CHAPTER 5 Differentiation

In this chapter we give a theoretical treatment of differentiation and related concepts, most or all of which will be familiar from the standard calculus course. Three of the most useful results are the Mean Value Theorem which is treated in §29, L'Hospital's Rule which is treated in §30, and Taylor's Theorem which is given in §31.

§28 Basic Properties of the Derivative

The reader may wish to review the theory of limits treated in §20.

28.1 Definition.
Let f be a real-valued function defined on an open interval containing a point a. We say f is *differentiable at* a, or f *has a derivative at* a, if the limit

$$\lim_{x\to a}\frac{f(x)-f(a)}{x-a}$$

K.A. Ross, *Elementary Analysis: The Theory of Calculus*, Undergraduate Texts in Mathematics, DOI 10.1007/978-1-4614-6271-2_5,

exists and is finite. We will write $f'(a)$ for the derivative of f at a:

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \tag{1}$$

whenever this limit exists and is finite.

Generally speaking, we will be interested in f' as a function in its own right. The domain of f' is the set of points at which f is differentiable; thus $\mathrm{dom}(f') \subseteq \mathrm{dom}(f)$.

Example 1
The derivative of the function $g(x) = x^2$ at $x = 2$ was calculated in Example 2 of §20:

$$g'(2) = \lim_{x \to 2} \frac{x^2 - 4}{x - 2} = \lim_{x \to 2} (x + 2) = 4.$$

We can calculate $g'(a)$ just as easily:

$$g'(a) = \lim_{x \to a} \frac{x^2 - a^2}{x - a} = \lim_{x \to a} (x + a) = 2a.$$

This computation is even valid for $a = 0$. We may write $g'(x) = 2x$ since the name of the variable a or x is immaterial. Thus the derivative of the function given by $g(x) = x^2$ is the function given by $g'(x) = 2x$, as every calculus student knows. □

Example 2
The derivative of $h(x) = \sqrt{x}$ at $x = 1$ was calculated in Example 3 of §20: $h'(1) = \frac{1}{2}$. In fact, $h(x) = x^{1/2}$ for $x \geq 0$ and $h'(x) = \frac{1}{2}x^{-1/2}$ for $x > 0$; see Exercise 28.3. □

Example 3
Let n be a positive integer, and let $f(x) = x^n$ for all $x \in \mathbb{R}$. We show $f'(x) = nx^{n-1}$ for all $x \in \mathbb{R}$. Fix a in $\mathbb{R}$ and observe

$$f(x) - f(a) = x^n - a^n = (x - a)(x^{n-1} + ax^{n-2} + a^2x^{n-3} + \cdots + a^{n-2}x + a^{n-1}),$$

so

$$\frac{f(x) - f(a)}{x - a} = x^{n-1} + ax^{n-2} + a^2x^{n-3} + \cdots + a^{n-2}x + a^{n-1}$$

for $x \neq a$. It follows that

$$\begin{aligned} f'(a) &= \lim_{x\to a} \frac{f(x)-f(a)}{x-a} \\ &= a^{n-1} + aa^{n-2} + a^2a^{n-3} + \cdots + a^{n-2}a + a^{n-1} = na^{n-1}; \end{aligned}$$

we are using Theorem 20.4 and the fact that $\lim_{x\to a} x^k = a^k$ for k in $\mathbb{N}$. □

We first prove differentiability at a point implies continuity at the point. This may seem obvious from all the pictures of familiar differentiable functions. However, Exercise 28.8 contains an example of a function that is differentiable at 0 and of course continuous at 0 [by the next theorem], but is discontinuous at all other points.

28.2 Theorem.
If f is differentiable at a point a, then f is continuous at a.

Proof
We are given $f'(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a}$, and we need to prove $\lim_{x\to a} f(x) = f(a)$. We have

$$f(x) = (x-a)\frac{f(x)-f(a)}{x-a} + f(a)$$

for $x \in \operatorname{dom}(f)$, $x \neq a$. Since $\lim_{x\to a}(x-a) = 0$ and $\lim_{x\to a} \frac{f(x)-f(a)}{x-a}$ exists and is finite, Theorem 20.4(ii) shows $\lim_{x\to a}(x-a)\cdot \frac{f(x)-f(a)}{x-a} = 0$. Therefore $\lim_{x\to a} f(x) = f(a)$, as desired. ■

We next prove some results about sums, products, etc. of derivatives. Let us first recall why the product rule is *not* $(fg)' = f'g'$ [as many naive calculus students wish!] even though the product of limits does behave as expected:

$$\lim_{x\to a}(f_1f_2)(x) = \left[\lim_{x\to a} f_1(x)\right] \cdot \left[\lim_{x\to a} f_2(x)\right]$$

provided the limits on the right side exist and are finite; see Theorem 20.4(ii). The difficulty is that the limit for the derivative of the product is not the product of the limits of the derivatives, i.e.,

$$\frac{f(x)g(x)-f(a)g(a)}{x-a} \neq \frac{f(x)-f(a)}{x-a} \cdot \frac{g(x)-g(a)}{x-a}.$$

The correct product rule is obtained by shrewdly writing the left hand side in terms of $\frac{f(x)-f(a)}{x-a}$ and $\frac{g(x)-g(a)}{x-a}$ as in the proof of Theorem 28.3(iii) below.

28.3 Theorem.
Let f and g be functions that are differentiable at the point a. Each of the functions cf [c a constant], $f+g$, fg and f/g is also differentiable at a, except f/g if $g(a) = 0$ since f/g is not defined at a in this case. The formulas are

(i) $(cf)'(a) = c \cdot f'(a)$;

(ii) $(f+g)'(a) = f'(a) + g'(a)$;

(iii) *[product rule]* $(fg)'(a) = f(a)g'(a) + f'(a)g(a)$;

(iv) *[quotient rule]* $(f/g)'(a) = [g(a)f'(a) - f(a)g'(a)]/g^2(a)$ *if* $g(a) \neq 0$.

Proof

(i) By definition of cf we have $(cf)(x) = c \cdot f(x)$ for all $x \in \mathrm{dom}(f)$; hence

$$(cf)'(a) = \lim_{x \to a} \frac{(cf)(x) - (cf)(a)}{x - a} = \lim_{x \to a} c \cdot \frac{f(x) - f(a)}{x - a} = c \cdot f'(a).$$

(ii) This follows from the identity

$$\frac{(f+g)(x) - (f+g)(a)}{x - a} = \frac{f(x) - f(a)}{x - a} + \frac{g(x) - g(a)}{x - a}$$

upon taking the limit as $x \to a$ and applying Theorem 20.4(i).

(iii) Observe

$$\frac{(fg)(x) - (fg)(a)}{x - a} = f(x)\frac{g(x) - g(a)}{x - a} + g(a)\frac{f(x) - f(a)}{x - a}$$

for $x \in \mathrm{dom}(fg)$, $x \neq a$. We take the limit as $x \to a$ and note that $\lim_{x \to a} f(x) = f(a)$ by Theorem 28.2. We obtain [again using Theorem 20.4]

$$(fg)'(a) = f(a)g'(a) + g(a)f'(a).$$

(iv) Since $g(a) \neq 0$ and g is continuous at a, there exists an open interval I containing a such that $g(x) \neq 0$ for $x \in I$. For $x \in I$ we can write

$$(f/g)(x) - (f/g)(a) = \frac{f(x)}{g(x)} - \frac{f(a)}{g(a)} = \frac{g(a)f(x) - f(a)g(x)}{g(x)g(a)}$$
$$= \frac{g(a)f(x) - g(a)f(a) + g(a)f(a) - f(a)g(x)}{g(x)g(a)},$$

so

$$\frac{(f/g)(x) - (f/g)(a)}{x-a}$$
$$= \left\{ g(a)\frac{f(x)-f(a)}{x-a} - f(a)\frac{g(x)-g(a)}{x-a} \right\} \frac{1}{g(x)g(a)}$$

for $x \in I$, $x \neq a$. Now we take the limit as $x \to a$ to obtain (iv); note that $\lim_{x\to a} \frac{1}{g(x)g(a)} = \frac{1}{g^2(a)}$. ■

Example 4
Let m be a positive integer, and let $h(x) = x^{-m}$ for $x \neq 0$. Then $h(x) = f(x)/g(x)$ where $f(x) = 1$ and $g(x) = x^m$ for all x. By the quotient rule,

$$h'(a) = \frac{g(a)f'(a) - f(a)g'(a)}{g^2(a)} = \frac{a^m \cdot 0 - 1 \cdot ma^{m-1}}{a^{2m}}$$
$$= \frac{-m}{a^{m+1}} = -ma^{-m-1}$$

for $a \neq 0$. If we write n for $-m$, then we see the derivative of x^n is nx^{n-1} for negative integers n as well as for positive integers. The result is also trivially valid for $n = 0$. For fractional exponents, see Exercise 29.15. □

28.4 Theorem [Chain Rule].
If f is differentiable at a and g is differentiable at $f(a)$, then the composite function $g\circ f$ is differentiable at a and we have $(g\circ f)'(a) = g'(f(a)) \cdot f'(a)$.

Discussion. Here is a faulty "proof" which nevertheless contains the essence of a valid proof. We write

$$\frac{g\circ f(x) - g\circ f(a)}{x-a} = \frac{g(f(x)) - g(f(a))}{f(x)-f(a)} \cdot \frac{f(x)-f(a)}{x-a} \tag{1}$$

for $x \neq a$. Since $\lim_{x\to a} f(x) = f(a)$, we have

$$\lim_{x\to a} \frac{g(f(x)) - g(f(a))}{f(x) - f(a)} = \lim_{y\to f(a)} \frac{g(y) - g(f(a))}{y - f(a)} = g'(f(a)). \tag{2}$$

We also have $\lim_{x\to a} \frac{f(x)-f(a)}{x-a} = f'(a)$, so (1) shows $(g \circ f)'(a) = g'(f(a)) \cdot f'(a)$.

The main problem with this "proof" is that $f(x) - f(a)$ in Eq. (2) might be 0 for x arbitrarily close to a. If, however, $f(x) \neq f(a)$ for x near a, then this proof can be made rigorous. If $f(x) = f(a)$ for some x's near a, the "proof" cannot be repaired using (2). In fact, Exercise 28.5 gives an example of differentiable functions f and g for which $\lim_{x\to 0} \frac{g(f(x))-g(f(0))}{f(x)-f(0)}$ is meaningless.

We now give a rigorous proof, but we will use the sequential definition of a limit. The proof reflects ideas in the article by Stephen Kenton [36].

Proof

The hypotheses include the assumptions that f is defined on an open interval J containing a, and g is defined on an open interval I containing $f(a)$. It is easy to check $g \circ f$ is defined on some open interval containing a (see Exercise 28.13), so by taking J smaller if necessary, we may assume $g \circ f$ is defined on J.

By Definitions 20.3(a) and 20.1, it suffices to consider a sequence (x_n) in $J \setminus \{a\}$ where $\lim_n x_n = a$ and show

$$\lim_{n\to\infty} \frac{g \circ f(x_n) - g \circ f(a)}{x_n - a} = g'(f(a)) \cdot f'(a). \tag{3}$$

For each n, let $y_n = f(x_n)$. Since f is continuous at $x = a$, we have $\lim_n y_n = f(a)$. For $f(x_n) \neq f(a)$,

$$\frac{(g \circ f)(x_n) - (g \circ f)(a)}{x_n - a} = \frac{g(y_n) - g(f(a))}{y_n - f(a)} \cdot \frac{f(x_n) - f(a)}{x_n - a}. \tag{4}$$

Case 1. Suppose $f(x) \neq f(a)$ for x near a. Then $y_n = f(x_n) \neq f(a)$ for large n, so taking the limit in (4), as $n \to \infty$, we obtain $(g \circ f)'(a) = g'(f(a)) \cdot f'(a)$.

Case 2. Suppose $f(x) = f(a)$ for x arbitrarily close to a. Then there is a sequence (z_n) in $J \setminus \{a\}$ such that $\lim_n z_n = a$ and $f(z_n) = f(a)$ for all n. Then

$$f'(a) = \lim_{n\to\infty} \frac{f(z_n) - f(a)}{z_n - a} = \lim_{n\to\infty} \frac{0}{z_n - a} = 0,$$

and it suffices to show $(g \circ f)'(a) = 0$. We do have

$$\lim_{n\to\infty} \frac{g(f(z_n)) - g(f(a))}{z_n - a} = \lim_{n\to\infty} \frac{0}{z_n - a} = 0,$$

but this only assures us that $(g \circ f)'(a) = 0$ *provided* we know this derivative exists. To help us prove this, we note $g'(f(a))$ exists and so the difference quotients $\frac{g(y)-g(f(a))}{y-f(a)}$ are bounded near $f(a)$. Thus, replacing the open interval I by a smaller one if necessary, there is a constant $C > 0$ so that

$$\left| \frac{g(y) - g(f(a))}{y - f(a)} \right| \le C \quad \text{for} \quad y \in I \setminus f(a).$$

Therefore

$$\left| \frac{(g \circ f)(x_n) - (g \circ f)(a)}{x_n - a} \right| \le C \left| \frac{f(x_n) - f(a)}{x_n - a} \right| \quad \text{for large} \quad n, \qquad (5)$$

which is clear if $f(x_n) = f(a)$ and otherwise follows from Eq. (4). Since $f'(a) = 0$, the right side of (5) tends to 0 as $n \to \infty$; thus the left side also tends to 0. Since (x_n) is any sequence in $J \setminus \{a\}$ converging to a, we conclude $(g \circ f)'(a) = 0$, as desired. ■

It is worth emphasizing that if f is differentiable on an interval I and if g is differentiable on $\{f(x) : x \in I\}$, then $(g \circ f)'$ is exactly the function $(g' \circ f) \cdot f'$ on I.

Example 5

Let $h(x) = \sin(x^3 + 7x)$ for $x \in \mathbb{R}$. The reader can undoubtedly verify $h'(x) = (3x^2 + 7)\cos(x^3 + 7x)$ for $x \in \mathbb{R}$ using some automatic technique learned in calculus. Whatever the automatic technique, it is justified by the chain rule. In this case, $h = g \circ f$ where $f(x) = x^3 + 7x$ and $g(y) = \sin y$. Then $f'(x) = 3x^2 + 7$ and $g'(y) = \cos y$ so that

$$h'(x) = g'(f(x)) \cdot f'(x) = [\cos f(x)] \cdot f'(x) = [\cos(x^3 + 7x)] \cdot (3x^2 + 7).$$

We do not want the reader to unlearn the automatic technique, but the reader should be aware that the chain rule stands behind it. □

Exercises

28.1 For each of the following functions defined on $\mathbb{R}$, give the set of points at which it is *not* differentiable. Sketches will be helpful.

(a) $e^{|x|}$ **(b)** $\sin|x|$
(c) $|\sin x|$ **(d)** $|x| + |x-1|$
(e) $|x^2 - 1|$ **(f)** $|x^3 - 8|$

28.2 Use the *definition* of derivative to calculate the derivatives of the following functions at the indicated points.

(a) $f(x) = x^3$ at $x = 2$;

(b) $g(x) = x + 2$ at $x = a$;

(c) $f(x) = x^2 \cos x$ at $x = 0$;

(d) $r(x) = \frac{3x+4}{2x-1}$ at $x = 1$.

28.3 **(a)** Let $h(x) = \sqrt{x} = x^{1/2}$ for $x \geq 0$. Use the *definition* of derivative to prove $h'(x) = \frac{1}{2}x^{-1/2}$ for $x > 0$.

(b) Let $f(x) = x^{1/3}$ for $x \in \mathbb{R}$ and use the definition of derivative to prove $f'(x) = \frac{1}{3}x^{-2/3}$ for $x \neq 0$.

(c) Is the function f in part (b) differentiable at $x = 0$? Explain.

28.4 Let $f(x) = x^2 \sin\frac{1}{x}$ for $x \neq 0$ and $f(0) = 0$.

(a) Use Theorems 28.3 and 28.4 to show f is differentiable at each $a \neq 0$ and calculate $f'(a)$. Use, without proof, the fact that $\sin x$ is differentiable and that $\cos x$ is its derivative.

(b) Use the definition to show f is differentiable at $x = 0$ and $f'(0) = 0$.

(c) Show f' is not continuous at $x = 0$.

28.5 Let $f(x) = x^2 \sin\frac{1}{x}$ for $x \neq 0$, $f(0) = 0$, and $g(x) = x$ for $x \in \mathbb{R}$.

(a) Observe f and g are differentiable on $\mathbb{R}$.

(b) Calculate $f(x)$ for $x = \frac{1}{\pi n}$, $n = \pm 1, \pm 2, \ldots$.

(c) Explain why $\lim_{x\to 0} \frac{g(f(x))-g(f(0))}{f(x)-f(0)}$ is meaningless.

28.6 Let $f(x) = x \sin \frac{1}{x}$ for $x \neq 0$ and $f(0) = 0$. See Fig. 19.3.

(a) Observe f is continuous at $x = 0$ by Exercise 17.9(c).

(b) Is f differentiable at $x = 0$? Justify your answer.

28.7 Let $f(x) = x^2$ for $x \geq 0$ and $f(x) = 0$ for $x < 0$.

(a) Sketch the graph of f.

(b) Show f is differentiable at $x = 0$. *Hint*: You will have to use the definition of derivative.

(c) Calculate f' on $\mathbb{R}$ and sketch its graph.

(d) Is f' continuous on $\mathbb{R}$? differentiable on $\mathbb{R}$?

28.8 Let $f(x) = x^2$ for x rational and $f(x) = 0$ for x irrational.

(a) Prove f is continuous at $x = 0$.

(b) Prove f is discontinuous at all $x \neq 0$.

(c) Prove f is differentiable at $x = 0$. *Warning*: You cannot simply claim $f'(x) = 2x$.

28.9 Let $h(x) = (x^4 + 13x)^7$.

(a) Calculate $h'(x)$.

(b) Show how the chain rule justifies your computation in part (a) by writing $h = g \circ f$ for suitable f and g.

28.10 Repeat Exercise 28.9 for the function $h(x) = (\cos x + e^x)^{12}$.

28.11 Suppose f is differentiable at a, g is differentiable at $f(a)$, and h is differentiable at $g \circ f(a)$. State and prove the chain rule for $(h \circ g \circ f)'(a)$. *Hint*: Apply Theorem 28.4 twice.

28.12 **(a)** Differentiate the function whose value at x is $\cos(e^{x^5 - 3x})$.

(b) Use Exercise 28.11 or Theorem 28.4 to justify your computation in part (a).

28.13 Show that if f is defined on an open interval containing a, if g is defined on an open interval containing $f(a)$, and if f is continuous at a, then $g \circ f$ is defined on an open interval containing a.

28.14 Suppose f is differentiable at a. Prove
(a) $\lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = f'(a)$, **(b)** $\lim_{h \to 0} \frac{f(a+h) - f(a-h)}{2h} = f'(a)$.

28.15 Prove Leibniz' rule

$$(fg)^{(n)}(a) = \sum_{k=0}^{n} \binom{n}{k} f^{(k)}(a) g^{(n-k)}(a)$$

provided both f and g have n derivatives at a. Here $h^{(j)}$ signifies the jth derivative of h so that $h^{(0)} = h$, $h^{(1)} = h'$, $h^{(2)} = h''$, etc. Also, $\binom{n}{k}$ is the binomial coefficient that appears in the binomial expansion; see Exercise 1.12. *Hint*: Use mathematical induction. For $n = 1$, apply Theorem 28.3(iii).

28.16 Let f be a function defined on an open interval I containing a. Show $f'(a)$ exists if and only if there is a function $\epsilon(x)$ defined on I such that

$$f(x) - f(a) = (x - a)[f'(a) - \epsilon(x)] \quad \text{and} \quad \lim_{x \to a} \epsilon(x) = 0.$$

§29 The Mean Value Theorem

Our first result justifies the following strategy in calculus: To find the maximum and minimum of a continuous function f on an interval $[a, b]$ it suffices to consider (a) the points x where $f'(x) = 0$; (b) the points where f is not differentiable; and (c) the endpoints a and b. These are the candidates for maxima and minima.

29.1 Theorem.
If f is defined on an open interval containing x_0, if f assumes its maximum or minimum at x_0, and if f is differentiable at x_0, then $f'(x_0) = 0$.

Proof
We suppose f is defined on (a, b) where $a < x_0 < b$. Since either f or $-f$ assumes its maximum at x_0, we may assume f assumes its maximum at x_0.

Assume first that $f'(x_0) > 0$. Since

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0},$$

there exists $\delta > 0$ such that $a < x_0 - \delta < x_0 + \delta < b$ and

$$0 < |x - x_0| < \delta \quad \text{implies} \quad \frac{f(x) - f(x_0)}{x - x_0} > 0; \tag{1}$$

see Corollary 20.7. If we select x so that $x_0 < x < x_0 + \delta$, then (1) shows $f(x) > f(x_0)$, contrary to the assumption that f assumes its maximum at x_0. Likewise, if $f'(x_0) < 0$, there exists $\delta > 0$ such that

$$0 < |x - x_0| < \delta \quad \text{implies} \quad \frac{f(x) - f(x_0)}{x - x_0} < 0. \tag{2}$$

If we select x so that $x_0 - \delta < x < x_0$, then (2) implies $f(x) > f(x_0)$, again a contradiction. Thus we have $f'(x_0) = 0$. ■

Our next result is fairly obvious except for one subtle point: one needs to know or believe that a continuous function on a closed interval assumes its maximum and minimum. We proved this in Theorem 18.1 using the Bolzano-Weierstrass theorem.

29.2 Rolle's Theorem.
Let f be a continuous function on $[a, b]$ that is differentiable on (a, b) and satisfies $f(a) = f(b)$. There exists [at least one] x in (a, b) such that $f'(x) = 0$.

Proof
By Theorem 18.1, there exist $x_0, y_0 \in [a, b]$ such that $f(x_0) \le f(x) \le f(y_0)$ for all $x \in [a, b]$. If x_0 and y_0 are both endpoints of $[a, b]$, then f is a constant function [since $f(a) = f(b)$] and $f'(x) = 0$ for *all* $x \in (a, b)$. Otherwise, f assumes either a maximum or a minimum at a point x in (a, b), in which case $f'(x) = 0$ by Theorem 29.1. ■

The Mean Value Theorem tells us that a differentiable function on $[a, b]$ must somewhere have its derivative equal to the slope of the line connecting $(a, f(a))$ to $(b, f(b))$, namely $\frac{f(b)-f(a)}{b-a}$. See Fig. 29.1.

29.3 Mean Value Theorem.
Let f be a continuous function on $[a, b]$ that is differentiable on (a, b). Then there exists [at least one] x in (a, b) such that

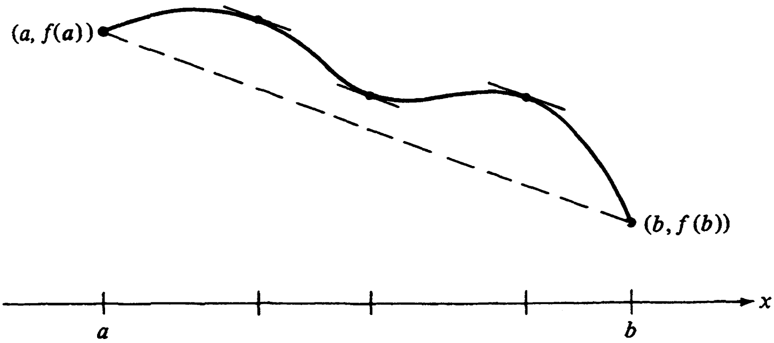

FIGURE 29.1

$$f'(x) = \frac{f(b) - f(a)}{b - a}. \tag{1}$$

Note that Rolle's Theorem is the special case of the Mean Value Theorem where $f(a) = f(b)$.

Proof

Let L be the function whose graph is the straight line connecting $(a, f(a))$ to $(b, f(b))$, i.e., the dotted line in Fig. 29.1. Observe $L(a) = f(a)$, $L(b) = f(b)$ and $L'(x) = \frac{f(b)-f(a)}{b-a}$ for all x. Let $g(x) = f(x) - L(x)$ for $x \in [a, b]$. Clearly g is continuous on $[a, b]$ and differentiable on (a, b). Also $g(a) = 0 = g(b)$, so $g'(x) = 0$ for some $x \in (a, b)$ by Rolle's Theorem 29.2. For this x, we have $f'(x) = L'(x) = \frac{f(b)-f(a)}{b-a}$. ■

29.4 Corollary.

Let f be a differentiable function on (a, b) such that $f'(x) = 0$ for all $x \in (a, b)$. Then f is a constant function on (a, b).

Proof

If f is not constant on (a, b), then there exist x_1, x_2 such that

$$a < x_1 < x_2 < b \quad \text{and} \quad f(x_1) \neq f(x_2).$$

By the Mean Value Theorem, for some $x \in (x_1, x_2)$ we have $f'(x) = \frac{f(x_2)-f(x_1)}{x_2-x_1} \neq 0$, a contradiction. ■

29.5 Corollary.
Let f and g be differentiable functions on (a, b) such that $f' = g'$ on (a, b). Then there exists a constant c such that $f(x) = g(x) + c$ for all $x \in (a, b)$.

Proof
Apply Corollary 29.4 to the function $f - g$. ■

Corollary 29.5 is important for integral calculus because it guarantees all antiderivatives, alias indefinite integrals, for a function differ by a constant. Old integral tables and modern computational software programs provide formulas like

$$\int x^2 \cos x \, dx = 2x \cos x + (x^2 - 2) \sin x + C.$$

It is straightforward to show the derivative of each function $2x \cos x + (x^2 - 2) \sin x + C$ is in fact $x^2 \cos x$. Corollary 29.5 shows that *these are the only antiderivatives of* $x^2 \cos x$.

We need some terminology in order to give another useful corollary of the Mean Value Theorem.

29.6 Definition.
Let f be a real-valued function defined on an interval I. We say f is *strictly increasing on I* if

$$x_1, x_2 \in I \quad \text{and} \quad x_1 < x_2 \quad \text{imply} \quad f(x_1) < f(x_2),$$

strictly decreasing on I if

$$x_1, x_2 \in I \quad \text{and} \quad x_1 < x_2 \quad \text{imply} \quad f(x_1) > f(x_2),$$

increasing on I if

$$x_1, x_2 \in I \quad \text{and} \quad x_1 < x_2 \quad \text{imply} \quad f(x_1) \leq f(x_2),$$

and *decreasing on I* if

$$x_1, x_2 \in I \quad \text{and} \quad x_1 < x_2 \quad \text{imply} \quad f(x_1) \geq f(x_2).$$

Example 1
The functions e^x on $\mathbb{R}$ and $\sqrt{x}$ on $[0, \infty)$ are strictly increasing. The function $\cos x$ is strictly decreasing on $[0, \pi]$. The signum function and the postage-stamp function in Exercises 17.10 and 17.16 are increasing functions but not strictly increasing functions. □

29.7 Corollary.
Let f be a differentiable function on an interval (a,b). Then
(i) *f is strictly increasing if $f'(x) > 0$ for all $x \in (a,b)$;*
(ii) *f is strictly decreasing if $f'(x) < 0$ for all $x \in (a,b)$;*
(iii) *f is increasing if $f'(x) \geq 0$ for all $x \in (a,b)$;*
(iv) *f is decreasing if $f'(x) \leq 0$ for all $x \in (a,b)$.*

Proof
(i) Consider x_1, x_2 where $a < x_1 < x_2 < b$. By the Mean Value Theorem, for some $x \in (x_1, x_2)$ we have

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(x) > 0.$$

Since $x_2 - x_1 > 0$, we see $f(x_2) - f(x_1) > 0$ or $f(x_2) > f(x_1)$.
The remaining cases are left to Exercise 29.8. ■

Exercise 28.4 shows the derivative f' of a differentiable function f need not be continuous. Nevertheless, like a continuous function, f' has the intermediate value property [see Theorem 18.2].

29.8 Intermediate Value Theorem for Derivatives.
Let f be a differentiable function on (a,b). If $a < x_1 < x_2 < b$, and if c lies between $f'(x_1)$ and $f'(x_2)$, there exists [at least one] x in (x_1, x_2) such that $f'(x) = c$.

Proof
We may assume $f'(x_1) < c < f'(x_2)$. Let $g(x) = f(x) - cx$ for $x \in (a,b)$. Then we have $g'(x_1) < 0 < g'(x_2)$. Theorem 18.1 shows g assumes its minimum on $[x_1, x_2]$ at some point $x_0 \in [x_1, x_2]$. Since

$$g'(x_1) = \lim_{y \to x_1} \frac{g(y) - g(x_1)}{y - x_1} < 0,$$

$g(y) - g(x_1)$ is negative for y close to and larger than x_1. In particular, there exists y_1 in (x_1, x_2) such that $g(y_1) < g(x_1)$. Therefore g does not take its minimum at x_1, so we have $x_0 \neq x_1$. Similarly, there exists y_2 in (x_1, x_2) such that $g(y_2) < g(x_2)$, so $x_0 \neq x_2$. We have shown x_0 is in (x_1, x_2), so $g'(x_0) = 0$ by Theorem 29.1. Therefore $f'(x_0) = g'(x_0) + c = c$. ■

We next show how to differentiate the inverse of a differentiable function. Let f be a one-to-one differentiable function on an open interval I. By Theorem 18.6, f is strictly increasing or strictly decreasing on I, and by Corollary 18.3 the image $f(I)$ is an interval J. The set J is the domain of f^{-1} and

$$f^{-1} \circ f(x) = x \quad \text{for} \quad x \in I; \qquad f \circ f^{-1}(y) = y \quad \text{for} \quad y \in J.$$

The formula for the derivative of f^{-1} is easy to obtain [or remember] from the Chain Rule: $x = f^{-1} \circ f(x)$, so

$$1 = (f^{-1})'(f(x)) \cdot f'(x) \quad \text{for all} \quad x \in I.$$

If $x_0 \in I$ and $y_0 = f(x_0)$, then we can write $1 = (f^{-1})'(y_0) \cdot f'(x_0)$ or

$$(f^{-1})'(y_0) = \frac{1}{f'(x_0)} \quad \text{where} \quad y_0 = f(x_0).$$

This is *not* a proof because the Chain Rule requires the functions, f^{-1} and f in this case, be differentiable. We assumed f is differentiable, but we must *prove* f^{-1} is also differentiable. In addition, observe $f'(x_0)$ might be 0 [consider $f(x) = x^3$ at $x_0 = 0$], so our final result will have to avoid this possibility.

29.9 Theorem.
Let f be a one-to-one continuous function on an open interval I, and let $J = f(I)$. If f is differentiable at $x_0 \in I$ and if $f'(x_0) \neq 0$, then f^{-1} is differentiable at $y_0 = f(x_0)$ and

$$(f^{-1})'(y_0) = \frac{1}{f'(x_0)}.$$

Proof
Note that J is also an open interval. We have $\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = f'(x_0)$. Since $f'(x_0) \neq 0$ and since $f(x) \neq f(x_0)$ for x near x_0, we can write

$$\lim_{x \to x_0} \frac{x - x_0}{f(x) - f(x_0)} = \frac{1}{f'(x_0)}; \tag{1}$$

see Theorem 20.4(iii). Let $\epsilon > 0$. By (1) and Corollary 20.7, there exists $\delta > 0$ such that

$$0 < |x - x_0| < \delta \quad \text{implies} \quad \left| \frac{x - x_0}{f(x) - f(x_0)} - \frac{1}{f'(x_0)} \right| < \epsilon. \tag{2}$$

Let $g = f^{-1}$ and observe that g is continuous at y_0 by Theorems 18.6 and 18.4 [or Exercise 18.11]. Hence there exists $\eta > 0$ [lower case Greek eta] such that

$$0 < |y - y_0| < \eta \quad \text{implies} \quad |g(y) - g(y_0)| < \delta, \quad \text{i.e.,} \quad |g(y) - x_0| < \delta. \tag{3}$$

Combining (3) and (2) we obtain

$$0 < |y - y_0| < \eta \quad \text{implies} \quad \left| \frac{g(y) - x_0}{f(g(y)) - f(x_0)} - \frac{1}{f'(x_0)} \right| < \epsilon.$$

Since $\frac{g(y)-x_0}{f(g(y))-f(x_0)} = \frac{g(y)-g(y_0)}{y-y_0}$, this shows

$$\lim_{y \to y_0} \frac{g(y) - g(y_0)}{y - y_0} = \frac{1}{f'(x_0)}.$$

Hence $g'(y_0)$ exists and equals $\frac{1}{f'(x_0)}$. ■

Example 2
Let n be a positive integer, and let $g(y) = \sqrt[n]{y} = y^{1/n}$. If n is even, the domain of g is $[0, \infty)$ and, if n is odd, the domain is $\mathbb{R}$. In either case, g is strictly increasing and its inverse is $f(x) = x^n$; here $\text{dom}(f) = [0, \infty)$ if n is even. Consider y_0 in $\text{dom}(g)$ where $y_0 \neq 0$, and write $y_0 = x_0^n$ where $x_0 \in \text{dom}(f)$. Since $f'(x_0) = nx_0^{n-1}$, Theorem 29.9 shows

$$g'(y_0) = \frac{1}{nx_0^{n-1}} = \frac{1}{ny_0^{(n-1)/n}} = \frac{1}{n} y_0^{1/n-1}.$$

This shows the function g is differentiable for $y \neq 0$ and the rule for differentiating x^n holds for exponents of the form $1/n$; see also Exercise 29.15. □

Theorem 29.9 applies to the various inverse functions encountered in calculus. We give one example.

Example 3
The function $f(x) = \sin x$ is one-to-one on $[-\frac{\pi}{2}, \frac{\pi}{2}]$, and it is traditional to use the inverse g of f restricted to this domain; g is usually denoted Sin^{-1} or arcsin. Note that $\text{dom}(g) = [-1, 1]$. For $y_0 = \sin x_0$ in $(-1, 1)$ where $x_0 \in (-\frac{\pi}{2}, \frac{\pi}{2})$, Theorem 29.9 shows $g'(y_0) = \frac{1}{\cos x_0}$.

Since $1 = \sin^2 x_0 + \cos^2 x_0 = y_0^2 + \cos^2 x_0$ and $\cos x_0 > 0$, we may write

$$g'(y_0) = \frac{1}{\sqrt{1 - y_0^2}} \quad \text{for} \quad y_0 \in (-1, 1).$$

□

Exercises

29.1 Determine whether the conclusion of the Mean Value Theorem holds for the following functions on the specified intervals. If the conclusion holds, give an example of a point x satisfying (1) of Theorem 29.3. If the conclusion fails, state which *hypotheses* of the Mean Value Theorem fail.

(a) x^2 on $[-1, 2]$,
(b) $\sin x$ on $[0, \pi]$,
(c) $|x|$ on $[-1, 2]$,
(d) $\frac{1}{x}$ on $[-1, 1]$,
(e) $\frac{1}{x}$ on $[1, 3]$,
(f) $\text{sgn}(x)$ on $[-2, 2]$.

The function sgn is defined in Exercise 17.10.

29.2 Prove $|\cos x - \cos y| \le |x - y|$ for all $x, y \in \mathbb{R}$.

29.3 Suppose f is differentiable on $\mathbb{R}$ and $f(0) = 0$, $f(1) = 1$ and $f(2) = 1$.

(a) Show $f'(x) = \frac{1}{2}$ for some $x \in (0, 2)$.

(b) Show $f'(x) = \frac{1}{7}$ for some $x \in (0, 2)$.

29.4 Let f and g be differentiable functions on an open interval I. Suppose a, b in I satisfy $a < b$ and $f(a) = f(b) = 0$. Show $f'(x) + f(x)g'(x) = 0$ for some $x \in (a, b)$. *Hint*: Consider $h(x) = f(x)e^{g(x)}$.

29.5 Let f be defined on $\mathbb{R}$, and suppose $|f(x) - f(y)| \le (x - y)^2$ for all $x, y \in \mathbb{R}$. Prove f is a constant function.

29.6 Give the equation of the straight line used in the proof of the Mean Value Theorem 29.3.

29.7 **(a)** Suppose f is twice differentiable on an open interval I and $f''(x) = 0$ for all $x \in I$. Show f has the form $f(x) = ax + b$ for suitable constants a and b.

(b) Suppose f is three times differentiable on an open interval I and $f''' = 0$ on I. What form does f have? Prove your claim.

29.8 Prove (ii)–(iv) of Corollary 29.7.

29.9 Show $ex \leq e^x$ for all $x \in \mathbb{R}$.

29.10 Let $f(x) = x^2 \sin(\frac{1}{x}) + \frac{x}{2}$ for $x \neq 0$ and $f(0) = 0$.

(a) Show $f'(0) > 0$; see Exercise 28.4.

(b) Show f is not increasing on any open interval containing 0.

(c) Compare this example with Corollary 29.7(i).

29.11 Show $\sin x \leq x$ for all $x \geq 0$. *Hint*: Show $f(x) = x - \sin x$ is increasing on $[0, \infty)$.

29.12 **(a)** Show $x < \tan x$ for all $x \in (0, \frac{\pi}{2})$.

(b) Show $\frac{x}{\sin x}$ is a strictly increasing function on $(0, \frac{\pi}{2})$.

(c) Show $x \leq \frac{\pi}{2} \sin x$ for $x \in [0, \frac{\pi}{2}]$.

29.13 Prove that if f and g are differentiable on $\mathbb{R}$, if $f(0) = g(0)$ and if $f'(x) \leq g'(x)$ for all $x \in \mathbb{R}$, then $f(x) \leq g(x)$ for $x \geq 0$.

29.14 Suppose f is differentiable on $\mathbb{R}$, $1 \leq f'(x) \leq 2$ for $x \in \mathbb{R}$, and $f(0) = 0$. Prove $x \leq f(x) \leq 2x$ for all $x \geq 0$.

29.15 Let r be a nonzero rational number $\frac{m}{n}$ where n is a positive integer, m is any nonzero integer, and m and n have no common factors. Let $h(x) = x^r$ where $\text{dom}(h) = [0, \infty)$ if n is even and $m > 0$, $\text{dom}(h) = (0, \infty)$ if n is even and $m < 0$, $\text{dom}(h) = \mathbb{R}$ if n is odd and $m > 0$, and $\text{dom}(h) = \mathbb{R} \setminus \{0\}$ if n is odd and $m < 0$. Show $h'(x) = rx^{r-1}$ for $x \in \text{dom}(h)$, $x \neq 0$. *Hint*: Use Example 2.

29.16 Use Theorem 29.9 to obtain the derivative of the inverse $g = \text{Tan}^{-1} = \arctan$ of f where $f(x) = \tan x$ for $x \in (-\frac{\pi}{2}, \frac{\pi}{2})$.

29.17 Let f and g be differentiable on an open interval I and consider $a \in I$. Define h on I by the rules: $h(x) = f(x)$ for $x < a$, and $h(x) = g(x)$ for $x \geq a$. Prove h is differentiable at a if and only if both $f(a) = g(a)$ and $f'(a) = g'(a)$ hold. *Suggestion*: Draw a picture to see what is going on.

29.18 Let f be differentiable on $\mathbb{R}$ with $a = \sup\{|f'(x)| : x \in \mathbb{R}\} < 1$.

(a) Select $s_0 \in \mathbb{R}$ and define $s_n = f(s_{n-1})$ for $n \geq 1$. Thus $s_1 = f(s_0)$, $s_2 = f(s_1)$, etc. Prove (s_n) is a convergent sequence. *Hint*: To show that (s_n) is Cauchy, first show $|s_{n+1} - s_n| \leq a|s_n - s_{n-1}|$ for $n \geq 1$.

(b) Show f has a *fixed point*, i.e., $f(s) = s$ for some s in $\mathbb{R}$.

§30 * L'Hospital's Rule

In analysis one frequently encounters limits of the form

$$\lim_{x\to s}\frac{f(x)}{g(x)}$$

where s signifies a, a^+, a^-, ∞ or $-\infty$. See Definition 20.3 concerning such limits. The limit exists and is simply $\frac{\lim_{x\to s} f(x)}{\lim_{x\to s} g(x)}$ provided the limits $\lim_{x\to s} f(x)$ and $\lim_{x\to s} g(x)$ exist and are finite and provided $\lim_{x\to s} g(x) \neq 0$; see Theorem 20.4. If these limits lead to an indeterminate form such as $\frac{0}{0}$ or $\frac{\infty}{\infty}$, then L'Hospital's rule can often be used. Moreover, other indeterminate forms, such as $\infty-\infty$, 1^∞, ∞^0, 0^0 or $0\cdot\infty$, can usually be reformulated so as to take the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$; see Examples 5–9. Before we state and prove L'Hospital's rule, we will prove a generalized mean value theorem.

30.1 Generalized Mean Value Theorem.
Let f and g be continuous functions on $[a,b]$ that are differentiable on (a,b). Then there exists [at least one] x in (a,b) such that

$$f'(x)[g(b)-g(a)] = g'(x)[f(b)-f(a)]. \tag{1}$$

This result reduces to the standard Mean Value Theorem 29.3 when g is the function given by $g(x)=x$ for all x.

Proof
The trick is to look at the difference of the two quantities in (1) and hope Rolle's Theorem will help. Thus we define

$$h(x) = f(x)[g(b)-g(a)] - g(x)[f(b)-f(a)];$$

it suffices to show $h'(x)=0$ for some $x\in(a,b)$. Note

$$h(a) = f(a)[g(b)-g(a)] - g(a)[f(b)-f(a)] = f(a)g(b)-g(a)f(b)$$

and

$$h(b) = f(b)[g(b)-g(a)] - g(b)[f(b)-f(a)] = -f(b)g(a)+g(b)f(a) = h(a).$$

Clearly h is continuous on $[a,b]$ and differentiable on (a,b), so Rolle's Theorem 29.2 shows $h'(x)=0$ for at least one x in (a,b). ■

Our proof of L'Hospital's rule below is somewhat wordy but is really quite straightforward. It is based on the elegant presentation in Rudin [62]. Many texts give more complicated proofs.

30.2 L'Hospital's Rule.
Let s signify a, a^+, a^-, ∞ or $-\infty$ where $a \in \mathbb{R}$, and suppose f and g are differentiable functions for which the following limit exists:

$$\lim_{x \to s} \frac{f'(x)}{g'(x)} = L. \tag{1}$$

If

$$\lim_{x \to s} f(x) = \lim_{x \to s} g(x) = 0 \tag{2}$$

or if

$$\lim_{x \to s} |g(x)| = +\infty, \tag{3}$$

then

$$\lim_{x \to s} \frac{f(x)}{g(x)} = L. \tag{4}$$

Note that the hypothesis (1) includes some implicit assumptions: f and g must be defined and differentiable "near" s and $g'(x)$ must be *nonzero* "near" s. For example, if $\lim_{x \to a^+} \frac{f'(x)}{g'(x)}$ exists, then there is an interval (a, b) on which f and g are differentiable and g' is nonzero. The requirement that g' be nonzero is crucial; see Exercise 30.7.

Proof
We first make some reductions. The case of $\lim_{x \to a}$ follows from the cases $\lim_{x \to a^+}$ and $\lim_{x \to a^-}$, since $\lim_{x \to a} h(x)$ exists if and only if the limits $\lim_{x \to a^+} h(x)$ and $\lim_{x \to a^-} h(x)$ exist and are equal; see Theorem 20.10. In fact, we restrict our attention to $\lim_{x \to a^+}$ and $\lim_{x \to -\infty}$, since the other two cases are treated in an entirely analogous manner. Finally, we are able to handle these cases together in view of Remark 20.11.

We assume $a \in \mathbb{R}$ or $a = -\infty$. We will show that if $-\infty \le L < \infty$ and $L_1 > L$, then there exists $\alpha_1 > a$ such that

$$a < x < \alpha_1 \quad \text{implies} \quad \frac{f(x)}{g(x)} < L_1. \tag{5}$$

A similar argument [which we omit] shows that if $-\infty < L \le \infty$ and $L_2 < L$, then there exists $\alpha_2 > a$ such that

$$a < x < \alpha_2 \quad \text{implies} \quad \frac{f(x)}{g(x)} > L_2. \tag{6}$$

We now show how to complete the proof *using* (5) and (6); (5) will be proved in the next paragraph. If L is finite and $\epsilon > 0$, we can apply (5) to $L_1 = L + \epsilon$ and (6) to $L_2 = L - \epsilon$ to obtain $\alpha_1 > a$ and $\alpha_2 > a$ satisfying

$$a < x < \alpha_1 \quad \text{implies} \quad \tfrac{f(x)}{g(x)} < L + \epsilon,$$
$$a < x < \alpha_2 \quad \text{implies} \quad \tfrac{f(x)}{g(x)} > L - \epsilon.$$

Consequently if $\alpha = \min\{\alpha_1, \alpha_2\}$ then

$$a < x < \alpha \quad \text{implies} \quad \left| \frac{f(x)}{g(x)} - L \right| < \epsilon;$$

in view of Remark 20.11 this shows $\lim_{x \to a^+} \frac{f(x)}{g(x)} = L$ [if $a = -\infty$, then $a^+ = -\infty$]. If $L = -\infty$, then (5) and the fact that L_1 is arbitrary show $\lim_{x \to a^+} \frac{f(x)}{g(x)} = -\infty$. If $L = \infty$, then (6) and the fact that L_2 is arbitrary show $\lim_{x \to a^+} \frac{f(x)}{g(x)} = \infty$.

It remains for us to consider $L_1 > L \ge -\infty$ and show there exists $\alpha_1 > a$ satisfying (5). Let (a, b) be an interval on which f and g are differentiable and on which g' never vanishes. Theorem 29.8 shows either g' is positive on (a, b) or else g' is negative on (a, b). The former case can be reduced to the latter case by replacing g by $-g$. So we assume $g'(x) < 0$ for $x \in (a, b)$, so that g is strictly decreasing on (a, b) by Corollary 29.7. Since g is one-to-one on (a, b), $g(x)$ can equal 0 for at most one x in (a, b). By choosing b smaller if necessary, we may assume g never vanishes on (a, b). Now select K

so that $L < K < L_1$. By (1) there exists $\alpha > a$ such that

$$a < x < \alpha \quad \text{implies} \quad \frac{f'(x)}{g'(x)} < K.$$

If $a < x < y < \alpha$, then Theorem 30.1 shows

$$\frac{f(x) - f(y)}{g(x) - g(y)} = \frac{f'(z)}{g'(z)} \quad \text{for some} \quad z \in (x, y).$$

Therefore

$$a < x < y < \alpha \quad \text{implies} \quad \frac{f(x) - f(y)}{g(x) - g(y)} < K. \tag{7}$$

If hypothesis (2) holds, then

$$\lim_{x \to a^+} \frac{f(x) - f(y)}{g(x) - g(y)} = \frac{f(y)}{g(y)},$$

so (7) shows

$$\frac{f(y)}{g(y)} \leq K < L_1 \quad \text{for} \quad a < y < \alpha;$$

hence (5) holds in this case. If hypothesis (3) holds, then $\lim_{x \to a^+} g(x) = +\infty$ since g is strictly decreasing on (a, b). Also $g(x) > 0$ for $x \in (a, b)$ since g never vanishes on (a, b). We multiply both sides of (7) by $\frac{g(x)-g(y)}{g(x)}$, which is positive, to see

$$a < x < y < \alpha \quad \text{implies} \quad \frac{f(x) - f(y)}{g(x)} < K \cdot \frac{g(x) - g(y)}{g(x)}$$

and hence

$$\frac{f(x)}{g(x)} < K \cdot \frac{g(x) - g(y)}{g(x)} + \frac{f(y)}{g(x)} = K + \frac{f(y) - Kg(y)}{g(x)}.$$

We regard y as fixed and observe

$$\lim_{x \to a^+} \frac{f(y) - Kg(y)}{g(x)} = 0.$$

Hence there exists $\alpha_2 > a$ such that $\alpha_2 \leq y < \alpha$ and

$$a < x < \alpha_2 \quad \text{implies} \quad \frac{f(x)}{g(x)} \leq K < L_1.$$

Thus again (5) holds. ■

Example 1

If we assume familiar properties of the trigonometric functions, then $\lim_{x\to 0} \frac{\sin x}{x}$ is easy to calculate by L'Hospital's rule:

$$\lim_{x\to 0} \frac{\sin x}{x} = \lim_{x\to 0} \frac{\cos x}{1} = \cos(0) = 1. \tag{1}$$

Note $f(x) = \sin x$ and $g(x) = x$ satisfy the hypotheses in Theorem 30.2. This particular computation is really dishonest, because the limit (1) is needed to *prove* the derivative of $\sin x$ is $\cos x$. This fact reduces to the assertion that the derivative of $\sin x$ at 0 is 1, i.e., to the assertion

$$\lim_{x\to 0} \frac{\sin x - \sin 0}{x - 0} = \lim_{x\to 0} \frac{\sin x}{x} = 1.$$

□

Example 2

We calculate $\lim_{x\to 0} \frac{\cos x - 1}{x^2}$. L'Hospital's rule will apply provided the limit $\lim_{x\to 0} \frac{-\sin x}{2x}$ exists. but $\frac{-\sin x}{2x} = -\frac{1}{2}\frac{\sin x}{x}$ and this has limit $-\frac{1}{2}$ by Example 1. We conclude

$$\lim_{x\to 0} \frac{\cos x - 1}{x^2} = -\frac{1}{2}.$$

□

Example 3

We show $\lim_{x\to\infty} \frac{x^2}{e^{3x}} = 0$. As written we have an indeterminate of the form $\frac{\infty}{\infty}$. By L'Hospital's rule, this limit will exist provided $\lim_{x\to\infty} \frac{2x}{3e^{3x}}$ exists and by L'Hospital's rule again, this limit will exist provided $\lim_{x\to\infty} \frac{2}{9e^{3x}}$ exists. The last limit is 0, so we conclude $\lim_{x\to\infty} \frac{x^2}{e^{3x}} = 0$. □

Example 4

Consider $\lim_{x\to 0^+} \frac{\log_e x}{x}$ if it exists. By L'Hospital's rule, this appears to be

$$\lim_{x\to 0^+} \frac{1/x}{1} = +\infty$$

and yet this is *incorrect*. The difficulty is that we should have checked the hypotheses. Since $\lim_{x\to 0^+} \log_e x = -\infty$ and $\lim_{x\to 0^+} x = 0$, neither of the hypotheses (2) or (3) in Theorem 30.2 hold. To find the limit, we rewrite $\frac{\log_e x}{x}$ as $-\frac{\log_e(1/x)}{x}$. It is easy to show $\lim_{x\to 0^+}\frac{\log_e(1/x)}{x}$ will agree with $\lim_{y\to\infty} y\log_e y$ provided the latter limit exists; see Exercise 30.4. It follows that $\lim_{x\to 0^+}\frac{\log_e x}{x} = -\lim_{y\to\infty} y\log_e y = -\infty$. □

The next five examples illustrate how indeterminate limits of various forms can be modified so that L'Hospital's rule applies.

Example 5
Consider $\lim_{x\to 0^+} x\log_e x$. As written this limit is of the indeterminate form $0\cdot(-\infty)$ since $\lim_{x\to 0^+} x = 0$ and $\lim_{x\to 0^+}\log_e x = -\infty$. By writing $x\log_e x$ as $\frac{\log_e x}{1/x}$ we obtain an indeterminate of the form $\frac{-\infty}{\infty}$, so we may apply L'Hospital's rule:

$$\lim_{x\to 0^+} x\log_e x = \lim_{x\to 0^+}\frac{\log_e x}{\frac{1}{x}} = \lim_{x\to 0^+}\frac{\frac{1}{x}}{-\frac{1}{x^2}} = -\lim_{x\to 0^+} x = 0.$$

We could also write $x\log_e x$ as $\frac{x}{1/\log_e x}$ to obtain an indeterminate of the form $\frac{0}{0}$. However, an attempt to apply L'Hospital's rule only makes the problem more complicated:

$$\lim_{x\to 0^+} x\log_e x = \lim_{x\to 0^+}\frac{x}{\frac{1}{\log_e x}} = \lim_{x\to 0^+}\frac{1}{\frac{-1}{x(\log_e x)^2}} = -\lim_{x\to 0^+} x(\log_e x)^2.$$

□

Example 6
The limit $\lim_{x\to 0^+} x^x$ is of the indeterminate form 0^0. We write x^x as $e^{x\log_e x}$ [remember $x = e^{\log_e x}$] and note that $\lim_{x\to 0^+} x\log_e x = 0$ by Example 5. Since $g(x) = e^x$ is continuous at 0, Theorem 20.5 shows

$$\lim_{x\to 0^+} x^x = \lim_{x\to 0^+} e^{x\log_e x} = e^0 = 1.$$

□

Example 7
The limit $\lim_{x\to\infty} x^{1/x}$ is of the indeterminate form ∞^0. We write $x^{1/x}$ as $e^{(\log_e x)/x}$. By L'Hospital's rule

$$\lim_{x\to\infty}\frac{\log_e x}{x} = \lim_{x\to\infty}\frac{\frac{1}{x}}{1} = 0.$$

Theorem 20.5 now shows $\lim_{x\to\infty} x^{1/x} = e^0 = 1$. □

Example 8

The limit $\lim_{x\to\infty}(1-\frac{1}{x})^x$ is indeterminate of the form 1^∞. Since

$$\left(1-\frac{1}{x}\right)^x = e^{x\log_e(1-1/x)}$$

we evaluate

$$\lim_{x\to\infty} x\log_e\left(1-\frac{1}{x}\right) = \lim_{x\to\infty}\frac{\log_e\left(1-\frac{1}{x}\right)}{\frac{1}{x}} = \lim_{x\to\infty}\frac{\left(1-\frac{1}{x}\right)^{-1}x^{-2}}{-x^{-2}}$$

$$= \lim_{x\to\infty} -\left(1-\frac{1}{x}\right)^{-1} = -1.$$

So by Theorem 20.5 we have

$$\lim_{x\to\infty}\left(1-\frac{1}{x}\right)^x = e^{-1},$$

as should have been expected since $\lim_{n\to\infty}(1-\frac{1}{n})^n = e^{-1}$. □

Example 9

Consider $\lim_{x\to 0} h(x)$ where

$$h(x) = \frac{1}{e^x-1} - \frac{1}{x} = (e^x-1)^{-1} - x^{-1} \quad \text{for} \quad x \neq 0.$$

Neither of the limits $\lim_{x\to 0}(e^x-1)^{-1}$ or $\lim_{x\to 0} x^{-1}$ exists, so $\lim_{x\to 0} h(x)$ is not an indeterminate form as written. However, $\lim_{x\to 0^+} h(x)$ is indeterminate of the form $\infty - \infty$ and $\lim_{x\to 0^-} h(x)$ is indeterminate of the form $(-\infty) - (-\infty)$. By writing

$$h(x) = \frac{x - e^x + 1}{x(e^x-1)}$$

the limit $\lim_{x\to 0} h(x)$ becomes an indeterminate of the form $\frac{0}{0}$. By L'Hospital's rule this should be

$$\lim_{x\to 0}\frac{1-e^x}{xe^x+e^x-1},$$

which is still indeterminate $\frac{0}{0}$. Note $xe^x + e^x - 1 \neq 0$ for $x \neq 0$ so that the hypotheses of Theorem 30.2 hold. Applying L'Hospital's rule again, we obtain

$$\lim_{x\to 0}\frac{-e^x}{xe^x+2e^x} = -\frac{1}{2}.$$

Note we have $xe^x + 2e^x \neq 0$ for x in $(-2, \infty)$. We conclude $\lim_{x\to 0} h(x) = -\frac{1}{2}$. □

Exercises

30.1 Find the following limits if they exist.

(a) $\lim_{x\to 0} \frac{e^{2x}-\cos x}{x}$ **(b)** $\lim_{x\to 0} \frac{1-\cos x}{x^2}$

(c) $\lim_{x\to\infty} \frac{x^3}{e^{2x}}$ **(d)** $\lim_{x\to 0} \frac{\sqrt{1+x}-\sqrt{1-x}}{x}$

30.2 Find the following limits if they exist.

(a) $\lim_{x\to 0} \frac{x^3}{\sin x - x}$ **(b)** $\lim_{x\to 0} \frac{\tan x - x}{x^3}$

(c) $\lim_{x\to 0}(\frac{1}{\sin x} - \frac{1}{x})$ **(d)** $\lim_{x\to 0}(\cos x)^{1/x^2}$

30.3 Find the following limits if they exist.

(a) $\lim_{x\to\infty} \frac{x-\sin x}{x}$ **(b)** $\lim_{x\to\infty} x^{\sin(1/x)}$

(c) $\lim_{x\to 0^+} \frac{1+\cos x}{e^x-1}$ **(d)** $\lim_{x\to 0} \frac{1-\cos 2x-2x^2}{x^4}$

30.4 Let f be a function defined on some interval $(0, a)$, and define $g(y) = f(\frac{1}{y})$ for $y \in (a^{-1}, \infty)$; here we set $a^{-1} = 0$ if $a = \infty$. Show $\lim_{x\to 0^+} f(x)$ exists if and only if $\lim_{y\to\infty} g(y)$ exists, in which case these limits are equal.

30.5 Find the limits

(a) $\lim_{x\to 0}(1 + 2x)^{1/x}$ **(b)** $\lim_{y\to\infty}(1 + \frac{2}{y})^y$

(c) $\lim_{x\to\infty}(e^x + x)^{1/x}$

30.6 Let f be differentiable on some interval (c, ∞) and suppose $\lim_{x\to\infty}[f(x) + f'(x)] = L$, where L is finite. Prove $\lim_{x\to\infty} f(x) = L$ and $\lim_{x\to\infty} f'(x) = 0$. *Hint*: $f(x) = \frac{f(x)e^x}{e^x}$.

30.7 This example is taken from [65, p. 188] and is due to Otto Stolz [64]. The requirement in Theorem 30.2 that $g'(x) \neq 0$ for x "near" s is important. In a careless application of L'Hospital's rule in which the zeros of g' "cancel" the zeros of f', erroneous results can be obtained. For $x \in \mathbb{R}$, let

$$f(x) = x + \cos x \sin x \quad \text{and} \quad g(x) = e^{\sin x}(x + \cos x \sin x).$$

(a) Show $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} g(x) = +\infty$.

(b) Show $f'(x) = 2(\cos x)^2$ and $g'(x) = e^{\sin x} \cos x[2\cos x + f(x)]$.

(c) Show $\frac{f'(x)}{g'(x)} = \frac{2e^{-\sin x}\cos x}{2\cos x + f(x)}$ if $\cos x \neq 0$ and $x > 3$.

(d) Show $\lim_{x\to\infty} \frac{2e^{-\sin x}\cos x}{2\cos x + f(x)} = 0$ and yet the limit $\lim_{x\to\infty} \frac{f(x)}{g(x)}$ does *not* exist.

§31 Taylor's Theorem

31.1 Discussion.
Consider a power series with radius of convergence $R > 0$ [R may be $+\infty$]:

$$f(x) = \sum_{k=0}^{\infty} a_k x^k. \tag{1}$$

By Theorem 26.5, f is differentiable in the interval $|x| < R$ and

$$f'(x) = \sum_{k=1}^{\infty} k a_k x^{k-1}.$$

The same theorem shows f' is differentiable for $|x| < R$ and

$$f''(x) = \sum_{k=2}^{\infty} k(k-1) a_k x^{k-2}.$$

Continuing in this way, we find the nth derivative $f^{(n)}$ exists for $|x| < R$ and

$$f^{(n)}(x) = \sum_{k=n}^{\infty} k(k-1)\cdots(k-n+1) a_k x^{k-n}.$$

In particular,

$$f^{(n)}(0) = n(n-1)\cdots(n-n+1)a_n = n!a_n.$$

This relation even holds for $n = 0$ if we make the convention $f^{(0)} = f$ and recall the convention $0! = 1$. Since $f^{(k)}(0) = k!a_k$, the original power series (1) has the form

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k, \qquad |x| < R. \tag{2}$$

As suggested at the end of §26, we now begin with a function f and seek a power series for f. The last paragraph shows f should possess derivatives of all orders at 0, i.e., $f'(0), f''(0), f'''(0), \ldots$ should all exist. For such f, formula (2) might hold for some $R > 0$, in which case we have found a power series for f. □

One can consider Taylor series that are not centered at 0, just as we did with the power series in Example 7 on 190.

31.2 Definition.
Let f be a function defined on some open interval containing c. If f possesses derivatives of all orders at c, then the series

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(c)}{k!}(x-c)^k \tag{1}$$

is called the *Taylor series for f about c*. For $n \geq 1$, *remainder $R_n(x)$* is defined by

$$R_n(x) = f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(c)}{k!}(x-c)^k. \tag{2}$$

Of course the remainder R_n depends on f and c, so a more accurate notation would be something like $R_n(f;c;x)$. The remainder is important because, for any x,

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(c)}{k!}(x-c)^k \quad \text{if and only if} \quad \lim_{n\to\infty} R_n(x) = 0.$$

We will show in Example 3 that f need not be given by its Taylor series, i.e., that $\lim_{n\to\infty} R_n(x) = 0$ can fail. Since we want to know when f is given by its Taylor series, our various versions of Taylor's Theorem all concern the nature of the remainder R_n.

31.3 Taylor's Theorem.
Let f be defined on (a,b) where $a < c < b$; here we allow $a = -\infty$ or $b = \infty$. Suppose the nth derivative $f^{(n)}$ exists on (a,b). Then for each $x \neq c$ in (a,b) there is some y between c and x such that

$$R_n(x) = \frac{f^{(n)}(y)}{n!}(x-c)^n.$$

The proof we give is due to James Wolfe[1] [71]; compare Exercise 31.6.

Proof

Fix $x \neq c$ and $n \geq 1$. Let M be the unique solution of

$$f(x) = \sum_{k=0}^{n-1} \frac{f^{(k)}(c)}{k!}(x-c)^k + \frac{M(x-c)^n}{n!} \tag{1}$$

and observe we need only show

$$f^{(n)}(y) = M \quad \text{for some} \quad y \quad \text{between} \quad c \quad \text{and} \quad x. \tag{2}$$

[To see this, replace M by $f^{(n)}(y)$ in Eq. (1) and recall the definition of $R_n(x)$.] To prove (2), consider the difference

$$g(t) = \sum_{k=0}^{n-1} \frac{f^{(k)}(c)}{k!}(t-c)^k + \frac{M(t-c)^n}{n!} - f(t). \tag{3}$$

A direct calculation shows $g(c) = 0$ and $g^{(k)}(c) = 0$ for $k < n$. Also $g(x) = 0$ by the choice of M in (1). By Rolle's theorem 29.2 we have $g'(x_1) = 0$ for some x_1 between c and x. Since $g'(c) = 0$, a second application of Rolle's theorem shows $g''(x_2) = 0$ for some x_2 between c and x_1. Again, since $g''(c) = 0$ we have $g'''(x_3) = 0$ for some x_3 between c and x_2. This process continues until we obtain x_n between c and x_{n-1} such that $g^{(n)}(x_n) = 0$. From (3) it follows that $g^{(n)}(t) = M - f^{(n)}(t)$ for all $t \in (a,b)$, so (2) holds with $y = x_n$. ■

31.4 Corollary.

Let f be defined on (a,b) where $a < c < b$. If all the derivatives $f^{(n)}$ exist on (a,b) and are bounded by a single constant C, then

$$\lim_{n\to\infty} R_n(x) = 0 \quad \textit{for all} \quad x \in (a,b).$$

Proof

Consider x in (a,b). From Theorem 31.3 we see

$$|R_n(x)| \leq \frac{C}{n!}|x-c|^n \quad \text{for all} \quad n.$$

[1] My undergraduate real analysis teacher in 1954–1955.

Since $\lim_{n\to\infty} \frac{|x-c|^n}{n!} = 0$ by Exercise 9.15, we conclude $\lim_{n\to\infty} R_n(x) = 0$. ■

Example 1

We assume the familiar differentiation properties of e^x, $\sin x$, etc.

(a) Let $f(x) = e^x$ for $x \in \mathbb{R}$. Then $f^{(n)}(x) = e^x$ for all $n = 0, 1, 2, \ldots$, so $f^{(n)}(0) = 1$ for all n. The Taylor series for e^x about 0 is

$$\sum_{k=0}^{\infty} \frac{1}{k!} x^k.$$

For any bounded interval $(-M, M)$ in $\mathbb{R}$ all the derivatives of f are bounded [by e^M, in fact], so Corollary 31.4 shows

$$e^x = \sum_{k=0}^{\infty} \frac{1}{k!} x^k \quad \text{for all} \quad x \in \mathbb{R}.$$

(b) If $f(x) = \sin x$ for $x \in \mathbb{R}$, then

$$f^{(n)}(x) = \begin{cases} \cos x & n = 1, 5, 9, \ldots \\ -\sin x & n = 2, 6, 10, \ldots \\ -\cos x & n = 3, 7, 11, \ldots \\ \sin x & n = 0, 4, 8, 12, \ldots; \end{cases}$$

thus

$$f^{(n)}(0) = \begin{cases} 1 & n = 1, 5, 9, \ldots \\ -1 & n = 3, 7, 11, \ldots \\ 0 & \text{otherwise.} \end{cases}$$

Hence the Taylor series for $\sin x$ about 0 is

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1}.$$

The derivatives of f are all bounded by 1, so

$$\sin x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1} \quad \text{for all} \quad x \in \mathbb{R}.$$

□

Example 2

In Example 2 of §26 we used Abel's theorem to prove

$$\log_e 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \cdots. \tag{1}$$

Here is another proof, based on Taylor's Theorem. Let $f(x) = \log_e(1+x)$ for $x \in (-1, \infty)$. Differentiating, we find

$$f'(x) = (1+x)^{-1}, \qquad f''(x) = -(1+x)^{-2}, \qquad f'''(x) = 2(1+x)^{-3},$$

etc. A simple induction argument shows

$$f^{(n)}(x) = (-1)^{n+1}(n-1)!(1+x)^{-n} \quad \text{for} \quad n \geq 1. \tag{2}$$

In particular, $f^{(n)}(0) = (-1)^{n+1}(n-1)!$, so the Taylor series for f about 0 is

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} x^k = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \cdots.$$

We also could have obtained this Taylor series using Example 1 in §26, but we need formula (2) anyway. We now apply Theorem 31.3 with $a = -1$, $b = +\infty$, $c = 0$ and $x = 1$. Thus for each n there exists $y_n \in (0,1)$ such that $R_n(1) = \frac{f^{(n)}(y_n)}{n!}$. Equation (2) shows

$$R_n(1) = \frac{(-1)^{n+1}(n-1)!}{(1+y_n)^n n!};$$

hence

$$|R_n(1)| = \frac{1}{(1+y_n)^n n} < \frac{1}{n} \quad \text{for all} \quad n.$$

Therefore $\lim_{n\to\infty} R_n(1) = 0$, so the Taylor series for f, evaluated at $x = 1$, converges to $f(1) = \log_e 2$, i.e., (1) holds. □

The next version of Taylor's theorem gives the remainder in integral form. The proof uses results from integration theory that should be familiar from calculus; they also appear in the next chapter.

31.5 Taylor's Theorem.

Let f be defined on (a,b) where $a < c < b$, and suppose the nth derivative $f^{(n)}$ exists and is continuous on (a,b). Then for $x \in (a,b)$

we have

$$R_n(x) = \int_c^x \frac{(x-t)^{n-1}}{(n-1)!} f^{(n)}(t)\,dt. \tag{1}$$

Proof

For $n = 1$, Eq. (1) asserts

$$R_1(x) = f(x) - f(c) = \int_c^x f'(t)\,dt;$$

this holds by Theorem 34.1. For $n \geq 2$, we repeatedly apply integration by parts, i.e., we use mathematical induction. So, assume (1) above holds for some n, $n \geq 1$. We evaluate the integral in (1) by Theorem 34.2, using $u(t) = f^{(n)}(t)$, $v'(t) = \frac{(x-t)^{n-1}}{(n-1)!}$, so that $u'(t) = f^{(n+1)}(t)$ and $v(t) = -\frac{(x-t)^n}{n!}$. We obtain $R_n(x)$

$$\begin{aligned} &= u(x)v(x) - u(c)v(c) - \int_c^x v(t)u'(t)\,dt \\ &= f^{(n)}(x)\cdot 0 + f^{(n)}(c)\frac{(x-c)^n}{n!} + \int_c^x \frac{(x-t)^n}{n!} f^{(n+1)}(t)\,dt. \end{aligned} \tag{2}$$

The definition of R_{n+1} in Definition 31.2 shows

$$R_{n+1}(x) = R_n(x) - \frac{f^{(n)}(c)}{n!}(x-c)^n; \tag{3}$$

hence from (2) we see that (1) holds for $n+1$. ■

31.6 Corollary.

If f is as in Theorem 31.5, then for each x in (a, b) different from c there is some y between c and x such that

$$R_n(x) = (x-c)\cdot\frac{(x-y)^{n-1}}{(n-1)!} f^{(n)}(y). \tag{1}$$

This form of R_n is known as Cauchy's form of the remainder.

Proof

We suppose $x < c$, the case $x > c$ being similar. The Intermediate Value Theorem for Integrals 33.9 shows

$$\frac{1}{c-x}\int_x^c \frac{(x-t)^{n-1}}{(n-1)!} f^{(n)}(t)\,dt = \frac{(x-y)^{n-1}}{(n-1)!} f^{(n)}(y) \tag{2}$$

for some y in (x, c). Since the integral in (2) equals $-R_n(x)$ by Theorem 31.5, formula (1) holds. ■

Recall that, for a nonnegative integer n, the binomial theorem tells us

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$$

where $\binom{n}{0} = 1$ and

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n(n-1)\cdots(n-k+1)}{k!} \quad \text{for} \quad 1 \leq k \leq n.$$

Let $a = x$ and $b = 1$; then

$$(1+x)^n = 1 + \sum_{k=1}^{n} \frac{n(n-1)\cdots(n-k+1)}{k!} x^k.$$

This result holds for some values of x even if the exponent n is not an integer, provided we allow the series to be an infinite series.

We next prove this using Corollary 31.6. Our proof is motivated by that in [55].

31.7 Binomial Series Theorem.
If $\alpha \in \mathbb{R}$ and $|x| < 1$, then

$$(1+x)^\alpha = 1 + \sum_{k=1}^{\infty} \frac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{k!} x^k. \tag{1}$$

Proof
For $k = 1, 2, 3, \ldots$, let $a_k = \frac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{k!}$. If α is a nonnegative integer, then $a_k = 0$ for $k > \alpha$ and (1) holds for all x as noted in our discussion prior to this theorem. Henceforth we assume α is not a nonnegative integer so that $a_k \neq 0$ for all k. Since

$$\lim_{k\to\infty} \left| \frac{a_{k+1}}{a_k} \right| = \lim_{k\to\infty} \left| \frac{\alpha - k}{k+1} \right| = 1,$$

the series in (1) has radius of convergence 1; see Theorem 23.1 and Corollary 12.3. Lemma 26.3 shows the series $\sum ka_k x^{k-1}$ also has radius of convergence 1, so it converges for $|x| < 1$. Hence

$$\lim_{n\to\infty} na_n x^{n-1} = 0 \quad \text{for} \quad |x| < 1. \tag{2}$$

Let $f(x) = (1+x)^\alpha$ for $|x| < 1$. For $n = 1, 2, \ldots$, we have

$$f^{(n)}(x) = \alpha(\alpha-1)\cdots(\alpha-n+1)(1+x)^{\alpha-n} = n!a_n(1+x)^{\alpha-n}.$$

Thus $f^{(n)}(0) = n!a_n$ for all $n \geq 1$, and the series in (1) is the Taylor series for f.

For the remainder of this proof, x is a fixed number satisfying $|x| < 1$. By Corollary 31.6, with $c = 0$, for each n, there is y_n between 0 and x such that

$$R_n(x) = x \cdot \frac{(x-y_n)^{n-1}}{(n-1)!} f^{(n)}(y_n) = x \cdot \frac{(x-y_n)^{n-1}}{(n-1)!} n!a_n(1+y_n)^{\alpha-n}.$$

So

$$|R_n(x)| \leq |x| \left|\frac{x-y_n}{1+y_n}\right|^{n-1} n|a_n|(1+y_n)^{\alpha-1}. \tag{3}$$

We easily show that

$$\left|\frac{x-y}{1+y}\right| \leq |x| \quad \text{if} \quad y \quad \text{is between} \quad 0 \quad \text{and} \quad x;$$

indeed, note $y = xz$ for some $z \in [0,1]$, so

$$\left|\frac{x-y}{1+y}\right| = \left|\frac{x-xz}{1+xz}\right| = |x| \cdot \left|\frac{1-z}{1+xz}\right| \leq |x|$$

since $1 + xz \geq 1 - z$. Therefore, (3) implies

$$|R_n(x)| \leq |x|^n n|a_n|(1+y_n)^{\alpha-1}.$$

The sequence $(1+y_n)^{\alpha-1}$ is bounded, because the continuous function $y \to (1+y)^{\alpha-1}$ is bounded on $[-|x|, |x|]$, so from (2) and Exercise 8.4, we conclude $\lim_n R_n(x) = 0$, so that (1) holds. ■

We next give an example of a function f whose Taylor series exists but does not represent the function. The function f is *infinitely differentiable on* $\mathbb{R}$, i.e., derivatives of all order exist at all points

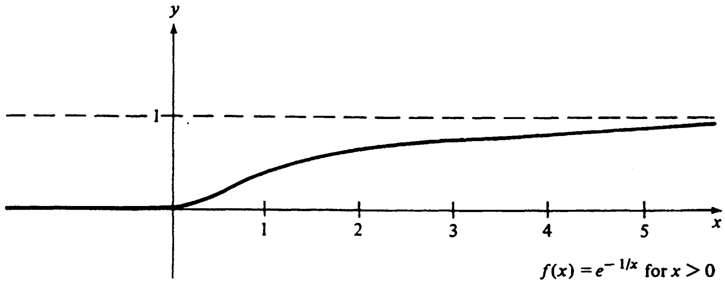

FIGURE 31.1

of $\mathbb{R}$. The example may appear artificial, but the existence of such functions [see also Exercise 31.4] is vital to the theory of distributions, an important theory related to work in differential equations and Fourier analysis.

Example 3

Let $f(x) = e^{-1/x}$ for $x > 0$ and $f(x) = 0$ for $x \le 0$; see Fig. 31.1. Clearly f has derivatives of all orders at all $x \ne 0$. We will prove

$$f^{(n)}(0) = 0 \quad \text{for} \quad n = 0, 1, 2, 3, \ldots. \tag{1}$$

Hence the Taylor series for f is identically zero, so f does not agree with its Taylor series in any open interval containing 0. First we show for each n there is a polynomial p_n of degree $2n$ such that

$$f^{(n)}(x) = e^{-1/x} p_n(1/x) \quad \text{for} \quad x > 0. \tag{2}$$

This is obvious for $n = 0$; simply set $p_0(t) = 1$ for all t. And this is easy for $n = 1$ and $n = 2$; the reader should check (2) holds with $n = 1$ and $p_1(t) = t^2$ and (2) holds with $n = 2$ and $p_2(t) = t^4 - 2t^3$. To apply induction, we assume the result is true for n and write

$$p_n(t) = a_0 + a_1 t + a_2 t^2 + \cdots + a_{2n} t^{2n} \quad \text{where} \quad a_{2n} \ne 0.$$

Then for $x > 0$ we have

$$f^{(n)}(x) = e^{-1/x} \left[\sum_{k=0}^{2n} \frac{a_k}{x^k} \right],$$

and a single differentiation yields

$$f^{(n+1)}(x) = e^{-1/x}\left[0 - \sum_{k=1}^{2n} \frac{ka_k}{x^{k+1}}\right] + \left[\sum_{k=0}^{2n} \frac{a_k}{x^k}\right] e^{-1/x} \cdot \left(\frac{1}{x^2}\right).$$

The assertion (2) is now clear for $n+1$; in fact, the polynomial p_{n+1} is evidently

$$p_{n+1}(t) = -\sum_{k=1}^{2n} ka_k t^{k+1} + \left[\sum_{k=0}^{2n} a_k t^k\right] \cdot (t^2),$$

which has degree $2n+2$.

We next prove (1) by induction. Assume $f^{(n)}(0) = 0$ for some $n \geq 0$. We need to prove

$$\lim_{x\to 0} \frac{f^{(n)}(x) - f^{(n)}(0)}{x - 0} = \lim_{x\to 0} \frac{1}{x} f^{(n)}(x) = 0.$$

Obviously $\lim_{x\to 0^-} \frac{1}{x} f^{(n)}(x) = 0$ since $f^{(n)}(x) = 0$ for all $x < 0$. By Theorem 20.10 it suffices to verify

$$\lim_{x\to 0^+} \frac{1}{x} f^{(n)}(x) = 0.$$

In view of (2), it suffices to show

$$\lim_{x\to 0^+} e^{-1/x} q\left(\frac{1}{x}\right) = 0$$

for any polynomial q. In fact, since $q(1/x)$ is a finite sum of terms of the form $b_k(1/x)^k$, where $k \geq 0$, it suffices to show

$$\lim_{x\to 0^+} \left(\frac{1}{x}\right)^k e^{-1/x} = 0 \quad \text{for fixed} \quad k \geq 0.$$

As noted in Exercise 30.4, it suffices to show

$$\lim_{y\to\infty} y^k e^{-y} = \lim_{y\to\infty} \frac{y^k}{e^y} = 0, \tag{3}$$

and this can be verified via k applications of L'Hospital's Rule 30.2.

□

As we observed in Remark 2.4 on page 12, Newton's method allows one to solve some equations of the form $f(x) = 0$.

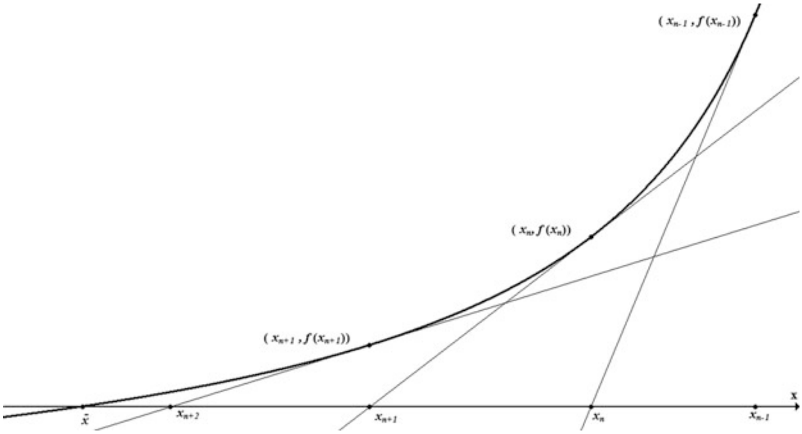

FIGURE 31.2

31.8 Newton's Method.
Newton's method for finding an approximate solution to $f(x) = 0$ is to begin with a reasonable initial guess x_0 and then compute

$$x_n = x_{n-1} - \frac{f(x_{n-1})}{f'(x_{n-1})} \quad \text{for} \quad n \geq 1.$$

Here x_n is chosen so that $(x_n, 0)$ lies on the line through $(x_{n-1}, f(x_{n-1}))$ having slope $f'(x_{n-1})$. See Fig. 31.2. Often the sequence (x_n) converges rapidly to a solution of $f(x) = 0$. □

31.9 Secant Method.
A similar approach to approximating solutions of $f(x) = 0$ is to start with two reasonable guesses x_0 and x_1 and then compute

$$x_n = x_{n-1} - \frac{f(x_{n-1})(x_{n-2} - x_{n-1})}{f(x_{n-2}) - f(x_{n-1})} \quad \text{for} \quad n \geq 2.$$

Here x_n is chosen so that $(x_n, 0)$ lies on the line through $(x_{n-1}, f(x_{n-1}))$ and $(x_{n-2}, f(x_{n-2}))$. See Fig. 31.3. □

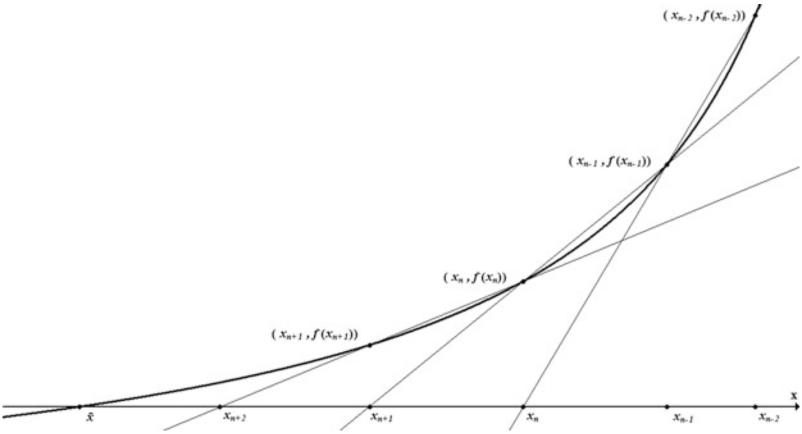

FIGURE 31.3 secant method

Example 4

Consider the equation $x^{10} + x - 1 = 0$. Let $f(x) = x^{10} + x - 1$. The equation $f(x) = 0$ will have at least two solutions, because $f(0) = -1$ and $\lim_{x\to\infty} f(x) = \lim_{x\to-\infty} f(x) = +\infty$. In fact, it has exactly two solutions, because $f'(x) = 10x^9 + 1$ is 0 at only one point, $y = -1/\sqrt[9]{10} \approx -0.774$, so that f decreases on $(-\infty, y)$ and increases on (y, ∞).

We will illustrate both approximation methods, starting with Newton's method. Since $f(0) = -1$ and $f(1) = 1$. a good initial guess for one of the solutions would be $x_0 = 1$. We omit the calculations, which we did on the spreadsheet Excel. With $x_0 = 1$ and Newton's method, we obtained $x_1 \approx 0.909090909$, $x_2 \approx 0.852873482$, $x_3 \approx 0.836193375$, $x_4 \approx 0.835083481$ and $x_5 \approx 0.835079043$. All subsequent approximations equaled x_5's, so this is surely the value to 9-place accuracy.

Since $f(-1) = -1$, the other solution is less than -1, and since $f(-2) = 1{,}021$, a good initial guess would be close to -1. But we started at $x_0 = -2$ to see what happens. We obtained $x_1 \approx -1.800546982$, $x_2 \approx -1.621810423$, $x_3 \approx -1.462802078$, $x_4 \approx -1.324100397$, $x_5 \approx -1.209349769$, $x_6 \approx -1.126853632$, $x_7 \approx -1.085350048$, $x_8 \approx -1.076156798$, $x_9 \approx -1.075766739$, and $x_{10} \approx -1.075766066$. All further values of x_n approximated x_{10}. Of course, $x_0 = -2$ was a poor choice, and $x_0 = -1.2$ would have been more sensible. With this choice, $x_5 \approx -1.075766066$.

As we will see, with the secant method the convergence is a bit slower. To find the positive solution, we started with $x_0 = 2$ and $x_1 = 1$. We obtained $x_9 \approx 0.835079043$, so it took about twice as many iterations to get 9-place accuracy as with Newton's method. For the other solution, we started with $x_0 = -2$ and $x_1 = -1$, and we obtained 9-place accuracy at $x_8 \approx -1.075766066$. If you look at a graph, you will see why choosing guesses on each side of the true solution can be efficient. If we had started with $x_0 = -2$ and $x_1 = -1.5$, we would not have reach 9-place accuracy until x_{12}. □

Unlike much of calculus, there isn't a single easily-found and easily-applied theorem justifying the use of either of these methods. We'll prove two such theorems, Theorems 31.12 and 31.13. They might be of value when studying the theory or when writing a program to implement a method. But for simple problems, it is much more efficient to simply assume and hope that the method will succeed, and then use common sense to assess whether the method worked.

The secant method predates Newton's method, but it does not seem as well known. In fact, the secant method is an example of the regula falsi (false position) method for approximating solutions of equations, which even appears in the Phind Papyrus attributed to the scribe Ahmes. It works in many examples, but a computer implementation would need to avoid a premature attempt to divide by 0. One advantage to this method is that the derivatives of f might be very complicated or not even exist. Another advantage to this method is that it can be used to find approximate solutions of equations, like $\cos x - x = 0$, when studying functions in algebra or trigonometry. The methodology also might be useful when motivating the sequential definition of the derivative.

We now work toward a theorem providing hypotheses that assure us that Newton's method works. We will need a technical lemma about sequences.

31.10 Lemma.
Let (a_n) be a sequence of nonnegative numbers, and let C and δ be positive numbers satisfying $C\delta < 1$.

(a) *If $a_0 \le \delta$ and $a_n \le Ca_{n-1}^2$ for $n \ge 1$, then*

$$a_n \le (C\delta)^{2^n-1}a_0 \quad \textit{for all} \quad n \ge 0.$$

(b) *If $\max\{a_0, a_1\} \le \delta$ and $a_n \le C\max\{a_{n-1}, a_{n-2}\}^2$ for $n \ge 2$, then*

$$\max\{a_{2n}, a_{2n+1}\} \le (C\delta)^{2^n-1}\max\{a_0, a_1\} \quad \textit{for all} \quad n \ge 0.$$

Proof
We prove (a) by mathematical induction, noting first that it is obvious for $n = 0$ and that $a_1 \le Ca_0^2 \le C\delta a_0$. In general, if $a_n \le (C\delta)^{2^n-1}a_0$, then

$$a_{n+1} \le Ca_n^2 \le C[(C\delta)^{2^n-1}a_0]^2 \le C^{2^{n+1}-1}\delta^{2^{n+1}-2}\delta a_0 = (C\delta)^{2^{n+1}-1}a_0.$$

For (b), we write $M = \max\{a_0, a_1\}$, so that $M \le \delta$ and we need to show

$$\max\{a_{2n}, a_{2n+1}\} \le (C\delta)^{2^n-1}M \quad \text{for all} \quad n \ge 0.$$

This is trivial for $n = 0$. In general, if we have $\max\{a_{2n}, a_{2n+1}\} \le (C\delta)^{2^n-1}M$, then

$$a_{2n+2} \le C\max\{a_{2n+1}, a_{2n}\}^2 \le C[(C\delta)^{2^n-1}M]^2$$

$$= C^{2^{n+1}-1}\delta^{2^{n+1}-2}M^2 \le C^{2^{n+1}-1}\delta^{2^{n+1}-2}\delta M = (C\delta)^{2^{n+1}-1}M,$$

and

$$a_{2n+3} \le C\max\{a_{2n+2}, a_{2n+1}\}^2 \le C\max\{(C\delta)^{2^{n+1}-1}M, (C\delta)^{2^n-1}M\}^2$$

$$\le C[(C\delta)^{2^n-1}M]^2 = C^{2^{n+1}-1}\delta^{2^{n+1}-2}M^2 \le C^{2^{n+1}-1}\delta^{2^{n+1}-2}\delta M = (C\delta)^{2^{n+1}-1}M.$$

Thus $\max\{a_{2n+2}, a_{2n+3}\} \le (C\delta)^{2^{n+1}-1}M$, and the induction proof of (b) is complete. ■

31.11 Discussion.
We assume f is defined and differentiable on an open interval $J = (c, d)$. A glance at the definition

$$x_n = x_{n-1} - \frac{f(x_{n-1})}{f'(x_{n-1})}$$

suggests it is reasonable to assume $|f'|$ is bounded away from 0 on J. This implies f' is always positive or always negative [Intermediate Value Theorem for Derivatives 29.8 on page 236]. Therefore f is strictly increasing or strictly decreasing on J, and f is one-to-one [Corollary 29.7]. In addition, we assume f'' exists on J and $|f''|$ is bounded above.

Suppose there exists $\delta_0 > 0$ with the property that $f(\overline{x}) = 0$ for some $\overline{x}$ in the closed interval $I = [c + \delta_0, d - \delta_0] \subset J$. We will show that if the initial value x_0 of Newton's sequence is sufficiently close to $\overline{x}$, then the iterates x_n in Newton's method converge to $\overline{x}$. To see how this goes, we assume for now that each x_n is indeed in the interval J. From the definition, we have

$$x_n - \overline{x} = x_{n-1} - \overline{x} - \frac{f(x_{n-1})}{f'(x_{n-1})} = \frac{f'(x_{n-1}) \cdot (x_{n-1} - \overline{x}) - f(x_{n-1})}{f'(x_{n-1})}.$$

By Taylor's Theorem 31.3 on page 250, with $n = 2$, $c = x_{n-1}$ and $x = \overline{x}$, we obtain

$$0 = f(\overline{x}) = f(x_{n-1}) + f'x_{n-1}) \cdot (\overline{x} - x_{n-1}) + \frac{1}{2} f''(y_n) \cdot (\overline{x} - x_{n-1})^2$$

for some y_n between x_{n-1} and $\overline{x}$. Hence

$$f'(x_{n-1}) \cdot (x_{n-1} - \overline{x}) - f(x_{n-1}) = \frac{1}{2} f''(y_n) \cdot (\overline{x} - x_{n-1})^2,$$

so

$$x_n - \overline{x} = \frac{f''(y_n)}{2f'(x_{n-1})} (x_{n-1} - \overline{x})^2.$$

Therefore

$$|x_n - \overline{x}| \leq \frac{\sup\{|f''(x)| : x \in J\}}{2 \inf\{|f'(x)| : x \in J\}} |x_{n-1} - \overline{x}|^2,$$

so that

$$|x_n - \overline{x}| \leq C |x_{n-1} - \overline{x}|^2,$$

where $C = \frac{\sup\{|f''(x)|:x\in J\}}{2\inf\{|f'(x)|:x\in J\}}$. Now select $\delta > 0$ so that $C\delta < 1$. By Lemma 31.10(a), with $a_n = |x_n - \overline{x}|$, we could conclude

$$|x_n - \overline{x}| \le (C\delta)^{2^n-1}|x_0 - \overline{x}| \quad \text{for all} \quad n \ge 0, \tag{1}$$

if we could arrange for $|x_0 - \overline{x}| = a_0 \le \delta$.

The inequalities (1) are our goal, and would assure us the sequence (x_n) converges to $\overline{x}$, but we need to backtrack and determine how to be sure each x_n is in the interval J, as well as arrange for $|x_0 - \overline{x}| \le \delta$. In addition to requiring $C\delta < 1$, we will need to choose δ so that $2\delta \le \delta_0$. Finally, we select x_0 in I so that $|f(x_0)| < m\delta$ where $m = \inf\{|f'(x)| : x \in J\}$. Note $(x_0 - 2\delta, x_0 + 2\delta) \subseteq (x_0 - \delta_0, x_0 + \delta_0) \subseteq J$. We also have

$$|x_0 - \overline{x}| < \delta, \tag{2}$$

because by the Mean Value Theorem 29.3, there is y_0 between x_0 and $\overline{x}$ such that

$$|x_0 - \overline{x}| = \frac{|f(x_0) - f(\overline{x})|}{|f'(y_0)|} = \frac{|f(x_0)|}{|f'(y_0)|} \le \frac{|f(x_0)|}{m} < \delta.$$

By (1) and (2), for each n, we have $|x_n - \overline{x}| \le |x_0 - \overline{x}| < \delta$; thus $|x_n - x_0| \le |x_n - \overline{x}| + |\overline{x} - x_0| < 2\delta$, so that each x_n is in the interval J. We summarize what we have.

31.12 Theorem.
Consider a function f having a zero $\overline{x}$ on an interval $J = (c, d)$, and assume f'' exists on J. Assume $|f''|$ is bounded above on J and $|f'|$ is bounded away from 0 on J. Choose $\delta_0 > 0$ so that $I = [c + \delta_0, d - \delta_0] \subset J$ is a nondegenerate interval containing $\overline{x}$ and so that $[c + \delta_0, d - \delta_0] \subseteq J$. Let

$$C = \frac{\sup\{|f''(x)| : x \in J\}}{2\inf\{|f'(x)| : x \in J\}},$$

and select $\delta > 0$ so that $2\delta \le \delta_0$ and $C\delta < 1$. Let $m = \inf\{|f'(x)| : x \in J\}$. Consider any x_0 in I satisfying $|f(x_0)| < m\delta$. Then the sequence of iterates given by Newton's method,

$$x_n = x_{n-1} - \frac{f(x_{n-1})}{f'(x_{n-1})} \quad \textit{for} \quad n \ge 1,$$

is a well-defined sequence and converges to $\overline{x}$*. Also,*

$$|x_n - \overline{x}| \leq C|x_{n-1} - \overline{x}|^2 \quad \textit{for} \quad n \geq 1, \quad \textit{and} \tag{1}$$

$$|x_n - \overline{x}| \leq (C\delta)^{2^n - 1}|x_0 - \overline{x}| \quad \textit{for} \quad n \geq 0. \tag{2}$$

In view of (1), the convergence of (x_n) is said to be "quadratic convergence." Here is an analogous theorem for the secant method. Comparing its conclusion with that in Theorem 31.12, it appears that it will take about twice as many iterations using the secant method to get the same accuracy as using Newton's method. This is what happened in Example 4 on page 260.

31.13 Theorem.
Notation and hypotheses are as in Theorem 31.12, except we set

$$C = \frac{3\sup\{|f''(x)| : x \in J\}}{2\inf\{|f'(x)| : x \in J\}},$$

and we consider distinct x_0 *and* x_1 *in* I *satisfying*

$$\max\{|f(x_0)|, |f(x_1)|\} < m\delta. \tag{1}$$

As before $\delta > 0$ *is chosen so the* $2\delta \leq \delta_0$ *and* $C\delta < 1$*.*

The sequence (x_n) *of iterates given by the secant method,*

$$x_n = x_{n-1} - \frac{f(x_{n-1})(x_{n-2} - x_{n-1})}{f(x_{n-2}) - f(x_{n-1})} \quad \textit{for} \quad n \geq 2, \tag{2}$$

is well defined and converges to $\overline{x}$*. Also,*

$$|x_n - \overline{x}| \leq C \cdot \max\{|x_{n-1} - \overline{x}|, |x_{n-2} - \overline{x}|\}^2 \quad \textit{for all} \quad n \geq 2, \tag{3}$$

and for $n \geq 0$*, we have*

$$\max\{|x_{2n} - \overline{x}|, |x_{2n+1} - \overline{x}|\} \leq (C\delta)^{2^n - 1}\max\{|x_0 - \overline{x}|, |x_1 - \overline{x}|\}. \tag{4}$$

Proof
As in Discussion 31.11, (1) implies

$$\max\{|x_0 - \overline{x}|, |x_1 - \overline{x}|\} < \delta. \tag{5}$$

Since

$$x_n - \overline{x} = x_{n-1} - \overline{x} - \frac{f(x_{n-1})(x_{n-2} - x_{n-1})}{f(x_{n-2}) - f(x_{n-1})} \quad \text{for} \quad n \geq 2,$$

the Mean Value Theorem shows there is w_n between x_{n-1} and x_{n-2} so that

$$x_n - \overline{x} = x_{n-1} - \overline{x} - \frac{f(x_{n-1})}{f'(w_n)}.$$

By Taylor's Theorem 31.3 on page 250, with $n = 2$, $c = \overline{x}$ and $x = x_{n-1}$, we have

$$f(x_{n-1}) = f(\overline{x}) + f'(\overline{x}) \cdot (x_{n-1} - \overline{x}) + \frac{1}{2} f''(y_n) \cdot (x_{n-1} - \overline{x})^2$$

$$= f'(\overline{x}) \cdot (x_{n-1} - \overline{x}) + \frac{1}{2} f''(y_n) \cdot (x_{n-1} - \overline{x})^2$$

for some y_n between x_{n-1} and $\overline{x}$. Hence

$$x_n - \overline{x} = (x_{n-1} - \overline{x}) - \frac{f'(\overline{x}) \cdot (x_{n-1} - \overline{x}) + f''(y_n) \cdot (x_{n-1} - \overline{x})^2/2}{f'(w_n)}$$

$$= \frac{(x_{n-1} - \overline{x}) f'(w_n) - f'(\overline{x}) \cdot (x_{n-1} - \overline{x}) - f''(y_n) \cdot (x_{n-1} - \overline{x})^2/2}{f'(w_n)}$$

$$= \frac{x_{n-1} - \overline{x}}{f'(w_n)} \left\{ [f'(w_n) - f'(\overline{x})] - \frac{f''(y_n) \cdot (x_{n-1} - \overline{x})}{2} \right\}.$$

By the Mean Value Theorem, applied to f', we obtain

$$x_n - \overline{x} = \frac{x_{n-1} - \overline{x}}{f'(w_n)} \left\{ f''(z_n) \cdot (w_n - \overline{x}) - \frac{f''(y_n) \cdot (x_{n-1} - \overline{x})}{2} \right\} \tag{6}$$

for some z_n between w_n and $\overline{x}$.

In general, $a < c < b$ implies $|c - x| \leq \max\{|a - x|, |b - x|\}$ for all x in $\mathbb{R}$ [Exercise 31.12]. Applying this, with $a = \min\{x_{n-1}, x_{n-2}\}$, $b = \max\{x_{n-1}, x_{n-2}\}$, $c = w_n$ and $x = \overline{x}$, we obtain

$$|w_n - \overline{x}| \leq \max\{|x_{n-1} - \overline{x}|, |x_{n-2} - \overline{x}|\}.$$

Using this and (6), above we conclude

$$|x_n - \overline{x}| \leq \frac{3}{2} \frac{\sup\{f''(x) : x \in J\}}{\inf\{f'(x) : x \in J\}} \max\{|x_{n-1} - \overline{x}|, |x_{n-2} - \overline{x}|\}^2,$$

so that (3) holds. Substituting $a_n = |x_n - \overline{x}|$ into (3) and (5), we see that Lemma 31.10(b) implies the inequalities in (4). Inequalities (4) and (5) imply $|x_n - x_0| < 2\delta$ for all n, so each x_n is in J, as needed for the preceding argument. This completes the proof. ■

Exercises

31.1 Find the Taylor series for $\cos x$ and indicate why it converges to $\cos x$ for all $x \in \mathbb{R}$.

31.2 Repeat Exercise 31.1 for the functions $\sinh x = \frac{1}{2}(e^x - e^{-x})$ and $\cosh x = \frac{1}{2}(e^x + e^{-x})$.

31.3 In Example 2, why did we apply Theorem 31.3 instead of Corollary 31.4?

31.4 Consider a, b in $\mathbb{R}$ where $a < b$. Show there exist infinitely differentiable functions f_a, g_b, $h_{a,b}$ and $h^*_{a,b}$ on $\mathbb{R}$ with the following properties. You may assume, without proof, that the sum, product, etc. of infinitely differentiable functions is again infinitely differentiable. The same applies to the quotient provided the denominator never vanishes.

(a) $f_a(x) = 0$ for $x \le a$ and $f_a(x) > 0$ for $x > a$. *Hint*: Let $f_a(x) = f(x-a)$ where f is the function in Example 3.

(b) $g_b(x) = 0$ for $x \ge b$ and $g_b(x) > 0$ for $x < b$.

(c) $h_{a,b}(x) > 0$ for $x \in (a,b)$ and $h_{a,b}(x) = 0$ for $x \notin (a,b)$.

(d) $h^*_{a,b}(x) = 0$ for $x \le a$ and $h^*_{a,b}(x) = 1$ for $x \ge b$. *Hint*: Use the function $\frac{f_a}{f_a + g_b}$.

31.5 Let $g(x) = e^{-1/x^2}$ for $x \ne 0$ and $g(0) = 0$.

(a) Show $g^{(n)}(0) = 0$ for all $n = 0, 1, 2, 3, \ldots$. *Hint*: Use Example 3.

(b) Show the Taylor series for g about 0 agrees with g only at $x = 0$.

31.6 An older proof of Theorem 31.3 goes as follows, which we outline for $c = 0$. Assume $x > 0$, let M be as in the proof of Theorem 31.3, and let

$$F(t) = f(t) + \sum_{k=1}^{n-1} \frac{(x-t)^k}{k!} f^{(k)}(t) + M \cdot \frac{(x-t)^n}{n!}$$

for t in $[0, x]$.

(a) Show F is differentiable on $[0, x]$ and

$$F'(t) = \frac{(x-t)^{n-1}}{(n-1)!}[f^{(n)}(t) - M].$$

(b) Show $F(0) = F(x)$.

(c) Apply Rolle's Theorem 29.2 to F to obtain y in $(0, x)$ such that $f^{(n)}(y) = M$.

31.7 Show the sequence in Exercise 9.5 comes from Newton's method when solving $f(x) = x^2 - 2 = 0$.

31.8 **(a)** Show $x^4 + x^3 - 1 = 0$ has exactly two solutions.

(b) Use Newton's method or the secant method to find the solutions to $x^4 + x^3 - 1 = 0$ to six-place accuracy.

31.9 Exercise 18.6 asked for a proof that $x = \cos x$ for some x in $(0, \frac{\pi}{2})$. Use Newton's method to find x to six-place accuracy. *Hint*: Apply Newton's method to $f(x) = x - \cos x$.

31.10 Exercise 18.7 asked for a proof that $xe^x = 2$ for some x in $(0, 1)$. Use Newton's method to find x to six-place accuracy.

31.11 Suppose f is differentiable on (a, b), f' is bounded on (a, b), f' never vanishes on (a, b), and the sequence (x_n) in (a, b) converges to $\overline{x}$ in (a, b). Show that if

$$x_n = x_{n-1} - \frac{f(x_{n-1})}{f'(x_{n-1})} \quad \text{for all} \quad n \geq 1,$$

then $f(\overline{x}) = 0$.

31.12 This result will complete the proof of Theorem 31.13. Show that if $a < c < b$, then $|c - x| \leq \max\{|a - x|, |b - x|\}$ for all $x \in \mathbb{R}$. *Hint*: Consider two cases: $x \leq c$ and $x > c$.