7 CHAPTER Capstone

§37 * A Discussion of Exponents and Logarithms

In this book we have carefully developed the theory, but have been casual about using the familiar exponential, logarithmic and trigonometric functions in examples and exercises. Most readers probably found this an acceptable approach, since they are comfortable with these basic functions. In this section, we indicate three ways to develop the exponential and logarithmic functions assuming only the axioms in Chap. 1 and the theoretical results in later chapters. We will provide proofs for the third approach.

Recall, for x in $\mathbb{R}$ and a positive integer n, x^n is the product of x by itself n times. For $x \neq 0$, we have the convention $x^0 = 1$. And for $x \neq 0$ and negative integers $-n$ where $n \in \mathbb{N}$, we define x^{-n} to be the reciprocal of x^n, i.e., $x^{-n} = (x^n)^{-1}$.

37.1 Piecemeal Approach.
This approach starts with Example 2 on page 238 and Exercise 29.15 where it is shown that x^r is meaningful whenever $x > 0$ and r is

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rational, i.e., $r \in \mathbb{Q}$. Moreover,

$$\text{if} \quad h(x) = x^r, \quad \text{then} \quad h'(x) = rx^{r-1}.$$

The algebraic properties $x^r x^s = x^{r+s}$ and $(xy)^r = x^r y^r$ can be verified for $r, s \in \mathbb{Q}$ and positive x and y. For any $t \in \mathbb{R}$ and $x > 0$, we *define*

$$x^t = \sup\{x^r : r \in \mathbb{Q} \text{ and } r \leq t\}$$

and

$$\left(\frac{1}{x}\right)^t = (x^t)^{-1}.$$

This defines x^t for $x > 0$. It can be shown that with this definition x^t is finite and the algebraic properties mentioned above still hold. Further, it can be shown that $h(x) = x^t$ is differentiable and $h'(x) = tx^{t-1}$.

Next we can consider a fixed $b > 0$ and the function B defined by $B(x) = b^x$ for $x \in \mathbb{R}$. The function B is differentiable and $B'(x) = c_b B(x)$ for some constant c_b. We elaborate on this last claim. In view of Exercise 28.14, we can write

$$B'(x) = \lim_{h \to 0} \frac{b^{x+h} - b^x}{h} = b^x \cdot \lim_{h \to 0} \frac{b^h - 1}{h}$$

provided these limits exist. Some analysis shows that the last limit does exist, so

$$B'(x) = c_b B(x) \quad \text{where} \quad c_b = \lim_{h \to 0} \frac{1}{h}[b^h - 1].$$

It turns out that $c_b = 1$ for a certain b, known universally as e. Since B is one-to-one if $b \neq 1$, B has an inverse function L which is named $L(y) = \log_b y$. Since B is differentiable, Theorem 29.9 on page 237 can be applied to show L is differentiable and

$$L'(y) = \frac{1}{c_b y}.$$

Finally, the familiar properties of $\log_b$ can be established for L. □

When all the details are supplied, the above approach is very tedious. It has one, and only one, merit; it is direct without any tricks. One could call it the "brute force approach." The next two approaches begin with some well defined mathematical object [either a power series or an integral] and then work backwards to develop the familiar properties of exponentials and logarithms. In both instances, for motivation we will draw on more advanced facts we believe but which have *not* been established in this book.

37.2 Exponential Power Series Approach.
This approach is adopted in two of our favorite books: [16, §4.9] and [62, Chap. 8]. As noted in Example 1 of §31, we believe

$$e^x = \sum_{k=0}^{\infty} \frac{1}{k!} x^k$$

though we have *not* proved this, since we have not even defined exponentials yet. In this approach, we *define*

$$E(x) = \sum_{k=0}^{\infty} \frac{1}{k!} x^k, \tag{1}$$

and we *define* $e = E(1)$. The series here has radius of convergence $+\infty$ [Example 1, §23], and E is differentiable on $\mathbb{R}$ [Theorem 26.5]. It is easy [Exercise 26.5] to show $E' = E$. The fundamental property

$$E(x+y) = E(x)E(y) \tag{2}$$

can be established using only the facts observed above. Actually [62] uses a theorem on multiplication of absolutely convergent series, but [16] avoids this. Other properties of E can be quickly established. In particular, E is strictly increasing on $\mathbb{R}$ and has an inverse L. Theorem 29.9 on page 237 assures us L is differentiable and $L'(y) = \frac{1}{y}$. For rational r and $x > 0$, x^r was defined in Exercise 29.15. Applying that exercise and the chain rule to $g(x) = L(x^r) - rL(x)$, we find $g'(x) = 0$ for $x > 0$. Hence g is a constant function. Since $g(1) = 0$, we conclude

$$L(x^r) = rL(x) \quad \text{for} \quad r \in \mathbb{Q} \quad \text{and} \quad x > 0. \tag{3}$$

For $b > 0$ and rational r, (3) implies

$$b^r = E(L(b^r)) = E(rL(b)).$$

Because of this, we *define*

$$b^x = E(xL(b)) \quad \text{for} \quad x \in \mathbb{R}.$$

The familiar properties of exponentials and their inverses [logarithms!] are now easy to prove. $\square$

The choice between the approach just outlined and the next approach, which we will present in some detail, is really a matter of taste and depends on the appeal of power series. One genuine advantage to the exponential approach is that the series in (1) defining E is equally good for defining $E(z) = e^z$ for complex numbers z.

37.3 Logarithmic Integral Approach.
Let us attempt to solve $f' = f$ where f never vanishes; we expect to obtain $E(x) = e^x$ as one of the solutions. This simple differential equation can be written

$$\frac{f'}{f} = 1. \tag{1}$$

In view of the chain rule, if we could find L satisfying $L'(y) = \frac{1}{y}$, then Eq. (1) would simplify to

$$(L \circ f)' = 1,$$

so one of the solutions would satisfy

$$L \circ f(y) = y.$$

In other words, one solution f of (1) would be an inverse to L where $L'(y) = \frac{1}{y}$. But by the Fundamental Theorem of Calculus II [Theorem 34.3 on page 294], we know such a function L exists. Since we also expect $L(1) = 0$, we *define*

$$L(y) = \int_1^y \frac{1}{t}\, dt \quad \text{for} \quad y \in (0, \infty).$$

We use this definition to prove the basic facts about logarithms and exponentials. $\square$

37.4 Theorem.

(i) *The function L is strictly increasing, continuous and differentiable on $(0,\infty)$. We have*

$$L'(y) = \frac{1}{y} \quad \textit{for} \quad y \in (0,\infty).$$

(ii) $L(yz) = L(y) + L(z)$ *for* $y, z \in (0,\infty)$.

(iii) $L(\frac{y}{z}) = L(y) - L(z)$ *for* $y, z \in (0,\infty)$.

(iv) $\lim_{y\to\infty} L(y) = +\infty$ *and* $\lim_{y\to 0^+} L(y) = -\infty$.

Proof

It is trivial to show the function $f(t) = t$ is continuous on $\mathbb{R}$, so its reciprocal $\frac{1}{t}$ is continuous on $(0,\infty)$ by Theorem 17.4. It is easy to see L is strictly increasing, and the rest of (i) follows immediately from Theorem 34.3.

Assertion (ii) can be proved directly [Exercise 37.1]. Alternatively, fix z and consider $g(y) = L(yz) - L(y) - L(z)$. Since $g(1) = 0$, it suffices to show $g'(y) = 0$ for $y \in (0,\infty)$ [Corollary 29.4]. But since z is fixed, we have

$$g'(y) = \frac{z}{yz} - \frac{1}{y} - 0 = 0.$$

To check (iii), note $L(\frac{1}{z}) + L(z) = L(\frac{1}{z} \cdot z) = L(1) = 0$, so that $L(\frac{1}{z}) = -L(z)$ and

$$L\left(\frac{y}{z}\right) = L\left(y \cdot \frac{1}{z}\right) = L(y) + L\left(\frac{1}{z}\right) = L(y) - L(z).$$

To see (iv), first observe $L(2) > 0$ and $L(2^n) = n \cdot L(2)$ in view of (ii). Thus $\lim_{n\to\infty} L(2^n) = +\infty$. Since L is increasing, it follows that $\lim_{y\to\infty} L(y) = +\infty$. Likewise $L(\frac{1}{2}) < 0$ and $L((\frac{1}{2})^n) = n \cdot L(\frac{1}{2})$, so $\lim_{y\to 0^+} L(y) = -\infty$. ■

The Intermediate Value Theorem 18.2 on page 134 shows L maps $(0,\infty)$ *onto* $\mathbb{R}$. Since L is a strictly increasing function, it has a continuous strictly increasing inverse [Theorem 18.4 on page 137] and the inverse has domain $\mathbb{R}$.

37.5 Definition.
We denote the function inverse to L by E. Thus

$$E(L(y)) = y \quad \text{for} \quad y \in (0, \infty)$$

and

$$L(E(x)) = x \quad \text{for} \quad x \in \mathbb{R}.$$

We also define $e = E(1)$ so that $\int_1^e \frac{1}{t}\,dt = 1$.

37.6 Theorem.

(i) *The function E is strictly increasing, continuous and differentiable on $\mathbb{R}$. Then*

$$E'(x) = E(x) \quad \textit{for} \quad x \in \mathbb{R}.$$

(ii) $E(u + v) = E(u)E(v)$ *for* $u, v \in \mathbb{R}$.

(iii) $\lim_{x\to\infty} E(x) = +\infty$ *and* $\lim_{x\to-\infty} E(x) = 0$.

Proof
All of (i) follows from Theorem 37.4 in conjunction with Theorem 29.9 on page 237. In particular,

$$E'(x) = \frac{1}{L'(E(x))} = \frac{1}{\frac{1}{E(x)}} = E(x).$$

If $u, v \in \mathbb{R}$, then $u = L(y)$ and $v = L(z)$ for some $y, z \in (0, \infty)$. Then $u + v = L(yz)$ by (ii) of Theorem 37.4, so

$$E(u + v) = E(L(yz)) = yz = E(L(y))E(L(z)) = E(u)E(v).$$

Assertion (iii) follows from (iv) of Theorem 37.4 [Exercise 37.2]. ■

Consider $b > 0$ and $r \in \mathbb{Q}$, say $r = \frac{m}{n}$ where $m, n \in \mathbb{Z}$ and $n > 0$. It is customary to write b^r for that positive number a such that $a^n = b^m$. By (ii) of Theorem 37.4, we have $nL(a) = mL(b)$; hence

$$b^r = a = E(L(a)) = E\left(\frac{1}{n} \cdot nL(a)\right) = E\left(\frac{1}{n} \cdot mL(b)\right) = E(rL(b)).$$

This motivates our next definition and also shows the definition is compatible with the usage of fractional powers in algebra.

37.7 Definition.
For $b > 0$ and x in $\mathbb{R}$, we define

$$b^x = E(xL(b)).$$

Since $L(e) = 1$, we have $e^x = E(x)$ for all $x \in \mathbb{R}$.

37.8 Theorem.
Fix $b > 0$.

(i) *The function $B(x) = b^x$ is continuous and differentiable on $\mathbb{R}$.*
(ii) *If $b > 1$, then B is strictly increasing; if $b < 1$, then B is strictly decreasing.*
(iii) *If $b \neq 1$, then B maps $\mathbb{R}$ onto $(0, \infty)$.*
(iv) *$b^{u+v} = b^u b^v$ for $u, v \in \mathbb{R}$.*

Proof
Exercise 37.3. ■

If $b \neq 1$, the function B has an inverse function.

37.9 Definition.
For $b > 0$ and $b \neq 1$, the inverse of $B(x) = b^x$ is written $\log_b$. The domain of $\log_b$ is $(0, \infty)$ and

$$\log_b y = x \quad \text{if and only if} \quad b^x = y.$$

Note $\log_e y = L(y)$ for $y > 0$.

37.10 Theorem.
Fix $b > 0$, $b \neq 1$.

(i) *The function $\log_b$ is continuous and differentiable on $(0, \infty)$.*
(ii) *If $b > 1$, $\log_b$ is strictly increasing; if $b < 1$, $\log_b$ is strictly decreasing.*
(iii) *$\log_b(yz) = \log_b y + \log_b z$ for $y, z \in (0, \infty)$.*
(iv) *$\log_b(\frac{y}{z}) = \log_b y - \log_b z$ for $y, z \in (0, \infty)$.*

Proof
This follows from Theorem 37.4 and the identity $\log_b y = \frac{L(y)}{L(b)}$ [Exercise 37.4]. Note $L(b)$ is negative if $b < 1$. ■

The function $E(x) = e^x$ has now been rigorously developed and, as explained in Example 1 of §31, we have

$$E(x) = \sum_{k=0}^{\infty} \frac{1}{k!} x^k.$$

In particular, $e = \sum_{k=0}^{\infty} \frac{1}{k!}$. Also

37.11 Theorem.

$$e = \lim_{h \to 0} (1+h)^{1/h} = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n.$$

Proof

We only need to verify the first equality. Since $L'(1) = 1$ by Theorem 37.4(i), and since $L(1) = 0$, we have

$$1 = \lim_{h \to 0} \frac{L(1+h) - L(1)}{h} = \lim_{h \to 0} \frac{1}{h} L(1+h) = \lim_{h \to 0} L\left((1+h)^{1/h}\right).$$

Since E is continuous, we can apply Theorem 20.5 on page 158 to the function $f(h) = L((1+h)^{1/h})$ to obtain

$$\lim_{h \to 0} (1+h)^{1/h} = \lim_{h \to 0} E\left(L\left((1+h)^{1/h}\right)\right) = E(1) = e.$$

■

37.12 Trigonometric Functions.

Either approach 37.2 or 37.3 can be modified to rigorously develop the trigonometric functions. They can also be developed using the exponential functions for *complex* values since

$$\sin x = \frac{1}{2i}[e^{ix} - e^{-ix}], \quad \text{etc.};$$

see [62, Chap. 8]. A development of the trigonometric functions analogous to approach 37.3 can proceed as follows. Since we believe

$$\arcsin x = \int_0^x (1 - t^2)^{-1/2}\, dt,$$

we can define $A(x)$ as this integral and obtain $\sin x$ from this. Then $\cos x$ and $\tan x$ are easy to obtain. In this development, the number π is defined to be $2 \int_0^1 (1 - t^2)^{-1/2}\, dt$. □

In the exercises, use results proved in Theorem 37.4 and subsequent theorems but not the material discussed in 37.1 and 37.2.

Exercises

37.1 Prove directly that

$$\int_1^{yz} \frac{1}{t}\,dt = \int_1^{y} \frac{1}{t}\,dt + \int_1^{z} \frac{1}{t}\,dt \quad \text{for} \quad y, z \in (0, \infty).$$

37.2 Prove (iii) of Theorem 37.6.

37.3 Prove Theorem 37.8.

37.4 Consider $b > 0$, $b \neq 1$. Prove $\log_b y = \frac{L(y)}{L(b)}$ for $y \in (0, \infty)$.

37.5 Let p be any real number and define $f(x) = x^p$ for $x > 0$. Show f is differentiable and $f'(x) = px^{p-1}$; compare Exercise 29.15. *Hint*: $f(x) = E(pL(x))$.

37.6 Show $x^p y^p = (xy)^p$ for $p \in \mathbb{R}$ and positive x, y.

37.7 **(a)** Show that if $B(x) = b^x$, then $B'(x) = (\log_e b)b^x$.

(b) Find the derivative of $\log_b$.

37.8 For $x > 0$, let $f(x) = x^x$. Show $f'(x) = [1 + \log_e x] \cdot x^x$.

37.9 **(a)** Show $\log_e y < y$ for $y > 1$.

(b) Show

$$\frac{\log_e y}{y} < \frac{2}{\sqrt{y}} \quad \text{for} \quad y > 1. \quad \textit{Hint}: \quad \log_e y = 2 \log_e \sqrt{y}.$$

(c) Use part (b) to prove $\lim_{y \to \infty} \frac{1}{y} \log_e y = 0$. This neat little exercise is based on the paper [28].

§38 * Continuous Nowhere-Differentiable Functions

The history of continuous nowhere-differentiable functions is long and fascinating. Examples of such functions were constructed as early as 1831, by Bernard Bolzano. But a version wasn't published until

Karl Weierstrass did so in 1875 [69]; he had given examples in lectures before that and formally presented his result on July 18, 1872. These early examples tended to be of the form $\sum_{n=0}^{\infty} b^n \cos(a^n \pi x)$ or $\sum_{n=1}^{\infty} \frac{\sin(n^2 \pi x)}{n^2}$. The delay in publicizing these examples is due to the suspicion of mathematicians of the era that the definition of "function" should not be so general as to allow such monstrosities.

Excellent brief accounts are described in William Dunham's book [21, Chap. 9] and David Bressoud's book [13, §6.4]. Dunham asserts that Hermite, Poincare and Picard were appalled by Weierstrass's "pathological example." Bressoud has a quotation from Henri Poincaré that refers to these abnormal functions as counterexamples to the "flaws in our forefathers' reasoning…". There is also a quote of Hermite referring to the "plague of functions that do not have derivatives". He had no idea how bad a plague; see Theorem 38.3 on page 350 and the paragraph preceding it.

An especially thorough and detailed history can be found in Johan Thim's Master's thesis [66].

Karl Stromberg [65, Example (4.8), page 174] provides an elegant variant of van der Waerden's example [68], published in 1930. We repeat Stromberg's example almost verbatim because we don't see how to improve it.

38.1 Van der Waerden's Example.

We provide an example of a continuous function on $\mathbb{R}$ that is nowhere differentiable. Let $\phi(x) = |x|$ for $|x| \le 2$ and extend it to be 4-periodic, i.e., $\phi(x + 4q) = \phi(x)$ for all $x \in \mathbb{R}$ and $q \in \mathbb{Z}$. Since the graph of ϕ has slope ± 1 on intervals between even integers, we have

$$|\phi(s) - \phi(t)| = |s - t| \tag{1}$$

provided there is no even integer in the open interval (s, t). The graph of ϕ is similar to that of g in Fig. 25.1 on page 204; in fact, $\phi(x) = 2g(\frac{x}{2})$ for all x.

The continuous nowhere-differentiable function f is defined by

$$f(x) = \sum_{n=1}^{\infty} f_n(x) \quad \text{where} \quad f_n(x) = 4^{-n}\phi(4^n x).$$

Since each f_n is continuous and $\sum_{n=1}^{\infty} 4^{-n} \cdot 2 < \infty$, the Weierstrass M-test 25.7 on page 205 and Theorem 25.5 show f is continuous on $\mathbb{R}$.

Fix a in $\mathbb{R}$; we will show f is not differentiable at a by showing there is a sequence (h_k) of nonzero numbers satisfying

$$h_k \to 0 \quad \text{and} \quad \lim_{k\to\infty} \frac{f(a+h_k) - f(a)}{h_k} \quad \textbf{does not exist.} \tag{2}$$

Until Eq. (5), the integer $k \geq 1$ is fixed. There exists $\epsilon_k = \pm 1$ so that no even integers lie between $4^k a$ and $4^k a + \epsilon_k$. From (1), we have

$$|\phi(4^k a + \epsilon_k) - \phi(4^k a)| = 1.$$

In fact, for $n \leq k$, there are no even integers between $4^n a$ and $4^n a + 4^{n-k}\epsilon_k$, because if $4^n a < 2q < 4^n a + 4^{n-k}\epsilon_k$ for some $q \in \mathbb{Z}$, multiplying through by the integer 4^{k-n} would yield $4^k a < 4^{k-n} \cdot 2q < 4^k a + \epsilon_k$. Thus, by (1) again,

$$|\phi(4^n a + 4^{n-k}\epsilon_k) - \phi(4^n a)| = 4^{n-k} \quad \text{for} \quad 1 \leq n \leq k. \tag{3}$$

Since ϕ is 4-periodic, we also have

$$|\phi(4^n a + 4^{n-k}\epsilon_k) - \phi(4^n a)| = 0 \quad \text{for} \quad n > k. \tag{4}$$

Combining (3) and (4) with the definition of f_n, we conclude

$$|f_n(a + 4^{-k}\epsilon_k) - f_n(a)| = 4^{-k} \quad \text{for} \quad n \leq k \quad \text{and} = 0 \text{ for} \quad n > k.$$

We now let $h_k = 4^{-k}\epsilon_k$ so that

$$|f_n(a + h_k) - f_n(a)| = 4^{-k} \quad \text{for} \quad n \leq k \quad \text{and} = 0 \text{ for} \quad n > k.$$

Therefore the difference quotient,

$$\begin{aligned} \frac{f(a+h_k) - f(a)}{h_k} &= \sum_{n=1}^{k} \frac{f_n(a+h_k) - f_n(a)}{h_k} \\ &= \sum_{n=1}^{k} \frac{f_n(a+h_k) - f_n(a)}{4^{-k}\epsilon_k} = \sum_{n=1}^{k} (\pm 1), \end{aligned} \tag{5}$$

is an even integer for even k and an odd integer for odd k. Therefore the difference quotients in (5) cannot converge, as $k \to \infty$. Since $\lim_k h_k = 0$, this proves (2) so that f is not differentiable at a. $\square$

38.2 Remark.
For later use, note the function ϕ in the preceding proof satisfies $\sup\{|\phi(y)| : y \in \mathbb{R}\} = 2$, and for any y in $\mathbb{R}$ we have

$$|\phi(y+h) - \phi(y)| = h \quad \text{for sufficiently small} \quad h > 0. \tag{1}$$

We've seen that constructing continuous nowhere-differentiable functions can be a challenge. Now we take a more abstract approach. In the discussion after Definition 22.6 on page 183, it is observed that the space $C(\mathbb{R})$ of bounded continuous functions on $\mathbb{R}$ is a complete metric space using the metric

$$d(f,g) = \sup\{|f(y) - g(y)| : y \in \mathbb{R}\} \quad \text{for all} \quad f, g \in C(\mathbb{R}).$$

Recall from Example 7(b) on page 174 that the irrational numbers are much more prevalent than the rational numbers, because the set $\mathbb{Q}$ is of first category in the complete metric space $\mathbb{R}$. Similarly, as we will see in the next theorem, the continuous nowhere-differentiable functions are more prevalent than the differentiable ones. One might say mathematicians of the 19th century viewed the continuous nowhere-differentiable functions as the "irrational functions."

As just noted, the next theorem shows "most" functions in $C(\mathbb{R})$ are nowhere differentiable. The proof uses the Baire Category Theorem 21.8 in the enrichment Sect. 21; see page 173. In 1931, S. Banach and S. Mazurkiewicz independently published this result in Studia Mathematica.

38.3 Theorem.
Let $\mathcal{D}$ consist of the functions f in $C(\mathbb{R})$ that have a derivative at at least one point. Then $\mathcal{D}$ is first category in $C(\mathbb{R})$.

Proof
Consider f in $\mathcal{D}$. Then for some x in $\mathbb{R}$, the derivative $f'(x)$ exists. Choose m so that $|x| \le m$ and $|f'(x)| < m$. There exists δ in $(0,1)$ so that

$$\left| \frac{f(x+h) - f(x)}{h} - f'(x) \right| < m - |f'(x)| \quad \text{for all} \quad h \in (0, \delta).$$

Select a positive integer $n \geq m$ satisfying $\frac{1}{n} < \delta$. Then

$$x \in [-n, n] \quad \text{and} \quad |f(x+h) - f(x)| \leq hn \quad \text{for} \quad h \in \left(0, \frac{1}{n}\right). \quad (1)$$

Thus f belongs to $\mathcal{F}_n$, which is defined to be all f in $C(\mathbb{R})$ such that some x in $\mathbb{R}$ satisfies (1) above. We have just shown $\mathcal{D} \subseteq \cup_{n=1}^{\infty} \mathcal{F}_n$, so by the Baire Category Theorem 21.8 on page 173, it suffices to show each $\mathcal{F}_n$ is closed in $C(\mathbb{R})$ and is nowhere dense in $C(\mathbb{R})$.

To see $\mathcal{F}_n$ is closed, consider a sequence (f_k) in $\mathcal{F}_n$ converging to f in $C(\mathbb{R})$. As noted after Definition 22.6, this means $f_k \to f$ uniformly on $\mathbb{R}$. We will show f is in $\mathcal{F}_n$. For each k, there exists x_k so that

$$x_k \in [-n, n] \quad \text{and} \quad |f_k(x_k+h) - f_k(x_k)| \leq hn \text{ for } h \in \left(0, \frac{1}{n}\right). \quad (2)$$

By the Bolzano-Weierstrass Theorem 11.5, a subsequence of (x_k) converges to some x in $[-n, n]$; to avoid subsequence notation, we will assume the sequence (x_k) itself converges to x. Since $x_k \to x$ and $f_k \to f$ uniformly, we have $f_k(x_k+h) \to f(x+h)$ for each $h \in [0, \frac{1}{n})$; see Exercise 24.17 on page 200. For each h in $(0, \frac{1}{n})$, we take the limit in (2) as $k \to \infty$ to obtain (1). Hence f belongs to $\mathcal{F}_n$, and $\mathcal{F}_n$ is closed.

To show $\mathcal{F}_n$ is nowhere dense, it suffices to show no ball $B_r(g) = \{f \in C(\mathbb{R}) : d(f, g) < r\}$ is a subset of any $\mathcal{F}_n$, i.e., every such ball contains a function in $C(\mathbb{R}) \setminus \mathcal{F}_n$. By the corollary to the Weierstrass Approximation Theorem 27.5 on page 220, there is a polynomial p such that $\sup\{|g(y) - p(y)| : y \in [-n-1, n+1]\} < \frac{r}{2}$, $p(-n-1) = g(-n-1)$ and $p(n+1) = g(n+1)$. The function $\overline{p}$ on $\mathbb{R}$, defined by $\overline{p}(y) = p(y)$ for $|y| \leq n+1$ and $\overline{p}(y) = g(y)$ for $|y| \geq n+1$, is in $C(\mathbb{R})$ and satisfies $d(g, \overline{p}) < \frac{r}{2}$. We choose $K \geq n$ so that it also satisfies $K > \sup\{|\overline{p}'(y)| : y \in [-n-1, n+1]\}$. Then for each x in $[-n, n]$, the Mean Value Theorem 29.3 on page 233 implies

$$|\overline{p}(x+h) - \overline{p}(x)| \leq hK \quad \text{for sufficiently small} \quad h > 0. \quad (3)$$

A modification ψ of the sawtooth function ϕ in van der Waerden's Example 38.1 on page 348 satisfies

$$\sup\{|\psi(y)| : y \in \mathbb{R}\} \leq \frac{r}{2} \quad (4)$$

and

$$|\psi(x+h) - \psi(x)| > 2Kh \quad \text{for sufficiently small} \quad h > 0, \tag{5}$$

for all x in $\mathbb{R}$. In fact, using Remark 38.2, it is easy to check $\psi(y) = \frac{r}{4}\phi(\frac{12Ky}{r})$ works. Since $d(\overline{p}, g) < \frac{r}{2}$, inequality (4) shows $\overline{p}+\psi$ belongs to $B_r(g)$. From (5) and (3), we see

$$|(\overline{p}+\psi)(x+h) - (\overline{p}+\psi)(x)| > hK \geq hn$$

for each $x \in [-n, n]$ and sufficiently small $h > 0$, which shows $\overline{p}+\psi$ cannot be in $\mathcal{F}_n$. ■

In Theorem 38.5 below, we give another interesting application of the Baire Category Theorem 21.8. It is due to Corominas and Sunyer Balaguer [17, 1954]. See also Boas [9, Chap. 1, §10], Phillips [52, Chap. 4, §3, Theorem 18], and Donoghue [20, § I.11]. Interesting generalizations can be found in [17] and Krüppel [38]. There are relatively straightforward versions for holomorphic complex-valued functions; see, e.g., Burckel [14, Example 5.64].

First, we give an easy result. Henceforth, when we refer to an open interval (α, β), we allow $\alpha = -\infty$ and $\beta = \infty$, so that in particular (α, β) may be $\mathbb{R}$. As in Sect. 31, we write $f^{(n)}$ for the nth derivative of f.

38.4 Proposition.
Let f be a function on an open interval (α, β), and suppose there is an integer n such that $f^{(n)}(x) = 0$ for all x. Then f agrees with a polynomial on (α, β) of degree at most $n-1$.

Proof
This can be seen by integrating the function $f^{(n)}$ n times. At each step, one obtains a polynomial of one higher degree; see Corollary 29.5 and Exercise 29.7. ■

We write $C^\infty((\alpha, \beta))$ for the space of functions f infinitely differentiable on (α, β), i.e., all the derivatives $f^{(n)}$ exist on (α, β). Here we agree $f^{(0)} = f$.

38.5 Theorem.
Suppose f is in $C^\infty((\alpha,\beta))$, and suppose for each x in (α,β), there is an integer n_x (depending on x) such that $f^{(n_x)}(x)=0$. Then f is a polynomial on (α,β).

We begin with a lemma.

38.6 Lemma.
Suppose x_0 is in $\mathbb{R}$, p and q are polynomials, and f is a C^∞ function defined on an interval containing x_0 that agrees with p on an interval $(x_0-h,x_0]$ and agrees with q on an interval $[x_0,x_0+h)$. Then $p=q$.

Proof
Assume $p\neq q$, and let m be the degree of the polynomial $p-q$. Then $(p-q)^{(m)}(x_0)\neq 0$, contradicting $p^{(m)}(x_0)=f^{(m)}(x_0)=q^{(m)}(x_0)$. ■

Proof of Theorem 38.5
It suffices to show that f agrees with a polynomial on every closed subinterval $[c,d]$ of (α,β). To see this is sufficient, consider a decreasing sequence (c_n) with limit α and an increasing sequence (d_n) with limit β, where $c_1<d_1$. For each n, let p_n be a polynomial that agrees with f on $[c_n,d_n]$. Then each p_n agrees with p_1 on $[c_1,d_1]$. Hence $p_n=p_1$ for all n.[1] It follows that $f=p_1$ on (α,β).

Step 1. In this step, we show that every nondegenerate closed subinterval[2] I of $[c,d]$ contains an open subinterval J on which f is a polynomial. For each $k\geq 0$, we let $E_k=\{x\in I : f^{(k)}(x)=0\}$. Then, by hypothesis $I=\bigcup_{k\geq 0}E_k$, and by the continuity of the function f and all of its derivatives, each E_k is closed. By the Baire Category Theorem 21.8 on page 173, for some index n, E_n has nonempty interior. Thus, for such an index, there is an open subinterval $J\subseteq E_n$ of I such that $f^{(n)}(x)=0$ for all $x\in J$. By Proposition 38.4, f agrees with a polynomial on J, as claimed.

---

[1]Polynomials that agree on an infinite set are equal; otherwise their difference would be a nonzero polynomial with infinitely many zeros.

[2]Nondegenerate here means that its endpoints are not equal.

Step 2. In this step, we identify maximal nondegenerate closed subintervals of $[c, d]$ on which the restriction of f is a polynomial. For any polynomial p that agrees with f on an open interval containing a point x_0 in $[c, d]$, let

$$a_p(x_0) = \inf\{x \in [c, x_0] : f = p \quad \text{on} \quad [x, x_0]\}$$

and

$$b_p(x_0) = \sup\{x \in [x_0, d] : f = p \quad \text{on} \quad [x_0, x]\}.$$

Let $I_p(x_0) = [a_p(x_0), b_p(x_0)]$, and note $a_p(x_0) < b_p(x_0)$ and $I_p(x_0) \subseteq [c, d]$. Then

- **(a)** $I_p(x_0)$ is the maximal closed subinterval of $[c, d]$ containing x_0 on which f agrees with p;
- **(b)** $(a_p(x_0), b_p(x_0))$ is the maximal open subinterval of $[c, d]$ containing x_0 on which f agrees with p;
- **(c)** The closed intervals $I_p(x_0)$ are either disjoint or identical.

To check assertion (c), assume $I_{p_0}(x_0) \cap I_{p_1}(x_1) \neq \varnothing$, where $x_0 \leq x_1$. Clearly $a_{p_1}(x_1) \leq b_{p_0}(x_0)$. If $b_{p_0}(x_0) = a_{p_1}(x_1)$, then f agrees with p_0 to the left of this value, and f agrees with p_1 to the right of this value. But then Lemma 38.6 shows $p_0 = p_1$, so the closed intervals involved aren't maximal. Therefore $a_{p_1}(x_1) < b_{p_0}(x_0)$, and the polynomials p_1 and p_0 agree on the open interval $(a_{p_1}(x_1), b_{p_0}(x_0))$. Hence $p_0 = p_1$. Now f agrees with $p_0 = p_1$ on the closed interval $[a_{p_0}(x_0), b_{p_1}(x_1)]$, so by maximality,

$$[a_{p_0}(x_0), b_{p_1}(x_1)] = [a_{p_0}(x_0), b_{p_0}(x_0)] = [a_{p_1}(x_1), b_{p_1}(x_1)].$$

Therefore $I_{p_0}(x_0) = [a_{p_0}(x_0), b_{p_0}(x_0)] = [a_{p_1}(x_1), b_{p_1}(x_1)] = I_{p_1}(x_1)$, and (c) holds.

Step 3. Our goal is to show all the closed intervals in Step 2 are the same, i.e., all equal $[c, d]$, which would prove the theorem. Assume, on the contrary, that there exist x_1, x_2 in $[c, d]$ and polynomials p_1, p_2 so that $I_{p_1}(x_1) \neq I_{p_2}(x_2)$. We assume $x_1 < x_2$. Then $b_{p_1}(x_1) < a_{p_2}(x_2)$ and we will focus on $[b_{p_1}(x_1), a_{p_2}(x_2)]$, which we will write as $[b, a]$, so

$$I_{p_1}(x_1) \neq I_{p_2}(x_2) \quad \text{and} \quad [b, a] = [b_{p_1}(x_1), a_{p_2}(x_2)]. \tag{1}$$

Consider the nondegenerate closed intervals from Step 2 that are subintervals of (b, a). Since these intervals are disjoint, there are only countably many of them. Moreover, there cannot be finitely many of them, because if there were, there would be an open interval in (b, a) that is disjoint from the finite union of them. By Step 1, there would be another closed interval as described in Step 2, a contradiction.

We relabel the countably infinite disjoint family of maximal *open* intervals in Step 2, that are subsets of (b, a), by (a_k, b_k). Observe $\bigcup_k (a_k, b_k)$ is dense in $[b, a]$; otherwise there is a nonempty open interval in $[b, a] \setminus \bigcup_k (a_k, b_k)$ and by applying Step 1 we could expand our family. Let $K = [b, a] \setminus \bigcup_k (a_k, b_k)$, which is clearly closed. Let E be the set of endpoints of the intervals $[b, a]$ and all (a_k, b_k). Observe that $E \subseteq K$. Note that, by (1), b and a are right and left endpoints, respectively, of maximal open intervals.

Step 4. In this step, we show that K is a perfect set, as defined in Discussion 21.10 on page 174, i.e., every point x in K is the limit of a sequence of points in $K \setminus \{x\}$. Otherwise, for some x in K and $h > 0$ we have $K \cap (x - h, x + h) = \{x\}$. The point x cannot be both a left and a right endpoint in E, so we may assume, say, that x is not a left endpoint in E. Then $x \neq a$ and we may suppose that $(x, x+h) \subseteq (b, a)$. Since $(x, x+h)$ intersects the dense set $\bigcup_k [a_k, b_k]$, we have $(x, x + h) \cap [a_k, b_k] \neq \varnothing$ for some k. Then $a_k < x + h$ and $x < b_k$. Since $x \neq a_k$, x is not in $\{a_k\} \cup (a_k, b_k) = [a_k, b_k)$. Since $x < b_k$, we have $x < a_k < x + h$ and a_k belongs to $(x, x + h)$. Thus $a_k \in K \cap (x, x + h)$, a contradiction. Thus K is perfect.

Step 5. In this step, we obtain a contradiction to our hypotheses by showing there is $x_0 \in K$ satisfying $f^{(n)}(x_0) \neq 0$ for all $n \geq 0$. This will prove the theorem.

Claim 1. Let J be an open interval in $[b, a]$ satisfying $J \cap K \neq \varnothing$, and consider an integer $n \geq 0$. Then exists an $x \in J \cap K$ such that $f^{(n)}(x) \neq 0$.

Assume that J is as specified, but that n_0 satisfies $f^{(n_0)}(x) = 0$ for all $x \in J \cap K$. We show by induction

$$f^{(n)}(x) = 0 \quad \text{for} \quad n \geq n_0 \quad \text{and all} \quad x \in J \cap K. \tag{2}$$

Suppose (2) holds for some $n \geq n_0$, and consider $x \in J \cap K$. Since K is perfect, there is a sequence (x_k) in $K \setminus \{x\}$ such that $\lim_k x_k = x$.

Since J is open, we may suppose that all x_k are also in J. Hence

$$f^{(n+1)}(x) = \lim_{k\to\infty} \frac{f^{(n)}(x_k) - f^{(n)}(x)}{x_k - x} = \lim_{k\to\infty} \frac{0-0}{x_k - x} = 0;$$

i.e., (2) holds for $n+1$. By induction, we conclude that (2) holds as stated.

On the other hand, if $x \in J \setminus K$ then x belongs to one of the maximal open intervals (a_k, b_k). Since $J \cap K \neq \varnothing$, there exists y in $J \setminus (a_k, b_k)$, so that $y \leq a_k$ or $y \geq b_k$, say $y \geq b_k$. Since $x < b_k$ and J is an interval, J also contains b_k. Thus b_k belongs to $J \cap K$. Let p_k be the polynomial that agrees with f on (a_k, b_k). Since x is in (a_k, b_k), we have $f^{(j)}(x) = p_k^{(j)}(x)$ for all $j \geq 0$. If p_k is the zero polynomial, then clearly $f^{(n_0)}(x) = p_k^{(n_0)}(x) = 0$. Otherwise, let n_k be the degree of the polynomial p_k, so that $p^{(n_k)}$ is the nonzero constant function $p^{(n_k)}(b_k)$. Since b_k is in $J \cap K$, (2) implies that $n_k < n_0$ and $f^{(n_0)}(x) = p^{(n_0)}(x) = 0$. Summarizing, $f^{(n_0)}(x) = 0$ for all x in the open interval J, so f agrees with a polynomial on J by Proposition 38.4 and $J \subseteq \bigcup_k (a_k, b_k)$, contradicting $J \cap K \neq \varnothing$. This proves Claim 1.

Since each $f^{(n)}$ is continuous, Claim 1 implies:

Claim 2. If $n \geq 0$ and J is an open interval in $[b, a]$ satisfying $J \cap K \neq \varnothing$, then there is an open interval J' so that $\overline{J'} \subseteq J$, $J' \cap K \neq \varnothing$ and $f^{(n)}(x) \neq 0$ for all $x \in J' \cap K$.[3]

Applying Claim 2 to $J = (b, a)$, we obtain an open interval J_0 so that $J_0 \cap K \neq \varnothing$ and $f(x) \neq 0$ for $x \in J_0 \cap K$. Repeated use of the claim yields a sequence (J_n) of open intervals so that

$$J_0 \supseteq \overline{J_1} \supseteq J_1 \supseteq \overline{J_2} \supseteq J_2 \supseteq \cdots, \tag{3}$$

each $J_n \cap K \neq \varnothing$ and $f^{(n)}(x) \neq 0$ for all $x \in J_n \cap K$. Since $\bigcap_n J_n \cap K = \bigcap_n \overline{J_n} \cap K \neq \varnothing$, by Theorem 13.10 there is x_0 in K that belongs to all the intervals J_n. Therefore $f^{(n)}(x_0) \neq 0$ for all $n \geq 0$; this completes Step 5 and the proof of the theorem. ■

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[3]The notation $\overline{J'}$ represents the closure of the interval J'.

38.7 Corollary.
Suppose $-\infty < \alpha < \beta < \infty$. If f is continuous on $[\alpha, \beta]$, and if f satisfies the hypotheses of Theorem 38.5 on (α, β), then f agrees with a polynomial on $[\alpha, \beta]$.

Even though continuous nowhere-differentiable functions are plentiful, by Theorem 38.3, the next theorem shows every continuous function on $\mathbb{R}$ is arbitrarily close to nice functions.

38.8 Theorem.
Given a continuous function f on $\mathbb{R}$ (bounded or not) and $\epsilon > 0$, there is a piecewise linear function[4] g and a function ψ in $C^\infty(\mathbb{R})$ such that $\sup\{|f(x) - g(x)| : x \in \mathbb{R}\} < \epsilon$ *and* $\sup\{|f(x) - \psi(x)| : x \in \mathbb{R}\} < \epsilon$.

Proof
If you look at pictures of a couple of continuous functions f, as well as $f+\epsilon$ and $f-\epsilon$, as in Fig. 24.1 on page 195, you might conclude the existence of such a piecewise linear function is obvious. The only flaw in this thinking is that, as we know, general continuous functions can be quite complicated, and impossible to graph. For this reason, we include a proof.

Step 1. If f is continuous on $[a, b]$ and $\epsilon > 0$, then there is a piecewise linear function g on $[a, b]$ satisfying $\sup\{|f(x) - g(x)| : x \in [a, b]\} < \epsilon$. To see this, we use uniform continuity of f to obtain $\delta > 0$ so that

$$|s - t| < \delta, \quad s, t \in [a, b] \quad \text{implies} \quad |f(s) - f(t)| < \frac{\epsilon}{3}.$$

Select a partition $\{a = t_0 < t_1 < \cdots < t_n = b\}$ of $[a, b]$ with mesh $< \delta$. For each $k = 1, 2, \ldots, n$, let ℓ_k be the linear function satisfying $\ell_k(t_{k-1}) = f(t_{k-1})$ and $\ell_k(t_k) = f(t_k)$. We write m_k for $\min\{f(t_{k-1}), f(t_k)\}$ and M_k for $\max\{f(t_{k-1}), f(t_k)\}$. Then for x in $[t_{k-1}, t_k]$, we have

$$m_k - \frac{\epsilon}{3} < f(x) < M_k + \frac{\epsilon}{3} \quad \text{and} \quad M_k - m_k < \frac{\epsilon}{3}.$$

[4] A piecewise linear function on an interval is a continuous function that is linear on subintervals.

Thus $f(x)$ lies between $m_k - \frac{\epsilon}{3}$ and $m_k + \frac{2\epsilon}{3}$. The same is true for $\ell_k(x)$, and therefore

$$|f(x) - \ell_k(x)| < \epsilon \quad \text{for} \quad x \in [t_{k-1}, t_k].$$

The function g on $[a, b]$, defined by $g(x) = \ell_k(x)$ for x in $[t_{k-1}, t_k]$, satisfies $\sup\{|f(x) - g(x)| : x \in [a, b]\} < \epsilon$.

Step 2. Let ℓ be a linear function on an interval $[a, b]$. There is a monotonic function ψ in $C^\infty(\mathbb{R})$ so that $\psi(a) = \ell(a)$, $\psi(b) = \ell(b)$, all derivatives of ψ at a and b are 0, and

$$|\psi(x) - \ell(x)| \leq |\ell(b) - \ell(a)| \quad \text{for all} \quad x \in [a, b]. \tag{1}$$

We begin with the function $h = h^*_{a,b}$ in Exercise 31.4 on page 267. Then $h(x) = 0$ for $x \leq a$, $h(x) = 1$ for $x \geq b$, h is increasing on $[a, b]$, and all its derivative are 0 at a and at b. In fact,

$$h(x) = \frac{f(x-a)}{f(x-a) + f(b-x)}$$

where f is the function in Example 3 and Fig. 31.1 on page 257. From the quotient rule, and writing D for the denominator obtained using the quotient rule, we see

$$h'(x) = [f(b-x)f'(x-a) + f(x-a)f'(b-x)]/D(x).$$

This is nonnegative, so h is increasing. By looking at the right-hand derivatives of h at a, and the left-hand derivatives at b, we see $h^{(n)}(a) = h^{(n)}(b) = 0$ for all n in $\mathbb{N}$.[5] Let

$$\psi(x) = \ell(a) + [\ell(b) - \ell(a)] \cdot h(x) \quad \text{for} \quad x \in [a, b].$$

Condition (1) holds because, for x in $[a, b]$, both $\psi(x)$ and $\ell(x)$ lie between $\ell(a)$ and $\ell(b)$.

Step 3. Given a piecewise linear function g on $[a, b]$ and $\epsilon > 0$, there is a function ψ in $C^\infty(\mathbb{R})$ so that $\psi(a) = g(a)$, $\psi(b) = g(b)$, all derivatives of ψ at a and b are 0, and

$$\sup\{|\psi(x) - g(x)| : x \in [a, b]\} < \epsilon. \tag{2}$$

---

[5]Regarding why h, the quotient of two infinitely differentiable functions, also is infinitely differentiable, see Exercise 38.6.

As in Step 1, there is a partition $\{a = t_0 < t_1 < \cdots < t_n = b\}$ of $[a, b]$ such that $|g(t_k) - g(t_{k-1})| < \epsilon$ and such that g agrees on each $[t_{k-1}, t_k]$ with a linear function ℓ_k, $k = 1, 2, \ldots, n$. Apply Step 2 to each $[t_{k-1}, t_k]$ to obtain a function ψ_k in $C^\infty(\mathbb{R})$ so that $\psi_k(t_{k-1}) = \ell_k(t_{k-1}) = g(t_{k-1})$ and similarly $\psi_k(t_k) = g(t_k)$, all derivatives at t_{k-1} and t_k are 0, and

$$|\psi_k(x) - g(x)| \leq |\ell_k(t_k) - \ell_k(t_{k-1})| < \epsilon \quad \text{for all} \quad x \in [t_{k-1}, t_k].$$

The desired ψ is defined by $\psi(x) = \psi_k(x)$ for $x \in [t_{k-1}, t_k]$, $k = 1, 2, \ldots, n$.

Step 4. Consider a continuous function f on $\mathbb{R}$ and $\epsilon > 0$. By Step 1, for each k in $\mathbb{Z}$, there is a piecewise linear function g_k on the interval $[k, k+1]$ satisfying

$$\sup\{|g_k(x) - f(x)| : x \in [k, k+1]\} < \frac{\epsilon}{3}. \tag{3}$$

We apply Step 3 to each interval $[k, k+1]$ to obtain ψ_k in $C^\infty(\mathbb{R})$ so that $\psi_k(k) = f(k)$, $\psi_k(k+1) = f(k+1)$, all derivatives of ψ_k are 0 at k and $k+1$, and

$$\sup\{|\psi_k(x) - g_k(x)| : x \in [k, k+1]\} < \frac{\epsilon}{3}. \tag{4}$$

We obtain ψ in $C^\infty(\mathbb{R})$ by defining $\psi(x) = \psi_k(x)$ for x in $[k, k+1]$, $k \in \mathbb{Z}$. From (3) and (4) we conclude

$$\sup\{|\psi(x) - f(x)| : x \in \mathbb{R}\} \leq \frac{2\epsilon}{3} < \epsilon.$$

■

38.9 Corollary.
The space of all bounded piecewise linear functions and the space $C^\infty(\mathbb{R}) \cap C(\mathbb{R})$ of all bounded infinitely differentiable functions on $\mathbb{R}$ are dense in $C(\mathbb{R})$, using the distance $d(f, g) = \sup\{|f(y) - g(y)| : y \in \mathbb{R}\}$ defined prior to Theorem 38.3.

The next result is both a corollary to Theorem 38.8 and a lemma for the next result.

38.10 Lemma.
Consider continuous f on $[a, b]$, $\epsilon > 0$, and a constant $c > 0$. Then there is a piecewise linear function g on $[a, b]$ such that

$\sup\{|f(x) - g(x)| : x \in [a,b]\} < \epsilon$ *and the absolute value of the slope of each of its linear segments exceeds* c.

Proof

Before continuing, you might wish to reread the first paragraph of the proof of Theorem 38.8. Since, by that theorem, it now suffices to consider the case when f itself is a linear function ℓ, it will be easy to see why this lemma is true.

It suffices to consider the case when ℓ is increasing, and we may assume c exceeds the slope of ℓ. There is a $\delta > 0$ sufficiently small that one can replace any line segment from $(x, \ell(x))$ to $(x+\delta, \ell(x+\delta))$ by two line segments, one rising from $(x, \ell(x))$, with slope exceeding c, to the line $\ell + \frac{\epsilon}{2}$, and the other dropping down to $(x+\delta, f(x+\delta))$, at a slope less than $-c$. By replacing δ by a smaller δ, if necessary, we may assume δ divides $b - a$. Then there is partition $\{t_0 = a < t_1 < \cdots < t_n = b\}$ of $[a,b]$, with $t_k = t_{k-1} + \delta$ for each k, so that each line segment from $(t_{k-1}, \ell(t_{k-1}))$ to $(t_k, \ell(t_k))$ can be replaced by two line segments each with slope exceeding c in absolute value. The resulting piecewise linear function[6] g on $[a,b]$ is continuous because it agrees with $f = \ell$ at each t_k, and it satisfies

$$\sup\{|g(x) - f(x)| : x \in [a,b]\} = \frac{\epsilon}{2} < \epsilon.$$

This establishes the lemma. ∎

We now return to continuous nowhere-differentiable functions. So far as we know, until 1992 the construction of such functions has involved summing certain continuous functions, i.e., building them up from simpler functions. The next construction, due to Mark Lynch [44] in 1992, is quite different, using some topology and no infinite series. Lynch starts with certain subsets of the plane and shrinks them down to the graph of a continuous nowhere-differentiable function. He uses Example 6 on page 171, which states that a function on a compact subset of $\mathbb{R}$ is continuous if and only if its graph is compact. Later Lynch [45] generalized his technique to construct continuous functions that are differentiable only at the rationals.

[6] It is tempting to call this a "stegosaurus linear function."

Below, we simplify the technique in [45] just to obtain a continuous nowhere-differentiable function.

38.11 Mark Lynch's Construction.

Step 1. Let ℓ be a linear function on an interval $[a,b]$. Then its graph L is a line segment in the plane. We assume the slope of ℓ exceeds $2n$ in absolute value. We will construct a (closed) rhomboid[7] R about L with the following property:

> **Property $\mathbf{R}_n$:** If f is a function on $[a,b]$ and if the graph of f is contained in R, then for each $x \in [a,b]$ there is $y \in [a,b]$ such that $x \neq y$ and $|\frac{f(y)-f(x)}{y-x}| > n$.

First, we suppose the slope exceeds $2n$. Let $m = \frac{a+b}{2}$ be the midpoint of the interval $[a,b]$. Then we have

$$\frac{\ell(b)-\ell(m)}{b-a} = \frac{\ell(m)-\ell(a)}{b-a} > n;$$

thus there exists $\epsilon > 0$ so that

$$\frac{\ell(b)-\epsilon-[\ell(m)+\epsilon]}{b-a} > n \quad \text{and} \quad \frac{\ell(m)-\epsilon-[\ell(a)+\epsilon]}{b-a} > n. \tag{1}$$

The desired rhomboid R is bounded above and below by the graphs of $\ell+\epsilon$ and $\ell-\epsilon$, and bounded on the left and right by the vertical line segments connecting $(x,\ell(x)-\epsilon)$ and $(x,\ell(x)+\epsilon)$ for $x=a$ and for $x=b$. Consider x in $[a,b]$. If $x \leq m$, then using (1) we obtain

$$\frac{f(b)-f(x)}{b-x} \geq \frac{\ell(b)-\epsilon-[\ell(x)+\epsilon]}{b-a} \geq \frac{\ell(b)-\epsilon-[\ell(m)+\epsilon]}{b-a} > n,$$

so Property R_n holds with $y=b$. If $x \geq m$, then

$$\frac{f(x)-f(a)}{x-a} \geq \frac{\ell(x)-\epsilon-[\ell(a)+\epsilon]}{b-a} \geq \frac{\ell(m)-\epsilon-[\ell(a)+\epsilon]}{b-a} > n,$$

and Property R_n holds with $y=a$.

Now suppose the slope of ℓ is less than $-2n$. Then use $-\ell$ to obtain a rhomboid R' satisfying Property R_n. Let $R = -R'$. If the

[7] For us, a rhomboid is a parallelogram with unequal adjacent angles.

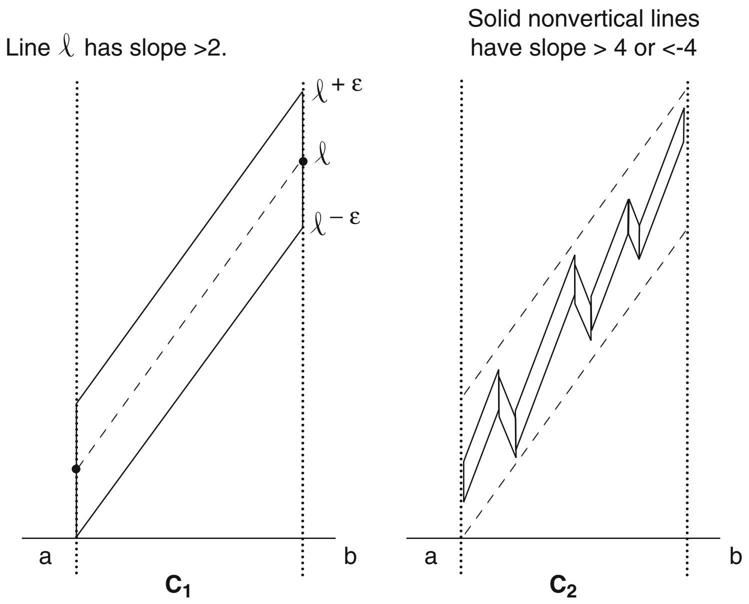

FIGURE 38.1

graph of f is contained in R, then the graph of $-f$ is contained in R', and Property R_n holds for $-f$ and R'. So it holds for f and R.

Step 2. Now we construct a continuous nowhere-differentiable function on $[0, 1]$. We begin with a piecewise linear function g_1 on $[0, 1]$ such that the absolute value of the slope of each segment exceeds 2. Note g_1 can be selected to be linear on $[a, b]$, as illustrated in Fig. 38.1, and observe the x and y axes are not drawn to scale. Let C_1 be the union of rhomboids surrounding the line segments of g_1, where each rhomboid satisfies Property R_n for $n = 1$. Then C_1 satisfies the following property for $n = 1$.

> **Property n:** If f is a function on $[0, 1]$ and if the graph of f is contained in C_n, then for each $x \in [0, 1]$ there is $y \in [0, 1]$ such that $0 < |y - x| \leq \frac{1}{n}$ and $\left|\frac{f(y)-f(x)}{y-x}\right| > n$.

For the inductive construction, suppose we have $C_1, \ldots, C_{n-1}$ satisfying $C_{n-1} \subseteq \cdots \subseteq C_1$, where each C_k is a union of rhomboids satisfying Property k. By Lemma 38.10, there is a piecewise linear function g_n on $[0,1]$ whose graph lies in the *interior* of C_{n-1} such that the absolute value of the slope of each segment exceeds $2n$. By breaking segments of g_n into smaller segments, if necessary, we may assume their domains, i.e., their projections onto the x-axis, each have length $\le \frac{1}{n}$. By Step 1, each such segment of g_n is included in a rhomboid satisfying Property R_n, with the additional conclusion $0 < |y - x| \le \frac{1}{n}$. Therefore C_n satisfies Property n. Moreover, since the graph of g_n lies in the interior of C_{n-1}, the construction in Step 1 shows (using a smaller ϵ if necessary) we can select the rhomboids so that their union C_n is a subset of C_{n-1} and so that the vertical line segments in C_n have length $\le \frac{1}{n}$.

Now let $C = \cap_{n=1}^{\infty} C_n$. Each set C_n is a closed and bounded set in the plane, hence compact. By Theorem 13.10 on page 89, C is compact and nonempty. Also, for each x in $[0,1]$, the sets $\{y \in [0,1] : (x,y) \in C_n\}$ are compact and nonempty and, since the vertical line segments in C_n have length $\le \frac{1}{n}$, their intersection $\{y \in [0,1] : (x,y) \in C\}$ has exactly one point. Thus C is the graph of a function f on $[0,1]$. Since its graph is compact, Example 6 on page 171 shows f is continuous. Finally, consider x in $[0,1]$. For each n, using Property n we obtain y_n in $[0,1]$ so that $0 < |y_n - x| \le \frac{1}{n}$ and $\left|\frac{f(y_n)-f(x)}{y_n - x}\right| > n$. Thus f is not differentiable at x. □

Exercises

38.1 Why hasn't anyone constructed a nowhere-differentiable continuous function by simply finding a suitably complicated Taylor series?

38.2 Show that if f is a function on $\mathbb{R}$ and $f^{(n)}$ is a polynomial for some n, then f is itself a polynomial.

38.3 Show that there is a differentiable function on $\mathbb{R}$ whose derivative is nowhere differentiable.

38.4 If f is nowhere differentiable and g is differentiable on $\mathbb{R}$, must $f + g$ be nowhere differentiable?

38.5 Consider continuous nowhere-differentiable functions f and g on $\mathbb{R}$.

(a) Must $f + g$ be nowhere differentiable?

(b) Show fg need not be nowhere differentiable. *Hint*: Let f be a nowhere-differentiable function satisfying $d(f, 1^*) < 1$, where 1^* denotes the constant function identically equal to 1 on $\mathbb{R}$. Explain why the reciprocal g of f is nowhere-differentiable on $\mathbb{R}$.

38.6 In Exercise 31.4 it was noted, without proof, that the sum, product, etc. of infinitely differentiable functions is again infinitely differentiable. Also, f/g is infinitely differentiable if both f and g are, and if g never vanishes. Verify the last claim. *Hint*: Show the nth derivative of f/g has the form h_n/g^{2^n} where h_n is infinitely differentiable.