Chapter 2
The Static of Fluids

The equilibrium of a fluid is certainly the most simple fluid "flow". However, not moving is not that easy for a fluid and we shall learn here, among other things, which conditions need to be satisfied for a fluid to remain in equilibrium.

2.1 The Equations of Static

If we let $\mathbf{v} = \mathbf{0}$, $\frac{\partial}{\partial t} = 0$ and $\sigma_{ij} = -p\delta_{ij}$ in (1.25) and (1.29), we find that mechanical and thermal equilibrium are governed by:

$$-\nabla P + \mathbf{f} = \mathbf{0} \tag{2.1}$$

$$\nabla \cdot (\chi \nabla T) + \mathcal{Q} = 0 \tag{2.2}$$

where $\mathbf{f}$ is an applied volumic force field and $\mathcal{Q}$ a heat source density. We immediately note that if $\mathbf{f}$ is zero then pressure is uniform.

The first important result from the above equations is that a static solution exists if, and only if, the external force can be derived from a potential. Thus, we may set $\mathbf{f} = -\nabla\phi_{ext}$ and solve for the pressure

$$P + \phi_{ext} = \text{Cst} .$$

This solution shows that isobars are identical to equipotential surfaces. We now know that if $\mathbf{f}$ is not the gradient of a potential no static solution exists. The fluid flows.

Equation (2.2) gives the temperature field. If the thermal conductivity is constant or a smooth function of the space coordinates, this equation has a solution.

M. Rieutord, *Fluid Dynamics*, Graduate Texts in Physics,
DOI 10.1007/978-3-319-09351-2_2

In most cases, **f** is proportional to the density ρ. Equations (2.1) and (2.2) need then to be completed by the equation of state:

$$P \equiv P(\rho, T)$$

The solution of the problem may be quite difficult, all the more that in general

$$\chi \equiv \chi(\rho, T)$$

2.2 Equilibrium in a Gravitational Field

The most common problem of fluid statics is certainly the one of a fluid at rest in a gravitational field. In this case

$$\mathbf{f} = -\rho \nabla \phi_g = \rho \mathbf{g}$$

where ϕ_g is the gravitational potential. The equation of mechanical equilibrium is then

$$\nabla P + \rho \nabla \phi_g = \mathbf{0} \tag{2.3}$$

which implies

$$\nabla \times (\frac{1}{\rho} \nabla P) = \mathbf{0} \quad \Longleftrightarrow \quad \nabla \rho \times \nabla P = \mathbf{0}$$

This identity shows that isochore surfaces (i.e. surfaces where ρ is constant) need to be identical to isobar surfaces for a static solution to exist. This condition leads to

$$P \equiv P(\rho)$$

where we recognize the case of a barotropic fluid.

The foregoing result shows that a fluid in static equilibrium is necessarily barotropic. Now, we also note that

$$\frac{1}{\rho} \nabla P + \nabla \phi_g = \mathbf{0}$$

but because $P \equiv P(\rho)$, then $\rho^{-1} \nabla P = \nabla \int dP/\rho$,[1]

[1]We should observe that $\boldsymbol{\nabla} \int dP/\rho = \left(\frac{d}{dP} \int \frac{dP}{\rho(P)} \right) \boldsymbol{\nabla} P = \frac{1}{\rho} \boldsymbol{\nabla} P$.

$$\Longrightarrow \quad \nabla(\int \frac{dP}{\rho} + \phi_g) = \mathbf{0}$$

$$\Longleftrightarrow \quad \int \frac{dP}{\rho} + \phi_g = \text{Cst} \tag{2.4}$$

A relation which determines the isobaric surfaces as a function of equipotentials.

2.2.1 Pascal Theorem

If we consider a fluid of constant density in a uniform gravity field, $\phi_g = gz$, the equation of mechanical equilibrium gives the relation

$$P + \rho g z = \text{Cst} \tag{2.5}$$

also known as Pascal[2] theorem. This relation shows that, in such a case, pressure only depends on the altitude z. We also see from this result that, in fluids at rest in a uniform gravity field, the difference of pressure between two points is just $\rho g h$, where h is the difference in their altitude.

A very direct application of this theorem is the barometer. For instance, the mercury barometer (see Fig. 2.1) is based on the fact that a column of mercury 76 cm high imposes a pressure difference similar to the atmospheric pressure at sea level.

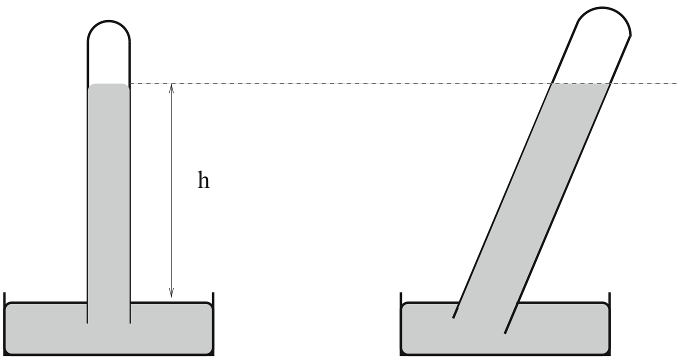


Fig. 2.1 The principle of a mercury barometer: the density of mercury is 1.36×10^4 kg/m^3 so that $\rho g h$ equals the atmospheric pressure (101,325 Pa) for h=76 cm. The void left by mercury is filled with mercury vapour but its pressure at room temperature is only 0.16 Pa, which is negligible compared to atmospheric pressure

[2]Blaise Pascal (1623–1662) was a French scientist and writer. As far as Physics is concerned, he is famous for his work on fluid's equilibria, *de l'Equilibre des liqueurs* and *de la Pesanteur de l'air* (the weight of air).

2.2.2 *Atmospheres*

Planetary atmospheres are a first application of the equilibria of fluids. The static solution is of course an approximation of an atmosphere. The Earth atmosphere is well known to be in constant evolution, with winds, clouds, etc. However, its mean vertical profile is not far from the static equilibrium. Here, we shall restrict ourselves to two very simple examples of atmosphere models: the isothermal and the isentropic ones. The latter will be compared to the actual Earth atmosphere.

2.2.2.1 The Isothermal Atmosphere

In some circumstances it is useful to simplify a model of atmosphere by assuming it being of constant temperature. Using the equation of state of ideal gases $P = \mathcal{R}_* \rho T$, which we combine with (2.4), we find the pressure profile

$$P(z) = P_0 e^{-z/z_0}$$

where $z_0 = \mathcal{R}_* T/g$ is called *the scale height* of the atmosphere. This expression shows that pressure, and hence density, decrease exponentially in an isothermal atmosphere. From the expression of z_0, we also see that the extension of such an atmosphere increases with temperature.

2.2.2.2 The Isentropic Atmosphere

The Earth atmosphere is far from being isothermal; everyone hiking in mountains has noticed that air temperature decreases with altitude. This is because the atmosphere of our planet is not very far from an isentropic state as we shall see now.

Thermodynamics gives a relation between the differential of enthalpy, entropy and pressure, namely

$$dh = Tds + dP/\rho \ .$$

For an isentropic fluid, $ds = 0$ and thus

$$dh = dP/\rho$$

This relation implies a similar one on all the partial derivatives so that we also have $\nabla h = \nabla P/\rho$. Mechanical equilibrium reads $\nabla P = \rho \mathbf{g}$, hence

$$\nabla h = \mathbf{g} \tag{2.6}$$

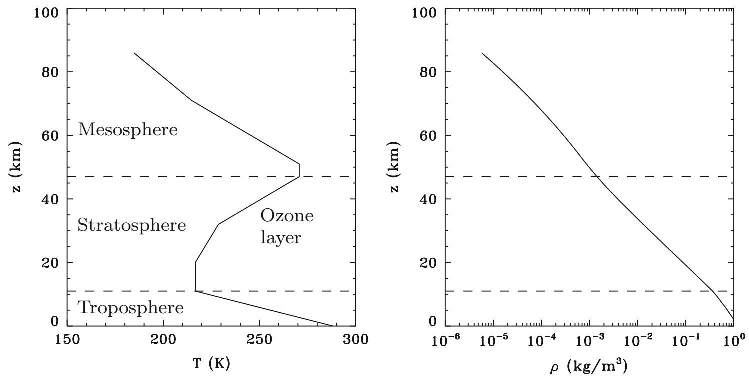


Fig. 2.2 Temperature and density profiles for the standard Earth atmosphere

This equation shows that the enthalpy gradient is just the local gravity. If the gas is ideal, then $h = c_p T$ and

$$\nabla T = \frac{\mathbf{g}}{c_p} \tag{2.7}$$

which demonstrates that the temperature gradient is, like gravity, constant and directed towards the ground. This means that temperature decreases with altitude.

Now, (2.7) can be easily solved since $\mathbf{g} = -g\mathbf{e}_z$. We find

$$T = T_0(1 - z/z_0) \tag{2.8}$$

where we introduced the ground temperature T_0 and, as before, the scale height which is now $z_0 = c_p T_0/g$. This quantity is only slightly different from the isothermal case if we take $T = T_0$. Pressure and density are derived from the relation $P^{1-\gamma}T^{\gamma} = \mathrm{Cst}$ valid for an isentropic ideal gas. They read

$$P = P_0(1 - z/z_0)^{\gamma/(\gamma-1)} \tag{2.9}$$

$$\rho = \rho_0(1 - z/z_0)^{1/(\gamma-1)} \tag{2.10}$$

These expressions show that the isentropic atmosphere has a finite height, given by z_0, unlike the isothermal atmosphere which is infinite. If we take standard values for the parameters, namely $T_0 = 289\,\mathrm{K}$, $g = 9.81\,\mathrm{m/s^2}$ and $c_p = 7/2\mathcal{R}$, we find $z_0 \simeq 30\,\mathrm{km}$. In fact, the atmosphere of the Earth is much more extended because isentropy is only approached in the troposphere (see Fig. 2.2 and the box on the standard atmosphere).

The gradient of temperature is found to be $-g/c_p = -9.8\,\mathrm{K/km}$, which represents a faster decrease than the actual atmosphere, which is close to -6.5 K/km. This comes from the simplifications that we adopted: in our model, the atmosphere is dry and in an isentropic state: there is no heat exchange between the fluid elements. Water vapour and heat exchanges reduce the temperature drop.

The standard atmosphere

The standard atmosphere has been defined for the needs in aeronautics and corresponds approximatively to the annual mean at a latitude of 40 degrees in North America. This atmosphere is defined up to an altitude of 86 km and is constructed with the temperature gradients defined in each layer of the model. Air is assumed to be an ideal gas with a mole mass of 28,9644 g, and located in a uniform gravity field with $g = 9.80665$ m/s^2.

Table 2.1 The standard atmosphere

Layers	Altitudes in km	∇T in K/km
Troposphere	0 – 11	−6.5
	11 – 20	0
Stratosphere	20 – 32	+1
	32 – 47	+2.8
	47 – 51	0
Mesosphere	51 – 71	−2.8
	71 – 86	−2.0

On ground (altitude z = 0 m), temperature is 15 °C (288.15 K) and pressure is 101325 Pa. Temperature decreases as 6.5 K/km up till 11 km, which is the upper limit of the *troposphere*. There, the temperature is 216.65 K (−56.5 °C). At this altitude the *stratosphere* begins and the temperature is first approximatively constant: this is the *tropopause*. The stratosphere contains two other layers like the famous ozone layer (20-32 km), and extends up to 47 km. Beyond and up to 86 km, one finds the mesosphere also divided into three layers (see Table 2.1). We should note that in the stratosphere, temperature increases and reaches a maximum of −2.5 °C near 50 km. This heating is essentially due to the absorption of solar UV radiation by the ozone molecules.

Beyond 86 km, we find the *thermosphere* where temperature increases again but density is so low that some part of the gas is always ionized. We touch here the *ionosphere* which extends up to 400 km, but this latter boundary is highly variable and rather fuzzy.

2.2.3 A Stratified Liquid Between Two Horizontal Plates

We now consider the equilibrium of a liquid inserted between two horizontal metallic plates. Such a device is used to study thermal convection in the laboratory (see Chap. 7). Here we shall describe the situation when the equilibrium of the fluid is stable and no convection occurs. To simplify we imagine that the two metallic plates are defined by the planes $z = 0$ and $z = d$, infinite horizontally. We also surmise that the metallic plates are perfect heat conductors and therefore impose the temperature to the fluid at these two heights. We denote these temperatures by T_b and T_t (bottom and top).

Liquids are weakly compressible; we introduced with (1.60) their simplified equation of state which we now use. Hence,

$$\rho = \rho_0(1 - \alpha(T - T_0))$$

where $\alpha > 0$ is the dilation coefficient which we assume to be constant. The thermal conductivity χ of the liquid is also assumed to be constant. With these assumptions the equations of mechanical and thermal equilibrium read:

$$\begin{cases} -\nabla P + \rho \mathbf{g} = \mathbf{0} \\ \nabla \cdot (\chi \nabla T) = 0 \end{cases} \quad \Rightarrow \quad \begin{cases} -\dfrac{dP}{dz} = \rho g \\ \dfrac{d^2 T}{dz^2} = 0 \end{cases}$$

where $\mathbf{g} = -g\mathbf{e}_z$ is the gravity. These equations can be easily solved and give the temperature, density and pressure profiles:

$$T(z) = T_b + (T_t - T_b)z/d$$

$$\rho(z) = \rho_b(1 - \alpha(T_t - T_b)z/d)$$

$$P(z) = P_b - \rho_b g z + \rho_b g \alpha (T_t - T_b) z^2/2d$$

The remarkable property of this system is that the temperature increase (or decreases) linearly with the altitude z. The stable situation corresponds to the increasing temperature. In this case light fluid is above dense fluid. The opposite case is obtained with a top plate cooler than the bottom one. As we shall see in Chap. 7, such a situation may be unstable if the temperature drop is strong enough. In such a case thermal convection takes place.

2.2.4 Rotating Self-gravitating Fluids ♠

Newton was the first to wonder about the shape of a rotating self-gravitating fluid. He was indeed interested in the shape of the Earth. This problem has then been tackled by the most renown mathematicians and physicists like Laplace, Jacobi, Riemann, Poincaré, Cartan, Chandrasekhar among the most famous. Recently, these results have been used in the theoretical approach of the dynamics of elliptical galaxies which may be viewed as a fluid of stars (see Binney and Tremaine, 1987).

Here we shall focus on the simplest of these kinds of problem: that of a fluid of constant density, self-gravitating and rotating uniformly like a solid body.

We first assume that the shape of such a system is that of an axisymmetric oblate ellipsoid and we look for the expression of its flatness as a function of its total mass M and angular velocity Ω. We shall verify afterwards that our assumption is indeed consistent with the solution.

It may be shown that the gravitational potential inside an ellipsoid of uniform density is given by

$$\Phi(r,z) = -\pi G\rho(Ia^2 - A_1 s^2 - A_3 z^2)$$

where a is the equatorial radius and also the semi-major axis of a meridional section. (s,φ,z) are the cylindrical coordinates. We denote by e the eccentricity of this meridional section. Constants I, A_1 and A_3 are defined by

$$I = 2\frac{\sqrt{1-e^2}}{e}\arcsin e$$

$$A_1 = \frac{\sqrt{1-e^2}}{e^2}\left(\frac{\arcsin e}{e} - \sqrt{1-e^2}\right), \quad A_3 = 2\frac{\sqrt{1-e^2}}{e^2}\left(\frac{1}{\sqrt{1-e^2}} - \frac{\arcsin e}{e}\right)$$

In a rotating frame the momentum equation reads:

$$-\nabla P - \rho\nabla\Phi - \rho\nabla\phi_c = \mathbf{0}$$

where $\phi_c = -\frac{1}{2}\Omega^2 s^2$ is the centrifugal potential. This equation shows that inside the body $P + \rho\Phi + \rho\phi_c$ is a constant. Since the pressure is vanishing at the surface, we have at this place

$$\Phi + \phi_c = \mathrm{Cst} \quad \Longleftrightarrow \quad \pi G\rho(A_1 s^2 + A_3 z^2) - \frac{1}{2}\Omega^2 s^2 = \mathrm{Cst}$$

which can be transformed into

$$\frac{s^2}{\pi G\rho A_3} + \frac{z^2}{\pi G\rho A_1 - \Omega^2/2} = \mathrm{Cst}$$

This equation describes the surface of the fluid. Since we assumed it to be an ellipsoid, we write it

$$\frac{s^2}{a^2} + \frac{z^2}{b^2} = 1$$

where a and b are the semi-major and semi-minor axis of the meridional ellipse, respectively. By simple identification, we get the relations

$$\begin{cases} a^2 = \text{Cst} \times \pi G \rho A_3 \\ b^2 = \text{Cst} \times (\pi G \rho A_1 - \Omega^2/2) \end{cases} \tag{2.11}$$

Taking the ratio of these quantities (to eliminate the constant) and remembering that $a^2 = b^2 + c^2$ in an ellipse, where c is the distance between the center and a focus, while $c = ae$, we find

$$\frac{\Omega^2}{2\pi G \rho} = A_1 - A_3(1 - e^2) \tag{2.12}$$

Using the expression of A_1 and A_3, one may notice that the eccentricity (or the flatness) of the ellipsoid depends only on the ratio Ω^2/ρ.

The volume of an ellipsoid is $\frac{4\pi}{3}abc$, where a, b and c are the three semi-major axis of the ellipses defining this volume. Because density is constant, the volume is easily related to the mass and (2.12) may be rewritten as:

$$\frac{2\Omega^2 a^3}{3GM} = \frac{\arcsin e}{e^3}(3 - 2e^2) - \frac{3\sqrt{1 - e^2}}{e^2} \tag{2.13}$$

This equations gives the eccentricity as a function of rotation for a given density. It needs a numerical solution. However, by plotting the right-hand side as a function of e, like in Fig. 2.3, we immediately see that the solution is not unique: For each ratio Ω^2/ρ two eccentricities are possible. A low one and a high one. The latter is in fact always that of an unstable configuration.

As Newton did at his time, we now focus on the case of slow rotation and therefore on small eccentricities. An expansion of the right-hand side of (2.13) yields the relation

$$\frac{\Omega^2 a^3}{GM} = \frac{4\varepsilon}{5}$$

where we used the flatness instead of the eccentricity. The flatness is defined as

$$\varepsilon = \frac{a - b}{a} = 1 - \sqrt{1 - e^2} \simeq e^2/2$$

Observing that the surface gravity of the sphere is $g = GM/a^2$, we find the expression of ε, namely

$$\varepsilon = \frac{5\Omega^2 a}{4g}$$

Applying this formula to the case of the Earth, where $M = 5.974 \times 10^{24}$ kg, $a = 6.378 \times 10^6$ m, $g = 9.8$ m/s^2 and $\Omega = 2\pi/24$ h, we obtain

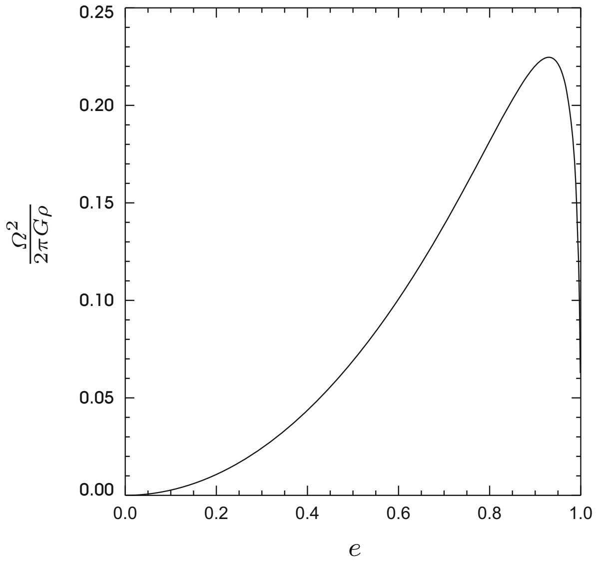


Fig. 2.3 The curve gives the value of the eccentricity of a MacLaurin ellipsoid when Ω^2/ρ, is known. The maximum, reached at $e = 0.929956$, shows that beyond some critical angular velocity (such that $\frac{\Omega^2}{2\pi G\rho} > 0.2246657$) no solution exists. In fact, an analysis of the stability of the configurations demonstrates that all solutions with $e \geq 0.9529$ are unstable, but if $0.81267 \leq e \leq 0.9529$ stable solutions exist only for an inviscid fluid. For rotations which give an eccentricity larger than 0.81267, stable solutions for a viscous fluid are triaxial Jacobi ellipsoids

$$\varepsilon_{\text{Earth}} = \frac{1}{232}$$

which is only slightly larger than the actual flatness $\varepsilon_{\text{Terre}} = 1/298$. The difference comes from the fact that the Earth is not homogeneous: central parts are much denser that the outer ones (the core of the Earth is essentially composed of iron, with a mean density of 10,500 kg/m^3 whereas the mantle is made of silicates and has a mean density of $\rho \sim 4{,}550$ kg/m^3). This central condensation of the mass makes the shape of the Earth closer to that of a sphere.

2.3 Some Properties of the Resultant Pressure Force

When fluids are in equilibrium, one of the local body forces is the pressure gradient. This mathematical expression of the pressure force, which thus derives from a potential, implies some simple properties when it is integrated over a given volume.

2.3.1 Archimedes Theorem

Let us consider a solid fully immersed in a fluid that is *in equilibrium* in a uniform gravity field **g**. We wish to compute the resultant of pressure forces exerted on its surface. By definition this is simply

$$\mathbf{F}_{res} = -\int_{(S)} P d\mathbf{S}$$

where the differential element $d\mathbf{S}$ is oriented towards the exterior of the solid. To evaluate this integral, we may observe that we can substitute to the solid an equivalent volume of fluid without changing the equilibrium of the fluid around the solid. Indeed, there exists an equilibrium distribution of pressure inside the volume occupied by the solid that perfectly matches the outer distribution of pressures. It is obtained by a mere continuation of the isobar surfaces inside (S) (see Fig. 2.4). Then, using the theorem of divergence (see (12.8)), the foregoing surface integral can be transformed into a volume integral, like

$$\mathbf{F}_{res} = -\int_{(V)} \nabla P \; dV$$

Then, using the equation of mechanical equilibrium (2.3), we obtain

$$\mathbf{F}_{res} = -\mathbf{g}\int_{(V)} \rho dV = -M_f \mathbf{g}$$

where M_f is the mass of the fluid substituted to the solid. Archimedes theorem can now be stated:

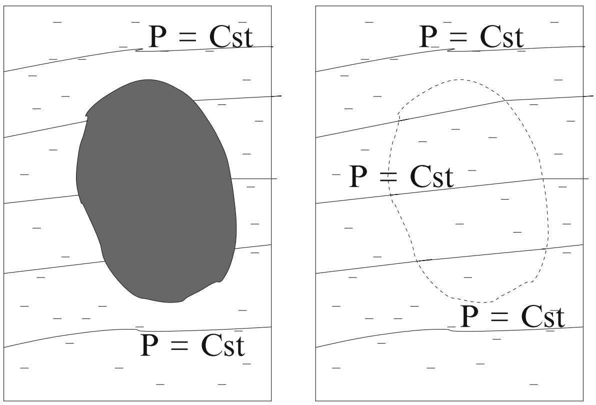


Fig. 2.4 Two equilibria of the fluid: with and without the solid

The resultant of pressure forces exerted on a volume V immersed in a fluid at equilibrium is equal and opposed to the weight of the displaced fluid.

This theorem can be applied in many situations. Note that ρ need not be constant. However, we see that it is crucial that the solid is completely surrounded by a fluid in mechanical equilibrium. This is because pressure needs to be continuous at the surface of the solid; some example where this is not the case are given in exercises.

2.3.2 The Centre of Buoyancy

A practical problem when considering the resultant of pressure forces is to know where to apply it. This is by definition the *centre of buoyancy*. When the buoyancy force is applied to it, it gives the same torque with respect to any point. In mathematical words, we need first the expression of the torque of pressure force with respect to an arbitrary point O:

$$-\int_{(S)} \mathbf{r} \times Pd\mathbf{S}$$

where $\mathbf{r} = \overrightarrow{OM}$, M being the current point. Let us play with this integral using (12.9) and (12.39); we rewrite it as

$$\int_{(V)} \nabla \times (P\mathbf{r})dV = \int_{(V)} \nabla P \times \mathbf{r}dV = \int_{(V)} \rho\mathbf{g} \times \mathbf{r}dV = \mathbf{g} \times \int_{(V)} \rho\mathbf{r}dV$$

where we now see the appearance of a new point C_b, defined as

$$\boldsymbol{OC}_{\mathbf{b}} = \frac{1}{M_f}\int_{(V)} \rho\mathbf{r}dV$$

We thus find that

$$\int_{(S)} -\overrightarrow{OM} \times Pd\mathbf{S} = \overrightarrow{OC_b} \times (-M_f\mathbf{g}) \qquad (2.14)$$

which means that the torque exerted by pressure forces is the same as the one exerted by the resultant of pressure forces applied to the barycentre of the displaced fluid. Two remarks are now in order:

- The torque of the buoyancy force is not modified if we apply this force on a point different than C_b provided that the new point is on a line defined by $\mathbf{g}$ and C_b.
- The point where the buoyancy force is applied exist only if the fluid is in equilibrium and if the pressure varies continuously around the solid (otherwise (2.14) is not valid).

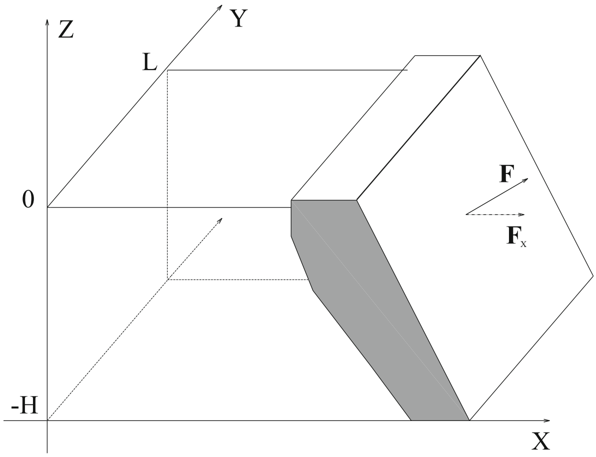


Fig. 2.5 Push on a dam

2.3.3 The Total Pressure on a Wall

The resultant pressure force exerted on a wall may easily be computed if one notices that the projection of the element $d\mathbf{S}$ on a plane whose normal is $\mathbf{e}_i$, is just $dS_i = d\mathbf{S}\cdot\mathbf{e}_i$. Hence,

$$F_i = \mathbf{e}_i \cdot \int_{(S)} Pd\mathbf{S} = \int_{(S_i)} PdS_i$$

where the integral is computed on the projected surface (S_i). If this surface is a rectangle of width L and height H, like in Fig. 2.5, and pressure is only a function of z, we find that

$$F_x = \int_{-H}^{0} (P_{atm} - \rho g z) L dz = LH\,(P_{atm} + \rho g H/2) = LH\,P(-H/2)$$

for an incompressible fluid.

2.4 Equilibria with Surface Tension

In Chap. 1 we pointed out that surface tension is a source of normal stress at the surface of liquids. This stress is at the origin of some specific figures of equilibrium that we shall investigate in broad lines (we refer the reader to more specialized work for a detailed discussion, e.g. de Gennes et al. 2004).

2.4.1 Some Specific Figures of Equilibrium

2.4.1.1 The Soap Bubble

This is certainly the most simple fluid equilibrium which involves surface tension. There, only pressure opposes to surface tension. Neglecting any effect of gravity, the equilibrium of a liquid in the thin film which makes a soap bubble is given by

$$P_{air} - \frac{2\gamma}{R} = P_{liq}, \qquad P_{liq} - \frac{2\gamma}{R'} = P_{atm}$$

where P_{air} is the air pressure inside the bubble. Because the envelope is very thin, $R \simeq R'$ and

$$P_{int} \approx P_{atm} + \frac{4\gamma}{R},$$

a formula which permits the measurement of surface tension of some liquid–gas interfaces.

2.4.1.2 The Catenoid

Let us imagine now a liquid film where pressure is the same on each side of the film. In such a situation

$$\Delta P = 0 = \gamma \left(\frac{1}{R_1} + \frac{1}{R_2} \right) \qquad \Longleftrightarrow \qquad \frac{1}{R_1} + \frac{1}{R_2} = 0$$

This equation defines a surface called *the catenoid* which is such that the sum of its radii of curvature is always zero; one radius is always negative (see Fig. 2.6).

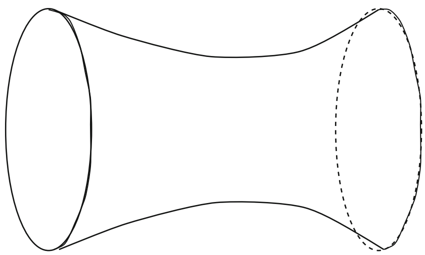


Fig. 2.6 The catenoid

2.4.2 Equilibrium of Liquid Wetting a Solid

The most spectacular effects of surface tension are certainly those associated with the wetting of solids. For instance, water raises in glass tube while mercury goes down (see Fig. 2.7). These different behaviours are the consequence of both the surface tension and the wetting properties of the solid by the liquid. These properties may be condensed in a single quantity ϑ, called the *wetting or contact angle*, in Young theory.[3] His theory assumes that the contact line gas–liquid–solid results from the equilibrium of three surface tensions: liquid–gas, liquid–solid and solid–gas. The angle between the liquid–gas and solid–liquid surfaces is called the contact angle ϑ (see Fig. 2.8). This theory gives a simple approach to very complex phenomena.

The equilibrium of the contact line yields *Young formula*:

$$\gamma_{\ell g} \cos\vartheta + \gamma_{\ell s} = \gamma_{sg} \tag{2.15}$$

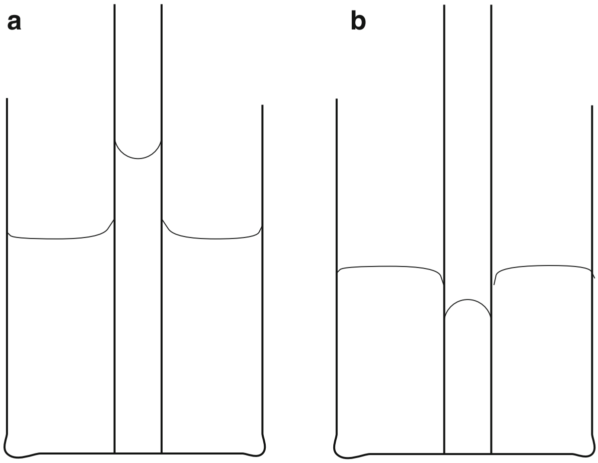


Fig. 2.7 *Upward* or *downward* displacement of a liquid due to the joint action of wetting and surface tension

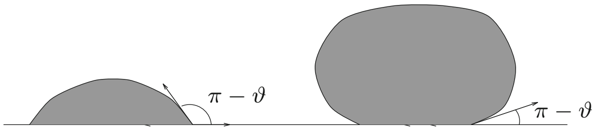


Fig. 2.8 The contact angle is ϑ but for the sake of clarity we show $\pi - \vartheta$

[3]Thomas Young (1773–1829) is well-known for his work in interferometry but he also studied the surface tension of liquids and the wetting of solids in 1805.

If $(\gamma_{sg} - \gamma_{\ell s})/\gamma_{\ell g} \simeq 1$, the contact angle is very small and the liquids wets the solid; if, on the contrary, $(\gamma_{sg} - \gamma_{\ell s})/\gamma_{\ell g} \simeq -1$, the contact angle is close to 180° and the liquid only weakly wets the solid. These two extreme cases are shown in Fig. 2.8. Now, what happens if $(\gamma_{sg} - \gamma_{\ell s})/\gamma_{\ell g} > 1$? Actually, no equilibrium is possible and the liquid spreads completely until it makes a very thin film: this is *total wetting*.

2.4.2.1 Jurin's Formula

Many of us have experienced the raise of water in a thin glass tube. This is a joint effect of surface tension and wetting. The contact angle imposes a negative curvature to the water's surface and thus a depression in the water inside the tube. Water thus raises.

We may easily determine this elevation of the liquid inside the tube if we assume that the meniscus has the shape of a spherical cap. Let r be the radius of the tube and ϑ the contact angle, then the radius of the spherical cap is $R = r/\cos\vartheta$. We infer the pressure difference between the liquid and the gas:

$$P_L = P_G - \frac{2\gamma\cos\vartheta}{r}$$

and the height of the raise

$$h = \frac{2\gamma\cos\vartheta}{\rho g r}. \tag{2.16}$$

This is Jurin's formula.[4] We should stress here that this formula is an approximation valid for small values of the radius only. It is not valid for large radii since the meniscus is no longer spherical.

Jurin's formula shows that capillary rise is maximum for a total wetting ($\vartheta = 0$) but may be negative for a pair of liquid–solid such that $\cos\vartheta < 0$. For instance, water, whose surface tension is $\gamma = 0.0728\,\mathrm{J/m^2}$ at 20 °C may rise or sink by 15 mm in a tube of 1 mm radius.

2.5 Exercises

1. *About buoyancy*
   (a) An ice cube floats in a glass of water. When the ice melts, what does the level of water in the glass do?

[4] J. Jurin (1684–1750) was an English physician and physicist.

(b) Same question if the ice cube contains a piece of metal inside (but still floats)?
(c) And with a piece of cork?
(d) Explain why a balloon filled with some light gas (less dense than ambient air) that starts to rise, will reach a well-defined altitude while a submarine, which starts to sink, sinks to the bottom of the sea.
(e) In a car, a child holds a balloon filled with helium at the end of a string. When the car starts, how does the balloon move?

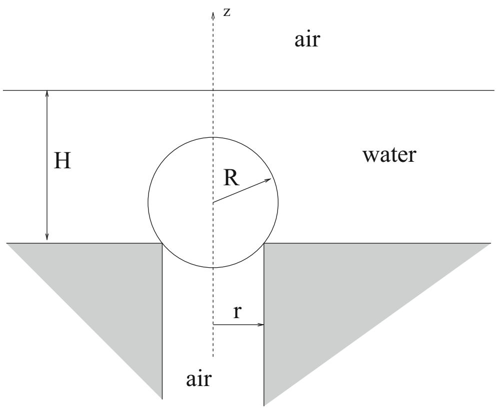
2. We consider a container filled with two immiscible liquids (oil and water for instance) and in a uniform gravity field.

(a) The two fluids being at rest, how do they settle in the container?
(b) What is the shape of the curve $P(z)$, the pressure as a function of the altitude z ($z = 0$ being the bottom of the container)?
(c) Oil and water densities are respectively $\rho_{\text{oil}} = 600\,\text{kg/m}^3$ and $\rho_{\text{water}} = 1{,}000\,\text{kg/m}^3$. A wooden sphere of density $\rho_{\text{wood}} = 900\,\text{kg/m}^3$ is left in this mixture; where is the equilibrium position of the sphere and what is the fraction of its volume inside water?

3. We consider a U-tube filled with water up to 10 cm from its bottom. The cross section of the tube is $1\,\text{cm}^2$. We then add $2\,\text{cm}^3$ of oil in one of the branches of the tube ($\rho_{\text{oil}}/\rho_{\text{water}} = 0.6$).

(a) At which height is the free surface of the oil?
(b) At which height is the interface oil–water?
(c) What is the height of water in the other branch?

4. A wooden sphere of density ρ and radius R is closing a circular hole of radius r at the bottom of a basin filled with water as shown in the figure below.



(a) Determine the force exerted by the sphere on the bottom of the basin.

(b) Give a numerical value using $\rho_{\text{water}} = 1{,}000\,\text{kg/m}^3$, $\rho = 850\,\text{kg/m}^3$, $H = 0.7\,\text{m}$, $R = 0.2\,\text{m}$, $r = 0.1\,\text{m}$, $g = 9.8\,\text{m/s}^2$.
(c) If the level of water is tunable, is there a value of this level which is such that the sphere rises to the surface before its top emerges?

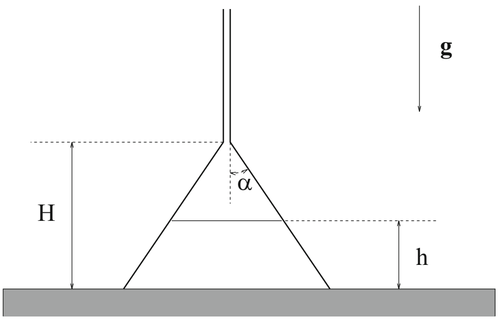
5. We wish to compute the flight altitude of a balloon filled with hydrogen and left in the atmosphere assumed isentropic. Let M_b be the mass of the balloon (the nacelle and the envelope), V_b its volume assumed to be fixed and M_H its mass of hydrogen. We recall that $\rho_{\text{air}} = \rho_0(1 - z/z_0)^{\gamma/(\gamma-1)}$ for the isentropic atmosphere

   (a) Which condition needs to be verified for the balloon to fly?
   (b) If this condition is fulfilled, find the altitude of the flying balloon.

6. We now assume that the envelope of the balloon is opened in its lower part. At take-off, a fraction of the volume of the envelope is filled with hydrogen which is in thermal equilibrium with the surrounding air. The volume of the envelope is assumed constant.

   (a) What can we say about the pressure of hydrogen in the balloon?
   (b) Show that the mass of hydrogen must exceed some critical value so that the balloon takes off?
   (c) Explain why the balloon reaches a well-defined altitude and give its expression.

7. Compute the temperature gradient at the equator of Jupiter assuming that its atmosphere is isentropic. The chemical composition is 85 % of molecular hydrogen and 15 % of helium. Jupiter's mass is 1.9×10^{27} kg, its radius 71,492 km and its rotation period 9.84 h.
8. A funnel is made of a tube with a very small cross section connected to a cone of aperture angle α. The funnel is put on a plane and filled with a liquid of density ρ as shown below.



(a) Compute the vertical component of the resultant of pressure forces as a function of α, of the height h of the liquid inside the funnel, H, ρ and g the gravity.
(b) We suppose the funnel is filled up to height H; show that the funnel must have a minimum mass M_e to be equilibrium. Express this mass as a function of the mass of the liquid M_l. What does happen if the mass of the fluid is larger?

9. *A polytropic model for the Sun*

(a) We assume that a star is a ball of gas in hydrostatic equilibrium. We recall that pressure $P(r)$ and gravity $g(r)$ at a distance r from the centre verify:

$$g(r) = \frac{GM(r)}{r^2} \qquad \text{and} \qquad \frac{dP}{dr} = -\rho g$$

where $\rho(r)$ is the density at r and $M(r)$ is the mass inside the sphere of radius r. We also assume that the gas verifies a polytropic equation of state, namely

$$P = K\rho^{1+1/n}$$

where K is a constant and n is the polytropic index of the gas. Setting $\rho = \rho_c\theta^n$, with ρ_c being the central density and θ a non-dimensional function that varies between 0 (at the surface) and 1 at the centre, show that θ obeys the following differential equation

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) + \theta^n = 0 \tag{2.17}$$

called Emden equation, where $\xi = r/r_0$ with

$$r_0 = \sqrt{\frac{(n+1)K}{4\pi G\rho_c^{1-1/n}}}$$

(b) Show that pressure may be written

$$P = P_c\theta^n$$

(c) Show that if mass and radius of the star are known then we may deduce its central density with

$$\rho_c = -\frac{\xi_1}{3\theta_1'}\langle\rho\rangle$$

where ξ_1 is the first root of function θ and θ_1' is the value of the derivative of this function at ξ_1. $\langle\rho\rangle$ is the mean density of the star (its mass divided by its volume).

(d) Show that central pressure reads

$$P_c = \frac{4\pi G\rho_c^2 r_0^2}{n+1}$$

(e) We now model the Sun by a polytrope of index $n = 3.37$. The numerical solution of Emden equation gives $\xi_1 = 8.686$ and $-\frac{\xi_1}{3\theta_1'} = 113.77$. Since the mass of the Sun is 2×10^{30} kg and its radius is 696×10^6 m, deduce the central density and pressure of the Sun according to this model.

(f) To derive the central temperature, we now assume that the solar plasma is an ideal gas. This gas is a mixture of protons, helium ions and electrons (other elements are neglected). We suppose that the mass fraction of helium is $\mathrm{Y} = 28\,\%$. Show that the mole mass of this mixture is

$$\mathcal{M} = \frac{4}{8-5Y}$$

grams per mole. Deduce the central temperature of the Sun according to that model. Compare with the values obtained from more realistic models: $\rho_c = 1.62\ 10^5$ kg/m^3, $P_c = 2.5\ 10^{16}$ Pa, $T_c = 1.57\ 10^7$ K.

Further Reading

For a deeper insight in the problems of wetting and capillarity, we refer the reader to de Gennes et al. (2004).

References

Binney, J., & Tremaine, S. (1987). *Galactic dynamics*. Princeton: Princeton University Press.
de Gennes, P.-G., Brochart-Wyart, F., & Quéré, D. (2004). *Capillarity and wetting phenomena: Drops, bubbles, pearls, waves*. Berlin: Springer.