Chapter 9
Turbulence

As we pointed it out in the first pages of this book, the understanding of turbulence remains one of the challenges of nowadays physics. The goal of this chapter is to introduce the reader to the main approaches that are used to deal with this difficult problem.

9.1 The Fundamental Problem of Turbulent Flows

9.1.1 How Can We Define Turbulence?

Defining a turbulent flow is not an easy matter because, to be precise, we need some notions that we shall develop below. However, if we are satisfied with general ideas, we can make it. First, let us observe that turbulent flows are quite disordered: a lot of vortices seem to be constantly appearing and disappearing in an essentially random way; this seems to be their main feature. To characterize this disorder, correlations are very useful. Let us introduce this notion in a simple way. If, in a turbulent flow, we record one component of the velocity at two distinct points A and B. The result is like the curves plotted in Fig. 9.1. These curves show an evolution of the physical quantities that looks like impredictable. To characterize the nature of this signal, one uses the autocorrelation function, which is the average over time of the product $V_A(t)V_A(t+T)$. The autocorrelation characterizes in some way the similarity of the function with itself at a different point. If the function changes nearly randomly, $V_A(t+T)$ is statistically independent of $V_A(t)$. Of course T cannot be too small, for the functions are continuous. The random nature appears when T is much larger than a specific time interval T_c which is called the *correlation time*. When this time is finite, the flow has a chaotic evolution. This is the case for a turbulent flow, but such a flow owns an "additional" chaos, namely a "spatial" chaos. Indeed, if we consider two points A and B at some large distance from one

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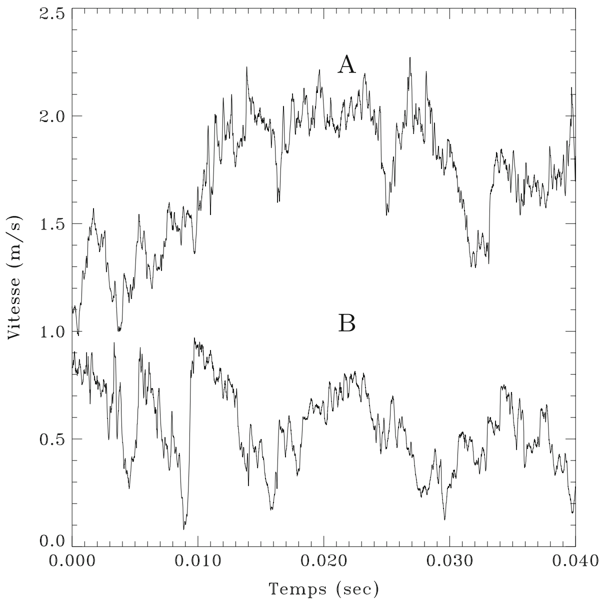

Fig. 9.1 Time evolution of the velocity at two distinct points A and B in a turbulent flow. No similarities between the two curves can be detected, except their random nature

another, the velocities at these points appear uncorrelated. Only when the distance is short enough, less than *the correlation length*, do the velocities show correlated values, so that $\langle V_A V_B \rangle \neq \langle V_A \rangle \langle V_B \rangle$, $\langle \ \rangle$ being a statistical average.

With these ideas in mind, it is easy to define turbulent flows: they are these flows where the correlation length is shorter that the size of the fluid domain and whose correlation time is shorter than the time scale we focus on. Turbulence thus appears as a fluid motion endowed with a "spatio-temporal chaos" or a "spatio-temporal decorrelation".

9.1.2 The Closure Problem of the Averaged Equations

The apparent random nature of a turbulent flow suggests that these flows should be studied using the tools of Statistics. We indeed suspect that only some mean quantities are really important for the understanding of the properties of the flow. Therefore, the effort should be concentrated on the derivation of the equations governing these mean quantities. This derivation raises a very difficult problem that has still not been circumvented. In order to present it in a simple way, let us consider the turbulent flow of some incompressible fluid. We write the equations of motion in a symbolic way:

$$\begin{cases} \partial_t v + \partial(vv) = -\nabla P + \Delta v \\ \nabla \cdot v = 0 \end{cases} \tag{9.1}$$

Let us now write the averaged equation. We use the averaging operator $\langle\ \rangle$ which we assume to commute with all the derivative operators. Applying this to the foregoing system, we get:

$$\begin{cases} \partial_t \langle v \rangle + \partial \langle vv \rangle = -\nabla \langle P \rangle + \Delta \langle v \rangle \\ \nabla \cdot \langle v \rangle = 0 \end{cases} \tag{9.2}$$

which governs the evolution of the mean velocity $\langle v \rangle$. However, this system is not closed since the quantity $\langle vv \rangle$ is unknown. It is usually different from $\langle v \rangle^2$. Thus, we need a new equation to constrain this quantity. The way to get this new equation is through (9.1) which is first multiplied by the velocity and then averaged. It turns out that we get a new equation like:

$$\frac{\partial \langle vv \rangle}{\partial t} + \partial \langle vvv \rangle = -\nabla \langle Pv \rangle + \langle v \Delta v \rangle$$

The important feature of this new equation is that it contains a term using $\langle vvv \rangle$, which is a triple correlation. Hence, the evolution of double correlations, like $\langle vv \rangle$, is controlled by the triple correlations and we easily guess that the evolution of the triple correlations depends on the one of fourth order correlations $\langle vvvv \rangle$ etc.

Hence, we always have a set of equations which contains a larger set of unknowns: this is the famous problem of the *closure* of the mean-field equations, for which the general solution is still missing.

9.2 The Tools

In order to continue this study, we now need to introduce the good tools which will allow us to deal with the random nature of turbulent flows. Very naturally, we shall borrow these tools to Statistical Physics.

9.2.1 Ensemble Averages

An ensemble average $\langle X \rangle$ of a quantity X is the average derived from N independent experiments measuring this quantity when N goes to infinity. Thus

$$\langle X \rangle = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} X_n$$

For the velocity field we have:

$$\langle \mathbf{v}(\mathbf{r},t)\rangle = \lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} \mathbf{v}_n(\mathbf{r},t)$$

Note that the mean field remains a function of space and time. We easily verify that taking the ensemble average of a quantity is an operation that commutes with all types of differentiation. Hence

$$\left\langle \frac{\partial \mathbf{v}}{\partial r} \right\rangle = \frac{\partial \langle \mathbf{v}\rangle}{\partial r}.$$

9.2.2 Probability Distributions

In the following of the chapter, the notion of probability distribution or that of probability density function will be used very often. We thus need to define the meaning of these tools.

Let us assume that we are analysing the pressure fluctuations and are interested in the probability of finding this quantity in the interval $]-\infty, x]$. We denote this probability $F_P(x)$, where the index P means that P is the random variable. The function F_P is called a *probability distribution function*. From this definition it turns out that

- $F_P(x)$ is not decreasing
- $F_P(x)$ is continuous on right
- $F_P(-\infty) = 0$ and $F_P(+\infty) = 1$.

If this function is differentiable, then $F'_P(x)$ is called the *probability density function*, often called the "pdf". Physically, it means that $F'_P(x)dx$ is the probability of finding P in the interval $]x, x+dx]$.

9.2.3 Moments and Cumulants

The moments of a probability distribution are the averages of the powers of the random variable:

$$M_n = \langle P^n \rangle = \int_{-\infty}^{+\infty} x^n F'_P(x)dx \tag{9.3}$$

The first order moment is just *the average or mean* also called *mathematical expectation or expected value*.

The moments of $P - \langle P \rangle$ are said to be *centered*. The variance is the centered moment of order two:

$$\left\langle (P - \langle P \rangle)^2 \right\rangle$$

and the root mean square is the square root of the variance.

The cumulant of order n of a random variable is the difference between the n-th order moment of this variable and the same moment of the gaussian distribution that has the same variance and the same mean as the given random variable.

9.2.4 Correlations and Structure Functions

In addition to the correlations that we already introduced at the beginning of this chapter, we shall also need n-points correlations which are the averages of a given function taken at n different points.

$$\langle f(x_1) f(x_2) \dots f(x_n) \rangle$$

As far as vectorial quantities are concerned, like the velocity, we may mix the components like in

$$\left\langle v_i(\mathbf{r}_1) v_j(\mathbf{r}_2) \cdots v_k(\mathbf{r}_n) \right\rangle$$

which leads to the definition of new tensors.

Structure functions are quantities like

$$S_n = \langle (f(\mathbf{r}_1) - f(\mathbf{r}_2))^n \rangle$$

which are easily related to two-points correlations.

9.2.5 Symmetries

The study of turbulent flows is much simplified when some symmetries are verified. Unfortunately, real flows usually own little symmetries or very approximately. Nevertheless, their use is very handy to reduce a given problem to its essential features.

Five symmetries may actually be verified by a turbulent flow:

- *Homogeneity*: This is the invariance of the properties of turbulence (all the moments for instance) with respect to space translations. This is a very strong hypothesis that is verified in only small regions. However, it generates important simplifications that are very useful to understand the problems generated by turbulence.

- *steadyness/stationarity*: This is invariance with respect to time translations. This is a less constraining hypothesis, but it is often difficult to really know when the flow is really steady on average.
- *Isotropy*: This is the invariance of the turbulence with respect to space rotation $\mathcal{O}(3)$. It is very useful in homogeneous turbulence in order to simplify the calculations (of course, it cannot exist in non-homogenous turbulence).
- *Parity*: This is the symmetry with respect to a point or a plane. One may see it as an invariance with respect to a change of sign of all the vectors. It cannot be reduced to a combination of rotations. It allows the elimination of quantities like helicity, which change sign when this symmetry is imposed.
- *Scale invariance*: This is the invariance with respect to transformations like $\mathbf{v}(\mathbf{r}, t) \to \lambda^h \mathbf{v}(\lambda \mathbf{r}, \lambda^{1-h} t)$. The solutions of Navier–Stokes equation satisfy this symmetry only if $h = -1$. This is just the similarity of flows at the same Reynolds number that we studied in Chap. 4. Now, if there is no viscosity, then h is not constrained (as long as $\lambda > 0$). In turbulent flows, which are usually at high Reynolds numbers, this symmetry may be verified in some range of scales.

In many textbooks, isotropic turbulence refers to a case which includes both isotropy and parity invariance. Here, we shall be more restrictive and always mention the use of parity invariance.

9.3 Two-Points Correlations

After this short presentation of the problem of turbulence and the basic tools that are used to deal with it, we shall now focus on the two-point correlations because they play an important role in the analysis of turbulent flows. To make this study easier we shall restrict ourselves to the case of *homogeneous turbulence* because this case owns all the universal properties of turbulence. Although more general, the non-homogeneous case is very dependent on this problem and can be discussed in a second step.

9.3.1 The Reynolds Stress

Let us come back to the mean-field equations using as in the introduction the case of an incompressible fluid. We now decompose the velocity field as:

$$\mathbf{v} = \langle \mathbf{v} \rangle + \mathbf{v}' \tag{9.4}$$

where $\langle \cdots \rangle$ is an ensemble average. The fluctuation $\mathbf{v}'$ is necessarily such that $\langle \mathbf{v}' \rangle = 0$. This decomposition is known as *Reynolds decomposition*. Inserting it in Navier–Stokes equation, we get

$$\rho \left(\frac{\partial \langle \mathbf{v} \rangle}{\partial t} + \frac{\partial \mathbf{v}'}{\partial t} + \langle \mathbf{v} \rangle \cdot \nabla \mathbf{v}' + \mathbf{v}' \cdot \nabla \langle \mathbf{v} \rangle + \langle \mathbf{v} \rangle \cdot \nabla \langle \mathbf{v} \rangle + \mathbf{v}' \cdot \nabla \mathbf{v}' \right) = -\nabla \langle P \rangle - \nabla P' + \mu \Delta \langle \mathbf{v} \rangle + \mu \Delta \mathbf{v}' \quad (9.5)$$

$$\nabla \cdot \langle \mathbf{v} \rangle + \nabla \cdot \mathbf{v}' = 0 \quad (9.6)$$

Averaging these equations, we get:

$$\rho \left(\frac{\partial \langle \mathbf{v} \rangle}{\partial t} + \langle \mathbf{v} \rangle \cdot \nabla \langle \mathbf{v} \rangle + \left\langle \mathbf{v}' \cdot \nabla \mathbf{v}' \right\rangle \right) = -\nabla \langle P \rangle + \mu \Delta \langle \mathbf{v} \rangle \quad (9.7)$$

$$\nabla \cdot \langle \mathbf{v} \rangle = 0 \quad (9.8)$$

These new equations contain a new quantity, namely $\langle \rho \mathbf{v}' \cdot \nabla \mathbf{v}' \rangle$ which is related to the *Reynolds stress tensor*:

$$[R] = \rho \left\langle \mathbf{v}' \otimes \mathbf{v}' \right\rangle$$

where $\otimes$ denote the tensorial product. The components of $[R]$ are

$$R_{ij} = \rho \langle v_i' v_j' \rangle$$

Now, (9.6) and (9.8) imply

$$\nabla \cdot \mathbf{v}' = 0 \quad (9.9)$$

so that

$$\rho \left\langle \mathbf{v}' \cdot \nabla \mathbf{v}' \right\rangle_i = \rho \partial_k \left\langle v_i' v_k' \right\rangle = \partial_k R_{ik}$$

if ρ is constant.[1]

[1] Let us note here that the true stress induced by the correlation $\langle \mathbf{v}' \otimes \mathbf{v}' \rangle$ is rather $-R_{ij}$ since the momentum equation (9.7) may also be written

$$\rho \frac{D \langle v_i \rangle}{Dt} = \partial_j \sigma_{ij}$$

with $\sigma_{ij} = -\langle P \rangle + \mu(\partial_i \langle v_j \rangle + \partial_j \langle v_i \rangle) - R_{ij}$. Note also that the Reynolds tensor is often defined as $\langle \mathbf{v}' \otimes \mathbf{v}' \rangle$.

We observed in the introduction of this chapter that the turbulence problem comes from our ignorance of how to express R_{ik} as a function of the mean-fields $\langle \mathbf{v} \rangle$ and $\langle P \rangle$. Modeling turbulence is therefore equivalent to finding a way of relating these quantities. As we may suspect it, no universal way is known yet. Understanding the present ones, which are all ad hoc at some level, requires a good knowledge of the properties of the Reynolds tensor. This demands a study of the velocity two-point correlations.

9.3.2 The Velocity Two-Point Correlations

The first step in studying the properties of the Reynolds tensor needs the investigation of a slightly more general quantity, namely

$$Q_{ij}(\mathbf{r}_A, \mathbf{r}_B) = \left\langle v'_i(\mathbf{r}_A) v'_j(\mathbf{r}_B) \right\rangle \tag{9.10}$$

which is the *second order tensor of the velocity (fluctuations) correlations* taken at two points A and B. In order to simplify the study, we assume that there is no mean flow, namely $\langle \mathbf{v} \rangle = \mathbf{0}$. Thus, the velocity and its fluctuations are identical.

In full generality, this tensor is a function of six space variables and two time variables. Again, we simplify the matter by assuming that the velocities are taken at the same time. Moreover we drop the time variable since we work on the spatial properties of the tensor. We also set $\mathbf{r}_A = \mathbf{x}$ and $\mathbf{r}_B = \mathbf{x}' = \mathbf{x} + \mathbf{r}$. Thus

$$Q_{ij}(\mathbf{x}, \mathbf{x}') = \left\langle v_i(\mathbf{x}) v_j(\mathbf{x} + \mathbf{r}) \right\rangle \tag{9.11}$$

The first (elementary) property of $[Q]$ is that

$$Q_{ij}(\mathbf{x}, \mathbf{x}') = Q_{ji}(\mathbf{x}', \mathbf{x}) \tag{9.12}$$

In addition, since we are working with an incompressible fluid, (9.9) is verified and

$$\partial_i Q_{ij} = \partial'_j Q_{ij} = 0 \tag{9.13}$$

where $\partial'_j = \partial/\partial x'_j$.

Let us now assume that the turbulence is homogeneous. Its properties are thus independent of the point that is considered. Consequently, Q_{ij} depends only on the difference between the two vectors $\mathbf{x}$ and $\mathbf{x}'$, so that $Q_{ij} \equiv Q_{ij}(\mathbf{r})$. We further note that

$$Q_{ij}(\mathbf{0}) = R_{ij}$$

and that $R_{ii} = Tr([Q])(0)$ is just twice the turbulent kinetic energy per unit mass.

The homogeneity of turbulence allows us to reduce the number of independent components of $[Q]$. Actually, as we shall see, they are only three. First of all, (9.12) implies that

$$Q_{ij}(\mathbf{r}) = Q_{ji}(-\mathbf{r}) \tag{9.14}$$

So that we are left with only six independent components. However, we have to build up a second order tensor that only depends on the vector $\mathbf{r}$. Since second order tensors are built up from tensor products using tensors of lower orders, the only possibility is

$$Q_{ij} = A(\mathbf{r})\delta_{ij} + B(\mathbf{r})\frac{r_i r_j}{r^2} + H(\mathbf{r})\epsilon_{ijk}\frac{r_k}{r} \tag{9.15}$$

The three functions $A(\mathbf{r})$, $B(\mathbf{r})$ and $H(\mathbf{r})$ are the three independent components. They are unknown, but if we use (9.14), we find that they verify

$$A(-\mathbf{r}) = A(\mathbf{r}), \quad B(-\mathbf{r}) = B(\mathbf{r}), \quad H(-\mathbf{r}) = H(\mathbf{r})$$

Furthermore, H is a pseudo-scalar.[2] One consequence is: if the turbulence is parity-invariant then H is zero. We shall see below that this quantity is related to helicity.

Now we may further constraint the unknown components of Q_{ij} using the relation of incompressibility. Equation (9.13) leads to

$$\nabla A + \mathbf{e}_r \nabla \cdot (B\mathbf{e}_r) + \nabla H \times \mathbf{e}_r = \mathbf{0}$$

Let us further restrict our discussion to that of *isotropic turbulence*. In this case, the functions no longer depend on the direction of $\mathbf{r}$, but only on $r = ||\mathbf{r}||$. Incompressibility then relates A and B

$$\frac{\partial A}{\partial r} + \frac{1}{r^2}\frac{\partial r^2 B}{\partial r} = 0$$

We now introduce the longitudinal and transversal velocity correlations. These quantities may indeed be measured experimentally, and they are usually used instead of A and B. The longitudinal component of the velocity is the one which is parallel to $\mathbf{r}$. We call it v_ℓ, and thus $v_\ell = \mathbf{v} \cdot \mathbf{r}/r$.

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[2] A pseudo-scalar is a scalar quantity the sign of which depends on the orientation of the vector basis. For instance, the determinant of three vectors (in three dimensions) is a pseudo-scalar. In our case, if $\mathbf{X}$ et $\mathbf{Y}$ are two vectors, from the definition of $[Q]$, $X_i Y_j Q_{ij}$ is a true scalar. Thus

$$A(\mathbf{X} \cdot \mathbf{Y})^2 + (\mathbf{r} \cdot \mathbf{X})(\mathbf{r} \cdot \mathbf{Y})B/r^2 + H\epsilon_{ijk}X_i Y_j r_k/r$$

is a true scalar. In this expression we see that the last term is the determinant of three vectors times H. Thus H is a pseudo-scalar.

The transversal component is just the remaining vector $\mathbf{v}^t = \mathbf{v} - v_\ell \mathbf{r}/r$. The longitudinal and transversal correlations are usually denoted f and g. From these definitions and using the expression (9.15) of Q_{ij}, we easily find that

$$f = Q_{ij} r_i r_j / r^2 \qquad \text{and} \qquad g = Q_{ij} (\mathbf{v}^t)_i (\mathbf{v}^t)_j / \|\mathbf{v}^t\|^2$$

so that

$$f = A + B \qquad \text{and} \qquad g = A$$

Incompressibility allows us to relate g to f, namely

$$g = \frac{1}{2r} \frac{\partial r^2 f}{\partial r} \tag{9.16}$$

Finally, we get the expression of Q_{ij} for the homogeneous isotropic turbulence of an incompressible fluid:

$$Q_{ij} = \frac{1}{2r} \frac{\partial r^2 f}{\partial r} \delta_{ij} - \frac{r}{2} \frac{\partial f}{\partial r} \frac{r_i r_j}{r^2} + H(\mathbf{r}) \epsilon_{ijk} \frac{r_k}{r} \tag{9.17}$$

We end this discussion with a final remark on the behaviour of $Q(\mathbf{r})$ when the distance between the point grows without bound. We said in the introduction that turbulence was also characterized by a finite correlation length L_c. Hence, if $r \gg L_c$ then the velocity correlations should be negligible. It is therefore legitimate to assume:

$$\lim_{r \to \infty} Q_{ij}(r) = 0 \tag{9.18}$$

9.3.3 Vorticity and Helicity Correlations

We shall need later another tensor, namely the one of vorticity correlations at two points:

$$\Omega_{ij} = \left\langle \omega_i(\mathbf{x}) \omega_j(\mathbf{x}') \right\rangle \tag{9.19}$$

Just like the velocity correlation tensor, this tensor depends on six independent variables $\mathbf{x}$ and $\mathbf{x}'$. However, $\omega_i = \epsilon_{ikl} \partial_k v_l$, but $\mathbf{x}$ and $\mathbf{x}'$ are independent; we may thus write:

$$\Omega_{ij} = \epsilon_{ikl} \epsilon_{jmn} \frac{\partial^2}{\partial x_k \partial x'_m} \left\langle v_l(\mathbf{x}) v_n(\mathbf{x}') \right\rangle = -\epsilon_{ikl} \epsilon_{jmn} \frac{\partial^2}{\partial r_k \partial r_m} Q_{ln}(\mathbf{r})$$

where we used the homogeneity of turbulence and the relations

$$\frac{\partial Q}{\partial x_j} = -\frac{\partial Q}{\partial r_j}, \qquad \frac{\partial Q}{\partial x'_j} = \frac{\partial Q}{\partial r_j}$$

since $\mathbf{r} = \mathbf{x}' - \mathbf{x}$. Now, using (12.1) together with incompressibility (9.13), the foregoing relation leads to the following expression:

$$\Omega_{ij} = \frac{\partial^2 Q_{kk}}{\partial r_i \partial r_j} - \Delta Q_{ji} - \delta_{ij}\Delta Q_{kk} \tag{9.20}$$

where Δ is the Laplacian.

Finally, we also need the cross-correlation vorticity-velocity, which is called helicity correlation or just mean helicity. This is

$$\begin{aligned}\langle \omega_i(\mathbf{x})v_i(\mathbf{x}')\rangle &= \epsilon_{ijk}\frac{\partial v_k}{\partial x_j}v_i(\mathbf{x}') = \epsilon_{ijk}\frac{\partial}{\partial x_j}\langle v_k(\mathbf{x})v_i(\mathbf{x}')\rangle \\ &= \epsilon_{ijk}\frac{\partial Q_{ki}}{\partial x_j} = \epsilon_{ijk}\frac{\partial Q_{ik}}{\partial r_j} = -\frac{\partial}{\partial r_j}\left(\frac{2H(\mathbf{r})r_j}{r}\right)\end{aligned} \tag{9.21}$$

This definition does not depend on the order of points. Indeed, if we exchange $\mathbf{x}$ and $\mathbf{x}'$, we find

$$\langle \omega_i(\mathbf{x}')v_i(\mathbf{x})\rangle = \epsilon_{ijk}\frac{\partial Q_{ik}(\mathbf{x},\mathbf{x}')}{\partial x'_j} = \epsilon_{ijk}\frac{\partial Q_{ik}(\mathbf{r})}{\partial r_j}$$

since $Q_{ik}(\mathbf{x},\mathbf{x}') = Q_{ik}(\mathbf{x}'-\mathbf{x}) = Q_{ik}(\mathbf{r})$.

9.3.4 The Associated Spectral Correlations

When we analysed instabilities, we found it convenient to decompose the unknowns on a basis of orthogonal functions. When we are dealing with turbulent flows such a decomposition is also useful. In the simple case of homogeneous turbulence, the Fourier basis is appropriate. The Fourier transform of the tensors describing the correlations own interesting properties which give another view of turbulence, in particular on its energy side.

As a first step we introduce the Fourier transform that are used to obtain the spectral quantities. Thus let us define $\hat{f}(\mathbf{k})$ the Fourier transform of a square integrable function $f(\mathbf{r})$ and the inverse transform. We have

$$\hat{f}(\mathbf{k}) = (2\pi)^{-3}\int f(\mathbf{r})e^{-i\mathbf{k}\cdot\mathbf{r}}d^3\mathbf{r} \qquad \text{and} \qquad f(\mathbf{r}) = \int \hat{f}(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{r}}d^3\mathbf{k}$$

Let us now introduce the Fourier transform of the two point velocity correlation tensor, namely:

$$\phi_{ij}(\mathbf{k}) = (2\pi)^{-3} \int Q_{ij}(\mathbf{r}) e^{-i\mathbf{k}\cdot\mathbf{r}} d^3\mathbf{r}$$

From (9.14), we easily show that

$$\phi_{ji}(\mathbf{k}) = \phi_{ij}(-\mathbf{k}) = \phi^*_{ij}(\mathbf{k}) \tag{9.22}$$

while incompressibility implies that

$$k_i \phi_{ij}(\mathbf{k}) = 0 \tag{9.23}$$

The relation (9.22) implies that the symmetric part of $[\phi]$ is real while the antisymmetric part is purely imaginary. Indeed, if we set:

$$\phi_{ij} = \frac{1}{2}(\phi_{ij} - \phi_{ji}) + \frac{1}{2}(\phi_{ij} + \phi_{ji}) = \hat{A}_{ij} + \hat{S}_{ij}$$

then (9.22) gives

$$\hat{A}^*_{ij} = -\hat{A}_{ij} \qquad \hat{S}^*_{ij} = \hat{S}_{ij} \tag{9.24}$$

Thus quite generally, we may write

$$\hat{A}_{ij} = i\epsilon_{ijn} a_n$$

where $\mathbf{a}$ is an unspecified real vector. However, incompressibility implies that $k_i \hat{A}_{ij} = 0$ and thus $\mathbf{a} \times \mathbf{k} = \mathbf{0}$. Hence, we can set $\mathbf{a} = h(\mathbf{k})\mathbf{k}$ and

$$\hat{A}_{ij} = i\epsilon_{ijn} k_n h(\mathbf{k}) \tag{9.25}$$

The function $h(\mathbf{k})$ is a pseudo-scalar related, as we may guess, to the helicity of turbulence. Let us indeed take the Fourier transform of $H(\mathbf{r})$. One can show (see exercises) that (9.21) yields

$$\hat{H} = i\epsilon_{ijn} k_j \phi_{in}$$

Using (9.25), we also get $\hat{H} = 2k^2 h(\mathbf{k})$. Thus the antisymmetric part of ϕ_{ij} is just proportional to the Fourier transform of helicity correlations. We thus write

$$\hat{A}_{ij} = \frac{i\hat{H}(\mathbf{k})}{2k^2} \epsilon_{ijn} k_n$$

We can treat in a similar way the symmetric part of ϕ_{ij}, since $\hat{S}_{ij}$ is a symmetric tensor that depends only on $\mathbf{k}$. Its most general form is therefore

$$\hat{S}_{ij}(\mathbf{k}) = \hat{e}(\mathbf{k})\delta_{ij} + \hat{g}(\mathbf{k})k_i k_j$$

Again, incompressibility can be used to simplify the expression and we obtain

$$\hat{S}_{ij} = \hat{e}(\mathbf{k}) \left(\delta_{ij} - \frac{k_i k_j}{k^2} \right) \tag{9.26}$$

We shall see below that the function $\hat{e}(\mathbf{k})$ is related to the kinetic energy spectrum of the turbulence. Finally, we may write the general form of ϕ_{ij} as:

$$\phi_{ij} = \hat{e}(\mathbf{k}) P_{ij} + \frac{i\hat{H}(\mathbf{k})}{2k^2} \epsilon_{ijn} k_n \tag{9.27}$$

In this latter expression we introduced $P_{ij} = \delta_{ij} - k_i k_j / k^2$ also called the *projection tensor*. Indeed, if $\mathbf{a}$ is some vector, $P_{ij} a_j$ is a vector which lies in a plane perpendicular to $\mathbf{k}$ since $k_i P_{ij} a_j = 0$. This tensor often appears when one deals with incompressible fluids, since the continuity equation implies that the Fourier transform of the velocity belongs to this plane.

In the same way we dealt with velocity correlations, we can consider vorticity correlations. Let $Z_{ij}(\mathbf{k})$ be the Fourier transform of Ω_{ij}. Using (9.20), $Z_{ij}(\mathbf{k})$ can be expressed as a function of $\phi_{ij}(\mathbf{k})$, namely

$$Z_{ij} = P_{ij} k^2 \phi_{nn} - k^2 \phi_{ji} \tag{9.28}$$

9.3.5 Spectra

We alluded above to the relation between the kinetic energy density and the Reynolds stress tensor which trace is just twice this quantity. Now, we may focus on the spectral energy density per unit mass, that is on the kinetic energy which is contained in the wavenumber interval $[k, k + dk]$. This quantity is $E(k)$ and it is defined by

$$E_{turb} = \frac{1}{2} \left\langle v^2 \right\rangle = \int_0^{+\infty} E(k) dk \tag{9.29}$$

We shall now relate this quantity to ϕ_{ij}. Indeed,

$$E_{turb} = \frac{1}{2} Q_{ii}(0) = \frac{1}{2} \int \phi_{ii}(\mathbf{k}) d^3\mathbf{k} = \frac{1}{2} \int_0^{+\infty} k^2 dk \int_{(4\pi)} \phi_{ii}(\mathbf{k}) d\Omega_k$$

$$\Longrightarrow \quad E(k) = \frac{1}{2} k^2 \int_{(4\pi)} \phi_{ii}(\mathbf{k}) d\Omega_k \tag{9.30}$$

where $d\Omega_k$ is the elementary solid angle in the Fourier space.

In a similar way, we can define the enstrophy spectrum $Z(k)$ writing

$$Z = \frac{1}{2}\langle\omega^2\rangle = \int_0^{+\infty} Z(k)dk\,;$$

where we see that enstrophy is analogous to turbulent kinetic energy, but using vorticity instead of the velocity field. Similarly as for kinetic energy, we write

$$Z(k) = \frac{1}{2}k^2 \int_{(4\pi)} Z_{ii}(\mathbf{k})d\Omega_k$$

Using (9.28) we deduce that $Z_{ii} = k^2\phi_{ii}$. This allows us to relate the enstrophy and kinetic energy spectra by

$$Z(k) = k^2 E(k) \tag{9.31}$$

Finally, we may also introduce the helicity spectrum $H(k)$ such that

$$\langle\boldsymbol{\omega}\cdot\mathbf{v}\rangle = \int_0^{+\infty} H(k)dk \tag{9.32}$$

9.3.6 The Isotropic Case

We shall specialize a little more our discussion by focusing on the important case of isotropic turbulence.

In this case the tensor $[Q]$ depends only on the distance r between the two points. We introduce the function $R(r) = \frac{1}{2}Q_{ii}(r)$ which is just half the trace of $[Q]$. Let us note that the value of R at $r = 0$ is just the local mean kinetic energy per unit mass, $R(0) = E_{turb}$.

If the turbulence is isotropic, then the Fourier transform of Q_{ij} is independent of the direction of the wavevector $\mathbf{k}$, thus

$$E(k) = 2\pi k^2\phi_{ii}(k) = 4\pi k^2\hat{e}(k) \tag{9.33}$$

following (9.30). If we observe that $H(k) = 4\pi k^2\hat{H}(k)$, then

$$\phi_{ij} = \frac{E(k)}{4\pi k^2}P_{ij} + \frac{iH(k)}{8\pi k^4}\epsilon_{ijn}k_n \tag{9.34}$$

The expression of $R(r)$ with respect to $E(k)$ may be derived using the expression of ϕ_{ii}. Indeed,

$$R(r) = \frac{1}{2}\int \phi_{ii}(k)e^{i\mathbf{k}\cdot\mathbf{r}}d^3\mathbf{k} = \frac{1}{2}\int \phi_{ii}(k)e^{ikr\cos\theta}k^2dk\sin\theta d\theta d\varphi$$

from which it turns out that, after integration on the angular variables and use of (9.33),

$$R(r) = \int_0^\infty E(k)\frac{\sin kr}{kr}dk \tag{9.35}$$

Another property of R is its symmetry with respect to the origin $R(-r) = R(r)$ (see (9.14)). The values of its (even) derivatives at the origin are also related to $E(k)$. Namely

$$\left(\frac{\partial^{2n}R}{\partial r^{2n}}\right)_{r=0} = \int_0^\infty E(k)\frac{\partial^{2n}}{\partial r^{2n}}\left(\frac{\sin kr}{kr}\right)_{r=0}dk = \frac{(-1)^n}{2n+1}\int_0^\infty k^{2n}E(k)dk$$

where we used that

$$\left(\frac{d^{2n}}{dx^{2n}}\frac{\sin x}{x}\right)_{x=0} = \frac{(-1)^n}{2n+1}$$

In particular, the second order derivative verifies

$$\left(\frac{\partial^2 R}{\partial r^2}\right)_{r=0} = -\frac{1}{3}\int_0^\infty k^2E(k)dk = -\frac{Z}{3} \tag{9.36}$$

which shows that it is related to the local enstrophy Z. It also emphasizes the fact that velocity correlations are, as expected, maximum at $r = 0$ since the derivative is zero and the second derivative is negative.

Relation (9.35) can be inverted (cf exercises) and yields the following relation:

$$E(k) = \frac{2}{\pi}\int_0^\infty kr\sin kr\ R(r)dr \tag{9.37}$$

which shows that if the wavelength of the Fourier mode is much larger than the correlation length, then

$$E(k) = \frac{2k^2}{\pi}\int_0^\infty r^2R(r)dr$$

since for all the values where $R(r)$ is non-zero, $kr \ll 1$ and thus $\sin kr \sim kr$.

This result shows that whatever the dynamics, the infrared behaviour of a three-dimensional kinetic energy spectrum follows a k^2-law. As may be guessed, the exponent depends on the dimension of space (see Sect. 9.9).

9.3.7 Triple Correlations

The next step in our investigation of turbulence leads us to now examine the triple correlations since we know that they control the evolution of double correlations. As for these correlations, we consider the triple correlations in two points. A priori, we expect two types of triple correlations, namely

$$S_{ijk} = \langle v_i(\mathbf{x})v_j(\mathbf{x})v_k(\mathbf{x}+\mathbf{r})\rangle \qquad \text{and} \qquad S'_{ijk} = \langle v_i(\mathbf{x})v_j(\mathbf{x}+\mathbf{r})v_k(\mathbf{x}+\mathbf{r})\rangle \tag{9.38}$$

However, when the turbulence is homogeneous, these two quantities are related by

$$S'_{ijk}(\mathbf{r}) = S_{jki}(-\mathbf{r})$$

Thus, only one type of triple correlations exists for homogenous turbulence.

The tensor $[S]$ has some interesting properties that deserve some discussion. It is obviously symmetric with respect to the first two indices. If the turbulence is parity invariant, then it should be invariant if we inverse all the axis of coordinates.

$$\begin{aligned} S_{ijk}(\mathbf{r}) &= \langle v_i(\mathbf{x})v_j(\mathbf{x})v_k(\mathbf{x}+\mathbf{r})\rangle \\ &= \langle(-v_i(-\mathbf{x}))(-v_j(-\mathbf{x}))(-v_k(-\mathbf{x}-\mathbf{r}))\rangle = -S_{ijk}(-\mathbf{r}) \end{aligned} \tag{9.39}$$

The third equality comes from the homogeneity of turbulence. Thus, the S_{ijk} are anti-symmetric with respect to the origin, and

$$S_{ijk}(\mathbf{0}) = 0 \tag{9.40}$$

As expected the one-point triple correlations are zero in a homogeneous and parity-invariant turbulence. It would not be the case with helical turbulence.

As for the double correlations, we shall reduce the expression of $[S]$ to a single scalar function: the longitudinal triple correlation:

$$k(\mathbf{r}) = \langle v_\ell(\mathbf{x})^2 v_\ell(\mathbf{x}+\mathbf{r})\rangle \tag{9.41}$$

We first express $S_{ijk}(\mathbf{r})$ with the tensors δ_{ij} and r_i, taking into account the symmetry with respect to the first two indices. Thus

$$S_{ijk}(\mathbf{r}) = A(r)r_ir_jr_k + B(r)(r_i\delta_{jk} + r_j\delta_{ik}) + C(r)\delta_{ij}r_k$$

The three functions A, B and C can be expressed with $k(r)$ if we assume the isotropy of turbulence and the fluid's incompressibility. Isotropy implies:

$$S_{iik} = 0 \quad \Longleftrightarrow \quad r^2 A + 2B + 3C = 0 \tag{9.42}$$

since S_{iik} is the average of a vector which therefore cannot indicate any privileged direction. Thus it is zero. Incompressibility implies:

$$\partial_k S_{ijk} = 0 \tag{9.43}$$

A short manipulation leads to

$$(rA' + 5A + \frac{2B'}{r})r_i r_j + (2B + rC' + 3C)\delta_{ij} = 0.$$

Thus, with (9.42), we get three relations:

$$\begin{cases} rA' + 5A + \frac{2B'}{r} = 0 \\ 2B + rC' + 3C = 0 \\ r^2 A + 2B + 3C = 0 \end{cases} \tag{9.44}$$

According to its definition,

$$k(r) = r^3 A + r(2B + C)$$

This equation combined with (9.42) leads to $C = -k/2r$, while the last two equations of (9.44) give $A = C'/r$. Thus, it turns out that $A = (k - rk')/2r^3$ and $B = (2k + rk')/4r$. Finally,

$$S_{ijk}(r) = \left(\frac{k - rk'}{2r^3}\right) r_i r_j r_k + \left(\frac{2k + rk'}{4r}\right)(r_i \delta_{jk} + r_j \delta_{ik}) - \frac{k}{2r}\delta_{ij} r_k \tag{9.45}$$

We easily show from (9.39) that the function k is antisymmetric: $k(r) = -k(-r)$ and thus $k(0) = 0$; in addition $k'(0) = 0$. This result may be shown as follows:

$$\begin{aligned} \left.\frac{\partial k}{\partial r}\right)_{r=0} &= \left\langle v_\ell^2(\mathbf{x}) \left(\frac{\partial}{\partial r} v_\ell(\mathbf{x} + \mathbf{r})\right)_{r=0} \right\rangle = \left\langle \left(v_\ell^2(\mathbf{x} + \mathbf{r}) \frac{\partial}{\partial r} v_\ell(\mathbf{x} + \mathbf{r})\right)_{r=0} \right\rangle \\ &= \frac{1}{3} \left\langle \left(\frac{\partial}{\partial r} v_\ell^3(\mathbf{x} + \mathbf{r})\right)_{r=0} \right\rangle = \frac{1}{3} \left\langle \left(\frac{\partial}{\partial x} v_\ell^3(\mathbf{x} + \mathbf{r})\right)_{r=0} \right\rangle = \frac{1}{3}\frac{\partial}{\partial x}\left\langle v_\ell^3 \right\rangle = 0 \end{aligned}$$

since $\left\langle v_\ell^3 \right\rangle$ is independent of $\mathbf{x}$ because of the homogeneity of turbulence.

9.4 Length Scales in Turbulent Flows

In order to characterize more completely turbulent flows, we need to precise the various spatial scales which control their dynamics.

9.4.1 Taylor and Integral Scales

The largest scale is the correlation length or *the integral scale*. It is defined by

$$\ell_0 = \int_0^\infty R(r)dr/R(0) \tag{9.46}$$

Using (9.35) and the fact that $\int_0^\infty \frac{\sin x}{x} dx = \pi/2$, we find the other following expression:

$$\ell_0 = \frac{\pi}{2} \frac{\int_0^\infty k^{-1} E(k)dk}{\int_0^\infty E(k)dk} \ . \tag{9.47}$$

It shows that the integral scale is the mean wavelength weighted by the spectral density of kinetic energy. This scales therefore points to the most energetic structures of a turbulent flow. We shall come back to it in the next section.

Another scale, also very useful to characterize turbulent flows, is the *Taylor scale*. It is defined by:

$$\ell_T = \sqrt{\frac{E}{Z}} = \sqrt{\frac{\langle v^2 \rangle}{\langle \omega^2 \rangle}} \tag{9.48}$$

or

$$\ell_T = \left(\frac{\int_0^\infty E(k)dk}{\int_0^\infty k^2 E(k)dk} \right)^{1/2} \tag{9.49}$$

Using (9.36), we see that this scale is related to the second derivative of the velocity autocorrelation since

$$\ell_T = \sqrt{-\frac{R(0)}{3R''(0)}}$$

The definition of this scale shows that it characterizes the velocity gradients: broadly speaking, this scale shows the mean size of vortices.

9.4.2 *The Dissipation Scale*

Let us first consider a chunk of fluid of unit mass, within a turbulent flow. Without any forcing, the turbulence would decay thanks to viscous dissipation. After some time, it would disappear altogether, the kinetic energy (of turbulence) being transformed into internal energy. In a steady state, turbulence is stationary because some energy is injected and compensates the losses by viscous dissipation. We shall denote by $\langle\varepsilon\rangle$ the power injected per unit mass into the turbulence (i.e. into the random like fluctuations of the flow). In a homogeneous and stationary turbulence, this quantity is a constant and because of the conservation of energy, this is also the power dissipated by unit mass.

If we observe that in the spectral space, the viscous force is proportional to $k^2\hat{v}_k$, we easily guess that the small scales (large k) are the scales where most of the dissipation occurs. Let us now assume that this dissipation comes from a single wavenumber k_D. Conservation of energy implies that

$$\langle\varepsilon\rangle \sim \nu k_D^2 \hat{v}^2(k_D) \tag{9.50}$$

for orders of magnitude. However, if $\hat{v}(k_D)$ is in the dissipative range, its associated Reynolds number is of order unity, thus $\hat{v}(k_D) \sim \nu k_D$. We easily derive from (9.50) that

$$k_D \sim \left(\frac{\langle\varepsilon\rangle}{\nu^3}\right)^{1/4} \tag{9.51}$$

With the wavenumber k_D, one usually associates the scale $\ell_D = 1/k_D$ called the *dissipation scale* or the *Kolmogorov scale*. This scale separates the spectrum into two domains: the one where viscosity dominates $\ell \ll \ell_D$ and the one where this force may be neglected $\ell \gg \ell_D$.

9.5 Universal Turbulence

After the long foregoing presentation of some kinematic aspects of turbulence, we shall now get closer to the real difficulties associated with turbulent flows, namely their dynamics. To get a first idea of it, we follow the work of Andrei Kolmogorov which was published in 1941. This pioneering work suggested for the first time the idea of a universal turbulence, which would be independent of the instabilities that maintain it. This ideal state has been investigated by Kolmogorov and his theory is often referred to as "K41", an acronym that we shall also use below.

9.5.1 Kolmogorov Theory

9.5.1.1 The Hypothesis

The basic idea of Kolmogorov is that there exist a universal state of turbulence that may be observed when we consider the flow in a box much smaller than the scales where the instabilities are working. In other words, a box following the mean flow and much smaller than the integral scale. Within this box, turbulence is characterized only by $\langle\varepsilon\rangle$ according to Kolmogorov who introduced this quantity. Kolmogorov also suggested that in this box the turbulence is (almost) homogeneous and isotropic and so that it should meet two hypothesis:

- **H1: First similarity hypothesis.** The structure functions for the velocity within an isotropic homogeneous turbulence just depend on $\langle\varepsilon\rangle$ and ν.
- **H2: Second similarity hypothesis.** If the distance between the points is large compared to the dissipation scale, then the structure functions just depend on $\langle\varepsilon\rangle$.

Kolmogorov concentrated on the structure functions because of the a priori idea that they are less sensitive to a large-scale flow hence to some non homogeneity.

A first consequence of these hypothesis is the existence of scaling laws when the distances are large compared to the dissipation scale. We shall come back to this when discussing the idea of intermittency.

9.5.1.2 The Kolmogorov Spectrum

We shall now derive the kinetic energy spectrum $E(k)$ under Kolmogorov's second hypothesis, namely when the effects of viscosity are negligible.

According to H2, $E(k)$ depends only on k and $\langle\varepsilon\rangle$. We thus need to build a quantity of the same dimension as $E(k)$ using k and $\langle\varepsilon\rangle$ only. Let us first observe that $\langle\varepsilon\rangle$ is a specific kinetic energy per unit time. Thus, if we use the velocity v_k typical of the scale $1/k$, dimensionally speaking

$$\langle\varepsilon\rangle \sim k v_k^3$$

However, still dimensionally, $v_k^2 \sim kE(k)$. Combining these two expressions, we find that

$$E(k) = \langle\varepsilon\rangle^{2/3} k^{-5/3} f(\langle\varepsilon\rangle, k)$$

where f is dimensionless. Moreover, the argument of f must be dimensionless too. But there isn't any combination of k and $\langle\varepsilon\rangle$ that is dimensionless. Therefore f is a constant C_K which is called the *Kolmogorov constant*. Scientists have tried very hard to determine the value of this constant, but this is a difficult task. Its value[3] seems to be close to 1.6.

Finally, the spectrum reads:

$$E(k) = C_K \langle \varepsilon \rangle^{2/3} k^{-5/3} \tag{9.52}$$

This is commonly called the *Kolmogorov spectrum*, even if its expression has first been given by Obukhov. The foregoing analysis is valid only for some range of scales: this range is known as the *inertial range*. These scales are much smaller than the integral scale but much larger than the dissipation one. The name "inertial range" comes from the idea that only inertial terms, like $(\mathbf{v}\cdot\nabla)\mathbf{v}$, and pressure terms are important in the dynamics of these scales.

The foregoing approach leads to a new view of turbulence through the evolution of the energy. Let us first observe that within the inertial range the kinetic energy is conserved: indeed, if we imagine a volume of fluid which would not exchange any mass with its surroundings and whose motion would be due to Fourier modes within the inertial range, for these modes viscous action is negligible and therefore the kinetic energy remains constant. On the contrary, the energy of modes with $k > k_D$ is rapidly transformed into heat by viscous friction. In a steady state, such a loss must be compensated. The energy is provided by the scales of the inertial range, however there is no energy source there. Thus, we must consider still larger scales, namely those where the forces driving the turbulence are working. These scales are generally in the domain $k \leq k_0$, also called the *injection range*.

Hence, the picture of the *turbulent cascade* is emerging: Energy injected in the large scales by the instabilities leading to the turbulent flow, is progressively carried through smaller and smaller scales until it reaches the dissipative range where it is transformed into heat. The physical picture behind this spectral dynamics is the repeated breaking of vortices. When the size of these structures is small enough, they are removed by viscosity. In Fig. 9.2 we show the kinetic energy spectrum in an ideal view and using real data. In Fig. 9.3 we illustrate the cascade process in the real space.

We should however be careful not to take this picture as an exact view of the reality. This is, unfortunately, only a partial view of it as we shall see below.

Let us now come back to the various scales that we introduced in 9.4 and let us plot them in Fig. 9.2a. Calculating the integral scale from (9.47) with the functions used to plot Fig. 9.2a, gives $k_0 = 1/\ell_0 \sim 0.6$ which is very close to the maximum value that we fixed at $k = 1$. As expected this scale is the one of the most energetic structures.

[3]First experimental values as those given by Monin and Yaglom (1975) are around 1.5. Recent measurements in the atmospheric boundary layer by Cheng et al. (2010) give 1.56. Numerical experiments have long given values around 2 (e.g. Vincent and Meneguzzi 1991), but recently it has been understood that the numerical resolution was an important issue. The latest results obtained with the very high resolution numerical simulations are getting closer to experimental values (Kaneda et al. 2003).

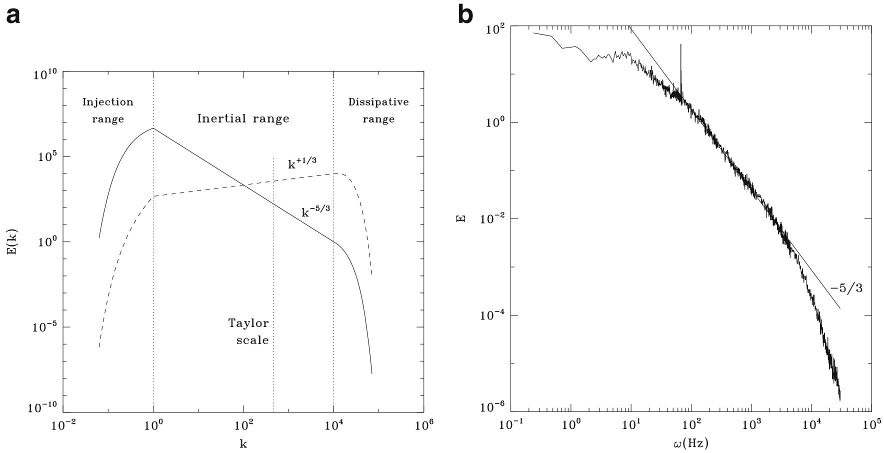

Fig. 9.2 (**a**) Idealized view of the different regions of the kinetic energy spectrum of a turbulent flow. The dashed line shows the enstrophy spectrum. (**b**) A kinetic energy spectrum derived from an experiment on turbulence with Helium. Note that the law $\omega^{-5/3}$ is visible on almost two decades (courtesy H. Willaime)

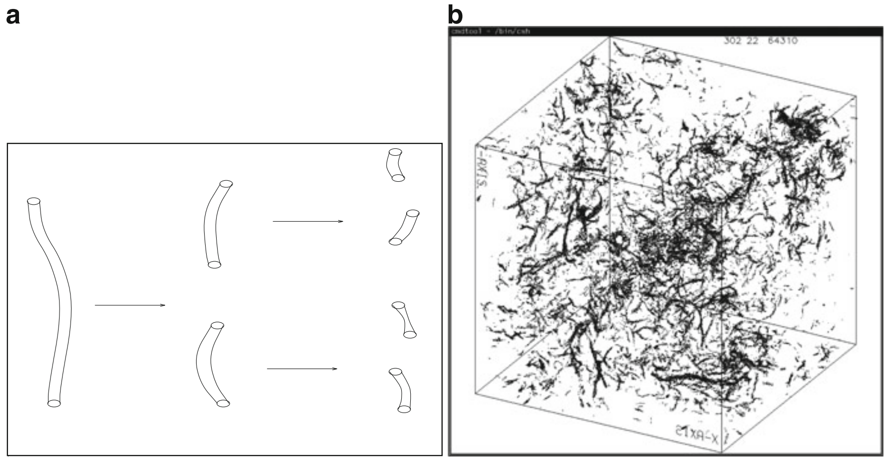

Fig. 9.3 (**a**) An illustration of the cascade of energy towards the small scales: vortices split into smaller and smaller pieces. (**b**) The vorticity field as computed by a direct numerical simulation of isotropic turbulence at Re$\sim$ 1000 from Vincent and Meneguzzi (1991)

Let us now estimate the Taylor scale. We derive its value from (9.49) assuming that the dissipation scale is much smaller than the integral one, namely $k_D \gg k_0$. We may show (see the exercises), that the order of magnitude of this scale is

$$\ell_T \sim \ell_0^{1/3} \ell_D^{2/3} \tag{9.53}$$

The Taylor scale is therefore a kind of geometric mean between the integral and dissipation scales with more weight to the latter. The wavenumber $k_T = 1/\ell_T$ is in the middle of the inertial range as shown in Fig. 9.2. This scale therefore characterizes more specifically the "Universal Turbulence". The Reynolds number associated with this scale is usually taken as

$$\mathrm{Re}_\lambda = \frac{v_0 \ell_T}{\nu} \tag{9.54}$$

Using (9.53) and $\langle \varepsilon \rangle \sim v_0^3/\ell_0$, we see that $\mathrm{Re}_\lambda = \sqrt{\mathrm{Re}_0}$ where Re_0 is the Reynolds number associated with the integral scale.

9.5.2 Dynamics in the Spectral Space

The foregoing discussion is essentially qualitative and we may wonder what kind of constraints are given by the equations of motion as far as the spectral quantities are concerned. To investigate this point let us write the Navier–Stokes equation and that of mass conservation in the spectral space. We have

$$\begin{cases} \partial_t \hat{v}_i + ik_j \widehat{v_i v_j} = -ik_i \hat{p} - \nu k^2 \hat{v}_i \\ k_i \hat{v}_i = 0 \end{cases} \tag{9.55}$$

where the hat is for the Fourier transform. We then project this equation on the plane perpendicular to $\mathbf{k}$ thanks to the projector tensor P_{ij} (see 9.27). Thus

$$(\partial_t + \nu k^2)\hat{v}_i = -iP_{ij}k_n \widehat{v_j v_n} = -\frac{i}{2} P_{ijn} \widehat{v_j v_n} \tag{9.56}$$

where we set $P_{ijn} = P_{ij}k_n + P_{in}k_j$.

These expressions show that the evolution of the Fourier component $\hat{v}_i(\mathbf{k})$ comes from the damping by viscosity on a time scale $1/(\nu k^2)$ and a forcing from all the components verifying:

$$\mathbf{p} + \mathbf{q} = \mathbf{k}, \quad \text{since} \quad \widehat{v_j v_n} = \int \hat{v}_j(\mathbf{p})\hat{v}_n(\mathbf{k} - \mathbf{p}) d^3\mathbf{p}$$
$$= \int \hat{v}_j(\mathbf{p})\hat{v}_n(\mathbf{q})\delta(\mathbf{p} + \mathbf{q} - \mathbf{k}) d^3\mathbf{p} d^3\mathbf{q}$$

These terms reflect a local interaction of Fourier modes when

$$\|\mathbf{p}\| \sim \|\mathbf{q}\| \sim \|\mathbf{k}\|$$

and a non-local interaction when

$$\|\mathbf{k}\| \ll \|\mathbf{p}\| \sim \|\mathbf{q}\|$$

Fig. 9.4 Interactions between Fourier modes

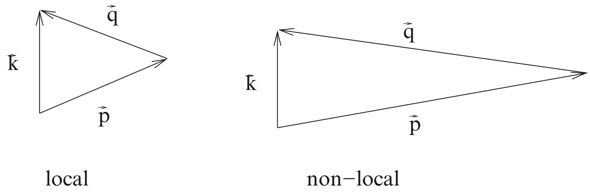

Figure 9.4 illustrates these two types of interactions.

Let us now focus on the evolution of the turbulence spectrum. We start from (9.56) and use the result of exercise 1. We easily show that

$$(\partial_t + 2\nu k^2)\phi_{ij}(\mathbf{k})\delta(\mathbf{k}-\mathbf{k}') = \frac{i}{2}\left\langle P_{ilm}\widehat{v_l v_m}^* \hat{v}'_j - P'_{jlm}\widehat{v_l v_m}'\hat{v}^*_i\right\rangle$$

Noting that the right-hand side is also proportional to $\delta(\mathbf{k}-\mathbf{k}')$, we find

$$(\partial_t + 2\nu k^2)\phi_{ii}(\mathbf{k}) = -\mathrm{Im}(P_{ilm}\left\langle \widehat{v_l v_m}^* \hat{v}_i\right\rangle)$$

which can be rewritten as

$$(\partial_t + 2\nu k^2)\phi_{ii}(\mathbf{k}) = -\mathrm{Im}\left[P_{ilm}\int \langle \hat{v}_i(\mathbf{k})\hat{v}_l(\mathbf{p})\hat{v}_m(\mathbf{q})\rangle\, \delta(\mathbf{k}+\mathbf{p}+\mathbf{q})dpdq\right] \tag{9.57}$$

The evolution of the spectral density of kinetic energy in the isotropic case is now driven by:

$$(\partial_t + 2\nu k^2)E(k,t) = T(k,t) \tag{9.58}$$

where $T(k,t)$ is called the transfer term. It comes from the nonlinear terms of the Navier–Stokes equation: it explicits the energy exchange between three Fourier modes when the wavenumbers of the triad are compatible ($\mathbf{k}+\mathbf{p}+\mathbf{q}=\mathbf{0}$).

We may wonder about the physical meaning of the transfer terms. Of course this is the spectral translation of the nonlinear interactions: more specifically, it expresses the mechanisms that allow one feature at a given scale to pump energy from structures at other scales. This mechanism is essentially non-local because it is intrinsically the result of instabilities. The mechanisms behind $T(k,t)$ are very complex: many instabilities, like the ones we analysed in Chap. 6, pump the energy from the large scales to the small ones. However, in turbulent flows the other way round is also possible: some large-scale flows may grow using the energy available in the small scales: this is a large-scale instability. When the turbulence is in a steady state, the transfer between scales is in both directions: towards the small scales and towards the large scales. Of course, the transfer to the small scales slightly dominates, so that the kinetic energy cascades on average from the large scales to the small scales with a rate equal to $\langle\varepsilon\rangle$.

9.5.3 The Dynamics in Real Space

9.5.3.1 Kármán–Howarth Equation

Some years before Kolmogorov proposed his new approach of turbulence, von Kármán and Howarth (1938) derived the first equation controlling the dynamics of homogeneous isotropic and symmetric turbulence. This equation relates the double and triple longitudinal correlation f and k. We now derive this equation and for that purpose we write the Navier–Stokes equation at two independent points:

$$\frac{\partial v_i}{\partial t} + \partial_k(v_k v_i) = -\partial_i p + \nu\Delta v_i$$
$$\frac{\partial v'_j}{\partial t} + \partial'_k(v'_k v'_j) = -\partial'_j p' + \nu\Delta' v'_j$$

where we simplified the expressions setting $\mathbf{v}' = \mathbf{v}(\mathbf{x}')$ and $\partial' = \partial/\partial x'$. We then multiply the first equation by v'_j and the second by v_i. We also note that $\partial_j v'_i = \partial'_j v_k = 0$. Summing the results and taking the average we finally get:

$$\frac{\partial Q_{ij}}{\partial t} + \partial_k(\left\langle v_i v'_j v'_k\right\rangle - \left\langle v_i v_k v'_j\right\rangle) = \partial_i\left\langle v'_j p\right\rangle - \partial_j\left\langle v_i p'\right\rangle + 2\nu\Delta Q_{ij}$$

where the space derivatives are taken with respect to $\mathbf{r}$. With the definition of S_{ijk} we have

$$\frac{\partial Q_{ij}}{\partial t} - \partial_k[S_{ikj}(\mathbf{r}) - S_{jki}(-\mathbf{r})] = 2\nu\Delta Q_{ij} + \partial_i\left\langle p(\mathbf{x})v_j(\mathbf{x}+\mathbf{r})\right\rangle - \partial_j\left\langle p(\mathbf{x})v_i(\mathbf{x}-\mathbf{r})\right\rangle$$

Using the antisymmetry of S_{ijk} and taking the trace of the equation we get:

$$\frac{\partial R}{\partial t} - \partial_k(S_{iki}) = 2\nu\Delta R \tag{9.59}$$

because $R = Q_{ii}/2$ and because pressure-velocity correlations disappear thanks to isotropy. However, from (9.45)

$$S_{iki} = \frac{1}{2r^4}\frac{\partial(r^4 k)}{\partial r}\, r_k$$

so that

$$\frac{\partial R}{\partial t} - \frac{1}{2r^2}\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial (r^4 k)}{\partial r}\right) = 2\nu\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right) \tag{9.60}$$

If we use longitudinal correlation rather than R, we can make the substitution

$$R = \frac{1}{2r^2}\frac{\partial r^3 f}{\partial r}\ , \tag{9.61}$$

Then the equation may be integrated and finally we get:

$$\frac{\partial f}{\partial t} - \frac{1}{r^4}\frac{\partial r^4 k}{\partial r} = 2\nu\left(\frac{\partial^2 f}{\partial r^2} + \frac{4}{r}\frac{\partial f}{\partial r}\right) \tag{9.62}$$

which is *the equation of Kármán–Howarth.*

When it is used at $r = 0$ this equation yields some additional informations. Indeed, we know that $f(0) = \left\langle v_l^2\right\rangle = \frac{1}{3}\left\langle v^2\right\rangle = \frac{2}{3}E_c$. However, at $r = 0$ (9.62) becomes

$$\frac{df(0)}{dt} = 2\nu\left(\frac{1}{r^4}\frac{\partial}{\partial r}r^4\frac{\partial f}{\partial r}\right)_{r=0} \tag{9.63}$$

which leads to

$$\frac{dE_c}{dt} = 15\nu f''(0)$$

The right-hand side represents the energy dissipation by viscosity, thus $-\langle\varepsilon\rangle$ by definition. Hence, we find that $\langle\varepsilon\rangle = -15\nu f''(0)$, but also that, using (9.36),

$$\langle\varepsilon\rangle = 2\nu Z = \nu\left\langle\omega^2\right\rangle \tag{9.64}$$

meaning that $\langle\varepsilon\rangle$ is directly proportional to enstrophy.

This latter equation gives a new interpretation of the Taylor scale. Indeed, we have:

$$\frac{dE_c}{dt} = -\langle\varepsilon\rangle = -\frac{\nu}{\ell_T^2}E_c$$

namely that the kinetic energy decreases on a time scale $\tau = \ell_T^2/\nu$. Thus, it is just like if turbulence was damped by viscosity but on an effective length-scale equal to the Taylor scale.

9.5.3.2 The Kolmogorov Equation

Kolmogorov manipulated furthermore Kármán–Howarth equation using the structure functions with longitudinal velocities, namely

$$S_2 = \left\langle (v_\ell(\mathbf{x}+\mathbf{r}) - v_\ell(\mathbf{x}))^2 \right\rangle \qquad \text{and} \qquad S_3 = \left\langle (v_\ell(\mathbf{x}+\mathbf{r}) - v_\ell(\mathbf{x}))^3 \right\rangle$$

These structure functions are easily expressed with f and k; indeed,

$$S_2 = 2(f(0) - f(\mathbf{r})) \qquad \text{and} \qquad S_3 = 6k(\mathbf{r})$$

If the turbulence is in a steady state, the viscous dissipation must be compensated by an energy source, the power of which is $\langle\varepsilon\rangle$. In freely decaying turbulence,

$$\frac{dE_c}{dt} = -\langle\varepsilon\rangle, \quad \text{while} \quad f(0) = \frac{2}{3}E_c$$

so that $df(0)/dt = -\frac{2}{3}\langle\varepsilon\rangle$. In a steady state, this loss is compensated by the same term with opposite sign. Hence,

$$\frac{df(0)}{dt} = 2\nu\left(\frac{1}{r^4}\frac{\partial}{\partial r}r^4\frac{\partial f}{\partial r}\right)_{r=0} + \frac{2}{3}\langle\varepsilon\rangle = 0$$

Since $\langle\varepsilon\rangle$ is a constant, we should find this term also in (9.62). The steady state version of the Karman–Howarth equation (9.62), is therefore:

$$-\frac{1}{r^4}\frac{\partial r^4 k}{\partial r} = 2\nu\left(\frac{\partial^2 f}{\partial r^2} + \frac{4}{r}\frac{\partial f}{\partial r}\right) + \frac{2}{3}\langle\varepsilon\rangle$$

After simple integrations and use of the structure functions instead of f and k, we find

$$4\langle\varepsilon\rangle + \frac{1}{2r^4}\frac{\partial(r^4 S_3)}{\partial r} = 6\nu\left(\frac{\partial^2 S_2}{\partial r^2} + \frac{4}{r}\frac{\partial S_2}{\partial r}\right)$$

which can be integrated after multiplication by r^4. We obtain

$$\frac{4}{5}\langle\varepsilon\rangle r + S_3 = 6\nu\frac{\partial S_2}{\partial r} \tag{9.65}$$

which is called the *Kolmogorov equation* This new equation is quite interesting since it shows that if $r \gg \ell_D$, namely if we are considering a length scale in the inertial range, then S_3 verifies the scaling law

$$S_3 = -\frac{4}{5}\langle\varepsilon\rangle r \tag{9.66}$$

called the "four-fifth law". This is a remarkable result as it is non-trivial and exact for universal turbulence (but see Frisch 1995, for a more thorough discussion).

The Log-Normal theory of Obukhov-Kolmogorov 1962

When we wrote the scaling laws (9.70) we only used two quantities: the scale r and the mean dissipation $\langle\varepsilon\rangle$. This lead us to the law $S_p \sim r^{p/3}\langle\varepsilon\rangle^{p/3}$. However, with the same dimensional arguments we could have written $S_p \sim r^{p/3}\langle\varepsilon^{p/3}\rangle$. But the two quantities $\langle\varepsilon\rangle^{p/3}$ and $\langle\varepsilon^{p/3}\rangle$ are not identical (except for $p = 3$) because dissipation is a fluctuating quantity. Landau was the first to point out this problem with the K41 theory. Hence, some years later, Obukhov and Kolmogorov (1962) proposed a modification of the K41 approach. This new model is now known as the *Log-normal theory*. This theory may be presented as follows.

First, OK62 define a dissipation ε_r, averaged over a ball of size r, namely

$$\varepsilon_r = \frac{1}{V}\int_{(V_r)} \varepsilon(\mathbf{x})dV$$

Obviously, $\langle\varepsilon_r\rangle^p$ is all the more different from $\langle\varepsilon_r^p\rangle$ that the fluctuations of ε_r are strong. However, these fluctuations increase when the volume decreases. Indeed, let us consider a flow with a very high Reynolds number. The velocity gradients may be very strong implying in some places very high values of viscous dissipation. Actually, we expect from the Kolmogorov spectrum that the fluctuations of dissipation are not bounded when the Reynolds number goes to infinity. OK62 thus proposed that the variance of the logarithm of ε_r is not bounded when L/r goes to infinity (L is a given large scale). They also assumed that this quantity obeys to a normal statistics (the probability density function is a gaussian). One may wonder why they considered the logarithm of ε_r ? This is because ε_r is a positive quantity which cannot follow a normal law, while the logarithm symmetrizes the points 0 and $+\infty$ by moving zero to $-\infty$. The normal law is symmetric with respect to the mean value, hence we may expect that it applies more precisely to the logarithm.[a] OK62 thus proposed this formulation of the variance, which completely defines, with the mean, a normal distribution,

$$\sigma_r^2 = A(x,t) + \mu \log(L/r) \qquad (9.67)$$

where μ is a supposed universal constant. Now we may wonder why a logarithm dependence has been chosen for the variance. Essentially, because power laws are expected for spectra or moments.[b]

For a log-normal law, one has

$$\left\langle\varepsilon_r^{p/3}\right\rangle = e^{p/3\langle\log\varepsilon_l\rangle + \frac{p^2\sigma_r^2}{18}} \qquad (9.68)$$

if we set $p = 3$ in this formula and if we use (9.67), we find $\langle\varepsilon_r\rangle = (L/r)^{\mu/2}e^{\langle\log\varepsilon_l\rangle + A/2}$, so that

$$\frac{\left\langle\varepsilon_r^{p/3}\right\rangle}{\langle\varepsilon_r\rangle^{p/3}} = C_p(x)\left(\frac{L}{r}\right)^{\mu p(p-3)/18}.$$

We can thus derive a new expression for the exponent ζ_p of the structure functions of order p:

$$\zeta_p = \frac{p}{3} - \mu p(p-3)/18 \qquad (9.69)$$

The first experimental measurements of ζ_p, obtained for small p's, gave $\mu \simeq 0.2$. Some years later, Arneodo et al. (1998) have shown that experimental data (obtained for $-10 \le p \le +10$) are well represented by a quadratic normal law with $\zeta_p = mp - \sigma^2 p^2/2$ with $m = 0.32$ and $\sigma^2 = 0.03$.

[a] However, this assumption is still approximate because there is no good reason that fluctuations towards small values are as probable as those towards high values.

[b] We should keep in mind that in 1962, the Kolmogorov spectrum had already been observed experimentally, and thus any new theory should reproduce this result.

9.5.4 Some Conclusions on Kolmogorov Theory

The foregoing discussion revealed to us some of the important properties of turbulence which we shall now summarize.

1. We first noticed that scales are important in a turbulent flow: the properties depend on the scale we are considering. Investigating the spectral side of turbulence, we could discriminate three different ranges of length scales: the integral, inertial and dissipative ranges.
2. We then understood that if a turbulent flow is very dependent of the instabilities at the integral scale, it may well be that within the inertial and dissipative ranges, turbulence reaches a universal state where (9.66) is certainly one of the first laws.
3. However, Kolmogorov's approach assumes rather strong hypothesis: homogeneity, isotropy, parity. Among the three assumptions, homogeneity is the strongest. If it is relaxed, there is some mean flow whose evolution is dictated by the transport properties of turbulence.
   These transport properties appear in the closure of averaged equations. We may have noticed that the Kármán–Howarth equation is not closed.
4. The Kolmogorov scenario "forgets" about the possible role of fluctuations of ε, which points to another side of turbulence, namely intermittency, to be discussed below.

9.6 Intermittency

9.6.1 Presentation

The intermittency of turbulence, which is sometimes called internal intermittency, is one of the ill-known sides of turbulence. We shall first present this phenomenon as it appears in the experiments.

Figure 9.5 shows a random function whose distribution function is gaussian, and its derivative. It also shows a plot of a record of the velocity of a turbulent flow as well as its derivative. The difference between these two random functions is quite clear: while we note that the gaussian random function and its derivative are rather similar, we see that the velocity and its derivative are quite different. In particular, the derivative of the velocity shows large amplitude fluctuations. Now, if we plot the probability density function of the velocity and the random function (Fig. 9.6), the difference between the gaussian random function and the velocity is even neater. The large amplitude events in a turbulent velocity field are more likely than if they were the results of a sum of random, uncorrelated events (which would lead to the gaussian distribution). We pinpoint here one of the true problem of turbulence: the phenomenon is really random in nature, but this chance is guided by the Navier–Stokes equation in a still obscure way.

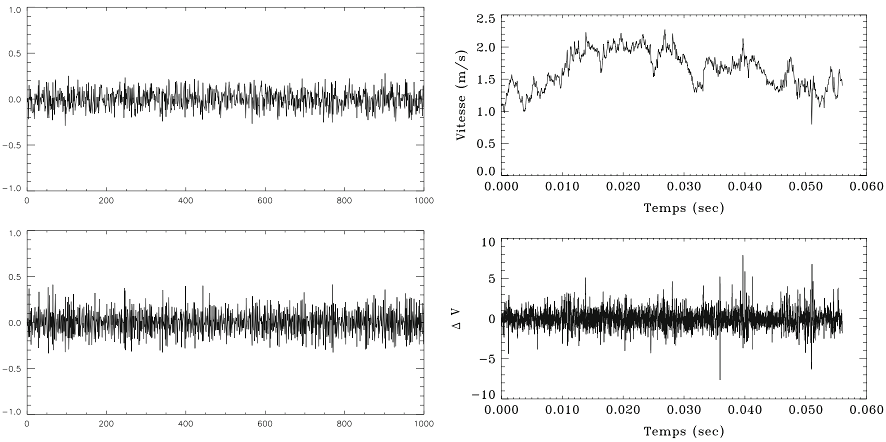

Fig. 9.5 On *left* (*top*) a random function with a gaussian distribution and its derivative (*bottom*). On *right* (*top*) the record of a turbulent velocity field and its derivative (*bottom*)

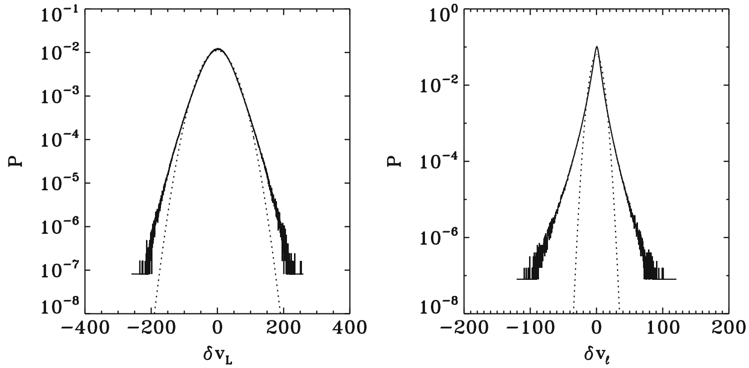

Fig. 9.6 The probability density function for the velocity difference between two points separated by a large scale L or by a small scale ℓ; these distributions have been derived using the data shown in Fig. 9.5. The normal, gaussian, distribution is shown as *dashed lines*

It is interesting to compare the random world of turbulence and the one of atoms and molecules within a gas. Indeed, the distribution of velocities of atoms or molecules of a gas in usual conditions is gaussian. This is a consequence of the fact that the velocity of a molecule at a given time results from the huge number of collisions that are statistically independent. Indeed, the central limit theorem states that the probability density function of a random variable that is the sum of an infinite number of independent random variable is a gaussian. Hence, the distribution of molecule velocities follows a normal statistics. We see that this statistical result is independent of the equation of motion of the molecules. In a turbulent flow the velocity at a given point is the combination of the influence of many vortices operating at various scales. In this respect many random processes contribute to the build up of the velocity field, but these processes are not independent: we know that long vortices tend to split thanks to instabilities and therefore correlations between scales are important.

However, intermittency does not only appear in the probability distributions; it also influences the scaling laws of structure functions. This is one aspect of the universal turbulence that we shall discuss now.

9.6.2 The Scaling Laws of Structure Functions

We already met the structure functions. These are important functions for many reasons: First they measure the relative velocity of two points of the flow: if the turbulence has a universal regime, such quantities will show it. Experimentally, it is difficult to create homogeneous and isotropic turbulence. The best approximation to this ideal situation is certainly grid turbulence,[4] which, in a frame comoving with the mean flow, is quasi-homogeneous and isotropic but is decaying with time. The structure functions eliminate the mean flow and are measurable quantities.

Using dimensional arguments, the structure function of order p may be written

$$S_p = \langle (v_\ell(\mathbf{x}+\mathbf{r}) - v_\ell(\mathbf{x}))^p \rangle = C_p(\langle\varepsilon\rangle r)^{p/3} \tag{9.70}$$

since the velocity scale is $(\langle\varepsilon\rangle r)^{1/3}$. The C_p's are constants which likely depend on the flow, except C_3 since

$$C_3 = -\frac{4}{5}$$

from (9.66). We should also note that C_2 is related to the Kolmogorov constant, and one may show, as an exercise, that when S_2 is proportional to $r^{2/3}$ then $E(k)$ is proportional to $k^{-5/3}$.

Let us now focus on variations of S_p with r. Setting

$$S_p \propto r^{\zeta_p}, \tag{9.71}$$

we see that the Kolmogorov theory implies that

$$\zeta_p = \frac{p}{3} \tag{9.72}$$

As shown by Fig. 9.7, experiments show a clear deviation to this suite of exponents with respect to the Kolmogorov one. This deviation is all the more marked that p is

[4]This is the turbulence which appears in the wake of a grid. It is homogeneous in the directions parallel to the grid

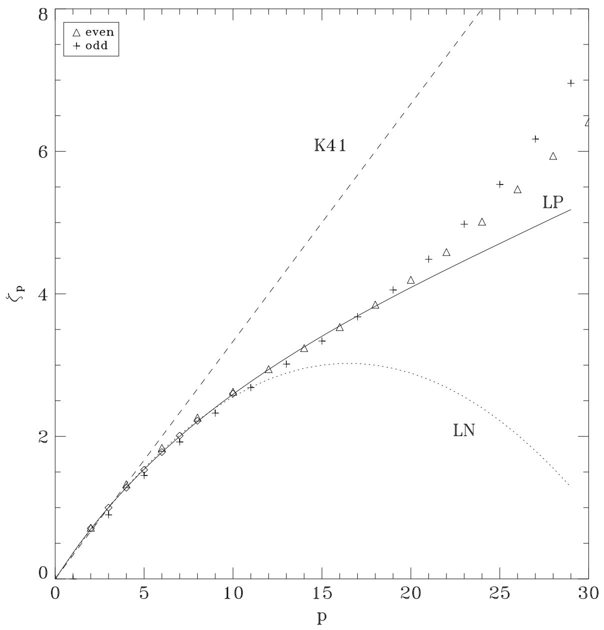

Fig. 9.7 The exponents ζ_p from various theories compared to some experimental data. Pluses and triangles are from a numerical simulation of Vincent and Meneguzzi (1991) and diamonds are from Benzi et al. (1993). The good fit of the Log-Poisson law (LP) is now understood as an effect of the small (not large!) value of the Reynolds number at Taylor scale used by the numerical simulation of Vincent and Meneguzzi (1991). Convergence to the Log-normal law would appear at very much higher Reynolds numbers. The straight line K41, is from the Kolmogorov theory and the LN curve shows the log-normal law with $\mu = 0.2$

high. However, large orders are sensitive to the wings of the probability distribution, that is to rare events. They are thus sensitive to the large amplitude events typical of the intermittency. The absence of intermittency in the K41 theory was soon noticed by Landau. Kolmogorov and Obukhov then proposed a modification of this theory, which is now known as the *Log-Normal Theory* (see the box). Unfortunately, this theory raised new questions and new theories have been developed (see the Log-Poisson box).

9.6.2.1 Two Properties of the Exponents

The exponents suite ζ_p verifies two general conditions:

$$\frac{d^2\zeta_p}{dp^2} \leq 0 \qquad \text{and} \qquad \zeta_{2p+2} \geq \zeta_{2p} \tag{9.73}$$

The first one comes from a Schwartz inequality verified by random variables. If A and B are two random variables, then

$$\langle AB\rangle \leq \left\langle A^2\right\rangle^{1/2}\left\langle B^2\right\rangle^{1/2}$$

taking $A = (v_\ell(\mathbf{x}+\mathbf{r}) - v_\ell(\mathbf{x}))^p$ and $B = (v_\ell(\mathbf{x}+\mathbf{r}) - v_\ell(\mathbf{x}))^q$, we get

$$S_{p+q} \leq \sqrt{S_{2p}S_{2q}}$$

If $S_p = A_p r^{\zeta_p}$, then

$$A_{p+q}r^{\zeta_{p+q}} \leq \sqrt{A_{2p}A_{2q}}\,r^{(\zeta_{2p}+\zeta_{2q})/2} \qquad \forall r \in \text{Inertial range} \tag{9.74}$$

The Log-Poisson Theory

Making more precise determinations of the exponents of structure functions has shown that neither the Kolmogorov theory, nor its Log-normal improvement could explain the variations of the ζ's with the order p. In 1994, She & Lévêque, She & Waymire and Dubrulle proposed a new approach which seemed to square much better with the experimental results available at the time (see Fig. 9.7).

This new approach was based on three hypothesis:

i. The structure function of order p verifies the scaling law:

$$S_p \sim r^{p/3}\left\langle \varepsilon_r^{p/3}\right\rangle$$

ii. The moments of the pdf of the energy dissipation obey the induction relation:

$$\frac{\left\langle \varepsilon_r^{p+1}\right\rangle}{\varepsilon_r^{\infty}\left\langle \varepsilon_r^{p}\right\rangle} = A_p\left(\frac{\left\langle \varepsilon_r^{p}\right\rangle}{\varepsilon_r^{\infty}\left\langle \varepsilon_r^{p-1}\right\rangle}\right)^{\beta}, \qquad \text{and} \qquad 0 < \beta < 1 \tag{9.75}$$

where A_p are constants and $\varepsilon_r^{\infty} = \lim_{p\to\infty}\left\langle \varepsilon_r^{p+1}\right\rangle / \left\langle \varepsilon_r^{p}\right\rangle$. We shall see that ε_r^{∞} is a quantity specific to the most intermittent structures. The relation (9.75) could be a hidden symmetry of the Navier–Stokes equation.

iii. When $r \longrightarrow 0$, $\varepsilon_r^{\infty} \sim r^{-2/3}$.

If we assume that $\left\langle \varepsilon_r^{p}\right\rangle$ verifies the scaling law $\left\langle \varepsilon_r^{p}\right\rangle \sim r^{\tau_p}$, then, it turns out from the definitions that $\zeta_p = p/3 + \tau_{p/3}$. the exponent τ_p measures the distance to the Kolmogorov law. Using its definition and the second hypothesis, we find the new relation

$$\tau_{p+2} - (1+\beta)\tau_{p+1} + \beta\tau_p + \frac{2}{3}(1-\beta) = 0$$

It is then convenient to set $\tau_p = -\frac{2}{3}p + 2 + f_p$. Finally,

$$f_{p+2} - f_{p+1} = \beta(f_{p+1} - f_p)$$

which is easily solved as

$$f_p = f_0 + A\left(\frac{1-\beta^p}{1-\beta}\right)$$

The initial conditions of this suite are given by the initial conditions on τ_p. By construction $\tau_1 = 0$ and we assume that $\tau_0 = 0$. This latter conditions is equivalent to the assumption that the volume of dissipative structures remains finite as viscosity tends to zero. We finally find

$$\zeta_p = \frac{p}{3} + \frac{2}{3}\left[\frac{1-\beta^{p/3}}{1-\beta} - \frac{p}{3}\right] \tag{9.76}$$

The curve which fits the experimental data so well is obtained for $\beta = 2/3$.[a] What is the meaning of this new exponent? Obviously, it characterizes the degree of intermittency of viscous dissipation. If $\beta \to 1$ then $\zeta \to p/3$: we find K41 again.

Dubrulle (1994) has shown that the second hypothesis (ii) could be inferred if the pdf of $\varepsilon_r/\varepsilon_r^\infty$ was assumed to be the convolution of a Log-Poisson law with another undetermined law. The hidden symmetry underlying (9.75) is therefore not very stringent. The following work of Arneodo et al. (1998) has shown that experimental data at very high Reynolds numbers ($\mathrm{Re}_\lambda \gtrsim 2000$) contradicted the Log-Poisson theory in favor of the Log-Normal approach. It seems that the Log-Poisson theory is more appropriate for $\mathrm{Re}_\lambda \lesssim 800$, a range of Reynolds numbers where the Log-Normal theory does not give very good results. Today (2013), it is believed that the Log-Normal theory applies when $\mathrm{Re}_\lambda \to \infty$, namely only asymptotically.

[a] We saw that the exponent ζ_2 was related to exponent of the energy density spectrum. The change implied by this new theory compared to the Kolmogorov one is very small: this exponent is now: $-\frac{5}{3} - 0.03$.

This inequality implies that

- if $r \gg 1$ then (9.74) is true only if $\zeta_{p+q} \le (\zeta_{2p} + \zeta_{2q})/2$, that is to say if the function $\zeta(p)$ is concave.[5]
- if $r \ll 1$ then (9.74) is true only if $\zeta_{p+q} \ge (\zeta_{2p} + \zeta_{2q})/2$ that is to say if the function $\zeta(p)$ is convex.

Experiments readily show that the suite $\zeta(p)$ is convex and therefore the second case is the right one. The relevant scale in the inertial domain is such that everywhere $r < 1$. We should thus take the integral scale as the unity.

The second condition, which demands that the suite of exponents with the same parity is non-decreasing was obtained by Frisch (1991). It comes from the fact that the velocity is bounded. Indeed, if V_{max} is that bounding value, then

$$S_{2p+2} = \left\langle (v_\ell(\mathbf{x}+\mathbf{r}) - v_\ell(\mathbf{x}))^{2p+2}\right\rangle \le \left\langle (v_\ell(\mathbf{x}+\mathbf{r}) - v_\ell(\mathbf{x}))^{2p}\right\rangle 4V_{max}^2 = 4V_{max}^2 S_{2p}$$

$$\Longrightarrow \quad A_{2p+2} r^{\zeta_{2p+2}} \le 4V_{max}^2 A_{2p} r^{\zeta_{2p}}$$

[5] A function f is concave, if the following inequality $f[(x+y)/2] \le (f(x)+f(y))/2$ is verified. For a continuous and derivable function, this inequality is equivalent to $f''(x) \ge 0$.

Since this inequality is valid for all $r = \|\mathbf{x} - \mathbf{x}'\|/\ell_0 \leq 1$, exponents naturally verify

$$\zeta_{2p+2} \geq \zeta_{2p}. \tag{9.77}$$

9.7 Theories for the Closure of Spectral Equations

Until now, all the dynamical equations of the mean fields have been left "open". Those written in the spectral space like (9.57) or those written in the real space like (9.62). Closing these equations is equivalent to expressing the third order moments as a function of those of lower order.

Several theoretical approaches have been devised to close the equations in the spectral space. Here, we shall present the main ideas of these approaches and refer the reader to the specialized textbooks (Lesieur 1990; Leslie 1973; McComb 1990) for more details.

9.7.1 The EDQNM Theory

EDQNM means "Eddy-Damped Quasi-Normal Markovian" which means that the statistics are assumed to be quasi-normal (close to the gaussian laws), Markovian (there is no memory effect), and Damped (some terms are purposely damped). This is probably one of the most popular closure in the spectral space. It was elaborated in the sixties and one of its conceptors, Steven Orszag, has written a masterful synthesis in the Les Houches Lectures of 1973.

The fundamentals of this approach are the followings: we need to get a closure of (9.57), which means that we have to relate the third order moments to the second order ones. However, we know that the evolution of the third order moments depends on those of the fourth order. At this point, the first hypothesis of quasi-normality interrupts the chain of equations. It is assumed indeed, that the statistics of the Fourier components is quasi-normal and hence obey to the Gaussian law. This law has the property that fourth order moments can be expressed with the second order ones. Thus, no hypothesis is made on the third order moments.

The quasi-normality hypothesis is simple: we just neglect the cumulants of fourth order. Unfortunately, this simplification has a disastrous consequence: the kinetic energy spectrum may become negative! The cure of that is to avoid the complete neglect of fourth order cumulant and to replace them by a damping term; hence the Eddy-Damped. This improved very much the theory, but still did not guarantee the positiveness of the energy. The Markovian constraint was then added, inferring that the turbulence has no memory effects.[6] One can then show that if the energy spectrum is positive at one time, then it is positive at all later times.

The EDQNM theory therefore simplifies turbulence on two crucial aspects:

- One assumes that the fourth order cumulants are damping terms for the third order correlations.
- There is no memory effect in the evolution of the spectral quantities.

This approach is interesting since it allows us to compute the evolution of the various spectral quantities. Hence, the way the Kolmogorov spectrum forms from some given initial conditions can be studied (with no intermittency of course!), and the relative simplicity of the method allows some generalization to more complex situation like helical turbulence, or turbulence with a background rotation.

9.7.2 The DIA

DIA means "Direct Interaction Approximation". This is another way of attacking turbulence theory which was developed by Kraichnan at the beginning of the sixties. It relies on a rather severe simplification of reality, which is a drawback, but it is self-consistent. Nevertheless, it allowed the scientists who investigated its consequences, to understand some important points for the theory of turbulence: for instance, the fact that the Kolmogorov spectrum is related to the invariance of the theory in random galilean transform. The book of Leslie (1973) gives a detailed description of this theory.

9.7.3 The Renormalization Group Approach

To end this short review of the closure theories, we should mention that of the renormalization group, which was inspired by the technics of statistical physics in the study of critical phenomena.

Let us assume that we can represent a turbulent flow by a discrete set of Fourier modes k-bounded from above by k_0 which is in the dissipative range ($k_0 \sim k_D$).

Now we cut the spectral domain in two parts by introducing a wavenumber k_1 slightly smaller than k_0 and we consider the fluid motions associated with the spectral domain $k_1 < k < k_0$. Thus we are considering fluid motions at a scale slightly larger than $1/k_0$. Since this range is in the dissipative range, we may linearize Navier–Stokes equation and solve for the evolution of these modes as a function of those in $0 < k \le k_1$. Now, the evolution of the modes in $0 < k \le k_1$

[6]Markovian processes are such that the probability of an event does not depend on the history of the process.

is also a function (but nonlinear) of those of the band $k_1 < k < k_0$; using the formal expression of the modes $k_1 < k < k_0$ as a function of the modes $0 < k \leq k_1$, we can derive an equation where only the mode of the band $0 < k \leq k_1$ intervene. Then the process can be iterated by replacing k_1 by a slightly smaller k_2. Progressively, the spectral band of the small scales is eliminated; at each iteration the viscosity is "renormalized", since the elimination of a range increases the dissipation of the remaining range.

The method's principle is quite simple, but its setting out is extremely difficult. The reader is referred to the textbook of McComb (1990) for a more thorough presentation of this approach.

9.8 Inhomogeneous Turbulence

In the foregoing sections we focused on the homogeneous turbulence case. This allowed us to be more familiar with the numerous concepts and problems that arise when studying a turbulent flow. Of course, turbulence in real flows is far from homogeneous and it is time now to make the jump in this new jungle...

To fix ideas, we consider the turbulent flow of an incompressible fluid that is in a statistically steady state. We rewrite the equations of the mean quantities (9.7) and (9.8):

$$\rho \partial_j \langle v_i \rangle \langle v_j \rangle + \partial_j R_{ij} = -\partial_i \langle p \rangle + \mu \Delta \langle v_i \rangle \qquad \text{and} \qquad \partial_i \langle v_i \rangle = 0$$

Contrary to the homogeneous case, the Reynolds stress tensor $R_{ij} = \langle \rho v'_i v'_j \rangle$ is no longer constant. We need to find a way to relate it to the mean flow $\langle \mathbf{v} \rangle$. The methods are called closure models on the Reynolds tensor. These models are said to be at zero, one or two equations, according to the number of equations that are solved simultaneously with the evolution of the mean velocity. They might for instance prescribe the evolution of the turbulent kinetic energy or the turbulent dissipation. We shall also say a word about models using a closure on the second order moments, where the evolution of all the components of the Reynolds tensor is computed.

9.8.1 A Short Review of the Closure Models

9.8.1.1 Models with Algebraic Prescriptions: Turbulent Viscosity

Facing the problem of the expression of the Reynolds tensor as a function of $\langle \mathbf{v} \rangle$, we may try to adopt the same reasoning that we used to determine the expression of the viscous stress, assuming that the role of small-scale turbulence is similar to that of the molecules of a Newtonian gas. Small scale turbulence is therefore assumed to diffuse momentum, heat, etc. Thus we write

$$-R_{ij} = -p_{turb}\delta_{ij} + \rho\nu_{turb}(\partial_i \langle v_j \rangle + \partial_j \langle v_i \rangle) \tag{9.78}$$

This simple closure is due to Joseph Boussinesq who introduced the idea of a turbulent viscosity as soon as 1877. In the foregoing expression, the turbulent pressure, p_{turb} can be determined after the velocity field when the fluid is incompressible. Thus, in this case, the crucial part of the model is the "viscous" shape of the tensor and the expression of the viscosity. We shall present two methods which are rather popular for the determination of the turbulent viscosity: the mixing-length theory which was devised by Prandtl, and the Smagorinsky approach devised in 1963.

Prandtl proposed (Prandtl 1925) that turbulent viscosity be the result of momentum transport by fluid elements with a velocity typical of the turbulent fluctuations. Namely, the fluid motion of velocity $\sqrt{\langle v'^2 \rangle}$ at a scale ℓ_M, the mixing length, are the engine of the turbulent diffusion. When the fluid elements have run this length, they vanish, releasing the quantities they carry. Of course this mixing-length is unknown and should be evaluated for every problem.

For a plane-parallel shear flow, Prandtl proposed that $\sqrt{\langle v'^2 \rangle} = \ell_M \left| \frac{d\langle v_x \rangle}{dy} \right|$ so that $\nu_T = \ell_M^2 \left| \frac{d\langle v_x \rangle}{dy} \right|$. This kind of approximation is unfortunately not general since the turbulent viscosity vanishes where the mean velocity gradient vanishes. This is obviously not the case for a turbulent jet: on its axis turbulent diffusion is certainly not vanishing. However, this assumption leads to very acceptable results as far as wall-turbulence is concerned.

Using the same concept, Smagorinsky (1963) proposed to model turbulent viscosity by a formula like

$$\nu_T = \Delta^2 \sqrt{\langle c_{ij} \rangle \langle c_{ij} \rangle}$$

where Δ is a length scale to be precised. We may note the similarity with the Prandtl approach. However, the idea of Smagorinsky is less ambitious: This expression is not meant to be used to determined a mean flow, but just to represent subgrid motions in a numerical simulation. Indeed, in numerical simulations of a turbulent flow, the small scales are generally not computed because of the implied computational cost. At the grid scale the Reynolds number is still large, and the effects of the dismissed scales have to be modeled, namely replaced by something. This is the role of subgrid models. The Smagorinsky model is one such models. Therefore, the length scale Δ is taken as the smallest resolved length (usually the mesh size).[7]

[7]Let us mention that usually subgrid scale models are not categorized in models of turbulence since they give a local prescription that can be used only in numerical simulation. However, their similarity with the mean-field approach is strong enough that we discuss them here.

9.8.1.2 The K-ε Model: A Model with Two Equations

We shall leave aside the models using just one additional equation (like the ones of turbulent kinetic energy), which are no longer used, and focus on one using two equations like the celebrated K-ε model.

The K-ε model was proposed by Launder and Spalding (1972). The assumption is that the Reynolds tensor is a function of both the large scale shear $\langle c_{ij}\rangle = \partial_i \langle v_j\rangle + \partial_j \langle v_i\rangle$ and the local strength of turbulence characterized by the turbulent kinetic energy K and the viscous dissipation ε (both taken per unit mass). This dependence is similar as (9.78), namely

$$-\left\langle v'_i v'_j\right\rangle = -\frac{2}{3}K\delta_{ij} + \nu_T\left(\partial_i \langle v_j\rangle + \partial_j \langle v_i\rangle\right) \tag{9.79}$$

where $\nu_T = c_\nu K^2/\varepsilon$. This expression reveals the assumptions of this model: first the turbulent pressure depends only on the turbulent kinetic energy and is equal to $\frac{2}{3}K$ while the turbulent viscosity is determined by both the kinetic energy and thus the viscous dissipation. c_ν is a dimensionless coefficient which is calibrated with experiments (one usually takes $c_\nu \sim 0.09$).

In such a model, the turbulent kinetic energy and the turbulent dissipation play a crucial role but need to be determined. The K-ε model proposes to compute them using equations that model their evolution. Hence, we write:

$$\begin{cases} \frac{\partial K}{\partial t} + \langle\mathbf{v}\rangle\cdot\nabla K = -\varepsilon + \frac{\nu_T}{2}\langle c_{ij}\rangle\langle c_{ij}\rangle + \nabla\cdot(\nu_T\nabla K) \\ \frac{\partial\varepsilon}{\partial t} + \langle\mathbf{v}\rangle\cdot\nabla\varepsilon = -c_2\varepsilon^2/K + \frac{c_1 K}{2}\langle c_{ij}\rangle\langle c_{ij}\rangle + \nabla\cdot(\nu_\varepsilon\nabla\varepsilon) \end{cases} \tag{9.80}$$

These equations come from the ones verified by the velocity fluctuations. Third order correlations or velocity-pressure correlations are then approximated to close the system.

The equation of velocity fluctuations is derived by combining (9.5) and (9.7):

$$\frac{\partial v'_i}{\partial t} + \langle v_j\rangle\partial_j v'_i + v'_j\partial_j\langle v_i\rangle + v'_j\partial_j v'_i - \left\langle v'_j\partial_j v'_i\right\rangle = -\frac{1}{\rho}\partial_i P' + \nu\partial_j\partial_j v'_i \tag{9.81}$$

Taking the dot product with v'_i and averaging, we find the equation governing the evolution of K:

$$\frac{\partial K}{\partial t} + \langle v_j\rangle\partial_j K + r_{ij}\partial_j\langle v_i\rangle + \overbrace{\left\langle v'_i v'_j\partial_j v'_i\right\rangle}^{\mathrm{I}} = -\partial_i\overbrace{\left\langle P' v'_i\right\rangle}^{\mathrm{II}} + \nu\overbrace{\left\langle v'_i\Delta v'_i\right\rangle}^{\mathrm{III}} \tag{9.82}$$

where $r_{ij} = R_{ij}/\rho$. In this equation, the three terms (I), (II) and (III) need to be modeled. The first of them can be rewritten $\left\langle v'_j\partial_j v'^2/2\right\rangle$: this is the advection of kinetic energy by the fluctuations of the velocity. Following an analogy with a purely diffusive process, the K-ε model assumes that this term can be represented by a turbulent diffusion, namely:

$$\left\langle v'_j \partial_j v'^2/2 \right\rangle = -\nabla \cdot (\nu_T \nabla K)$$

where the turbulent viscosity ν_T remains to be determined. If the turbulence is locally isotropic, we may neglect the correlations with pressure: $\left\langle P' v'_i \right\rangle \sim 0$. Similarly, we can also rewrite $\nu \left\langle v'_i \Delta v'_i \right\rangle = \partial_j \left\langle v'_i \sigma'_{ij} \right\rangle - \varepsilon$ and, assuming again local isotropy, $\left\langle v'_i \sigma'_{ij} \right\rangle \sim 0$, we find that the power of the viscous force is just the opposite of the viscous dissipation, as expected.[8] Finally, K verifies the following equation:

$$\frac{\partial K}{\partial t} + \left\langle v_j \right\rangle \partial_j K + r_{ij} \partial_j \left\langle v_i \right\rangle = \nabla \cdot (\nu_T \nabla K) - \varepsilon \tag{9.83}$$

Using (9.79), we find again the first equation of (9.80).

The equation governing the evolution of ε is much more difficult to derive and we leave the details of the derivation in an appendix of this chapter. It leads to the following expression:

$$\frac{\partial \varepsilon}{\partial t} + \langle v_k \rangle \partial_k \varepsilon + \left\langle c_{jk} \right\rangle \overbrace{\left\langle \nu c'_{ij} c'_{ik} \right\rangle}^{\text{I}} + \langle \Omega_{ik} \rangle \overbrace{\left\langle \nu c'_{ij} \Omega'_{kj} \right\rangle}^{\text{II}} + \overbrace{\left\langle \nu c'_{ij} v'_k \right\rangle}^{\text{III}} \partial_k \left\langle c_{ij} \right\rangle$$
$$+ \overbrace{\left\langle v'_k \partial_k \nu c'^2_{ij}/2 \right\rangle}^{\text{IV}} + 2\nu \overbrace{\left\langle c'_{ij} (\partial_j v'_k)(\partial_k v'_i) \right\rangle}^{\text{V}} = -2\nu \overbrace{\left\langle c'_{ij} \partial_i \partial_j p' \right\rangle}^{\text{VI}} + \nu^2 \overbrace{\left\langle c'_{ij} \Delta c'_{ij} \right\rangle}^{\text{VII}}$$

The seven numbered terms need a model. A first simplification is to assume that the turbulence is locally homogeneous, isotropic and parity invariant. This latter property with homogeneity eliminates (III) while isotropy zeroes terms (II) and (VI).[9] Hence, terms (I), (IV), (V) and (VII) need a more detailed model.

Term (I) is a second order tensor. It may be related to the large-scale shear $\left\langle c_{jk} \right\rangle$; making a Taylor expansion for weak shears (just like we did when dealing with Newtonian fluids), we get

$$\left\langle c_{jk} \right\rangle \left\langle \nu c'_{ij} c'_{ik} \right\rangle \sim -c_1 K \left\langle c_{jk} \right\rangle^2$$

where c_1 is a dimensionless constant. The turbulent kinetic energy is the quantity which characterizes turbulence.[10]

[8] We noted that $\sigma'_{ij} = \nu(\partial_j v'_i + \partial_i v'_j)$.

[9] See appendix for the demonstration.

[10] Indeed, the local properties of turbulence can only be, with this model, characterized by the two scalars K and ε. In the present case K is the only one dimensionly correct.

The fourth term (IV) is modeled is a similar way as the first term (I) of the K-equation, namely with a turbulent diffusion. We thus write

$$\left\langle v'_k \partial_k \nu c'^2_{ij}/2 \right\rangle \sim -\nabla \cdot (\nu_\varepsilon \nabla \varepsilon)$$

where ν_ε is a new turbulent diffusion, but for dissipation.

Finally, we are left with terms (V) and (VII). These two terms are slightly special since they are the only ones to remain in an isotropic homogeneous steady turbulence. In this latter case they compensate exactly. Hence, it is not necessary to separate their modelling, since their difference is the only important quantity. In the K-ε model these two terms are proportional to $\langle\varepsilon\rangle^2/K$, namely

$$\text{V} - \text{VII} \sim c_2 \frac{\varepsilon^2}{K}$$

Finally, these equations need to be completed by the expression of the turbulent diffusions ν_T and ν_ε. Their expression is obtained from the only scalar that has the same dimension as a diffusivity. This leads to the expressions

$$\nu_T \sim c_\nu \frac{K^2}{\varepsilon} \qquad \text{and} \qquad \nu_\varepsilon \sim c_\varepsilon \frac{K^2}{\varepsilon}$$

As before, the non-dimensional coefficients are calibrated with experiments and usually the following values are adopted:

$$c_\nu = 0.09, \quad c_\varepsilon = 0.07, \quad c_1 = 0.126, \quad c_2 = 1.92$$

The K-ε model tries to establish a relation between inhomogeneous turbulence and universal turbulence by assuming that, locally, the fluctuations of the mean flow are universal. Moreover, this local turbulence is assumed to behave like a Newtonian fluid with a variable viscosity.

These hypothesis are obviously very strong and one may wonder whether the turbulent fluid can behave as a Newtonian fluid even with a variable viscosity. This would mean a separation of scales that is not observed, even approximately. The local homogeneity also implicitly assumes a separation of the spatial scales.

In addition to these questions about the physical validity of the model, some other problems on the internal consistency of the model arise. For instance, the model allows a computation of the kinetic energy and viscous dissipation. These two quantities are positive and their evolution should preserve this positivity. Presently, there is no general proof that this is indeed the case. A few demonstrations, applying to some restricted cases and showing that this is true, may reassure us.

9.8.1.3 Second Order Closure Models

As we mentioned it previously, a two-equations model like the K-ε one implies a very restrictive form to the Reynolds tensor. It is assumed to be like that of a Newtonian fluid even if its viscosity is not locally determined. However, the turbulent fluid has no reason to be isotropic, and anisotropy is likely not a local function as well. It may result from the past time evolution of a fluid element (memory effect) or from distant interactions like those coming through the pressure. It therefore seems simpler, if we wish to get a more realistic description, to directly compute the evolution of R_{ij}, through equations like:

$$\frac{DR_{ij}}{Dt} = \cdots$$

These equations of course introduce the third order correlations, which need a new modeling. Even if this modelling is coarse, it is hoped that it will give realistic values of the R_{ij}, just like the turbulent viscosity model is able to give realistic mean flows in some cases. Hence, if the R_{ij} are better, the mean flow may be much better. Comparison of the results of these models with experiments seem to comfort this hope.

9.8.2 Examples: Turbulent Jets and Turbulent Plumes

We end this section with the analysis of two very common turbulent flows: those of jets and plumes. As shown in Fig. 9.8, these flows have a conical shape outside of

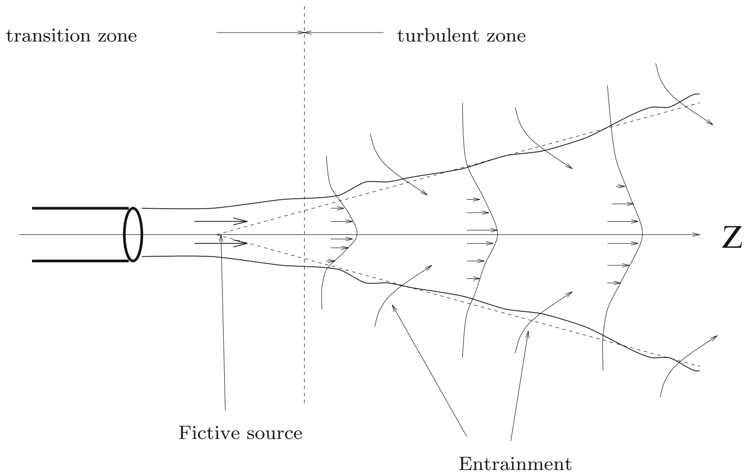

Fig. 9.8 A schematic view of a turbulent jet. The transition zone is the region where the shear instabilities give birth to the turbulence

which the turbulence is very low or absent. We shall see that this property, comes from the self-similarity of the solutions. Self-similarity is likely deeply rooted in turbulent flows.

Jets, plumes and wakes are often called free shear flows. Indeed, turbulence results from an imposed strong shear, actually a shear layer, which is, as we saw in Chap. 6, very unstable. The development of turbulence entrains the outer fluid inside the jet and the jet broadens as it progresses.

Let us assume that the fluid flow is self-similar, so that we may write the velocity field as:

$$\begin{cases} \langle v_r \rangle = V(z) g(r/b(z)) \\ \langle v_z \rangle = V(z) f(r/b(z)) \end{cases} \tag{9.84}$$

where we additionally assumed the jet axial symmetry. The dependence of the solutions with respect to $\xi = r/b(z)$, insures the similarity of the velocity profiles for all z. This profile has always the same shape, given by $f(\xi)$ for v_z, but its real width varies with the distance to the source z. Let us now assume incompressibility so that mass conservation implies:

$$\frac{\partial \langle v_r \rangle}{\partial r} + \frac{\langle v_r \rangle}{r} + \frac{\partial \langle v_z \rangle}{\partial z} = 0 \quad \Longleftrightarrow \quad \frac{1}{\xi}\frac{d(\xi g)}{d\xi} - b'(z)\xi f'(\xi) + \frac{V'(z)b(z)}{V(z)} f(\xi) = 0$$

where the prime indicates a derivative. The existence of solutions like (9.84) implies that the variables can be separated. This implies that $b'(z)$ and $B = V'(z)b(z)/V(z)$ are constant. This means that the width of the jet grows linearly. Then, noting that $V'(z)/V(z) \propto 1/z$ (taking the origin of z where b is zero), then $V(z)$ varies like z^α.

We need the equation of dynamics to infer α. Neglecting viscous effects, the steady mean flow verifies:

$$\frac{1}{r}\frac{\partial r \langle v_r \rangle \langle v_z \rangle}{\partial r} + \frac{\partial \langle v_z \rangle^2}{\partial z} = -\frac{1}{r}\frac{\partial r \langle v'_r v'_z \rangle}{\partial r} - \overbrace{\frac{\partial \langle v'^2_z \rangle + \langle P \rangle}{\partial z}}^{I}$$

The term (I) is usually neglected in this type of flow. It is indeed a pressure gradient which is usually small: this comes from the fact that the ambient pressure where the jet develops is constant. Indeed, the pressure is almost constant in a section of the jet, because the mean streamlines are straight lines (recall the results of Chap. 3). Then this equation may be rewritten to show explicitely the flux of momentum; integrating over r, we find

$$\frac{d}{dz}\int_0^\infty b(z)^2 V(z)^2 f(\xi)^2 \xi d\xi + [r \langle v_r \rangle \langle v_z \rangle + r \langle v'_r v'_z \rangle]_0^\infty = 0$$

The second term is zero since v_z vanishes at infinity and $r v_r$ is finite (see below). The correlation term is zero outside the turbulent zone. Finally, this equation shows that

$b(z)^2V(z)^2 = P^2$ where P^2 is a constant (the momentum flux), which characterizes the jet. Consequently, it turns out that the jet mean velocity decreases as $1/z$ and that $B = -b'$.

Now, we apply the same treatment to the mass conservation equation. We find that

$$\lim_{r\to\infty} r\,\langle v_r\rangle = -\left(\int_0^\infty f(\xi)\xi d\xi\right)\frac{d[b^2V(z)]}{dz} = -V(z)b(z)\underbrace{b'\int_0^\infty f(\xi)\xi d\xi}_{\alpha_j}$$

This expression shows that the radial velocity is proportional to the axial velocity $V(z)$ and to α_j which is called the *entrainment constant*.

The expression of α_j shows that it depends on the velocity profile $f(\xi)$ and on the aperture angle of the cone b'. In fact these two quantities are themselves dependent on the closure relations or the transport properties of turbulence. Many models have been proposed to explain the velocity profile of a turbulent jet, but none is completely satisfactory. Experiments show that in general the profile is a Gaussian. Thus, taking $f(\xi) = \exp\{-\xi^2\}$ is a good approximation. Measurements then give $\alpha_j = 0.054$ for the entrainment constant of jets. As an exercise, we may show that the entrainment constant and the aperture cone angle are functions of the velocity profile.

The self-similar turbulent jet with a gaussian profile thus obeys two simple equations:

$$\begin{cases} \dfrac{d[b^2V(z)]}{dz} = 2\alpha_j b(z)V(z) \\[2ex] \dfrac{d[b(z)^2V(z)^2]}{dz} = 0 \end{cases} \tag{9.85}$$

which translate respectively the conservation of mass and momentum. Their solution is obviously given by the proceeding laws: $b(z) = 2\alpha z$ and $V(z) = P/2\alpha z$. We note that the initial mass flux does not play any role in the solution. Actually, a short analysis of the solutions shows that the initial conditions are rapidly forgotten by the solution which quite quickly reaches the self-similar regime. In the final steady state the initial mass flux is a very small part of the actual mass flux, which has grown through entrainment of the surrounding fluid. The jet thus appears as generated by a pure source of momentum.

Let us now examine the case of a *turbulent plume*. The most common example is the smoke plume that raises over a chimney. The hot fluid raises in the atmosphere thanks to buoyancy which smoke particles render visible as a turbulent mixed flow.

The behaviour of the plume is very similar to that of the jet, but in this case, this is the initial enthalpy flux which controls the dynamics of the plume. Indeed, similarly as above, the initial mass flux and the initial momentum flux are forgotten by the flow. The initial mass flux disappears because of entrainment as in the turbulent jet, and the initial momentum flux also disappears because of the work of buoyancy that add new momentum to the flow. Thus the plume is made by a pure source of enthalpy.

Experiments show that the velocity profiles in plumes are close to those of jets. In addition to the mean velocity field, the plume is characterized by an "enthalpy jump" field δh, which measures the difference of enthalpy within the plume and outside the plume. As for the jets, self-similar solutions also exist and verify:

$$\begin{cases} \dfrac{d[b^2V]}{dz} = 2\alpha_p bV \\ \dfrac{d[b^2V^2]}{dz} = 2gb^2\Delta\rho \\ \dfrac{d[b^2\Delta\rho V]}{dz} = 0 \end{cases} \tag{9.86}$$

We derived these equations using the Boussinesq approximation and orienting the z-axis along the gravity field $\mathbf{g}$. We also note that another entrainment constant has been used because experiments say that $\alpha_p = 0.083$, which is different from the jet (why?).

The solutions of (9.86) are naturally power laws in z as prescribed by self-similarity. We easily find that:

$$b(z) = \frac{6\alpha_p}{5}z, \quad V(z) = \left(\frac{25gF_b}{24\alpha_p^2}\right)^{1/3} z^{-1/3}$$

$$\Delta\rho = \frac{5}{6}\left(\frac{5F_b^2}{9\alpha_p^4 g}\right)^{1/3} z^{-5/3} \tag{9.87}$$

where $b^2\Delta\rho V = F_b$ is the buoyancy flux. We also note that the aperture angle of the plume is quite similar to that of the jet, namely ~ 0.1 while the entrainment constant is somewhat different. This difference is certainly related to the transport of momentum and the active scalar δh.

9.9 Two-Dimensional Turbulence

Two-dimensional turbulence is quite different from its three-dimensional counterpart. Despite the strong approximation that is made (our world is three dimensional!), the turbulent flows in two dimensions deserve being studied because they enlight us on the dynamics of the Earth's atmosphere or oceans. Indeed, the fluid flows in these two thin layers of the Earth are almost two dimensional as soon as the scales considered largely exceeds the thickness of the layer (10 km for the atmosphere and 5 km for the oceans).

Two dimensionality implies new conservations law which strongly modify the dynamics, in particular when we consider the evolution of vorticity. Equation (3.41) indeed says that:

$$\frac{Df(\omega)}{Dt} = f'(\omega)\frac{D\omega}{Dt} = 0 \qquad \Longleftrightarrow \qquad \int_{(S)} f(\omega)dS = \mathrm{Cst}$$

where S is a surface advected by the fluid. The main consequence of this peculiarity is that the picture of the turbulent cascade is completely modified.

9.9.1 Spectra and Second Order Correlations

As in three dimensions, it is interesting to examine the properties of the homogeneous and isotropic turbulence.

The tensors Q and ϕ have the same definition, but just four components. If we observe that in two dimensions there is no helicity (vorticity being orthogonal to velocity), then following the same steps as when deriving (9.17), we find:

$$Q_{ij} = (rf)'\delta_{ij} - f'(r)r_i r_j/r \tag{9.88}$$

where $f(r)$ is the longitudinal correlation. Similarly, as for (9.27), we find:

$$\phi_{ij} = e(k)P_{ij} = \frac{E(k)}{\pi k}P_{ij} \tag{9.89}$$

using expression (9.92) below. Indeed, we still have

$$E_{turb} = \frac{1}{2}\left\langle v^2\right\rangle = \int_0^{+\infty} E(k)dk \tag{9.90}$$

which we relate to ϕ_{ij} by

$$E_{turb} = \frac{1}{2}Q_{ii}(0) = \frac{1}{2}\int \phi_{ii}(\mathbf{k})dS_k = \frac{1}{2}\int_0^{+\infty} kdk \int_{(2\pi)} \phi_{ii}(\mathbf{k})d\theta_k$$

$$\Longrightarrow \quad E(k) = \frac{1}{2}k\int_0^{2\pi} \phi_{ii}(\mathbf{k})d\theta_k \; . \tag{9.91}$$

In the isotropic case

$$E(k) = \pi k \phi_{ii}(k) \tag{9.92}$$

The function $R(r)$, defined as half the trace of $[Q]$, now reads,

$$R(r) = \frac{1}{2} Q_{ii}(r) = \frac{1}{2} \int \phi_{ii}(k) e^{i\mathbf{k}\cdot\mathbf{r}} dS_k = \frac{1}{2} \int \phi_{ii}(k) e^{ikr\cos\theta} k dk d\theta$$

The integration over the angular variable can easily be realized if we use the general expression of the zeroth order Bessel function, namely

$$J_0(z) = \frac{1}{\pi} \int_0^{\pi} \cos(z \cos\theta) d\theta \ .$$

Thus, we find

$$R(r) = \int_0^{\infty} E(k) J_0(kr) dk \tag{9.93}$$

Conversely

$$\phi_{ii}(\mathbf{k}) = \frac{1}{(2\pi)^2} \int Q_{ii}(\mathbf{r}) e^{-i\mathbf{k}\cdot\mathbf{r}} d^2\mathbf{r} = \frac{1}{\pi} \int_0^{+\infty} r R(r) J_0(kr) dr$$

gives the expression of the spectrum

$$E(k) = \int_0^{+\infty} kr J_0(kr) R(r) dr \tag{9.94}$$

As in the three-dimensional case, this expression allows us to derive the behaviour of the spectrum at the very large scales. In this case

$$E(k) \sim k \int_0^{+\infty} r R(r) dr \quad \text{as} \quad k \to 0$$

showing that the spectral kinetic energy density grows like k.

9.9.2 *Enstrophy Conservation and the Inverse Cascade*

In order to understand the implication of enstrophy conservation on the spectral properties of two-dimensional turbulence, it is convenient to consider a set-up where there would be only three Fourier modes of wavenumbers $k_1 < k_2 < k_3$, and energy E_1, E_2, E_3. The modes are assumed to be in nonlinear interactions, in a one-dimensional flow. Hence, we set $k_3 = k_1 + k_2$. Neglecting furthermore the effects of viscosity, energy and enstrophy of this system are constant. Thus energy variations must meet:

$$\delta E_1 + \delta E_2 + \delta E_3 = 0$$
$$k_1^2 \delta E_1 + k_2^2 \delta E_2 + k_3^2 \delta E_3 = 0$$

from which we easily find

$$\delta E_1 = -\frac{k_3^2 - k_2^2}{k_3^2 - k_1^2}\delta E_2 \qquad \text{and} \qquad \delta E_3 = -\frac{k_2^2 - k_1^2}{k_3^2 - k_1^2}\delta E_2$$

Now let us suppose that the energy of the intermediate mode of wavenumber k_2 decreases. Then energy of the two others increases, namely $\delta E_2 < 0 \implies \delta E_1 > 0$ and $\delta E_3 > 0$. If the modes are now spectrally close, for instance if $k_2 = \lambda k_1$ and $\lambda < 1 + \sqrt{3}$, then $\delta E_1 > \delta E_3$. This means that in this case the energy of the second mode is preferentially transferred to the first mode. This illustrate the case of an inverse cascade of energy. Enstrophy conservation together with nonlinear interactions tends to transfer energy towards the larger scales. On the other hand we may observe that simultaneously, enstrophy would rather tend to cascade to the small scales for, generally, $\delta Z_3 = k_3^2 \delta E_3 > \delta Z_1 = k_1^2 \delta E_1$.

Let us now focus on the shape of the spectra. The evolution of kinetic energy is guided by

$$\begin{aligned}\frac{dE_{turb}}{dt} =& \frac{d}{dt}\int_0^{+\infty} E(k,t)dk = -2\nu \int_0^{+\infty} k^2 E(k,t)dk \\ =& -2\nu \int_0^{+\infty} Z(k,t)dk = -\varepsilon(t)\end{aligned}$$

while enstrophy follows

$$\begin{aligned}\frac{dZ}{dt} =& \frac{d}{dt}\int_0^{+\infty} Z(k,t)dk = -2\nu \int_0^{+\infty} k^2 Z(k,t)dk \\ =& -2\nu \int_0^{+\infty} k^4 E(k,t)dk = -\zeta(t)\end{aligned}$$

This equation shows that enstrophy can only decrease, and thus remain bounded from above by its initial value. However if $\int_0^{+\infty} Z(k,t)dk$ is bounded, this implies that $\varepsilon \to 0$ when $\nu \to 0$. This means that ε cannot be used to determine the kinetic energy spectrum in the inertial range in two dimensions. We are left with $\zeta(t)$, namely the dissipation rate of enstrophy. Assuming that it is the quantity which controls the two-dimensional turbulence, then, using similar arguments as in three-dimensions, we find that

$$E(k,t) \propto \zeta(t)^{2/3} k^{-3} \tag{9.95}$$

The major consequence of this power law, is that the dissipation of kinetic energy $k^2 E \propto k^{-1}$ occurs in the large scales. Enstrophy dissipation, on the contrary, occurs in the small scales ($k^2 Z \propto k$).

Two-dimensional turbulence gives us a scenario that is quite different from its three-dimensional counterpart. Energy tends to accumulate in the large scale while enstrophy tends to be extracted by the small scales.

Numerical simulations have shown quite clearly what was going on in the physical space: vortices with similar vorticity (i.e. cyclonic or anti-cyclonic) tend to merge and form larger structures, signing the inverse cascade, while enstrophy, which is conserved by any fluid element, faces a filamentation producing smaller and smaller scales which are in the end erased by viscosity.

9.9.3 Turbulence with Rotation or Stratification

The shape of the container is not the only way to impose two-dimensionality to a flow. In Chap. 8, we saw that rotation, through the Coriolis acceleration could make a flow two-dimensional. In fact, two phenomena may also impose some two-dimensional dynamics to turbulence, i.e. a background rotation or a stable density stratification.

These two physical constraints can make a flow two-dimensional when the time-scale of the motion is much larger than those imposed either by rotation or stratification. Comparing time scales leads to the determination of the scale at which the effects of rotation or stratification start to be noticeable.

The turn-over time at a scale ℓ is $\tau_\ell = \ell/v_\ell$ but in Kolmogorov inertial range $v_\ell = (\ell\varepsilon)^{1/3}$, thus $\tau_\ell = \ell^{2/3}\varepsilon^{-1/3}$. This leads to the scale ℓ_t where the transition between three-dimensional and two-dimensional motions occurs:

$$\ell_t = \varepsilon^{1/2}\Omega^{-3/2} \qquad \text{or} \qquad \ell_t = \varepsilon^{1/2}N^{-3/2} \tag{9.96}$$

for respectively the rotating and stratified cases. For stably stratified fluids, ℓ_t is also known as Ozmidov scale.

9.10 Some Conclusions on Turbulence

To conclude this rather long chapter, I would like to present to the reader some ideas in order to better appreciate the way we are from a solution to the problems that we crossed in the course of this chapter.

The models such as the K-ε one, try to make a parallel between the "turbulent fluid" and a real fluid. Like the real fluid, the turbulent fluid would have a "pressure", a viscosity, etc. However, if such a way of doing is relevant, the first step is to describe correctly the equilibrium state, namely the "thermodynamics" of such a material. Such a step is still not accomplished although attempts have flourished in the literature (see Castaing 1989, 1996; Chorin 1991, for instance). However, let us admit that we succeeded. The next step is to derive the transport coefficients of the turbulence. As we did for the Newtonian fluid, one should analyse the response of the turbulence to weak perturbations. However, this is a formidable task. Indeed, unlike a standard fluid, which own just a single (very small) scale (the mean-free path, see Chap. 11), the turbulent fluid owns a very large number of scales that strongly interact. Pushing this idea to its end, we see that the turbulent fluid should be compared to some non-Newtonian fluid with and extremely complex, non-local, rheological law.

Presently, we may hope that the situation be not so dramatic and that among all the scales which intervene, just a small number are truly important, the others following the firsts. This possibility is not so unrealistic since in many cases, turbulent flows tend to self-similar situations, emphasizing scale invariance.

9.11 Exercises

1. a) Show that

$$\left\langle \hat{v}_i^*(\mathbf{k})\hat{v}_j(\mathbf{k}')\right\rangle = \phi_{ij}(\mathbf{k})\delta(\mathbf{k}-\mathbf{k}')$$

   where $\hat{v}$ represents the Fourier transform of the velocity fluctuations and δ is Dirac distribution.

   b) Show the following equality:

$$Z_{ij} = (2\pi)^{-3}\int \left\langle \omega_i'(\mathbf{x})\omega_j'(\mathbf{x}')\right\rangle e^{-i\mathbf{k}\cdot\mathbf{r}}d^3\mathbf{r} = \epsilon_{ilm}\epsilon_{jnp}k_l k_n \phi_{mp}(\mathbf{k})$$

   and then, that this expression leads to (9.28).

   c) Show the reciprocal relation of (9.35), namely

$$E(k) = \frac{2}{\pi}\int_0^\infty kr\sin kr\ R(r)dr$$

2. Following a similar way as in Sect. 9.4.1, show that the correlation length of the vorticity may be written as

$$\ell_z = \frac{\pi}{2}\frac{\int_0^\infty kE(k)dk}{\int_0^\infty k^2 E(k)dk}$$

Show that ℓ_z is also the dissipation scale ℓ_D.

3. Let us consider the relation linking the energy spectrum $E(k)$ and the scaling law of the structure function S_2.

a) From (9.35) and (9.61) show that the two-point correlation of the longitudinal components of the velocity f verifies

$$f(r) = 2\int_0^\infty \frac{\sin kr - kr\cos kr}{(kr)^3}E(k)dk \tag{9.97}$$

b) Retrieve that $f(0) = \frac{2}{3}E_{turb}$ and derive that

$$S_2(r) = 4\int_0^\infty \left[\frac{1}{3} - \frac{\sin kr - kr\cos kr}{(kr)^3}\right]E(k)dk \tag{9.98}$$

c) Show that if $E(k) = C_K\langle\varepsilon\rangle^{2/3}k^{-5/3}$ then $S_2 = C_2\langle\varepsilon\rangle^{2/3}r^{2/3}$ and that C_2 and C_K are proportional.

d) Using (9.76), find the difference between the She & Leveque exponent and the Kolmogorov exponent of the energy spectrum.

4. We assume that the energy spectrum of some turbulence is such that:

$$\begin{cases} E(k) = i(k) & k \le k_0 \\ E(k) = k^{-5/3} & k_0 \le k \le k_D \\ E(k) = d(k) & k \ge k_D \end{cases} \tag{9.99}$$

and that functions i and d verify:

$$i(k) \le k_0^{-5/3}$$

$$d(k) \le k_D{}^{-5/3}e^{-(k-k_D)/k_D}$$

Show that in these conditions, if $k_D \gg k_0$, then

$$\sqrt{\frac{8}{69}}(\ell_0\ell_D^2)^{1/3} \lesssim \ell_T \lesssim \frac{2\sqrt{5}}{3}(\ell_0\ell_D^2)^{1/3} \tag{9.100}$$

derive (9.53).

5. If the distribution of the logarithm of x obey to a normal law, show that

$$\langle x^p \rangle = e^{p\langle \ln x \rangle + p^2\sigma^2/2}$$

where σ^2 is the variance of the distribution. The first step is to show that if y is a random variable with a normal distribution and zero mean, then

$$\frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{+\infty} y^{2p} e^{-y^2/2\sigma^2} dy = (2p-1)!!\, \sigma^{2s}$$

where $(2p-1)!! = 1 \times 3 \times 5 \times 7 \times \ldots \times (2p-1)$.

6. *Turbulent jet.*

a) If we measure in Fig. 9.9, the half-aperture angle of the turbulent jet visualized by the water vapour (or rather the droplets of the condensing vapour), we find a value around 0.15 rd. What can we infer?

b) A ping-pong ball is placed in a turbulent jet directed upwards. Although the thrust of the jet is maximum on its axis we observe that the ball remains in the

Fig. 9.9 Steam jet at the outlet of a pressure cooker. Note the conical shape of the jet steam

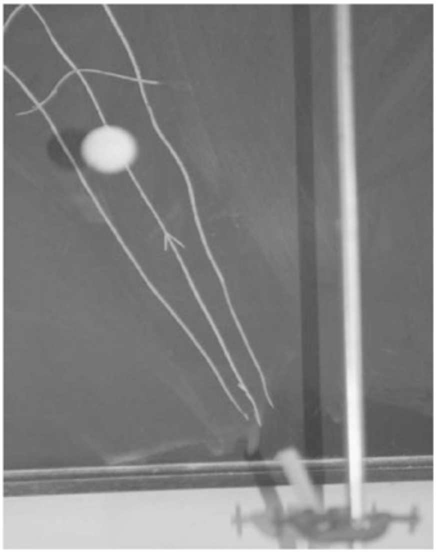

Fig. 9.10 Ping-pong ball sustained by an inclined turbulent air jet

jet, wandering around. We may even incline the jet about 30° without the ball fall (see Fig. 9.10). Explain.

Appendix: Complements for the K-ε Model

Let us start from the equation of the velocity fluctuations (9.81),

$$\frac{\partial v_i'}{\partial t} + \langle v_k \rangle \partial_k v_i' + v_k' \partial_k \langle v_i \rangle + v_k' \partial_k v_i' - \langle v_k' \partial_k v_i' \rangle = -\partial_i P' + \nu \Delta v_i'$$

We rewrite it for v_j', and write down the one for the fluctuations of the shear, namely $c_{ij}' = \partial_j v_i' + \partial_i v_j'$. We get

$$\begin{aligned}\frac{\partial c_{ij}'}{\partial t} &+ \langle v_k \rangle \partial_k c_{ij}' + \partial_j \langle v_k \rangle \partial_k v_i' + \partial_i \langle v_k \rangle \partial_k v_j' \\ &+ v_k' \partial_k \langle c_{ij} \rangle + \partial_j v_k' \partial_k \langle v_i \rangle + \partial_i v_k' \partial_k \langle v_j \rangle \\ &+ v_k' \partial_k c_{ij}' + \partial_j v_k' \partial_k v_i' + \partial_i v_k' \partial_k v_j' \\ &- \partial_j \partial_k R_{ik} - \partial_i \partial_k R_{jk} = -2\partial_i \partial_j p' + \nu \Delta c_{ij}'\end{aligned} \tag{9.101}$$

Since $\varepsilon = \frac{\nu}{2} \langle c_{ij}' c_{ij}' \rangle$, we now contract the foregoing equation with $\nu c_{ij}'$ and take the average. The first two terms can be rewritten as

$$\frac{\partial \varepsilon}{\partial t} + \langle v_k \rangle \partial_k \varepsilon$$

Then the four terms $\partial_j \langle v_k \rangle \partial_k v'_i + \partial_i \langle v_k \rangle \partial_k v'_j + \partial_j v'_k \partial_k \langle v_i \rangle + \partial_i v'_k \partial_k \left\langle v_j \right\rangle$ give

$$2\left\langle c'_{ij}\left(\partial_j \langle v_k \rangle \partial_k v'_i + \partial_i \langle v_k \rangle \partial_k v'_j\right)\right\rangle$$

because c'_{ij} is symmetric. They are further transformed into

$$\langle c'_{ij}\Omega'_{kj}\rangle \langle \Omega_{ik}\rangle + \left\langle c'_{ij}c'_{jk}\right\rangle \langle c_{ik}\rangle$$

where we set $\Omega_{ij} = \partial_i v_j - \partial_j v_i$. Noting that

$$c'_{ij}v'_k\partial_k c'_{ij} = v'_k\partial_k c'^2_{ij}/2 \qquad \text{and} \qquad c'_{ij}\left(\partial_j v'_k\partial_k v'_i + \partial_i v'_k \partial_k v'_j\right) = 2c'_{ij}\partial_j v'_k\partial_k v'_i$$

Then we get the equation that we were looking for, namely

$$\frac{\partial \varepsilon}{\partial t} + \langle v_k \rangle \partial_k \varepsilon + \overbrace{\langle \nu c'_{ij}\Omega'_{kj}\rangle}^{\text{(I)}} \langle \Omega_{ik}\rangle + \left\langle \nu c'_{ij}c'_{jk}\right\rangle \langle c_{ik}\rangle + \overbrace{\langle \nu c'_{ij}v'_k\rangle}^{\text{(II)}} \partial_k \langle c_{ij}\rangle$$
$$+\langle \nu v'_k\partial_k c'^2_{ij}/2\rangle + 2\nu \langle c'_{ij}\partial_j v'_k \partial_k v'_i\rangle = -2\nu \underbrace{\langle c'_{ij}\partial_i\partial_j p'\rangle}_{\text{(III)}} + \nu^2 \langle c'_{ij}\Delta c'_{ij}\rangle$$

Let us show now that local isotropy removes terms (I) and (III). If $\boldsymbol{\omega}$ is the vorticity, then

$$\Omega_{ij} = \epsilon_{ijk}\omega_k$$

thus (I) also reads

$$\epsilon_{kjl}\epsilon_{ikn}\left\langle \nu c'_{ij}\omega'_l\right\rangle \langle \omega_n\rangle = \left\langle c'_{ij}\omega'_i\right\rangle \left\langle \omega_j\right\rangle - \left\langle c'_{ii}\omega'_l\right\rangle \langle \omega_l\rangle$$

incompressibility implies that $c'_{ii} = 0$ and isotropy that $\left\langle c'_{ij}\omega'_i\right\rangle = 0$.

Term (III) is reshuffled as:

$$\left\langle c'_{ij}\partial_i\partial_j p'\right\rangle = \partial_i \left\langle c'_{ij}\partial_j p'\right\rangle - \left\langle \partial_i c'_{ij}\partial_j p'\right\rangle = \partial_i \left\langle c'_{ij}\partial_j p'\right\rangle - \partial_j \left\langle \partial_i c'_{ij} p'\right\rangle + \left\langle (\partial_i\partial_j c'_{ij}) p'\right\rangle$$

Isotropy makes the first two terms zero, while the last one disappears because $\nabla \cdot \mathbf{v}' = 0$.

Term (II) can be also eliminated if the turbulence is locally homogeneous and parity-invariant. In this case, we introduce $C_{ijk}(\mathbf{r}) = \langle c'_{ij}(\mathbf{x})v'_k(\mathbf{x}+\mathbf{r})\rangle$ and we show that like for S_{ijk} (see 9.39) we have $C_{ijk}(\mathbf{r}) = -C_{ijk}(-\mathbf{r})$ so that $C_{ijk}(\mathbf{0}) = 0$. Noting that $\langle c'_{ij}v'_k\rangle = C_{ijk}(\mathbf{0})$, the result follows.

Further Reading

There are numerous textbooks devoted to this very rich subject. A recent thorough review may be found in Davidson (2004). In a slightly more comprehensive style, we recommend the book of Frisch (1995) *Turbulence: the legacy of A. N. Kolmogorov* where the case of intermittency is well discussed. *Turbulence in fluids* by Lesieur (1990) presents at length the spectral side of turbulence, while *The physics of fluid turbulence* de McComb (1990) is a monograph focusing on the renormalization group approach (for more acquainted readers). The Les Houches lectures of Orszag (1973), is still a very good introduction to turbulence and to EDQNM in particular. Let us also mention some now classical work like the monograph of Leslie (1973) dealing with the DIA, the two volumes of Monin and Yaglom, reviewing the knowledge in 1975 or the *Turbulence* of Hinze (1959,1975) focused on the engineering approach of turbulence, like Tennekes and Lumley (1972) or Piquet (2001).

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