} = \int \left\langle \omega_i'(\mathbf{x})\omega_j'(\mathbf{x}')\right\rangle e^{-i\mathbf{k}\cdot\mathbf{r}}\frac{d^3\mathbf{r}}{(2\pi)^3}$$

or

$$= \epsilon_{ilm}\epsilon_{jnp} \int \left\langle \partial_l v'_m(\mathbf{x})\partial_n v'_p(\mathbf{x}')\right\rangle e^{-i\mathbf{k}\cdot\mathbf{r}} \frac{d^3\mathbf{r}}{(2\pi)^3}$$

But

$$\partial_l v'_m(\mathbf{x}) = \int ik_l \hat{v}_m e^{i\mathbf{k}\cdot\mathbf{x}} d^3\mathbf{k}, \quad \partial_n v'_p(\mathbf{x}) = \partial_n v'^*_p = -\int ik'_l \hat{v}_m e^{-i\mathbf{k}'\cdot\mathbf{x}} d^3\mathbf{k}'$$

Using the result of the previous exercise, we have

$$\left\langle \partial_l v'_m(\mathbf{x})\partial_n v'_p(\mathbf{x}')\right\rangle = \int k_l k_n \phi_{mp}(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{r}} d^3\mathbf{k}$$

Using this expression in that of Z_{ij} we get the requested expression. The final formula (9.28) is derived after an expansion of $\epsilon_{ilm}\epsilon_{jnp}$ (given in the complements of mathematics) and after the use of the incompressibility relation (9.23).

(c) We note that

$$\phi_{ii}(k) = \frac{1}{(2\pi)^3} \int Q_{ii}(r) e^{-ikr\cos\theta} r^2 dr \sin\theta d\theta d\varphi$$

Using $\int_0^\pi e^{-ikr\cos\theta} \sin\theta d\theta = 2\sin kr/kr$ and (9.33) we derive the requested result.

2. We start from the definition (9.46) and modify it into

$$\ell_Z = \Omega(0)^{-1} \int_0^\infty \Omega(r) dr$$

where $\Omega(r) = Q_{ii}(r) = -6\Delta R(r)$, Δ being the Laplacian. It is sufficient to use (9.35) and the expression of the Laplacian in spherical coordinates (see 12.45), to get the desired expression. To show the similarity between ℓ_D and ℓ_Z, we may consider the Kolmogorov spectrum where $k_D \gg k_0$. Its energy density is negligible beyond k_D. Then it is easy to show that $\ell_Z \sim \ell_D$.

3. (a) A short integration by part yields the result.

(b) A Taylor expansion of the functions at the origin yields $(\sin kr - kr\cos kr)/(kr)^3 \sim 1/3$. The result follows.

(c) From $E(k) = C_K\langle\varepsilon\rangle^{2/3}k^{-5/3}$, if we set $x = kr$ then

$$S_2 = C_K\langle\varepsilon\rangle^{2/3} r^{2/3} \int_0^\infty q(x) dx$$

One should check that the integral converges. C_2 and C_K are proportional since $C_2 = C_K \int_0^\infty q(x) x^{-5/3} dx$.

(d) If $S_2 \propto r^{2/3+m}$, then $E(k) \propto k^{-5/3-m}$. In the case of the law (9.76), $m = \frac{14}{9} - 2\left(\frac{2}{3}\right)^{2/3} = 0.02927$ if $\beta = 2/3$.

4. From the Definition (9.49) and after splitting over the various domains

$$\ell_T^2 = \frac{\displaystyle\int_0^{k_0} i(k)dk + \int_{k_0}^{k_D} k^{-5/3} + \int_{k_D}^{\infty} d(k)dk}{\displaystyle\int_0^{k_0} k^2 i(k)dk + \int_{k_0}^{k_D} k^{1/3}dk + \int_{k_D}^{\infty} k^2 d(k)dk}$$

Taking an upper bound of the numerator and a lower bound of the denominator, and then doing the opposite, we can construct a lower and upper bound for ℓ_T^2, namely:

$$\frac{2/3{k_0}^{-2/3}}{{k_0}^{4/3} + 3/4{k_D}^{4/3} + 5{k_D}^{4/3}} \leq \ell_T^2 \leq \frac{{k_0}^{-2/3} + 2/3({k_0}^{-2/3} - {k_D}^{-2/3}) + {k_D}^{-2/3}}{3/4({k_D}^{4/3} - k_0^{4/3})}$$

The result follows since $k_D \gg k_0$.

5. We note that

$$\langle x^p \rangle = \left\langle e^{p\ln x} \right\rangle = \left\langle e^{p(\ln x - \langle \ln x \rangle)} \right\rangle e^{p\langle \ln x \rangle}$$

setting $y = (\ln x - \langle \ln x \rangle)$, we have

$$\langle e^{py} \rangle = \sum_{n=0}^{+\infty} \frac{p^n \langle y^n \rangle}{n!}$$

however, y is a random variable with a normal distribution, thus $\langle y^n \rangle = 0$ if n is odd. It follows that

$$\langle e^{py} \rangle = \sum_{m=0}^{+\infty} \frac{p^{2m} \left\langle y^{2m} \right\rangle}{(2m)!}.$$

For a normal law of probability even order moments just depend on the variance of the random variable, namely $\left\langle y^{2m} \right\rangle = (2m-1)!!\sigma^{2m}$ (which can be demonstrated by induction) where

$$(2m-1)!!/(2m)! = 2^{-m}/m!\,.$$

It follows that

$$\langle e^{py} \rangle = \sum_{m=0}^{+\infty} \frac{1}{m!} \left(\frac{p^2\sigma^2}{2} \right)^m = e^{p^2\sigma^2/2}$$

hence the result.

Chapter 10

1. The electric current generated by free charges of density ρ_e comes from the matter motion and reads $\rho_e\mathbf{v}$. From Gauss equation $\nabla\cdot\mathbf{E} = \rho_e/\varepsilon_0$ and the induction equation $\partial_t\mathbf{B} = -\nabla\times\mathbf{E}$, we find the following order of magnitude relations:

$$\rho_e\mathbf{v} \sim \varepsilon_0 EV/L \sim V^2/c^2(B/\mu_0 L) \sim V^2/c^2\mathbf{j}$$

   Thus $j \gg \rho_e V$ and $|\mathbf{j}\times\mathbf{B}| \gg \rho_e VB \sim \rho_e E$. The Laplace force is always larger than the Coulomb force coming from the local electric field.
2. The mean magnetic field verifies:

$$\frac{\partial\mathbf{B}}{\partial t} = \nabla\times(\alpha\mathbf{B}) + \eta_{turb}\Delta\mathbf{B}$$

   Taking $\mathbf{B} = \mathbf{B}_0 \exp i(\omega t + \mathbf{k}\cdot\mathbf{r})$, we get

$$i\omega\mathbf{B} = \alpha i\mathbf{k}\times\mathbf{B} - \eta_{turb}k^2\mathbf{B}$$

   Taking the cross product of this expression with $\mathbf{k}$ and reporting the new expression in the first, we derive the dispersion relation:

$$(i\omega + \eta_{turb}k^2)^2 = \alpha^2 k^2$$

   which shows that all the waves whose wavenumber is smaller than $|\alpha|/\eta_{turb}$ are amplified.

Complements of Mathematics

1. To derive one expression from the preceding one we just need to contract two indices and note that $\delta_{ii} = 3$.
2. If [A] is symmetric then $A_{ij} = A_{ji}$, thus

$$\epsilon_{ijk}A_{jk} = \epsilon_{ijk}A_{kj} = -\epsilon_{ijk}A_{kj} = -\epsilon_{ijk}A_{jk}$$

   The reciprocal is also true: if $\epsilon_{ijk}A_{jk} = 0$ then

$$\epsilon_{lmi}\epsilon_{ijk}A_{jk} = 0 \Longleftrightarrow (\delta_{lj}\delta_{mk} - \delta_{lk}\delta_{mj})A_{jk} = 0 \Longleftrightarrow A_{lm} - A_{ml} = 0$$

   thus [A] is symmetric.

3. This is indeed a Sturm–Liouville problem. Identifying the terms, we find $p(x) = 1$, $q(x) = 0$ and $w(x) = 1$. The general solution is:

$$y(x) = Ae^{ax} + Be^{-ax}$$

with $a = \sqrt{-\lambda}$. The boundary conditions $y(0) = y(1) = 0$ imply that $A + B = 0$ and $\operatorname{sh} a = 0$. Thus $a = in\pi$ with $n \in \mathbb{N}$. The eigenvalue spectrum is therefore the set $\{n^2\pi^2 \mid n \in \mathbb{N}\}$. It is discrete and has a lower bound as expected.