Chapter 5
Waves in Fluids

5.1 Ideas on Disturbances

Disturbances play an important role in Physics and notably in Fluid Mechanics. Indeed all flows in Nature are constantly subjected to perturbations of various origin: thermal noise, variations of boundary conditions, etc. If the flow is stable, these disturbances are always damped: otherwise, some of them grow, up to the disappearance of the initial flow replaced by, perhaps, more stable one. The study of the stability of a flow therefore begins with the study of perturbations. However, before addressing the case of flow stability in the next chapter, we shall first concentrate on the simplest manifestation of disturbances, namely *the waves*. Their existence is indeed the first evidence that an equilibrium (or a steady state) has been slightly perturbed.

5.1.1 Equation of a Disturbance

Let us begin with the simple case of a disturbance affecting the steady flow $\mathbf{V}$ of an incompressible fluid. The fluid is in a domain D delimited by a solid wall ∂D on which $\mathbf{V} = \mathbf{0}$. The motion is generated by a force $\mathbf{f}$. The equations of the original steady flow are simply

$$\begin{cases} \rho\mathbf{V}\cdot\nabla\mathbf{V} = -\nabla P + \mu\Delta\mathbf{V} + \mathbf{f} \\ \nabla\cdot\mathbf{V} = 0 \\ \mathbf{V} = \mathbf{0} \qquad \text{on} \quad \partial D \end{cases} \tag{5.1}$$

M. Rieutord, *Fluid Dynamics*, Graduate Texts in Physics,
DOI 10.1007/978-3-319-09351-2_5

Given that $(\delta P, \delta \mathbf{v})$ is a disturbance of this flow, the total field $(P + \delta P, \mathbf{V} + \delta \mathbf{v})$ must also be a solution to the equations of motion

$$\begin{cases} \rho \left(\frac{\partial(\mathbf{V}+\delta\mathbf{v})}{\partial t} + (\mathbf{V} + \delta\mathbf{v}) \cdot \nabla(\mathbf{V} + \delta\mathbf{v}) \right) = -\nabla(P + \delta P) + \mu \Delta(\mathbf{V} + \delta\mathbf{v}) + \mathbf{f} \\ \nabla \cdot (\mathbf{V} + \delta\mathbf{v}) = 0 \\ \mathbf{V} + \delta\mathbf{v} = \mathbf{0} \qquad \text{on} \quad \partial D \end{cases} \tag{5.2}$$

We develop these terms and subtract (5.1), which leads to the following equations for the disturbances:

$$\begin{cases} \rho \left(\frac{\partial \delta\mathbf{v}}{\partial t} + \overbrace{\mathbf{V} \cdot \nabla\delta\mathbf{v}}^{\text{I}} + \overbrace{\delta\mathbf{v} \cdot \nabla\mathbf{V}}^{\text{II}} + \delta\mathbf{v} \cdot \nabla\delta\mathbf{v} \right) = -\nabla\delta P + \mu\Delta\delta\mathbf{v} \\ \nabla \cdot \delta\mathbf{v} = 0 \\ \delta\mathbf{v} = \mathbf{0} \qquad \text{on} \qquad \partial D \end{cases} \tag{5.3}$$

Terms I and II show that the disturbances do not obey the same equations as the original flow. Their evolution is indeed a function of the flow on which they appear and this dependence is the result of the nonlinearities of the original equations.

5.1.2 *Analysis of an Infinitesimal Disturbance*

The study of disturbances is done in successive steps. The first of these consists in analyzing the case of disturbances whose amplitude is infinitesimal: indeed, for these disturbances the equations are linear and therefore easy to resolve (in theory!). Two situations can thus occur: either we are seeking the evolution of disturbances in a homogeneous region of the flow (whose properties are independent of the local coordinates) and we make an analysis using plane waves, also called *local analysis*, or we are facing important spatial variations and we must do a *global analysis* (this is the case if the boundary conditions play a role).

5.1.2.1 Local Analysis

Local analysis is the easiest one because the form of the disturbances is known in advance. Let us consider the system (5.3); by linearizing it, we have

$$\begin{cases} \rho\left(\dfrac{\partial\delta\mathbf{v}}{\partial t} + \mathbf{V}\cdot\nabla\delta\mathbf{v} + \delta\mathbf{v}\cdot\nabla\mathbf{V}\right) = -\nabla\delta P + \nu\Delta\delta\mathbf{v} \\ \nabla\cdot\delta\mathbf{v} = 0 \\ \delta\mathbf{v} = \mathbf{0} \qquad \text{on} \qquad \partial D \end{cases} \tag{5.4}$$

for which we seek a solution of the plane wave type, namely

$$\delta\mathbf{v} = \delta\mathbf{v}_0\, e^{i(\omega t + \mathbf{k}\cdot\mathbf{r})} \qquad \text{and} \qquad \delta P = \delta P_0\, e^{i(\omega t + \mathbf{k}\cdot\mathbf{r})} \tag{5.5}$$

where $\mathbf{k}$ is the wave vector. Observing that the operators ∇ and $\frac{\partial}{\partial t}$ are transformed in the following manner:

$$\nabla \to i\mathbf{k} \qquad \text{and} \qquad \frac{\partial}{\partial t} \to i\omega\ ,$$

we immediately get the system

$$\begin{cases} i(\omega + \mathbf{k}\cdot\mathbf{V})\delta\mathbf{v}_0 + (\delta\mathbf{v}_0\cdot\nabla)\mathbf{V} = -\delta P_0 i\mathbf{k} - \nu k^2\mathbf{v}_0 \\ \mathbf{k}\cdot\delta\mathbf{v}_0 = 0 \end{cases} \tag{5.6}$$

The solution (5.5) and the relation (5.6) that follows are valid only if $\mathbf{V}$ and $\nabla\mathbf{V}$ are almost constant. This is obviously not the case in general: but if we limit ourselves to a small area of the flow, then these quantities are almost constants and the derivation that we did does make sense. This is the reason for which it is called local analysis. It is valid only if the wavelength λ of the disturbance is very small compared to the scale of the variations of $\mathbf{V}$ or $\nabla\mathbf{V}$; in other words

$$\lambda \ll \text{Min}\left[\frac{\|\mathbf{V}\|}{\|\nabla\mathbf{V}\|}, \frac{\|\nabla\mathbf{V}\|}{\|\nabla^2\mathbf{V}\|}\right] \tag{5.7}$$

Let us continue our analysis and give a matrix form to relation (5.6), namely

$$\begin{bmatrix} D_{xx} & D_{xy} & D_{xz} & ik_x \\ D_{yx} & D_{yy} & D_{yz} & ik_y \\ D_{zx} & D_{zy} & D_{zz} & ik_z \\ k_x & k_y & k_z & 0 \end{bmatrix} \begin{pmatrix} \delta v_{0,x} \\ \delta v_{0,y} \\ \delta v_{0,z} \\ \delta P_0 \end{pmatrix} = \mathbf{0} \tag{5.8}$$

This system has a non-zero solution if the determinant of the matrix is vanishing. Since each component D_{ij} of this matrix is a function of $\mathbf{k}$ and ω, we finally get the *dispersion relation* of the waves:

$$\det[D] = \mathcal{D}(\omega, \mathbf{k}) = 0 \tag{5.9}$$

We note that the dispersion relation is an implicit relation between ω and $\mathbf{k}$. We shall see in the next chapter that this form of the relation has important consequences as far as stability is concerned.

5.1.2.2 Global Analysis

When one cannot neglect the boundary conditions or the heterogeneities of the disturbed system, one cannot impose the plane wave form to the disturbances. Their partial differential equations (5.4) need to be solved directly.

If the solution $\mathbf{V}$, which we analyse, is stationary, we can look for disturbances in the form

$$\delta\mathbf{v}(\mathbf{r},t) = \mathbf{v}_\lambda(\mathbf{r})e^{\lambda t} \tag{5.10}$$

Such a solution is called an *eigenmode* of the system. It is associated with the eigenvalue λ, which is a complex number in general. The search for the eigenmodes of a system is also called *modal analysis*.

If we note that the system (5.4) may be written

$$\begin{cases} \mathcal{L}(\delta\mathbf{v}) = \lambda\delta\mathbf{v} & \text{in} \quad D \\ \delta\mathbf{v} = 0 & \text{on} \quad \partial D \end{cases} \tag{5.11}$$

the search of the eigenmodes is equivalent to finding the eigenfunctions of the operator $\mathcal{L}$ verifying the boundary conditions. Simultaneously, we determine the associated eigenvalues which give the point spectrum (the set of eigenvalues) of the operator $\mathcal{L}$. If the operator is compact[1] then the spectrum is discrete and each eigenvalue can be identified by a triplet of quantum numbers (ℓ, m, n).

The resolution of such a problem is difficult in general and must be carried out numerically. In the examples that we shall consider, we shall combine the local analysis and the global analysis so as to reduce the partial differential equation to ordinary differential equations. This is possible when the system owns symmetries.

5.1.3 Disturbances with Finite Amplitude

When the amplitude of the disturbances cannot be neglected, the problem becomes very complicated because of the nonlinearities of the equations. Several strategies are then possible.

[1] To say it in a simple way, compact (linear) operators are like matrices of finite dimension although the space on which they act is a space of functions and therefore of infinite dimension.

- The amplitudes are finite but small: we can develop the solution into powers of the amplitude.
- Several, very different, scales intervene in the problem and we are able to make a multi-scale expansion of the solution.

An example of each of these strategies will be given in Chap. 7 using the case of thermal convection.

5.1.4 Waves and Instabilities

In the following we analyse the simple case of disturbances that are neither amplified nor damped (or very little). They are waves which freely propagate in the fluid. When an amplification appears, ones speaks rather of an instability, the study of which is postponed to the following chapter.

5.2 Sound

5.2.1 Equation of Propagation

Sound waves are the simplest and the most frequent of the disturbances which propagate in a fluid. In order to study them, we assume that the undisturbed fluid is at rest, i.e. $\mathbf{V} = \mathbf{0}$. With regard to (5.3), we must take into account the compressibility of the fluid: sound does not exist in an incompressible environment!

We make the following expansion:

$$\begin{cases} P = P_0 + \delta P \\ T = T_0 + \delta T \\ \rho = \rho_0 + \delta\rho \\ \mathbf{v} = \mathbf{0} + \delta\mathbf{v} \end{cases} \tag{5.12}$$

We assume moreover that the fluid be *perfect* and initially at constant pressure, density and temperature. By neglecting all the nonlinear terms we get

$$\frac{\partial\delta\rho}{\partial t} + \rho_0 \nabla \cdot \delta\mathbf{v} = 0 \tag{5.13}$$

from the equation of continuity,

$$\rho_0 \frac{\partial\delta\mathbf{v}}{\partial t} = -\nabla\delta P \tag{5.14}$$

from the equation of momentum and

$$\frac{\partial \delta s}{\partial t} = 0 \tag{5.15}$$

from the equation of entropy. This last equation can be used immediately. Indeed, this equation implies $\delta s = f(x, y, z)$; but at $t = 0$ (or before the disturbance starts) $\delta s = 0$, and therefore the disturbance stays isentropic. We can thus write a relation between the fluctuations of ρ and P:

$$\delta P = \left(\frac{\partial P}{\partial \rho}\right)_s \delta\rho \tag{5.16}$$

where the partial derivative of the pressure is taken at constant entropy. Now, if we take the time derivative of (5.13) and combine it with (5.14) and (5.16), we get the following wave equation:

$$\Delta\delta\rho - \frac{1}{c_s^2}\frac{\partial^2\delta\rho}{\partial t^2} = 0 \qquad \text{with} \qquad c_s^2 = \left(\frac{\partial P}{\partial \rho}\right)_s \tag{5.17}$$

c_s is naturally identified with the speed of propagation of the disturbance. We easily verify by exercise that δP and $\delta\mathbf{v}$ obey the same wave equation.

If the gas is ideal, the speed of sound can be expressed as

$$c_s^2 = \frac{\gamma P_0}{\rho_0} = \gamma R_* T_0 \tag{5.18}$$

where γ and R_* are defined in Sect. 1.7.1 This equation shows that, for an ideal gas, c_s depends only on temperature. Let us calculate an order of magnitude of the speed of sound in the air at 300 K. For $\gamma = 1.4$, $R_* = 8.314/0.029$ J/kg and $T_0 = 300$ K, we find $c_s = 347$ m/s. We observe that sound propagates faster in hot gases and with small molecular mass. In hydrogen at 300 K, $M = 0.002$ kg/mole, the sound speed is $c_s = 1321$ m/s thus almost four times faster than in the air.

The sound speed is of the same order of magnitude as the rms[2] velocity of the molecules of the gas. Pressure is indeed due to collisions between molecules and pressure disturbances cannot go much faster than the molecules themselves!

5.2.2 *The Dispersion Relation*

In the medium that we have chosen the sound waves have a very simple dispersion relation; assuming

$$(\delta P, \delta\rho, \delta\mathbf{v}) \propto \exp(i\omega t - i\mathbf{k}\cdot\mathbf{r})$$

[2]For "root mean square"; this is the typical dispersion of molecules velocities in a gas.

we easily obtain *the dispersion relation*

$$\omega^2 = c_s^2 k^2 \tag{5.19}$$

which shows that the waves are not dispersives since the phase velocity ω/k is independent of k.

Let us now consider the orientation of the velocity field associated with the wave, and the wavevector **k**. We take back (5.14) which we transform into

$$\rho_0 i\omega\delta\mathbf{v} = i\mathbf{k}\delta P \tag{5.20}$$

This last relation shows that the velocity vector is parallel to the wave vector. One says that the wave is *longitudinal*.

5.2.3 Examples of Acoustic Modes in Wind Instruments

The study of sound waves naturally leads to the vast domain of acoustics. We shall just outline the subject by examining the acoustic oscillations associated with wind instruments.

5.2.3.1 The Flute

The flute is one of the oldest instruments and its principle is one of the simplest. It is based on the oscillation of an air column in a cylindrical pipe. In order to study this oscillation, we neglect the viscosity of the air and assume its motion to be one-dimensional. Taking the axis of the tube parallel to Ox, with the origin at one extremity and the other at $x = L$, we write that the velocity, the pressure, etc. are the superimposition of two plane waves propagating in opposite directions, namely

$$v_x = Ae^{i(\omega t - kx)} + Be^{i(\omega t + kx)}$$

$$\delta p = A'e^{i(\omega t - kx)} + B'e^{i(\omega t + kx)}$$

At the extremities of the tube, the pressure is fixed (it is the atmospheric pressure), so that the pressure disturbance vanishes there.[3] These two boundary conditions allow us to write

$$\delta p = 0 \quad \text{at} \quad x = 0 \qquad \Longrightarrow \qquad A' = -B'$$

$$\delta p = 0 \quad \text{at} \quad x = L \qquad \Longrightarrow \qquad A'e^{-ikL} + B'e^{ikL} = 0$$

[3]This is an idealization of course. In reality the fluctuations of pressure do not exactly vanish, but their amplitude is very small compared to the one inside the tube.

from which comes the relation

$$\sin kL = 0 \quad \Longleftrightarrow \quad k = \frac{n\pi}{L}, \quad n \in \mathbb{N} \tag{5.21}$$

(5.21) gives the wavelength of the acoustic modes of a fluid in a cylindrical cavity open at both ends, namely

$$\lambda = \frac{2L}{n}$$

The reader has certainly observed that we just determined an eigenmode of the air column since we took the boundary conditions into account. This example shows the utility of the local analysis, which, in some cases, is easily extended to the global one.

The frequency F of these modes is immediately obtained from the dispersion relation $\omega = kc_s$ with $\omega = 2\pi F$. We have therefore

$$F_n = \frac{nc_s}{2L} \tag{5.22}$$

Let us apply this result to a flute in which the lowest note ($n = 1$) is the C at 261.6 Hz. Its length should be (if $c_s = 347$ m/s) L= 66.3 cm, to be compared to the length of a modern transverse flute in C which is 67 cm. We also note that the next harmonic, $n = 2$, vibrates at a frequency exactly two times higher than the fundamental $n = 1$. Hence, if the player is able to excite the second harmonic, a new set of notes with a frequency twice higher, i.e. at the next octave of the fundamental, is available.

5.2.3.2 The Clarinet

The clarinet is another interesting instrument because it uses different boundary conditions: one of the extremities is closed and we must set the velocity to zero there.[4] We have

$$\begin{array}{llll} \delta p = 0 & \text{at} \quad x = 0 & \Longrightarrow & A' = -B' \\ \delta \mathbf{v} = \mathbf{0} & \text{at} \quad x = L & \Longrightarrow & Ae^{-ikL} + Be^{ikL} = 0 \end{array}$$

Now, additional relations between A, A', B and B' are necessary. These relations are given by (5.20), thus

$$A' = \rho_0 c_s A \qquad \text{and} \qquad B' = -\rho_0 c_s B$$

[4]This doesn't imply that $\delta p = 0$ because (5.20) doesn't apply to a superimposition of plane waves.

which implies that $A = B$. Moreover,

$$\cos kL = 0 \quad \Longleftrightarrow \quad k = \frac{(2n+1)\pi}{2L}, \quad n \in \mathbb{N} \tag{5.23}$$

The frequencies of the different harmonics are thus

$$F_0 = \frac{c_s}{4L}, \qquad F_1 = \frac{3c_s}{4L}, \ \ldots, \quad F_n = \frac{(2n+1)c_s}{4L}, \ldots$$

Let us apply this result to a real instrument like the B-flat clarinet. The fundamental is the D at 146.8 Hz. The calculation of its theoretical length is 59 cm to be compared to the real length of 60 cm. Contrary to the flute, the first harmonic ($n = 1$) is not the next octave, but a frequency that is three times the fundamental one ($n = 0$), namely the A at 440 Hz (for the player this is an octave plus a fifth or a perfect twelfth).

Because of this dispersion relation of the modes, this instrument is necessarily more complex to make. Other examples, like the oboe and the bassoon are studied in the exercises.

5.3 Surface Waves

A second category of very common waves in our environment is that of surface gravity waves, namely all the waves which agitate the surface of water planes. Contrary to sound waves, these waves are very dispersive, i.e. their phase and group velocity strongly depends on the wavelength. We would have a hard time making music if sound waves behaved that way!

5.3.1 Surface Gravity Waves

In order to understand the way in which these waves propagate, we must return to the boundary conditions ruling a free surface. We have seen in Chap. 1 (Sect. 1.8.1.2) that the surface obeys the kinematic condition

$$\frac{\partial S}{\partial t} + \mathbf{v} \cdot \nabla S = 0 \tag{5.24}$$

where $S = \text{Cst}$ is the equation of the given surface. To this condition we add the dynamic one, which imposes the continuity of the stress when we cross the surface. As we neglect viscosity, this last condition amounts to the continuity of the pressure. We shall also neglect the effects of surface tension which will be examined separately.

In order to treat this problem we make several simplifying hypotheses: we first assume that the fluid is incompressible and that its motion is vorticity free. This latter assumption means that the flow is driven by forces derived from a potential. To be more precise, we consider the case of an interface between air and water and simultaneously treat the motion of the two fluids. The equations which govern these motions are (3.22) and (3.23):

$$\begin{cases} \Delta\Phi_w = 0 \\ \dfrac{\partial\Phi_w}{\partial t} + \dfrac{1}{2}\mathbf{v}_w^2 + \dfrac{P_w}{\rho_w} + gz = \text{Cst} \end{cases} \tag{5.25}$$

$$\begin{cases} \Delta\Phi_a = 0 \\ \dfrac{\partial\Phi_a}{\partial t} + \dfrac{1}{2}\mathbf{v}_a^2 + \dfrac{P_a}{\rho_a} + gz = \text{Cst} \end{cases} \tag{5.26}$$

where the indices a and w refer to air and water respectively. We first look for one-dimensional solutions that propagate in the x-direction:

$$\Phi = \Phi(z)e^{ikx-i\omega t} \tag{5.27}$$

Laplace's equation then implies

$$\frac{\partial^2\Phi}{\partial z^2} - k^2\Phi = 0 \qquad \Longrightarrow \qquad \Phi = Ae^{kz} + Be^{-kz} \tag{5.28}$$

5.3.1.1 In Deep Water

We assume, as a first step, that the air occupies the upper half-space $z \geq 0$ while the water occupies the lower half-space $z \leq 0$. In this case

$$\Phi_a = A_a e^{-kz} \qquad \text{and} \qquad \Phi_w = A_w e^{kz} \tag{5.29}$$

Let us now consider the boundary conditions. The surface verifies an equation of the form

$$S(\mathbf{r}, t) = z - z_s(x, t) = 0$$

from which we derive that $\nabla S = \mathbf{e}_z - \frac{\partial z_s}{\partial x}\mathbf{e}_x$.

If we now assume that the amplitudes of motions are small, we find from (1.63), that

$$v_z = \frac{\partial z_s}{\partial t} \qquad \text{at} \quad z = 0 \tag{5.30}$$

neglecting second order terms. Finally, on the surface the velocity potential verifies

$$\frac{\partial \Phi}{\partial z} = \frac{\partial z_s}{\partial t} \qquad \text{at} \quad z = 0 \tag{5.31}$$

as given by (5.24) for small amplitude motions. We apply this relation to the air and the water and thus

$$k\Phi_w(0) = -i\omega z_s = -k\Phi_a(0) \tag{5.32}$$

which shows incidentally that $A_w = -A_a$. We now use the linearized equation of dynamics and the second boundary condition (continuity of pressure). At $z = 0$ we have

$$\begin{cases} -i\omega\rho_w\Phi_w + \rho_w g z_s + \delta P_w = 0 \\ -i\omega\rho_a\Phi_a + \rho_a g z_s + \delta P_a = 0 \end{cases} \tag{5.33}$$

by subtracting these two equations and by using (5.32) together with the fact that $\delta P_w = \delta P_a$, we get the sought-after dispersion relation:

$$\omega^2 = \frac{\rho_w - \rho_a}{\rho_w + \rho_a} gk \tag{5.34}$$

If we neglect the influence of the air we simply have:

$$\omega = \sqrt{gk} \tag{5.35}$$

We easily derive from this expression the phase and group velocities, namely

$$v_\varphi = \frac{\omega}{k} = \sqrt{\frac{\rho_w - \rho_a}{\rho_w + \rho_a}\frac{g}{k}}, \qquad v_g = \frac{\partial \omega}{\partial k} = \frac{1}{2} v_\varphi \tag{5.36}$$

These two relations show that the waves are dispersive: the waves with long wavelength are the fastest.

5.3.1.2 In Shallow Water

If the depth of the water is not infinite (and especially if it is smaller than the wavelength of the waves), the dispersion relation is much simplified.

Taking the bottom into account, which we assume to be flat and located in $z = -H$, we have to modify the solution (5.28) so that the boundary condition:

$$v_z = 0 \qquad \text{at} \qquad z = -H$$

is verified. We easily find that it implies that

$$\Phi_w(z) = 2A_w e^{-kH} \cosh[k(z+H)] = A'_w \cosh[k(z+H)]$$

Using this relation at the surface, it turns out that

$$A'_w k \sinh(kH) = -i\omega z_s = -kA_a \tag{5.37}$$

which replaces (5.32). The equation of pressure allows us, after minor calculations, to find the new dispersion relation

$$\omega^2 = \frac{(\rho_e - \rho_a)\tanh(kH)}{\rho_e + \rho_a \tanh(kH)} gk. \tag{5.38}$$

If we neglect the density of the air, it simplifies into

$$\omega^2 = gk \tanh(kH) \tag{5.39}$$

We find again the foregoing relation, (5.35), when $\lambda \ll H$ because then $kH \gg 1$ and $\tanh(kH) \sim 1$. On the other hand, if we take the opposite case where $\lambda \gg H$, that is in the case of *a shallow layer*, then $\tanh(kH) \sim kH$ and the dispersion relation (5.39) becomes

$$\omega^2 = gHk^2 \tag{5.40}$$

The phase velocity is

$$v_\phi = \sqrt{gH}$$

identical to the group velocity: *the waves are no longer dispersive.*

Some example of the use of these results may be found in the list of exercises.

5.3.2 Capillary Waves

When discussing the case of surface gravity waves, we voluntarily ignored the role of surface tension. We may wonder whether this simplification was justified or not. In order to evaluate the effects of this new phenomenon, we just need to modify the dynamic boundary conditions. Indeed, now

$$P_{water} = P_{air} + \gamma\left(\frac{1}{R_1} + \frac{1}{R_2}\right)$$

As before we linearize these equations and simplify the problem to two dimensions, thus $R_2 = \infty$. With the linear approximation we have

$$\frac{1}{R} = -\frac{\partial^2 z_s}{\partial x^2}$$

(see 12.12 in the complements of Mathematics). This equation allows us to write

$$P_{water} = P_{air} - \gamma \frac{\partial^2 z_s}{\partial x^2} \tag{5.41}$$

The relations (5.33) are thus transformed into

$$- i\omega(\rho_w \Phi_w - \rho_a \Phi_a) + (\rho_w - \rho_a) g z_s + \gamma k^2 z_s = 0 \tag{5.42}$$

from which we derive the following dispersion relation:

$$\omega^2 = \frac{\rho_w - \rho_a}{\rho_w + \rho_a} gk + \frac{\gamma k^3}{\rho_w + \rho_a} \tag{5.43}$$

which is also written under the form

$$\omega^2 = gk + \frac{\gamma}{\rho} k^3 \tag{5.44}$$

when we neglect the density of the air. In this last relation, we have of course assumed the depth of the liquid to be infinite (the case of finite depth is proposed as an exercise). It shows that the effects of surface tension are expected at short wavelengths. They dominate if

$$\frac{\gamma}{\rho} k^3 > gk \qquad \Longleftrightarrow \qquad \lambda < \lambda_t = 2\pi \sqrt{\frac{\gamma}{\rho g}}$$

For water, we find that $\lambda_t = 1.7$ cm. We may observe that when the surface tension dominates, the waves are also dispersive.

5.4 Internal Gravity Waves

Gravity waves or internal gravity waves (in order to distinguish them from surface gravity waves) are present in all the fluids that are stably stratified by gravitation (see Chap. 2 sect. 2.2.3 for a presentation of a stratified fluid). This type of situation is encountered frequently in our environment. For example, in a lake where the cold water is found at the bottom and the warm water, lighter, close to the surface. Such a situation is stable. All disturbances of this equilibrium give birth to waves which are the internal waves of gravity. For such waves the restoring force is the buoyancy force which has a privileged (vertical) direction. Hence, these waves propagate anisotropically.

In order to get more familiar with these waves, we consider the following idealized situation: a quasi-incompressible fluid (such as water) is in equilibrium under the effect of gravitation. Its temperature is supposed to increase in a linear manner with z (see Chap. 2, sect. 2.2.3). We suppose moreover that the variations of density associated with the variations of temperature are negligible except those generating the buoyancy force (this is *the Boussinesq approximation* that we shall thoroughly describe in Chap. 7).

Neglecting the effects of diffusion (viscosity and thermal conduction) and the nonlinear terms, the equations of disturbances are now

$$\begin{cases} \dfrac{\partial \delta \mathbf{v}}{\partial t} = -\dfrac{1}{\rho}\nabla \delta p + \dfrac{\delta\rho}{\rho}\mathbf{g} \\[2ex] \dfrac{\partial \delta T}{\partial t} + \delta\mathbf{v}\cdot\nabla T_0 = 0 \\[2ex] \nabla\cdot\delta\mathbf{v} = 0 \end{cases} \tag{5.45}$$

T_0 is the temperature of the equilibrium configuration that we assume to vary linearly with z. We set

$$T_0 = T_{00} + \beta z \qquad \text{with} \quad \beta > 0$$

If the fluid is a liquid (see 1.60),

$$\frac{\delta\rho}{\rho} = -\alpha\delta T$$

where α is the dilation coefficient of the fluid.

We are looking for a solution in the form of plane waves, thus setting

$$\delta f = f_0 \exp(i\omega t - \mathbf{k}\cdot\mathbf{r}),$$

we transform (5.45) into

$$\begin{cases} i\omega\delta\mathbf{v}_0 = i\mathbf{k}p/\rho + \alpha\delta T_0 g\mathbf{e}_z \\ i\omega\delta T_0 + \beta\delta v_{0z} = 0 \\ \mathbf{k}\cdot\delta\mathbf{v}_0 = 0 \end{cases} \tag{5.46}$$

Taking the dot product of the first equation with $\mathbf{k}$, we find that $ipk^2 = -\alpha\rho\delta T_0 g k_z$. We combine this equation with $i\omega\delta v_z = ik_z p/\rho + \alpha\delta T_0 g$ and $i\omega\delta T_0 + \beta\delta v_{0z} = 0$, in order to finally obtain the following dispersion relation:

$$\omega^2 = N^2 \frac{k_x^2 + k_y^2}{k^2} \tag{5.47}$$

where we have set $N = \sqrt{\alpha\beta g}$. N is a frequency, called *the Brunt-Väisälä frequency* . Its interpretation is simple: it is the frequency of the oscillations of a fluid element when it is slightly moved from its position of equilibrium. Often, it is written in an equivalent manner as

$$N^2 = -\frac{g}{\rho}\frac{d\rho}{dz}$$

We note that if the density gradient is of the opposite sign (density increases with height), this frequency is imaginary. This situation corresponds to an instability (the Rayleigh-Taylor instability—Sect. 6.3.2.1- or thermal convection—Chap. 7).

The dispersion relation (5.47) shows that the waves are anisotropic. If θ is the angle between the wavevector $\mathbf{k}$ and $\mathbf{e}_z$, then (5.47) reads

$$\omega^2 = N^2 \sin^2\theta$$

This relation clearly shows that the frequency of the wave depends on the direction of propagation and never exceeds N. In particular, such waves do not propagate vertically. We can calculate the group velocity

$$\mathbf{v}_g = \nabla_k\,\omega = N \left| \begin{array}{l} k_z^2/k^3 \\ 0 \\ -k_z k_s/k^3 \end{array} \right. = N(k^2\mathbf{e}_s - k_s\mathbf{k})/k^3 = N\frac{\mathbf{k}\times(\mathbf{e}_s\times\mathbf{k})}{k^3} \tag{5.48}$$

where we used cylindrical coordinates ($\mathbf{e}_s$ is the radial unit vector). From this relation, it turns out that $\mathbf{v}_g \cdot \mathbf{k} = 0$: *Energy propagates perpendicularly to the phase*. Such a property, stemming from the anisotropy of the background, is also shared by inertial waves (see Chap. 8).

We note that these waves are transversal, namely $\mathbf{k}\cdot\delta\mathbf{v} = 0$. This property is the consequence of mass conservation $\nabla\cdot\mathbf{v} = 0$ and is shared by all the waves propagating in an incompressible fluid.

5.5 Waves Associated with Discontinuities

Until now we neglected the nonlinear terms in the equations of disturbances. We just considered waves of infinitesimal amplitude. However, is this approximation always relevant? To answer this question we need to estimate the relative importance of nonlinear terms to linear ones. Linear and nonlinear terms are not unique, forcing us to be more specific. We shall therefore take the pressure term $-\nabla\delta P$ as typical of the linear part and the inertial one, $\rho(\delta\mathbf{v}\cdot\nabla)\delta\mathbf{v}$, typical of nonlinear ones. Hence, nonlinear effects are important when

$$\nabla\delta P \sim \rho(\delta\mathbf{v}\cdot\nabla)\delta\mathbf{v}$$

If the characteristic scale of motion is L (namely the wavelength), then the foregoing criterion becomes

$$\delta P \sim \rho(\delta\mathbf{v})^2$$

saying that the kinetic energy of the fluctuations is of the order of the pressure disturbances. If we consider sound waves, from (5.20) it turns out that

$$\omega\rho\delta v = k\delta P \implies \rho v_\phi \delta v \sim \delta P,$$

where v_ϕ is the phase velocity. The criterion is now

$$\delta v \sim v_\phi \tag{5.49}$$

Nonlinear effects are therefore important when the velocity of the fluid, associated with the passing wave, is of the same order as the wave velocity. In fact, writing (5.49), we introduced a dimensionless number

$$\mathrm{M} = \frac{V}{V_\phi} \tag{5.50}$$

which is just a *Mach number*. The most famous of these numbers is the ratio of the fluid velocity and the speed of sound. This is the number which is referred to when one speaks about the Mach number without any precision. The foregoing discussion shows that it may be defined for any type of waves.

5.5.1 Propagation of a Disturbance as a Function of the Mach Number

The difference between a flow with $\mathrm{M} < 1$ and a flow with $\mathrm{M} > 1$ is not just quantitative: it is also qualitative. The propagation of perturbations is very different in these two cases.

Let us consider a source of low amplitude waves (sound waves or gravity waves for instance) moving at a speed V while emitting waves that propagate with a phase velocity c in the fluid. Using a reference frame attached to the source, the space filled by the fluid appears very differently when the Mach number is changed. If this number is smaller than unity, waves can reach any point in this space; in the opposite case they are confined to the *Mach cone* (see Fig. 5.1). The transition $\mathrm{M} = 1$ defines

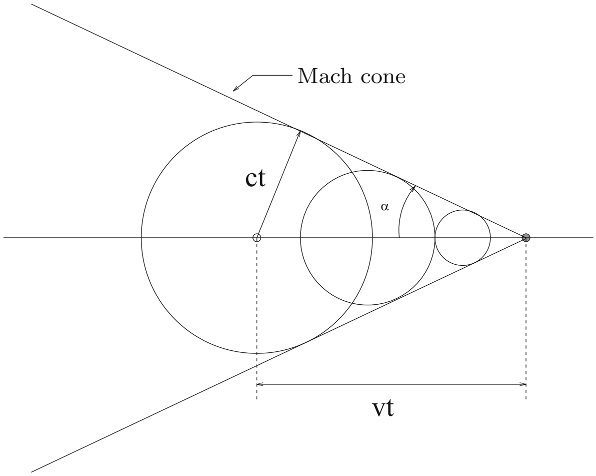


Fig. 5.1 The Mach cone formed by a source of periodic perturbations moving with a supersonic speed

the raise of a partition of space. Now, if we consider the example of a plane flying at a supersonic speed, air disturbances are of finite amplitude and usually produce a discontinuity.

These discontinuities are the consequence of the nonlinear evolution of the waves. In the case of a supersonic flight, the discontinuity is just the supersonic "bang", and in other words the shock wave. We shall see below that shock waves are part of a more general phenomenon which gathers all the waves resulting from a discontinuity. The other common example is the one of crunching water waves.

5.5.2 Equations for a Finite-Amplitude Sound Wave

The first step needed to study shock waves, is to write down the equations governing the evolution of a finite-amplitude sound wave. To simplify the matter, we restrict our discussion to the one-dimensional case. Although much simplified, this model represents fairly well the formation and propagation of a shock wave in a shock tube as we shall see below.

The equations of momentum and mass conservation are :

$$\begin{cases} \dfrac{\partial \rho}{\partial t} + \dfrac{\partial \rho u}{\partial x} = 0 \\[2ex] \dfrac{\partial u}{\partial t} + u\dfrac{\partial u}{\partial x} = -\dfrac{1}{\rho}\dfrac{\partial p}{\partial x} \end{cases} \tag{5.51}$$

In addition, we assume that the gas is isentropic so that $p \propto \rho^\gamma$. The sound speed $\sqrt{\gamma P/\rho}$ is a convenient variable. If we note that

$$\frac{d\rho}{\rho} = \frac{2}{\gamma - 1}\frac{dc}{c} \qquad \text{and} \qquad \frac{dp}{p} = \frac{2\gamma}{\gamma - 1}\frac{dc}{c}$$

then, (5.51) may be written as

$$\begin{cases} \left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x}\right)\frac{2c}{\gamma - 1} + c\frac{\partial u}{\partial x} = 0 \\ \left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x}\right)u + \frac{2c}{\gamma - 1}\frac{\partial c}{\partial x} = 0 \end{cases} \tag{5.52}$$

which may be mixed to yield:

$$\begin{cases} \left(\frac{\partial}{\partial t} + (u + c)\frac{\partial}{\partial x}\right)r = 0 \\ \left(\frac{\partial}{\partial t} + (u - c)\frac{\partial}{\partial x}\right)s = 0 \end{cases} \tag{5.53}$$

where we introduced

$$\begin{cases} r = \frac{u}{2} + \frac{c}{\gamma - 1} \\ s = \frac{u}{2} - \frac{c}{\gamma - 1} \end{cases} \tag{5.54}$$

called the *Riemann invariants.*

5.5.3 The Equations of Characteristics

Equation (5.53) are nonlinear and may be difficult to solve. Fortunately, these are a little simpler and often called *quasi-linear equations*. They can be solved qualitatively at least. For this purpose, we use the theory of characteristics. The reader who may not be familiar with this approach of partial differential equations, may have a look to Sect. 12.6.2 first.

The first result of characteristics theory to be used is the following. If r and s are solutions of (5.53), then r is constant along the characteristic curves of equation

$$\frac{dx}{dt} = u + c \tag{5.55}$$

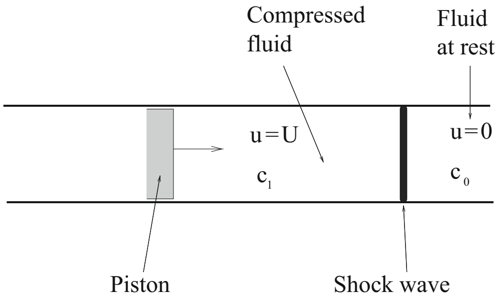


Fig. 5.2 Schematic view of a shock tube

while s is constant on the other characteristic curves

$$\frac{dx}{dt} = u - c \tag{5.56}$$

If the (5.53) were linear, the characteristic curves could be determined directly from (5.55) and (5.56); the initial conditions completely specify the solution. Here, u and c are unknown; nevertheless, we shall see that one can determine the shape of characteristics and understand the evolution of the solutions.

5.5.4 *Example: The Compression Wave*

We consider the following system: a piston inside a tube (of infinite length) starts at $t = 0$ and reach a constant velocity after a time t_a. The set-up is schematically drawn in Fig. 5.2.

Initially, the fluid is at rest: $u = 0$ and $c = c_0$ on the $t = 0$ line (see Fig. 5.3). The characteristics of s have a slope $\frac{dt}{dx} = -1/c_0$ on the x-axis (the $t = 0$ line), which they cut (see Fig. 5.3). But $s(x, 0) = -c_0/(\gamma - 1)$ is constant on the axis at $t = 0$ and therefore *it is constant everywhere in a region of the* (x, t) *plane bounded by the piston trajectory.* Using the definition of s on the piston where the gas velocity is U, we find that

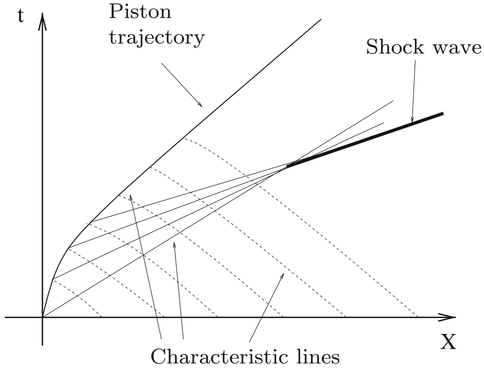
$$c = c_0 + \frac{\gamma - 1}{2} U > c_0 \tag{5.57}$$

Since the gas is isentropic and ideal $c \propto \rho^{(\gamma-1)/2}$, density increases when one gets closer to the piston.

In the same region we can write the other Riemann invariant as

$$r = s_0 + \frac{2c}{\gamma - 1} \tag{5.58}$$

Fig. 5.3 Characteristic lines in the (x, t) plane. The *solid lines* show r-characteristics and the *dotted lines* show the s-ones



where $s_0 = -c_0/(\gamma - 1)$. Thus, we find that *characteristics associated with* r *are straight lines*. Indeed, on its characteristic, r is constant and therefore c is constant from (5.58). Thus, u is also constant from the definition of r, (5.54). Hence, along an r-characteristic, $u + c$ is constant showing that these curves are straight lines.

Let us now consider characteristics emitted by the piston. They are straight lines verifying:

$$\frac{dt}{dx} = \frac{1}{c_0 + \frac{\gamma+1}{2}U}$$

The slope of these lines decreases with time since U increases. Consequently, the straight lines will cross somewhere. The function r is then no longer single valued and a discontinuity appears: the shock forms.

We may estimate the time by which the shock has formed. It is given by the point where the characteristic emitted at $t = 0$ (with a slope $1/c_0$) crosses the one when the piston reaches its asymptotic velocity after a time t_a. The slope is then $1/(c_0 + (\gamma + 1)U/2)$. We find that in this case the shock forms after a time t_c such that

$$t_c \sim \frac{2c_0 t_a}{(\gamma + 1)U}$$

Qualitatively, the formation of the discontinuity may be understood in the following manner: The acceleration of the piston increases the density in its vicinity. Sound waves move more rapidly in this denser region. A shock appears when the sound waves emitted in the compressed region overtake the ones emitted at $t = 0$, leading to a steepening of the wave front (see Fig. 5.4).

We would infer from the last formula that a shock forms whatever the conditions. This not the case of course. Indeed, our discussion neglects the dissipative effect as well as the finite length of the tube. If we still assume the infinite length of the tube, we may say that the shock will appear only if t_c is short enough, shorter that

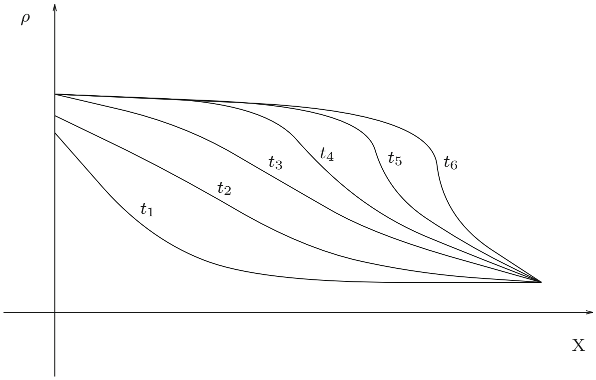


Fig. 5.4 Schematic evolution of density during the formation of a shock: note the steepening of the wave front. Similar shapes may be found for the pressure and temperature

$t_d \sim d^2/\nu$ where ν is the kinematic viscosity of the fluid. Inequality $t_c < t_d$ implies that

$$\frac{U}{t_a} \gtrsim \frac{c_0 \nu}{d^2}$$

showing that the piston acceleration must be large enough. Let us give a numerical example to fix ideas: if we take a cylindrical tube with a diameter of 3 cm, filled with air, then $\frac{U}{t_a} \gtrsim 5$ m/s^2. Thus, the piston needs to accelerate about half that of terrestrial gravity.

5.5.5 Interface and Jump Conditions

When the shock is formed, it may be described as a pure discontinuity. Indeed, its thickness is only a few mean-free paths which may be neglected macroscopically. However, not all the variables are discontinuous. For instance, the mass flux must be the same on each side of the shock. Thus, in a frame attached to the shock

$$\rho_1 \mathbf{v}_1 \cdot \mathbf{n} = \rho_2 \mathbf{v}_2 \cdot \mathbf{n} \tag{5.59}$$

where indices 1 and 2 refer to the upstream and downstream quantities. Let us first give some precisions about the up- and downstream regions. The flow goes from the upstream to the downstream, of course. The upstream region is the one of low pressure and supersonic velocity whereas the downstream one is of high pressure and subsonic velocity. The supersonic region may sometimes be qualified as “before” the shock since the shock wave may be seen as propagating, supersonically, in the low pressure region. The foregoing case of the shock tube is clear in this respect: the shock wave propagates in a fluid at rest. However, standing on the shock wave, we would see low pressure supersonic air rushing into the high pressure subsonic region!

The next interface condition which must be met at a discontinuity demands the balance of forces. More precisely, the flux of momentum through the shock wave must compensate the pressure jump. Hence,

$$p_1\mathbf{n} + \rho_1\mathbf{v}_1\ \mathbf{v}_1\cdot\mathbf{n} = p_2\mathbf{n} + \rho_2\mathbf{v}_2\ \mathbf{v}_2\cdot\mathbf{n}$$

Finally, the energy flux must also be continuous, thus

$$\frac{1}{2}v_1^2 + h_1 = \frac{1}{2}v_2^2 + h_2$$

where h is the enthalpy of the fluid. Let us demonstrate this latter relation, where the reader might wonder why the enthalpy comes into play. For this purpose we consider a cylindrical control surface whose generatrix lines are parallel to the flow and the ends of which are on each side of the shock. The energy flux entering the cylinder is just

$$(\frac{1}{2}v^2 + e)\rho\mathbf{v}\cdot\mathbf{n}\Big|_1$$

In a steady state, it differs from the outgoing flux by the power of forces applied on this volume. In our case we just need to consider pressure forces and their power $-p\mathbf{v}\cdot\mathbf{n}$. Thus,

$$(\frac{1}{2}v^2 + e)\rho\mathbf{v}\cdot\mathbf{n}\Big|_1 - (\frac{1}{2}v^2 + e)\rho\mathbf{v}\cdot\mathbf{n}\Big|_2 = -p\mathbf{v}\cdot\mathbf{n}|_1 + p\mathbf{v}\cdot\mathbf{n}|_2$$

After some rearrangements,

$$(\frac{1}{2}v^2 + e + \frac{p}{\rho})\rho\mathbf{v}\cdot\mathbf{n}\Big|_1 = (\frac{1}{2}v^2 + e + \frac{p}{\rho})\rho\mathbf{v}\cdot\mathbf{n}\Big|_2$$

Taking into account mass conservation, we find that

$$\frac{1}{2}v^2 + e + \frac{p}{\rho}$$

must be continuous; we note that $h = e + p/\rho$ is just the enthalpy.

Actually, the demonstration could be far shorter if we used Bernoulli theorem (3.7), which shows that

$$\frac{1}{2}v^2 + h$$

is constant along a streamline.

5.5.6 Relations Between Upstream and Downstream Quantities in an Orthogonal Shock

The foregoing relations are much simpler when the velocity field is orthogonal to the shock wave. The shock is said to be *normal* to underline the difference with the general case of an *oblique* shock. The conditions are now

$$\begin{cases} \rho_1 v_1 = \rho_2 v_2 \\ p_1 + \rho_1 v_1^2 = p_2 + \rho_2 v_2^2 \\ h_1 + v_1^2/2 = h_2 + v_2^2/2 \end{cases} \tag{5.60}$$

They may be used to rewrite the downstream quantities (index 2) as a function of the upstream ones. Using the upstream M_1 and downstream M_2 Mach numbers, one may show (see the demonstration at the end of the chapter) that

$$v_2 = v_1 \frac{(\gamma - 1)M_1^2 + 2}{(\gamma + 1)M_1^2} \tag{5.61}$$

$$\rho_2 = \rho_1 \frac{(\gamma + 1)M_1^2}{(\gamma - 1)M_1^2 + 2} \tag{5.62}$$

$$M_2 = \frac{M_1}{\sqrt{(v_1/v_2)^2 + (\gamma - 1)((v_1/v_2)^2 - 1)M_1^2/2}} \tag{5.63}$$

$$p_2/p_1 = 1 + \frac{2\gamma}{\gamma + 1}(M_1^2 - 1) \tag{5.64}$$

These relations allow us to determine the state of the fluid after crossing a shock wave. The upstream flow being supersonic, $M_1 > 1$, we immediately find the following inequalities:

$$v_2 < v_1, \qquad \rho_2 > \rho_1, \qquad p_2 > p_1$$

After crossing the shock wave the fluid slows down and is compressed. Pressure and density increase. Using (5.60c), which we rewrite

$$T_2 = T_1 + \frac{v_1^2 - v_2^2}{2c_p}$$

we also see that temperature increases. Obviously the downstream flow is subsonic and $M_2 < 1$. This inequality is not clear in (5.63), but becomes as such when this equation is transformed using the ratio of velocities (5.61). One then finds

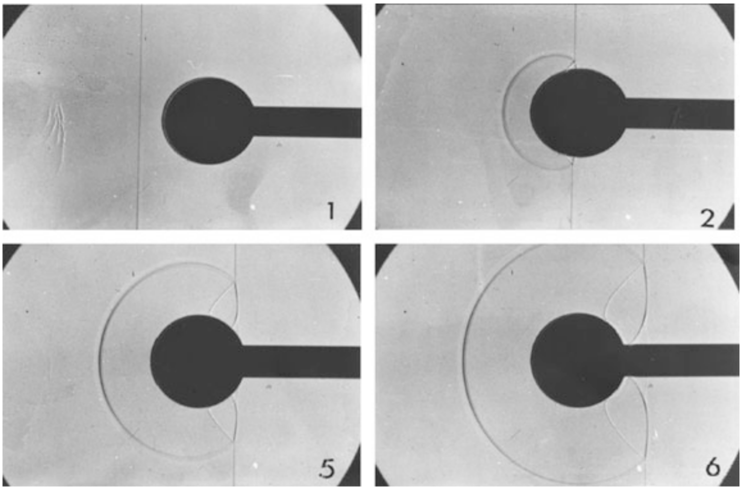


Fig. 5.5 Interaction of a plane shock wave with a cylinder. In (2), (5) and (6), we observe the evolution of the incident, reflected and refracted shock waves. We also observe (in 5 and 6) the appearance of a line joining the intersection point of the three waves and the cylinder; this is a "contact surface" where the pressure is continuous but where there is still an entropy and temperature jump

$$M_2 = \sqrt{\frac{(\gamma - 1)M_1^2 + 2}{2\gamma M_1^2 - \gamma + 1}} \tag{5.65}$$

showing the equivalence $M_1 > 1 \Longleftrightarrow M_2 < 1$ (Fig. 5.5).

To summarize, when going through a shock wave, a supersonic flow becomes subsonic and pressure, density, temperature increase. The temperature rise is not only the consequence of compression, but also that of the strong dissipation which occurs within the shock wave. Macroscopically, the velocity gradient are infinite but the volume of the dissipative region is vanishing; one may expect that a finite dissipation implies an *increase of entropy*.

Recalling the entropy expression (1.59), we can derive the entropy jump:

$$s_2 - s_1 = c_v \ln(p_2/p_1) - c_p \ln(\rho_2/\rho_1) = c_v \ln\left[\frac{p_2}{p_1}\left(\frac{\rho_1}{\rho_2}\right)^\gamma\right]$$

We may show that the entropy jump is always positive as expected in a dissipative process. For this purpose, we first express the entropy jump as a function of M_1. It turns out that

$$\Delta s = c_v \ln\left[\left(1 + \frac{2\gamma}{\gamma + 1}(M_1^2 - 1)\right)\left(\frac{(\gamma - 1)M_1^2 + 2}{(\gamma + 1)M_1^2}\right)^\gamma\right]$$

Let us use a Taylor expansion of this function in a neighbourhood of the threshold $M_1 = 1$ up to the third order. Setting $\varepsilon = M_1^2 - 1 \ll 1$, we have

$$\Delta s = \Delta s(1) + \Delta s'(1)\varepsilon + \Delta s''(1)\frac{\varepsilon^2}{2} + \Delta s'''(1)\frac{\varepsilon^3}{6} + \mathcal{O}(\varepsilon^4)$$

One may show that $\Delta s(1) = \Delta s'(1) = \Delta s''(1) = 0$ and that

$$\Delta s'''(1) = 4c_v\frac{\gamma(\gamma-1)}{(\gamma+1)^2}$$

hence

$$\Delta s = c_v\frac{2\gamma(\gamma-1)}{3(\gamma+1)^2}(M_1^2-1)^3 + \mathcal{O}(\varepsilon^4) \tag{5.66}$$

This expression shows that the entropy of the fluid increases when passing the shock, as soon as $M_1 > 1$. However, the jump is very small if M_1 is not very different from unity. This leads to a classification of shocks into weak and strong ones (see below).

Let us now show that the entropy jump is an increasing function of the Mach number. Setting $m = M_1^2$, we compute ds/dm. We have

$$\frac{1}{c_v}\frac{ds}{dm} = \frac{d}{dm}\left[\ln\left(1 + \frac{2\gamma}{\gamma+1}(m-1)\right) + \gamma\ln\left(\frac{(\gamma-1)m+2}{(\gamma+1)m}\right)\right]$$

$$\begin{aligned}\frac{1}{c_v}\frac{ds}{dm} &= \frac{2\gamma}{\gamma+1+2\gamma(m-1)} - \frac{2\gamma}{(\gamma-1)m^2+2m}\\ &= \frac{2\gamma(\gamma-1)(m-1)^2}{m(\gamma+1+2\gamma(m-1))((\gamma-1)m+2)}\end{aligned}$$

Thus $ds/dm > 0$ when $m > 1$. We also note that if M_1 goes to infinity, Δs also tends to infinity as $\ln M_1$.

5.5.7 *Strong and Weak Shocks*

The strength of a shock is usually measured by its pressure jump $(p_2 - p_1)/p_1$. We may note that this quantity is proportional to M_1^2 and is not bounded. This is not the case for the density or velocity jumps. When $M_1 \to \infty$

$$\frac{\rho_2}{\rho_1} \to \frac{\gamma+1}{\gamma-1} \qquad \text{and} \qquad \frac{v_2}{v_1} \to \frac{\gamma-1}{\gamma+1}$$

The weak shocks are defined as those where the variation of entropy is negligible. This distinction between weak and strong shocks is possible because of the very slow variation of entropy in the neighbourhood of $M_1 = 1$. If, for instance, $M_1 = 1.1$ then $\Delta s/s \sim 6 \times 10^{-5}$. If we neglect entropy variations less than 10%, shocks are weak as long as $M_1 \leq 1.46$.

5.5.8 Radiative Shocks

Some stars like cepheids or "RR Lyrae" show periodic variations of their luminosity (see Fig. 5.6). These variations are understood as the signature of radiative shock waves that propagate inside these stars as a result of the breaking-up of some acoustic waves. Radiative shock waves are much more powerful than the foregoing strong shocks. They are also called hypersonic shock waves because $M_1^2 \gg 1$. They occur in a low-density and hot medium. After the shock, matter is ionized by collisions but free electrons recombine with ions while emitting mainly ultraviolet radiations. Part of these photons propagate towards the upstream region and pre-heat the gas (see Fig. 5.7). This phenomenon makes the shock almost isothermal: the downstream gas cools efficiently by radiating photons after its compression. This is very different from the previous shocks where we assumed that the gas was evolving adiabatically after the shock. If we remember that pressure is proportional to density for an ideal isothermal gas, namely $P \propto \rho$, the adiabatic index would be $\gamma = 1$. Thus, the compression ratio $(\gamma + 1)/(\gamma - 1)$ may raise to infinity if the gas supports a quasi-isothermal compression. This ratio should be of order of M_1^2. In actual stellar models, typical shocks have a compression ratio of order 30. They are obviously in the hypersonic regime. Such shocks have been reproduced in the laboratory only very recently, thanks to the development of powerful lasers that can deposit a lot of energy in a very small volume.

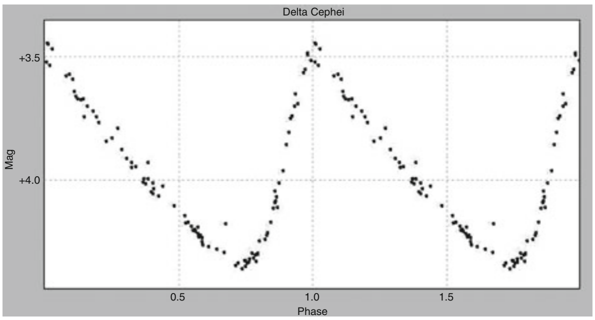


Fig. 5.6 Lumosity variations of the cepheid star δ Cephei. Its oscillation period is 5.37 days. The luminosity of the star varies by more than a factor two between minimum and maximum. This variation comes from an acoustic oscillation of the star. Its large amplitude leads to the formation of a radiative shock wave (source ThomasK Vbg)

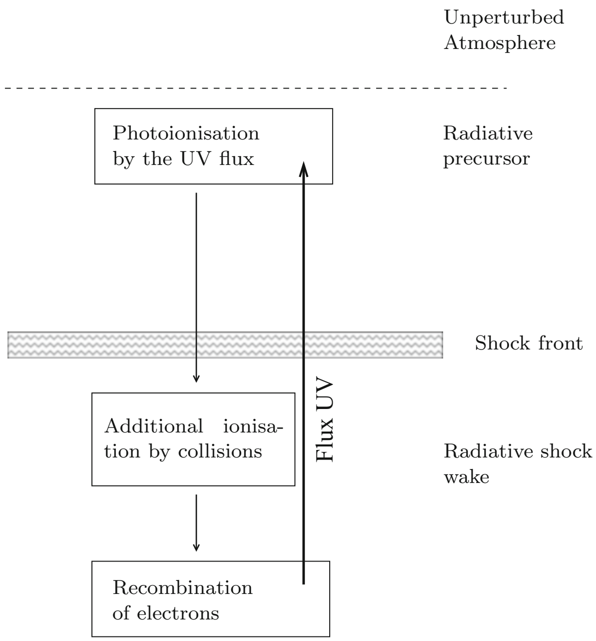


Fig. 5.7 Schematic view of a radiative shock in a star according to Gillet (2006)

5.5.9 The Hydraulic Jump

Another, very common type of discontinuity wave is the discontinuity of water depth in breaking water waves like in Fig. 5.8: this is the *hydraulic jump* (see the schematic view in Fig. 5.9). A connection can be easily made with sound waves if we consider waves propagating in shallow water. In this case, indeed, wave velocity is $\sqrt{gh}$ showing that in a similar way as in Fig. 5.4, the wave front steepens inevitably since the wave velocity increases like $\sqrt{h}$. In this case there is a direct analogy between depth h and temperature T of an ideal gas where the sound wave propagates.[5] However, the analogy, cannot be pushed too far, since gravity waves are necessarily two dimensional because of the incompressibility of the fluid. Another complication comes from the fact that these waves are naturally dispersive. The equality of phase and group velocity is only true asymptotically for wavelengths long compared to the depth. We shall see below that these dispersive effects can stop the steepening of the wave front and give rise to a solitary wave.

[5] Generally, we use the density as the analog of the depth [just compare (5.67) et (5.60a)], but the analog of the hydraulic jump is a shock wave in gas where $\gamma = 2$ in which case T and ρ are proportional.

Fig. 5.8 Hydraulic jumps from braking waves on a sand beach (Picture from the author)

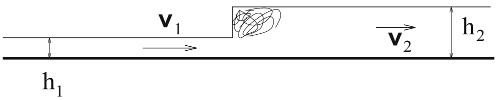


Fig. 5.9 A schematic view of a hydraulic jump

Let us examine the jump conditions of a hydraulic jump. Considering the fluid as incompressible, the conservation of the mass flux when crossing the hydraulic jump implies that:

$$v_1 h_1 = v_2 h_2 \tag{5.67}$$

in the most simple set-up where the velocity is assumed to be constant in the whole cross section of the flow.

The conservation of momentum leads to the same reasoning as for shock waves. The variation of the momentum flux must be compensated by the total pressure forces. Hence,

$$\rho(v_1^2 h_1 - v_2^2 h_2) + \int_0^{h_1} p(z)dz - \int_0^{h_2} p(z)dz = 0$$

Since we neglect vertical motions, the hydrostatic balance controls the z-dependence of the pressure. Thus,

$$p = \rho g(h - z)$$

where we assumed a zero pressure above the fluid. It yields

$$\int_0^{h_1} p(z)dz = \rho g \frac{h_1^2}{2} \qquad \text{and} \qquad \int_0^{h_2} p(z)dz = \rho g \frac{h_2^2}{2}$$

From these relations, a second jump condition connecting upstream and downstream quantities[6] can be derived:

$$v_1^2 h_1 + g h_1^2/2 = v_2^2 h_2 + g h_2^2/2 \tag{5.68}$$

With (5.67), we find out the ratio between upstream and downstream depth:

$$\frac{h_2}{h_1} = \frac{\sqrt{1 + 8\mathrm{Fr}_1^2} - 1}{2} \tag{5.69}$$

In this expression we introduced the Froude number:

$$\mathrm{Fr}_1 = \frac{v_1}{\sqrt{g h_1}}$$

This number quantify the ratio between the fluid velocity and the speed of waves. It is the analog of the Mach number for acoustic waves. When this number is larger than unity, the flow is *supercritical* or *torrential*. On the contrary, when $\mathrm{Fr} < 1$, the flow is said to be *subcritical* or *fluvial*. One may show as an exercise, that if the flow is supercritical in the upstream region, it is subcritical in the downstream region. Hence, we have the following equivalence:

$$\mathrm{Fr}_1 \gtreqless 1 \quad \Longleftrightarrow \quad \mathrm{Fr}_2 \lesseqgtr 1 \tag{5.70}$$

The first observation of a solitary wave

"I believe I shall best introduce this phænomenon by describing the circumstances of my own first acquaintance with it. I was observing the motion of a boat which was rapidly drawn along a narrow channel by a pair of horses, when the boat suddenly stopped - not so the mass of water in the channel which it had put in motion; it accumulated round the prow of the vessel in a state of violent agitation, then suddenly leaving it behind, rolled forward with great velocity, assuming the form of a large solitary elevation, a rounded, smooth and well-defined heap of water, which continued its course along the channel apparently without change of form or diminution of speed. I followed it on horseback, and overtook it still rolling on at a rate of some eight or nine miles an hour, preserving its original figure some thirty feet long and a foot to a foot and a half in height. Its height gradually diminished, and after a chase of one or two miles I lost it in the windings of the channel. Such, in the month of August 1834, was my first chance interview with that singular and beautiful phænomenon which I have called the Wave of Translation, a name which it now very generally bears;" From "Report on waves", *Rep. 14th Meet. Brit. Assoc. Adv. Sci., York,* 319–320 par S. Russell (1844).

[6] As for the shock waves, we use a frame attached to the discontinuity. Upstream and downstream regions are defined similarly.

As it may be guessed, the hydraulic jump is a dissipative structure: the hydraulic load decreases through a hydraulic jump. To show this, let us consider a streamline on the plane $z = 0$ and compare the energy per unit mass upstream and downstream. For this purpose we just need to check that

$$\frac{1}{2}v_1^2 + gh_1 \geq \frac{1}{2}v_2^2 + gh_2$$

Using expressions (5.86), demonstrated in the exercises, the preceding inequality implies

$$\left(\frac{h_2}{h_1} - \frac{h_1}{h_2}\right)\frac{h_1 + h_2}{4} + h_1 - h_2 \geq 0$$
$$\Longleftrightarrow (h_2 - h_1)^2 \geq 0$$

which is always true.

5.6 Solitary Waves ♠

Nonlinear effects have not always dramatic consequences such as the formation of a discontinuity. The steepening of the wave front can indeed be compensated by some dispersion effects which tend to spread the wave packet. When this balance occurs, one may observe a *solitary wave* which is remarkable for its stability.

The first observation of a solitary wave was made by Scott Russell in 1834 (see box and Fig. 5.10) on a surface gravity wave. In Sect. 5.3.1 we saw that these waves are dispersive. It is just in the asymptotic case of long wavelengths compared to the depth, that dispersion disappears. This property allows a control of the effects of dispersion by tuning the ratio of the wavelength to the depth. Another possible small parameter is obviously the amplitude of the wave. We shall see that when these two possibly small parameters are linked through a simple relation, one obtains a new equation, first derived by Korteweg and de Vries in 1895, which governs the motion of solitary waves.

5.6.1 The Korteweg and de Vries Equation

We first set again the general equations governing surface waves, still neglecting the effects of viscosity. We concentrate on the propagation of a wave in a one-dimension water basin of depth h. The motion is assumed irrotational and the velocity potential verifies:

$$\begin{cases} \Delta\Phi = 0 \\ \dfrac{\partial \Phi}{\partial t} + \dfrac{1}{2}\mathbf{v}^2 + \dfrac{P}{\rho} + gz = \text{Cst} \end{cases} \tag{5.71}$$

This system is completed by the following boundary conditions:

$$v_z = \frac{\partial \Phi}{\partial z} = 0 \qquad \text{at} \quad z = 0$$

$$\left.\begin{aligned} &\frac{\partial z_s}{\partial t} + \mathbf{v}\cdot\nabla z_s = v_z \\ &P = 0 \end{aligned}\right\} \qquad \text{at} \quad z = h + z_s(x,t)$$

where the pressure above the fluid has been set to zero. We first rewrite this system using non-dimensional variables. $\sqrt{gh}$ is a natural scale for the velocities and h for the lengths. Thus we set

$$\mathbf{v} = \sqrt{gh}\,\mathbf{u}, \qquad \Phi = h\sqrt{gh}\,\phi, \qquad t = \sqrt{h/g}\,\tau \qquad \text{and} \qquad z_s = h\,\zeta_s$$

We then get

$$\Delta\phi = 0 \tag{5.72}$$

Fig. 5.10 Repetition of Russell's observation of a solitary wave in Union Canal of Edinburgh during a conference at Heriot-Watt University (Nature, 3 August 1995)

$$\left.\begin{aligned}\frac{\partial \zeta_s}{\partial \tau} + \mathbf{u}\cdot\boldsymbol{\nabla}\zeta_s &= \frac{\partial \phi}{\partial z}\\ \frac{\partial \phi}{\partial \tau} + \frac{1}{2}(\boldsymbol{\nabla}\phi)^2 + \zeta_s &= 0\end{aligned}\right\} \quad \text{at} \quad z = 1 + \zeta_s(x,t) \tag{5.73}$$

$$\frac{\partial \phi}{\partial z} = 0 \quad \text{at} \quad z = 0 \tag{5.74}$$

where we substituted to the boundary condition $P = 0$ the equation of momentum taken on the surface, with the constant set to zero.

The next step is slightly delicate. We wish to introduce the small parameters and to consider the situations that are weakly nonlinear and weakly dispersive. Since the solitary wave seems to be a steady solution in some appropriate reference frame, we look for slowly evolving solutions in a new frame. We therefore introduce the following new form for the solutions:

$$\zeta_s \equiv \varepsilon\tilde{\zeta}_s(\tilde{\tau},\tilde{x}) \qquad \text{and} \qquad \phi = \varepsilon^{1/2}\tilde{\phi}(\tilde{\tau},\tilde{x})$$

where we set:

$$\tilde{x} = \varepsilon^{1/2}(x-\tau), \quad \tilde{\tau} = \varepsilon^{3/2}\tau$$

Theses new functions and new variables are sensitive to large scale or long time-scale variations only: x and τ need to vary a lot to yield significant variations on $\tilde{x}$ and $\tilde{\tau}$. With these new variables (5.72) now reads:

$$\frac{\partial^2 \tilde{\phi}}{\partial z^2} + \varepsilon\frac{\partial^2 \tilde{\phi}}{\partial \tilde{x}^2} = 0 \tag{5.75}$$

while (5.73) yields

$$\begin{aligned} &\frac{\partial \zeta_s}{\partial \tau} + \mathbf{u}\cdot\boldsymbol{\nabla}\zeta_s = \frac{\partial \phi}{\partial z}\\ \Longleftrightarrow \qquad &\frac{\partial \zeta_s}{\partial \tilde{\tau}}\frac{\partial \tilde{\tau}}{\partial \tau} + \frac{\partial \zeta_s}{\partial \tilde{x}}\frac{\partial \tilde{x}}{\partial \tau} + \frac{\partial \zeta_s}{\partial \tilde{x}}\frac{\partial \phi}{\partial \tilde{x}}\left(\frac{\partial \tilde{x}}{\partial x}\right)^2 = \frac{\partial \phi}{\partial z}\end{aligned}$$

where we took into account that $\tilde{x}$ depends on τ. We now deduce:

$$\left.\begin{aligned}\varepsilon^2\left(\frac{\partial \tilde{\zeta}_s}{\partial \tilde{\tau}} + \frac{\partial \tilde{\phi}}{\partial \tilde{x}}\frac{\partial \tilde{\zeta}_s}{\partial \tilde{x}}\right) - \varepsilon\frac{\partial \tilde{\zeta}_s}{\partial \tilde{x}} &= \frac{\partial \tilde{\phi}}{\partial z}\\ \varepsilon\frac{\partial \tilde{\phi}}{\partial \tilde{\tau}} - \frac{\partial \tilde{\phi}}{\partial \tilde{x}} + \frac{1}{2}\left(\frac{\partial \tilde{\phi}}{\partial z}\right)^2 + \frac{\varepsilon}{2}\left(\frac{\partial \tilde{\phi}}{\partial \tilde{x}}\right)^2 + \tilde{\zeta}_s &= 0\end{aligned}\right\} \quad \text{at} \quad z = 1 + \varepsilon\zeta(x,t) \tag{5.76}$$

and

$$\frac{\partial \tilde{\phi}}{\partial z} = 0 \qquad \text{at} \quad z = 0 \tag{5.77}$$

We then make the expansion of $\tilde{\zeta}_s$ and $\tilde{\phi}$ in powers of ε:

$$\tilde{\phi} = \sum_{n=0}^{\infty} \phi_n(\tilde{x}, z, \tilde{\tau})\varepsilon^n \qquad \text{and} \qquad \tilde{\zeta}_s = \sum_{n=0}^{\infty} \zeta_n(\tilde{x}, \tilde{\tau})\varepsilon^n$$

The first step consists in solving (5.75) for $\tilde{\phi}$. The first orders give:

$$\frac{\partial^2 \phi_0}{\partial z^2} = 0, \qquad \frac{\partial^2 \phi_1}{\partial z^2} + \frac{\partial^2 \phi_0}{\partial \tilde{x}^2} = 0, \qquad \frac{\partial^2 \phi_2}{\partial z^2} + \frac{\partial^2 \phi_1}{\partial \tilde{x}^2} = 0, \cdots$$

Using the boundary conditions in $z = 0$, we easily find a new expression for $\tilde{\phi}$, namely:

$$\tilde{\phi}(\tilde{x}, z, \tilde{\tau}) = c_0 + \varepsilon(c_1 - c_0'' z^2/2) + \varepsilon^2(c_0^{(4)} z^4/24 - c_1'' z^2/2 + c_2) + \mathcal{O}(\varepsilon^3)$$

where c_0, c_1, c_2 are functions of $\tilde{x}$ and $\tilde{\tau}$. The primes indicate the derivatives with respect to $\tilde{x}$. The first boundary condition, taken at order ε, gives:

$$\frac{\partial \zeta_0}{\partial \tilde{x}} = c_0''$$

and at order ε^2:

$$\frac{\partial \zeta_0}{\partial \tilde{\tau}} + c_0' \frac{\partial \zeta_0}{\partial \tilde{x}} - \frac{\partial \zeta_1}{\partial \tilde{x}} = -c_0'' \zeta_0 + c_0^{(4)}/6 - c_1''$$

Concerning the second boundary condition, it gives

$$\zeta_0 = c_0' \qquad \text{and} \qquad \frac{\partial c_0}{\partial \tilde{\tau}} - c_1' + c_0^{(3)}/2 + c_0'^2/2 + \zeta_1 = 0$$

These equations are used to obtain the equation controlling ζ_0, which we rename ζ. We thus find:

$$\frac{\partial \zeta}{\partial \tilde{\tau}} + \frac{3}{2}\zeta \frac{\partial \zeta}{\partial \tilde{x}} + \frac{1}{6}\frac{\partial^3 \zeta}{\partial \tilde{x}^3} = 0 \tag{5.78}$$

known as Korteweg and de Vries equation (or also KdV equation).

5.6.2 *The Solitary Wave*

This equation is solved by looking for a solution of the form $\zeta(\tilde{x} - \tilde{\tau})$. Hence,

$$-\zeta' + \frac{3}{2}\zeta\zeta' + \frac{1}{6}\zeta''' = 0$$

which we integrate once, and get:

$$-\zeta + \frac{3}{4}\zeta^2 + \frac{1}{6}\zeta'' = A$$

The constant of integration A is set to zero as we are interested in solutions vanishing at infinity. Multiplying this equation by ζ' and integrating again, we find:

$$-\frac{1}{2}\zeta^2 + \frac{1}{4}\zeta^3 + \frac{1}{12}\zeta'^2 = 0$$

As before, the constant of integration has been set to zero. This equation can be solve analytically since the variables can be separated.

$$\int \frac{d\zeta}{\zeta\sqrt{1 - \zeta/2}} = \sqrt{6}(\tilde{x} - \tilde{\tau})$$

Despite of its look, the integration of the left-hand side is very easy if we set

$$\zeta = \frac{2}{\cosh^2 \vartheta} \ ;$$

we immediately find that

$$\vartheta = -\sqrt{\frac{3}{2}}(\tilde{x} - \tilde{\tau}) \ .$$

$1/\cosh\vartheta$ is called the hyperbolic secant, and noted *sech*. Back to the dimensional variables, we have

$$\begin{aligned} z_s &= h(1 + \varepsilon\zeta) \\ &= h + 2h\varepsilon \operatorname{sech}^2\left[\sqrt{\frac{3\varepsilon}{2}}\,(x - (1 + \varepsilon)\tau)\right] \end{aligned}$$

We now introduce the wave amplitude $a = 2h\varepsilon$, the dimensional length and time scale:

$$z_s = h + a \operatorname{sech}^2\left[\sqrt{\frac{3a}{4h^3}}\left\{x - \sqrt{gh}\left(1 + \frac{a}{2h}\right)t\right\}\right] \tag{5.79}$$

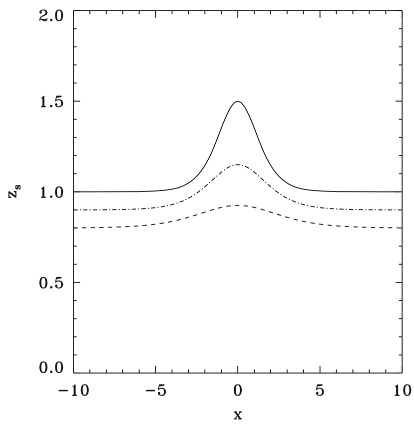


Fig. 5.11 This graph shows three solitary waves of amplitudes a, $a/2$, $a/4$ respectively

The solitary wave thus moves at a speed of:

$$c = \sqrt{gh}\left(1 + \frac{a}{2h}\right) \tag{5.80}$$

This velocity is only slightly different from the velocity of a gravity wave moving in shallow water if we take the total height $c \simeq \sqrt{g(h+a)}$.

We also observe that the horizontal scale of the wave (the width of the "bump") is given by

$$L = \sqrt{\frac{4h^3}{3a}} \tag{5.81}$$

We show in Fig. 5.11 the shape of the solitary wave for three amplitudes.

5.6.3 Elementary Analysis of the KdV Equation

The properties of the KdV equation are numerous and we could write a full book on it! Here, we content ourselves with an elementary analysis so as to appreciate the role of the various terms involved in this equation. Let us first focus on the linear term $\partial^3\zeta/\partial x^3$. Eliminating nonlinear terms, the KdV equation reads

$$\frac{\partial\zeta}{\partial\tau} + \frac{1}{6}\frac{\partial^3\zeta}{\partial x^3} = 0 \tag{5.82}$$

which we modify by changing of reference frame, namely $\zeta(x,\tau) = \zeta'(x-\tau,\tau)$. The equation for ζ' is

$$\frac{\partial\zeta'}{\partial\tau} + \frac{\partial\zeta'}{\partial x} + \frac{1}{6}\frac{\partial^3\zeta'}{\partial x^3} = 0.$$

A solution of the plane wave type gives the following dispersion relation:

$$\omega = k - k^3/6$$

It can be compared to (5.39) which is the dispersion relation of gravity waves in a basin of finite depth. If we expand (5.39) for the long wavelengths, it turns out that

$$\omega^2 \simeq ghk^2\left(1 - \frac{k^2h^2}{3}\right) \qquad \Longleftrightarrow \qquad \omega \simeq \sqrt{gh}k\left(1 - \frac{k^2h^2}{6}\right)$$

which is exactly the foregoing relation up to some dimensional coefficients. We see that the dispersive term $\partial^3\zeta/\partial x^3$ comes from the finite depth of the fluid.

We may now have a look to the way this term contributes to the spreading of the wave packet. For this purpose, we use a frame attached to the waves and the relation (5.82).

Let $\zeta_0(x)$ be the shape of the wave packet initially. We suppose that this shape is a kind of bell curve such that its Fourier transform $\hat{\zeta}_0$ exists. Taking the Fourier transform of (5.82), it turns out that

$$\hat{\zeta}(k,\tau) = \hat{\zeta}_0 e^{ik^3\tau/6}$$

hence

$$\zeta(x,\tau) = \int_{-\infty}^{+\infty} \hat{\zeta}_0 e^{ikx+ik^3\tau/6} dk$$

This expression is nicer if we observe that $e^{ik^3/3}$ is the Fourier transform of Airy's function $Ai(x)$, i.e.

$$e^{ik^3/3} = \int_{-\infty}^{+\infty} Ai(z)e^{-ikz} dz$$

After some easy manipulations, it turns out

$$\zeta(x,\tau) = \left(\frac{2}{\tau}\right)^{1/3} \int_{-\infty}^{+\infty} \zeta_0(z) Ai\left[\frac{x-z}{(\tau/2)^{1/3}}\right] dz$$

If the initial wave packet is strongly peaked and may be assimilated to a Dirac peak, then

$$\zeta(x,\tau) = \left(\frac{2}{\tau}\right)^{1/3} Ai\left[\frac{x}{(\tau/2)^{1/3}}\right] \tag{5.83}$$

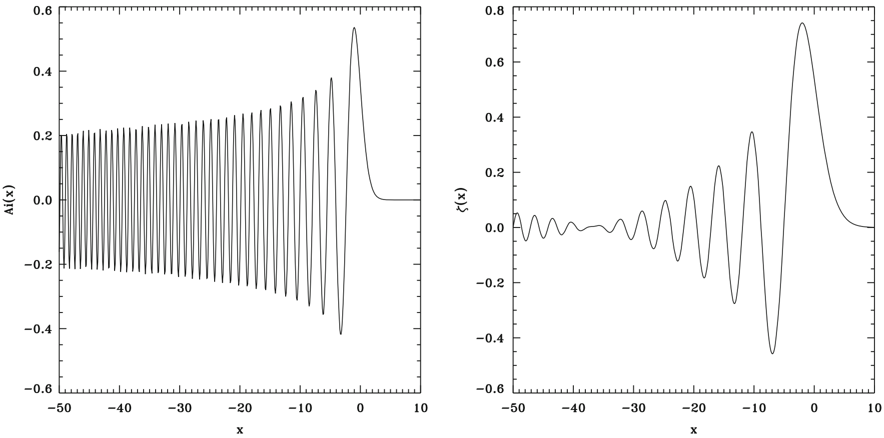


Fig. 5.12 (**a**) A plot of Airy's function. Schematically, this function behaves as $\cos x/(-x)^{1/4}$ when $x \to -\infty$ and as $e^{-2x^{3/2}/3}$ when $x \to +\infty$. (**b**) The convolution of Airy's function and a gaussian $\zeta_0(z) = e^{-10z^2}$

Airy's function and its convolution by a gaussian are shown in Fig. 5.12. With this figure, we note that energy is dispersed in the domain $]-\infty, 0]$ through a set of oscillations. The expression (5.83) shows the spreading of the wave packet: its width increases like $\tau^{1/3}$ (in fact the width of oscillations) whereas its amplitude decreases as $\tau^{-1/3}$.

Let us now examine the role of the nonlinear term. We leave aside the linear term and rewrite the KdV equation as

$$\frac{\partial \zeta}{\partial \tau} + \frac{3}{2}\zeta\frac{\partial \zeta}{\partial x} = 0 \tag{5.84}$$

This equation is of the same type as the one verified by Riemann's invariant. Thus we write the equation of characteristics, namely

$$\frac{dx}{d\tau} = \frac{3}{2}\zeta$$

which are straight lines since ζ is constant on such a line. If, at initial time, ζ has a bell shape, the construction of characteristics issued from the wave front immediately shows that a discontinuity will appear after a finite time (see Fig. 5.13).

Equation (5.84) is in fact of the same type as a famous equation in Fluid Mechanics, namely

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \nu\frac{\partial^2 u}{\partial x^2} \tag{5.85}$$

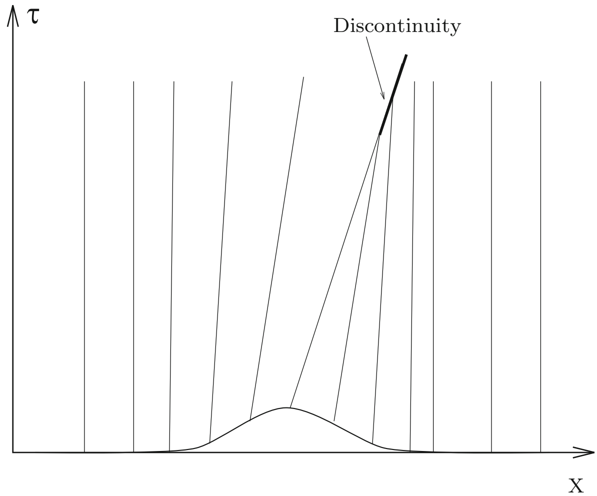


Fig. 5.13 A schematic view of the formation of a discontinuity through Burgers equation

which is called *Burgers equation*. This equation is just the equation of momentum of an incompressible fluid in which one would have neglected the pressure force. When viscosity is neglected (and time and length are appropriately scaled), (5.85) and (5.84) are equivalent. The foregoing reasoning shows that the solutions of Burgers equation without viscosity always form discontinuities.

5.6.4 Examples

Similarly as the observation of Scott Russell, there exist some natural phenomena where solitary waves appear. We shall mention two of them: the tidal bores and the tsunamis.

The tidal bore is the wave that propagates in an estuary, shallow enough, when the tide rises. This wave is usually first breaking and can be describe as a hydraulic jump. Getting upstream this hydraulic jump decreases and may give birth to a train of waves, which, like the wave observed by Russell, have a very long life time and are also solutions of the KdV equation: these are cnoidal waves. Tidal bores are very spectacular at the equinoxial high tides. In Europe, famous ones are in the Gironde in France and in the river Severn in England (see Fig. 5.14).

Tsunamis ("thunderstorm" wave in Japanese) designate the tidal waves which break on the coasts of the Pacific ocean (where they are the most common). Their origin is generally related to an earthquake. The seismic wave gives momentum to a large mass of water which may generate a solitary wave. Such a wave can cross the Pacific ocean without much damping. For instance, it is well-known that earthquakes occurring along the Alaska coast can generate a few hours later a tsunami on the Hawaiian shores. The wave has an horizontal scale which may reach

Fig. 5.14 The hydraulic jump made by the tidal bore of river Severn (photographed by D. H. Peregrine, in *An Album of Fluid Motion*, van Dyke 1982).

a hundred nautical miles (180 km). In this case, taking into account the depth of the Pacific ocean (5 km), the amplitude may be estimated to 5 m from (5.81). Its velocity may also be computed; it is close to $\sqrt{gh}$, which gives 800 km/h. Thus it crosses half of the Pacific ocean (4,800 km) in 6 h. When it arrives on a shore the steepening of the wave front may generate a water wall up to 20 or 30 m high.

5.7 Exercises

1. The bassoon and the oboe are two instruments whose air column is conical. Using the fact that a cone is a part of a sphere, rewrite the equation of disturbances and show that the eigenmodes obey the same dispersion relation as those of the flute. Compute the length of a bassoon whose gravest note is at 58.27 Hz (third B flat). Compare the result to its real length of 295 cm.
2. What is the frequency variation of the fundamental mode of a flute when the air temperature varies from 10 to 30 °C. Compare it to the change of frequency in a half-tone interval. The variation of the length of the tube is neglected and we recall that an octave is divided into twelve equal half-tones (tempered scale).
3. In a harbour along the coast of the Atlantic ocean, waves arrive periodically with a period of 15 s. What is their wavelength? their phase velocity? How long would it take for them to cross the Atlantic ocean (4,800 km)? We assume that the ocean is infinitely deep.
4. How long does it take for a wave of very long wavelength to cross the Atlantic ocean whose width is 4,800 km and depth 5 km? We give $g = 9.81$ m/s^2. Show that the Atlantic ocean is a resonant cavity for the tides; what are the consequences?

5. Derive the dispersion relation of capillary waves in the shallow water approximation.
6. a) From the jump conditions of a hydraulic jump, show that upstream and downstream velocities are of the form:

$$v_1 = \sqrt{g\frac{h_2}{h_1}\left(\frac{h_1+h_2}{2}\right)} \qquad \text{and} \qquad v_2 = \sqrt{g\frac{h_1}{h_2}\left(\frac{h_1+h_2}{2}\right)} \tag{5.86}$$

   b) Show that on each side of a hydraulic jump, Froude numbers are related by

$$\text{Fr}_2 = \text{Fr}_1\left(\frac{\sqrt{1+8\text{Fr}_1^2}-1}{2}\right)^{-3/2}$$

   Derive the equivalence (5.70).
7. Show that the following quantities

$$\int_{-\infty}^{+\infty}\zeta(x,\tau)dx \qquad \text{and} \qquad \int_{-\infty}^{+\infty}\zeta^2(x,\tau)dx$$

   are conserved by the KdV equation. What is the physical interpretation of these conservation laws?

Appendix: Jump Conditions

We give here the demonstration of the relations (5.61)–(5.64) relating upstream and downstream quantities in a normal shock. Let us recall that the enthalpy of an ideal gas is:

$$h = \frac{\gamma}{\gamma-1}\frac{p}{\rho} = \frac{c^2}{\gamma-1} \tag{5.87}$$

The energy relation can thus be written

$$c_1^2 + (\gamma-1)v_1^2/2 = x\gamma p_2/\rho_1 + x^2(\gamma-1)v_1^2/2$$

where we introduced $x = v_2/v_1$. The conservation of momentum (5.60b) reads now

$$p_2 = p_1 + \rho_1 v_1^2(1-x)$$

Combining the two foregoing equations, we find

$$(\gamma+1)x^2 - 2(\gamma+1/M_1^2)x + 2/M_1^2 + \gamma - 1 = 0$$

This second order equation necessarily has $x = 1$ as a solution (why?). Thus, factorizing it we get straightforwardly

$$(x-1)((\gamma+1)x - 2/M_1^2 - \gamma + 1) = 0$$

which gives the non-trivial solution sought after. From (5.60a), we derive (5.62), relating the densities.

The relation between upstream and downstream pressures comes from (5.60b) which we divide by p_1. Thus

$$\frac{p_2}{p_1} = 1 + \frac{\rho_1 v_1^2}{p_1}\left[1 - \frac{\rho_2}{\rho_1}\left(\frac{v_2}{v_1}\right)^2\right] = 1 + \frac{\gamma v_1^2}{c_1^2}\left(1 - \frac{v_2}{v_1}\right)$$

The desired expression is obtained using (5.61).

The relation on Mach numbers (5.63) comes from the equation on enthalpy. Using (5.87), we find

$$c_2^2 = c_1^2 + \frac{(\gamma-1)}{2}(v_1^2 - v_2^2)$$

Dividing this expression by v_2^2 we get (5.63).

Further Reading

The monograph "Waves in Fluids" of Lighthill (1978), cannot be ignored, but, at a less ambitious level, general books on fluid mechanics may be useful. As far as shock waves are concerned the reader may consult the monograph of Courant and Friedrichs (1976), *Supersonic flow and shock waves*, while the study of solitary waves may be followed up with the introduction of Drazin and Johnson (1989) and the more mathematical approach of solitons by Newell (1985).

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