6

C H A P T E R

Integration

This chapter serves two purposes. It contains a careful development of the Riemann integral, which is the integral studied in standard calculus courses. It also contains an introduction to a generalization of the Riemann integral called the Riemann-Stieltjes integral. The generalization is easy and natural. Moreover, the Riemann-Stieltjes integral is an important tool in probability and statistics, and other areas of mathematics.

§32   The Riemann Integral

The theory of the Riemann integral is no more difficult than several other topics dealt with in this book. The one drawback is that it involves some technical notation and terminology.

32.1 Definition.
Let f be a bounded function on a closed interval $[a, b]$.[1] For $S \subseteq [a, b]$, we adopt the notation

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[1]Here and elsewhere in this chapter, we assume $a < b$.

$$M(f, S) = \sup\{f(x) : x \in S\} \quad \text{and} \quad m(f, S) = \inf\{f(x) : x \in S\}.$$

A *partition of* $[a, b]$ is any finite ordered subset P having the form

$$P = \{a = t_0 < t_1 < \cdots < t_n = b\}.$$

The *upper Darboux sum* $U(f, P)$ of f with respect to P is the sum

$$U(f, P) = \sum_{k=1}^{n} M(f, [t_{k-1}, t_k]) \cdot (t_k - t_{k-1})$$

and the *lower Darboux sum* $L(f, P)$ is

$$L(f, P) = \sum_{k=1}^{n} m(f, [t_{k-1}, t_k]) \cdot (t_k - t_{k-1}).$$

Note

$$U(f, P) \le \sum_{k=1}^{n} M(f, [a, b]) \cdot (t_k - t_{k-1}) = M(f, [a, b]) \cdot (b - a);$$

likewise $L(f, P) \ge m(f, [a, b]) \cdot (b - a)$, so

$$m(f, [a, b]) \cdot (b - a) \le L(f, P) \le U(f, P) \le M(f, [a, b]) \cdot (b - a). \quad (1)$$

The *upper Darboux integral* $U(f)$ of f over $[a, b]$ is defined by

$$U(f) = \inf\{U(f, P) : P \text{ is a partition of } [a, b]\}$$

and the *lower Darboux integral* is

$$L(f) = \sup\{L(f, P) : P \text{ is a partition of } [a, b]\}.$$

In view of (1), $U(f)$ and $L(f)$ are real numbers.

We will prove in Theorem 32.4 that $L(f) \le U(f)$. This is not obvious from (1). [Why?] We say f is *integrable* on $[a, b]$ provided $L(f) = U(f)$. In this case, we write $\int_a^b f$ or $\int_a^b f(x)\,dx$ for this common value:

$$\int_a^b f = \int_a^b f(x)\,dx = L(f) = U(f). \quad (2)$$

Specialists call this integral the *Darboux integral*. Riemann's definition of the integral is a little different [Definition 32.8], but we will show in Theorem 32.9 that the definitions are equivalent. For this
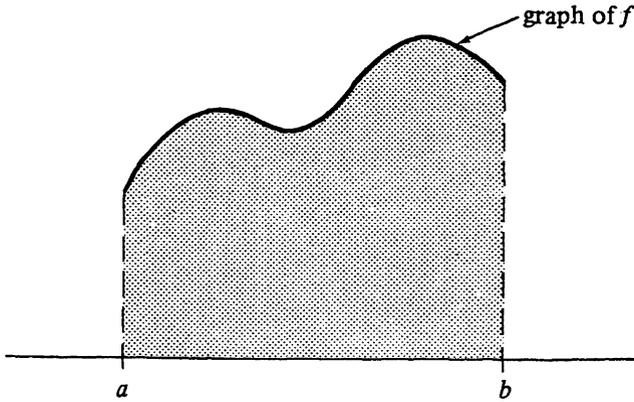
graph of f

a b

FIGURE 32.1

reason, we will follow customary usage and call the integral defined above the *Riemann integral.*

For nonnegative functions, $\int_a^b f$ is interpreted as the area of the region under the graph of f [see Fig. 32.1] for the following reason. Each lower Darboux sum represents the area of a union of rectangles inside the region, and each upper Darboux sum represents the area of a union of rectangles that contains the region. Moreover, $\int_a^b f$ is the unique number that is larger than or equal to all lower Darboux sums and smaller than or equal to all upper Darboux sums. Figure 19.2 on page 145 illustrates the situation for $[a, b] = [0, 1]$ and

$$P = \left\{ 0 < \frac{1}{n} < \frac{2}{n} < \cdots < \frac{n-1}{n} < 1 \right\}.$$

Example 1
The simplest function whose integral is not obvious is $f(x) = x^2$. Consider f on the interval $[0, b]$ where $b > 0$. For a partition

$$P = \{0 = t_0 < t_1 < \cdots < t_n = b\},$$

we have

$$U(f, P) = \sum_{k=1}^{n} \sup\{x^2 : x \in [t_{k-1}, t_k]\} \cdot (t_k - t_{k-1}) = \sum_{k=1}^{n} t_k^2 (t_k - t_{k-1}).$$

If we choose $t_k = \frac{kb}{n}$, then we can use Exercise 1.1 to calculate

$$U(f, P) = \sum_{k=1}^{n} \frac{k^2 b^2}{n^2} \left(\frac{b}{n} \right) = \frac{b^3}{n^3} \sum_{k=1}^{n} k^2 = \frac{b^3}{n^3} \cdot \frac{n(n+1)(2n+1)}{6}.$$

For large n, this is close to $\frac{b^3}{3}$, so we conclude $U(f) \leq \frac{b^3}{3}$. For the same partition we find

$$L(f, P) = \sum_{k=1}^{n} \frac{(k-1)^2 b^2}{n^2} \left(\frac{b}{n} \right) = \frac{b^3}{n^3} \cdot \frac{(n-1)(n)(2n-1)}{6},$$

so $L(f) \geq \frac{b^3}{3}$. Therefore $f(x) = x^2$ is integrable on $[0, b]$ and

$$\int_0^b x^2 \, dx = \frac{b^3}{3}.$$

Of course, any calculus student could have calculated this integral using a formula that is based on the Fundamental Theorem of Calculus; see Example 1 in §34. $\qquad\square$

Example 2

Consider the interval $[a, b]$, where $a < b$, and let $f(x) = 1$ for rational x in $[a, b]$, and let $f(x) = 0$ for irrational x in $[a, b]$. For any partition

$$P = \{a = t_0 < t_1 < \cdots < t_n = b\},$$

we have

$$U(f, P) = \sum_{k=1}^{n} M(f, [t_{k-1}, t_k]) \cdot (t_k - t_{k-1}) = \sum_{k=1}^{n} 1 \cdot (t_k - t_{k-1}) = b - a$$

and

$$L(f, P) = \sum_{k=1}^{n} 0 \cdot (t_k - t_{k-1}) = 0.$$

It follows that $U(f) = b - a$ and $L(f) = 0$. The upper and lower Darboux integrals for f do not agree, so f is not integrable! $\qquad\square$

We next develop some properties of the integral.

32.2 Lemma.
Let f be a bounded function on $[a, b]$. If P and Q are partitions of $[a, b]$ and $P \subseteq Q$, then

$$L(f, P) \leq L(f, Q) \leq U(f, Q) \leq U(f, P). \tag{1}$$

Proof
The middle inequality is obvious. The proofs of the first and third inequalities are similar, so we will prove

$$L(f, P) \leq L(f, Q). \tag{2}$$

An induction argument [Exercise 32.4] shows we may assume Q has only one more point, say u, than P. If

$$P = \{a = t_0 < t_1 < \cdots < t_n = b\},$$

then

$$Q = \{a = t_0 < t_1 < \cdots < t_{k-1} < u < t_k < \cdots < t_n = b\}$$

for some $k \in \{1, 2, \ldots, n\}$. The lower Darboux sums for P and Q are the same except for the terms involving t_{k-1} or t_k. In fact, their difference is

$$L(f, Q) - L(f, P) = m(f, [t_{k-1}, u]) \cdot (u - t_{k-1}) + m(f, [u, t_k]) \cdot (t_k - u)$$
$$-m(f, [t_{k-1}, t_k]) \cdot (t_k - t_{k-1}). \tag{3}$$

To establish (2) it suffices to show this quantity is nonnegative. Using Exercise 4.7(a), we see

$$\begin{aligned}
m(f&, [t_{k-1}, t_k]) \cdot (t_k - t_{k-1}) \\
&= m(f, [t_{k-1}, t_k]) \cdot \{(t_k - u) + (u - t_{k-1})\} \\
&\leq m(f, [u, t_k]) \cdot (t_k - u) + m(f, [t_{k-1}, u]) \cdot (u - t_{k-1}). \quad \blacksquare
\end{aligned}$$

32.3 Lemma.
If f is a bounded function on $[a, b]$, and if P and Q are partitions of $[a, b]$, then $L(f, P) \leq U(f, Q)$.

Proof
The set $P \cup Q$ is also a partition of $[a, b]$. Since $P \subseteq P \cup Q$ and $Q \subseteq P \cup Q$, we can apply Lemma 32.2 to obtain

$$L(f, P) \leq L(f, P \cup Q) \leq U(f, P \cup Q) \leq U(f, Q). \quad \blacksquare$$

32.4 Theorem.
If f is a bounded function on $[a, b]$, then $L(f) \leq U(f)$.

Proof
Fix a partition P of $[a, b]$. Lemma 32.3 shows $L(f, P)$ is a lower bound for the set

$$\{U(f, Q) : Q \text{ is a partition of } [a, b]\}.$$

Therefore $L(f, P)$ is less than or equal to the greatest lower bound [infimum!] of this set. That is

$$L(f, P) \leq U(f). \tag{1}$$

Now (1) shows that $U(f)$ is an upper bound for the set

$$\{L(f, P) : P \text{ is a partition of } [a, b]\},$$

so $U(f) \geq L(f)$. ∎

Note that Theorem 32.4 also follows from Lemma 32.3 and Exercise 4.8; see Exercise 32.5. The next theorem gives a "Cauchy criterion" for integrability.

32.5 Theorem.
A bounded function f on $[a, b]$ is integrable if and only if for each $\epsilon > 0$ there exists a partition P of $[a, b]$ such that

$$U(f, P) - L(f, P) < \epsilon. \tag{1}$$

Proof
Suppose first that f is integrable and consider $\epsilon > 0$. There exist partitions P_1 and P_2 of $[a, b]$ satisfying

$$L(f, P_1) > L(f) - \frac{\epsilon}{2} \quad \text{and} \quad U(f, P_2) < U(f) + \frac{\epsilon}{2}.$$

For $P = P_1 \cup P_2$, we apply Lemma 32.2 to obtain

$$U(f, P) - L(f, P) \leq U(f, P_2) - L(f, P_1)$$
$$< U(f) + \frac{\epsilon}{2} - \left[L(f) - \frac{\epsilon}{2} \right] = U(f) - L(f) + \epsilon.$$

Since f is integrable, $U(f) = L(f)$, so (1) holds.

Conversely, suppose for $\epsilon > 0$ the inequality (1) holds for some partition P. Then we have

$$U(f) \leq U(f, P) = U(f, P) - L(f, P) + L(f, P)$$
$$< \epsilon + L(f, P) \leq \epsilon + L(f).$$

Since ϵ is arbitrary, we conclude $U(f) \leq L(f)$. Hence we have $U(f) = L(f)$ by Theorem 32.4, i.e., f is integrable. ∎

The remainder of this section is devoted to establishing the equivalence of Riemann's and Darboux's definitions of integrability. Subsequent sections will depend only on items Definition 32.1 through Theorem 32.5. Therefore the reader who is content with the Darboux integral in Definition 32.1 can safely proceed directly to the next section.

32.6 Definition.
The *mesh* of a partition P is the maximum length of the subintervals comprising P. Thus if

$$P = \{a = t_0 < t_1 < \cdots < t_n = b\},$$

then

$$\mathrm{mesh}(P) = \max\{t_k - t_{k-1} : k = 1, 2, \ldots, n\}.$$

Here is another "Cauchy criterion" for integrability.

32.7 Theorem.
A bounded function f on $[a, b]$ is integrable if and only if for each $\epsilon > 0$ there exists a $\delta > 0$ such that

$$\mathrm{mesh}(P) < \delta \quad implies \quad U(f, P) - L(f, P) < \epsilon \tag{1}$$

for all partitions P of $[a, b]$.

Proof
The ϵ–δ condition in (1) implies integrability by Theorem 32.5.

Conversely, suppose f is integrable on $[a, b]$. Let $\epsilon > 0$ and select a partition

$$P_0 = \{a = u_0 < u_1 < \cdots < u_m = b\}$$

of $[a, b]$ such that

$$U(f, P_0) - L(f, P_0) < \frac{\epsilon}{2}. \tag{2}$$

Since f is bounded, there exists $B > 0$ such that $|f(x)| \leq B$ for all $x \in [a, b]$. Let $\delta = \frac{\epsilon}{8mB}$; m is the number of intervals comprising P_0.

To verify (1), we consider any partition

$$P = \{a = t_0 < t_1 < \cdots < t_n = b\}$$

with $\text{mesh}(P) < \delta$. Let $Q = P \cup P_0$. If Q has one more element than P, then a glance at (3) in the proof of Lemma 32.2 leads us to

$$L(f, Q) - L(f, P) \leq B \cdot \text{mesh}(P) - (-B) \cdot \text{mesh}(P) = 2B \cdot \text{mesh}(P).$$

Since Q has at most m elements that are not in P, an induction argument shows

$$L(f, Q) - L(f, P) \leq 2mB \cdot \text{mesh}(P) < 2mB\delta = \frac{\epsilon}{4}.$$

By Lemma 32.2 we have $L(f, P_0) \leq L(f, Q)$, so

$$L(f, P_0) - L(f, P) < \frac{\epsilon}{4}.$$

Similarly

$$U(f, P) - U(f, P_0) < \frac{\epsilon}{4},$$

so

$$U(f, P) - L(f, P) < U(f, P_0) - L(f, P_0) + \frac{\epsilon}{2}.$$

Now (2) implies $U(f, P) - L(f, P) < \epsilon$ and we have verified (1). ∎

Now we give Riemann's definition of integrability.

32.8 Definition.
Let f be a bounded function on $[a, b]$, and let $P = \{a = t_0 < t_1 < \cdots < t_n = b\}$ be a partition of $[a, b]$. A *Riemann sum* of f associated with the partition P is a sum of the form

$$\sum_{k=1}^{n} f(x_k)(t_k - t_{k-1})$$

where $x_k \in [t_{k-1}, t_k]$ for $k = 1, 2, \ldots, n$. The choice of x_k's is quite arbitrary, so there are infinitely many Riemann sums associated with a single function and partition.

The function f is *Riemann integrable* on $[a, b]$ if there exists a number r with the following property. For each $\epsilon > 0$ there exists $\delta > 0$ such that

$$|S - r| < \epsilon \tag{1}$$

for every Riemann sum S of f associated with a partition P having $\text{mesh}(P) < \delta$. The number r is the *Riemann integral* of f on $[a, b]$ and will be provisionally written as $\mathcal{R} \int_a^b f$.

32.9 Theorem.
A bounded function f on $[a, b]$ is Riemann integrable if and only if it is [Darboux] integrable, in which case the values of the integrals agree.

Proof
Suppose first that f is [Darboux] integrable on $[a, b]$ in the sense of Definition 32.1. Let $\epsilon > 0$, and let $\delta > 0$ be chosen so that (1) of Theorem 32.7 holds. We show

$$\left| S - \int_a^b f \right| < \epsilon \tag{1}$$

for every Riemann sum

$$S = \sum_{k=1}^n f(x_k)(t_k - t_{k-1})$$

associated with a partition P having $\text{mesh}(P) < \delta$. Clearly we have $L(f, P) \leq S \leq U(f, P)$, so (1) follows from the inequalities

$$U(f, P) < L(f, P) + \epsilon \leq L(f) + \epsilon = \int_a^b f + \epsilon$$

and

$$L(f, P) > U(f, P) - \epsilon \geq U(f) - \epsilon = \int_a^b f - \epsilon.$$

This proves (1); hence f is Riemann integrable and

$$\mathcal{R} \int_a^b f = \int_a^b f.$$

Now suppose f is Riemann integrable in the sense of Definition 32.8, and consider $\epsilon > 0$. Let $\delta > 0$ and r be as given in Definition 32.8. Select any partition

$$P = \{a = t_0 < t_1 < \cdots < t_n = b\}$$

with $\text{mesh}(P) < \delta$, and for each $k = 1, 2, \ldots, n$, select x_k in $[t_{k-1}, t_k]$ so that

$$f(x_k) < m(f, [t_{k-1}, t_k]) + \epsilon.$$

The Riemann sum S for this choice of x_k's satisfies

$$S \leq L(f, P) + \epsilon(b - a)$$

as well as

$$|S - r| < \epsilon.$$

It follows that

$$L(f) \geq L(f, P) \geq S - \epsilon(b - a) > r - \epsilon - \epsilon(b - a).$$

Since ϵ is arbitrary, we have $L(f) \geq r$. A similar argument shows $U(f) \leq r$. Since $L(f) \leq U(f)$, we see $L(f) = U(f) = r$. This shows f is [Darboux] integrable and

$$\int_a^b f = r = \mathcal{R} \int_a^b f.$$

∎

32.10 Corollary.
Let f be a bounded Riemann integrable function on $[a, b]$. Suppose (S_n) is a sequence of Riemann sums, with corresponding partitions P_n, satisfying $\lim_n \text{mesh}(P_n) = 0$. Then the sequence (S_n) converges to $\int_a^b f$.

Proof

Let $\epsilon > 0$. There is a $\delta > 0$ so that if S is a Riemann sum with corresponding partition P, and if $\text{mesh}(P) < \delta$, then

$$\left| S - \int_a^b f \right| < \epsilon.$$

Choose N so that $\text{mesh}(P_n) < \delta$ for $n > N$. Then

$$\left| S_n - \int_a^b f \right| < \epsilon \quad \text{for} \quad n > N.$$

Since $\epsilon > 0$ is arbitrary, this shows $\lim_n S_n = \int_a^b f$. ∎

32.11 Remark.
I recently had occasion to use the following simple observation. If one ignores the end intervals of the partitions, the "almost Riemann sums" so obtained still converge to the integral; see [59]. This arose because the intervals had the form $[a, b]$, but the partition points had the form $\frac{k}{n}$. Thus the partition points were nice and equally spaced, *except* for the end ones.

Exercises

32.1 Find the upper and lower Darboux integrals for $f(x) = x^3$ on the interval $[0, b]$. *Hint*: Exercise 1.3 and Example 1 in §1 will be useful.

32.2 Let $f(x) = x$ for rational x and $f(x) = 0$ for irrational x.

(a) Calculate the upper and lower Darboux integrals for f on the interval $[0, b]$.

(b) Is f integrable on $[0, b]$?

32.3 Repeat Exercise 32.2 for g where $g(x) = x^2$ for rational x and $g(x) = 0$ for irrational x.

32.4 Supply the induction argument needed in the proof of Lemma 32.2.

32.5 Use Exercise 4.8 to prove Theorem 32.4. Specify the sets S and T in this case.

32.6 Let f be a bounded function on $[a, b]$. Suppose there exist sequences (U_n) and (L_n) of upper and lower Darboux sums for f such that $\lim(U_n - L_n) = 0$. Show f is integrable and $\int_a^b f = \lim U_n = \lim L_n$.

32.7 Let f be integrable on $[a, b]$, and suppose g is a function on $[a, b]$ such that $g(x) = f(x)$ except for finitely many x in $[a, b]$. Show g is integrable and $\int_a^b f = \int_a^b g$. *Hint*: First reduce to the case where f is the function identically equal to 0.

32.8 Show that if f is integrable on $[a, b]$, then f is integrable on every interval $[c, d] \subseteq [a, b]$.

§33   Properties of the Riemann Integral

In this section we establish some basic properties of the Riemann integral and we show many familiar functions, including piecewise continuous and piecewise monotonic functions, are Riemann integrable.

A function is *monotonic* on an interval if it is either increasing or decreasing on the interval; see Definition 29.6.

33.1 Theorem.
Every monotonic function f on $[a, b]$ is integrable.

Proof
We assume f is increasing on $[a, b]$ and leave the decreasing case to Exercise 33.1. We also assume $f(a) < f(b)$, since otherwise f would be a constant function. Since $f(a) \le f(x) \le f(b)$ for all $x \in [a, b]$, f is clearly bounded on $[a, b]$. In order to apply Theorem 32.5, let $\epsilon > 0$ and select a partition $P = \{a = t_0 < t_1 < \cdots < t_n = b\}$ with mesh less than $\frac{\epsilon}{f(b) - f(a)}$. Then

$$U(f, P) - L(f, P) = \sum_{k=1}^{n} \{M(f, [t_{k-1}, t_k]) - m(f, [t_{k-1}, t_k])\} \cdot (t_k - t_{k-1})$$

$$= \sum_{k=1}^{n} [f(t_k) - f(t_{k-1})] \cdot (t_k - t_{k-1}).$$

Since mesh$(P) < \frac{\epsilon}{f(b) - f(a)}$, we have

$$U(f, P) - L(f, P) < \sum_{k=1}^{n} [f(t_k) - f(t_{k-1})] \cdot \frac{\epsilon}{f(b) - f(a)}$$

$$= [f(b) - f(a)] \cdot \frac{\epsilon}{f(b) - f(a)} = \epsilon.$$

Theorem 32.5 now shows f is integrable. ∎

33.2 Theorem.
Every continuous function f on $[a, b]$ is integrable.

Proof
Again, in order to apply Theorem 32.5, consider $\epsilon > 0$. Since f is uniformly continuous on $[a, b]$ by Theorem 19.2, there exists $\delta > 0$ such that

$$x, y \in [a, b] \quad \text{and} \quad |x - y| < \delta \quad \text{imply} \quad |f(x) - f(y)| < \frac{\epsilon}{b - a}. \quad (1)$$

Consider any partition $P = \{a = t_0 < t_1 < \cdots < t_n = b\}$ where

$$\max\{t_k - t_{k-1} : k = 1, 2, \ldots, n\} < \delta.$$

Since f assumes its maximum and minimum on each interval $[t_{k-1}, t_k]$ by Theorem 18.1, it follows from (1) above that

$$M(f, [t_{k-1}, t_k]) - m(f, [t_{k-1}, t_k]) < \frac{\epsilon}{b - a}$$

for each k. Therefore we have

$$U(f, P) - L(f, P) < \sum_{k=1}^{n} \frac{\epsilon}{b - a}(t_k - t_{k-1}) = \epsilon,$$

and Theorem 32.5 shows f is integrable. ∎

33.3 Theorem.
Let f and g be integrable functions on $[a, b]$, and let c be a real number. Then
   (i) *cf is integrable and $\int_a^b cf = c \int_a^b f$;*
   (ii) *$f + g$ is integrable and $\int_a^b (f + g) = \int_a^b f + \int_a^b g$.*

Exercise 33.8 shows fg, $\max(f,g)$ and $\min(f,g)$ are also integrable, but there are no formulas giving their integrals in terms of $\int_a^b f$ and $\int_a^b g$.

Proof

The proof of (i) involves three cases: $c > 0$, $c = -1$, and $c < 0$. Of course, (i) is obvious for $c = 0$.

Let $c > 0$ and consider a partition

$$P = \{a = t_0 < t_1 < \cdots < t_n = b\}$$

of $[a, b]$. A simple exercise [Exercise 33.2] shows

$$M(cf, [t_{k-1}, t_k]) = c \cdot M(f, [t_{k-1}, t_k])$$

for all k, so $U(cf, P) = c \cdot U(f, P)$. Another application of the same exercise shows $U(cf) = c \cdot U(f)$. Similar arguments show $L(cf) = c \cdot L(f)$. Since f is integrable, we have $L(cf) = c \cdot L(f) = c \cdot U(f) = U(cf)$. Hence cf is integrable and

$$\int_a^b cf = U(cf) = c \cdot U(f) = c \int_a^b f, \qquad c > 0. \tag{1}$$

Now we deal with the case $c = -1$. Exercise 5.4 implies $U(-f, P) = -L(f, P)$ for all partitions P of $[a, b]$. Hence we have

$$
\begin{aligned}
U(-f) &= \inf\{U(-f, P) : P \text{ is a partition of } [a, b]\} \\
&= \inf\{-L(f, P) : P \text{ is a partition of } [a, b]\} \\
&= -\sup\{L(f, P) : P \text{ is a partition of } [a, b]\} = -L(f).
\end{aligned}
$$

Replacing f by $-f$, we also obtain $L(-f) = -U(f)$. Since f is integrable, $U(-f) = -L(f) = -U(f) = L(-f)$; hence $-f$ is integrable and

$$\int_a^b (-f) = -\int_a^b f. \tag{2}$$

The case $c < 0$ is handled by applying (2), and then (1) to $-c$:

$$\int_a^b cf = -\int_a^b (-c)f = -(-c)\int_a^b f = c\int_a^b f.$$

To prove (ii) we will again use Theorem 32.5. Let $\epsilon > 0$. By Theorem 32.5 there exist partitions P_1 and P_2 of $[a, b]$ such that

$$U(f, P_1) - L(f, P_1) < \frac{\epsilon}{2} \quad \text{and} \quad U(g, P_2) - L(g, P_2) < \frac{\epsilon}{2}.$$

Lemma 32.2 shows that if $P = P_1 \cup P_2$, then

$$U(f, P) - L(f, P) < \frac{\epsilon}{2} \quad \text{and} \quad U(g, P) - L(g, P) < \frac{\epsilon}{2}. \qquad (3)$$

For any subset S of $[a, b]$, we have

$$\inf\{f(x) + g(x) : x \in S\} \geq \inf\{f(x) : x \in S\} + \inf\{g(x) : x \in S\},$$

i.e., $m(f + g, S) \geq m(f, S) + m(g, S)$. It follows that

$$L(f + g, P) \geq L(f, P) + L(g, P)$$

and similarly we have

$$U(f + g, P) \leq U(f, P) + U(g, P).$$

Therefore from (3) we obtain

$$U(f + g, P) - L(f + g, P) < \epsilon.$$

Theorem 32.5 now shows $f + g$ is integrable. Since

$$\int_a^b (f + g) = U(f + g) \leq U(f + g, P) \leq U(f, P) + U(g, P)$$

$$< L(f, P) + L(g, P) + \epsilon \leq L(f) + L(g) + \epsilon = \int_a^b f + \int_a^b g + \epsilon$$

and

$$\int_a^b (f + g) = L(f + g) \geq L(f + g, P) \geq L(f, P) + L(g, P)$$

$$> U(f, P) + U(g, P) - \epsilon \geq U(f) + U(g) - \epsilon = \int_a^b f + \int_a^b g - \epsilon,$$

we see that

$$\int_a^b (f + g) = \int_a^b f + \int_a^b g.$$

∎

33.4 Theorem.

(i) *If f and g are integrable on $[a,b]$ and if $f(x) \leq g(x)$ for x in $[a,b]$, then $\int_a^b f \leq \int_a^b g$.*

(ii) *If g is a continuous nonnegative function on $[a,b]$ and if $\int_a^b g = 0$, then g is identically 0 on $[a,b]$.*

Proof

(i) By Theorem 33.3, $h = g - f$ is integrable on $[a,b]$. Since $h(x) \geq 0$ for all $x \in [a,b]$, it is clear that $L(h, P) \geq 0$ for all partitions P of $[a,b]$, so $\int_a^b h = L(h) \geq 0$. Applying Theorem 33.3 again, we see

$$\int_a^b g = \int_a^b f + \int_a^b h \geq \int_a^b f.$$

(ii) Otherwise, since g is continuous, there is a nonempty interval $(c,d) \subseteq [a,b]$ and $\alpha > 0$ satisfying $g(x) \geq \alpha/2$ for $x \in (c,d)$. Then

$$\int_a^b g \geq \int_c^d g \geq \frac{\alpha}{2}(d-c) > 0,$$

contradicting $\int_a^b g = 0$. ∎

33.5 Theorem.

If f is integrable on $[a,b]$, then $|f|$ is integrable on $[a,b]$ and

$$\left| \int_a^b f \right| \leq \int_a^b |f|. \tag{1}$$

Proof

This follows easily from Theorem 33.4(i) provided we know $|f|$ is integrable on $[a,b]$. In fact, $-|f| \leq f \leq |f|$; therefore

$$-\int_a^b |f| \leq \int_a^b f \leq \int_a^b |f|,$$

which implies (1).

We now show $|f|$ is integrable, a point that was conveniently glossed over in Exercise 25.1. For any subset S of $[a,b]$, we have

$$M(|f|, S) - m(|f|, S) \leq M(f, S) - m(f, S) \tag{2}$$

by Exercise 33.6. From (2) it follows that

$$U(|f|, P) - L(|f|, P) \leq U(f, P) - L(f, P) \tag{3}$$

for all partitions P of $[a, b]$. By Theorem 32.5, for each $\epsilon > 0$ there exists a partition P such that

$$U(f, P) - L(f, P) < \epsilon.$$

In view of (3), the same remark applies to $|f|$, so $|f|$ is integrable by Theorem 32.5. ∎

33.6 Theorem.
Let f be a function defined on $[a, b]$. If $a < c < b$ and f is integrable on $[a, c]$ and on $[c, b]$, then f is integrable on $[a, b]$ and

$$\int_a^b f = \int_a^c f + \int_c^b f. \tag{1}$$

Proof
Since f is bounded on both $[a, c]$ and $[c, b]$, f is bounded on $[a, b]$. In this proof we will decorate upper and lower sums so that it will be clear which intervals we are dealing with. Let $\epsilon > 0$. By Theorem 32.5 there exist partitions P_1 and P_2 of $[a, c]$ and $[c, b]$ such that

$$U_a^c(f, P_1) - L_a^c(f, P_1) < \frac{\epsilon}{2} \quad \text{and} \quad U_c^b(f, P_2) - L_c^b(f, P_2) < \frac{\epsilon}{2}.$$

The set $P = P_1 \cup P_2$ is a partition of $[a, b]$, and it is obvious that

$$U_a^b(f, P) = U_a^c(f, P_1) + U_c^b(f, P_2) \tag{2}$$

with a similar identity for lower sums. It follows that

$$U_a^b(f, P) - L_a^b(f, P) < \epsilon,$$

so f is integrable on $[a, b]$ by Theorem 32.5. Also (1) holds because

$$\int_a^b f \leq U_a^b(f, P) = U_a^c(f, P_1) + U_c^b(f, P_2)$$

$$< L_a^c(f, P_1) + L_c^b(f, P_2) + \epsilon \leq \int_a^c f + \int_c^b f + \epsilon$$

and similarly $\int_a^b f > \int_a^c f + \int_c^b f - \epsilon$. ∎

Most functions encountered in calculus and analysis are covered by the next definition. However, see Exercises 33.10–33.12.

33.7 Definition.
A function f on $[a, b]$ is *piecewise monotonic* if there is a partition

$$P = \{a = t_0 < t_1 < \cdots < t_n = b\}$$

of $[a, b]$ such that f is monotonic on each interval (t_{k-1}, t_k). The function f is *piecewise continuous* if there is a partition P of $[a, b]$ such that f is uniformly continuous on each interval (t_{k-1}, t_k).

33.8 Theorem.
If f is a piecewise continuous function or a bounded piecewise monotonic function on $[a, b]$, then f is integrable on $[a, b]$.

Proof
Let P be the partition described in Definition 33.7. Consider a fixed interval $[t_{k-1}, t_k]$. If f is piecewise continuous, then its restriction to (t_{k-1}, t_k) can be extended to a continuous function f_k on $[t_{k-1}, t_k]$ by Theorem 19.5. If f is piecewise monotonic, then its restriction to (t_{k-1}, t_k) can be extended to a monotonic function f_k on $[t_{k-1}, t_k]$; for example, if f is increasing on (t_{k-1}, t_k), simply define

$$f_k(t_k) = \sup\{f(x) : x \in (t_{k-1}, t_k)\}$$

and

$$f_k(t_{k-1}) = \inf\{f(x) : x \in (t_{k-1}, t_k)\}.$$

In either case, f_k is integrable on $[t_{k-1}, t_k]$ by Theorem 33.1 or 33.2. Since f agrees with f_k on $[t_{k-1}, t_k]$ except possibly at the endpoints, Exercise 32.7 shows f is also integrable on $[t_{k-1}, t_k]$. Now Theorem 33.6 and a trivial induction argument show f is integrable on $[a, b]$. ∎

We close this section with a simple but useful result.

33.9 Intermediate Value Theorem for Integrals.
*If f is a continuous function on $[a, b]$, then for at least one x in (a, b)
we have*

$$f(x) = \frac{1}{b - a} \int_a^b f.$$

Proof
Let M and m be the maximum and minimum values of f on $[a, b]$.
If $m = M$, then f is a constant function and $f(x) = \frac{1}{b-a} \int_a^b f$ for
all $x \in [a, b]$. Otherwise, $m < M$ and by Theorem 18.1, there exist
distinct x_0 and y_0 in $[a, b]$ satisfying $f(x_0) = m$ and $f(y_0) = M$. Since
each function $M - f$ and $f - m$ is nonnegative and not identically 0,
Theorem 33.4(ii) shows $\int_a^b m < \int_a^b f < \int_a^b M$. Thus

$$m < \frac{1}{b - a} \int_a^b f < M,$$

and by the Intermediate Value Theorem 18.2 for continuous func-
tions, we have $f(x) = \frac{1}{b-a} \int_a^b f$ for some x between x_0 and y_0. Since
x is in (a, b), this completes the proof. ∎

33.10 Discussion.
An important question concerns when one can interchange limits and
integrals, i.e., when is

$$\lim_{n \to \infty} \int_a^b f_n(x)\, dx = \int_a^b \lim_{n \to \infty} f_n(x)\, dx \tag{1}$$

true? By Theorems 24.3 and 25.2, if the f_ns are continuous and
converge uniformly to $f = \lim_{n \to \infty} f_n$ on $[a, b]$, then f is continuous
and (1) holds. It turns out that if each f_n is just Riemann integrable
and $f_n \to f$ uniformly, then f is Riemann integrable and (1) holds;
see Exercise 33.9. What happens if $f_n \to f$ pointwise on $[a, b]$? One
problem is that f need not be integrable even if it is bounded and
each f_n is integrable.

Consider, for example, the non-integrable function f in Exam-
ple 2 on page 272: $f(x) = 1$ for rational x in $[a, b]$ and $f(x) = 0$ for
irrational x in $[a, b]$. Let $(x_k)_{k \in \mathbb{N}}$ be an enumeration of the rationals
in $[a, b]$, and define $f_n(x_k) = 1$ for $1 \le k \le n$ and $f_n(x) = 0$ for

all other x in $[a, b]$. Then $f_n \to f$ pointwise on $[a, b]$, and each f_n is integrable.

This example leaves open the possibility that (1) will hold provided all the functions f_n *and* the limit function f are integrable. However, Exercise 33.15 provides an example of a sequence (f_n) of functions on $[0, 1]$ converging pointwise to a function f, with all the functions integrable, and yet (1) does not hold. Nevertheless, there is an important theorem that does apply to sequences of functions that converge pointwise. □

33.11 Dominated Convergence Theorem.
Suppose (f_n) is a sequence of integrable functions on $[a, b]$ and $f_n \to f$ pointwise where f is an integrable function on $[a, b]$. If there exists an $M > 0$ such that $|f_n(x)| \leq M$ for all n and all x in $[a, b]$, then

$$\lim_{n \to \infty} \int_a^b f_n(x)\, dx = \int_a^b \lim_{n \to \infty} f_n(x)\, dx.$$

We omit the proof. An elementary proof of the Dominated Convergence Theorem is given by Jonathan W. Lewin [42]. Here is a corollary.

33.12 Monotone Convergence Theorem.
Suppose (f_n) is a sequence of integrable functions on $[a, b]$ such that $f_1(x) \leq f_2(x) \leq \cdots$ for all x in $[a, b]$. Suppose also that $f_n \to f$ pointwise where f is an integrable function on $[a, b]$. Then

$$\lim_{n \to \infty} \int_a^b f_n(x)\, dx = \int_a^b \lim_{n \to \infty} f_n(x)\, dx.$$

This follows from the Dominated Convergence Theorem, because there exists an $M > 0$ such that $|f_1(x)| \leq M$ and also $|f(x)| \leq M$ for all x in $[a, b]$. This implies $|f_n(x)| \leq M$ for all n and all x in $[a, b]$, since $-M \leq f_1(x) \leq f_n(x) \leq M$ for all x.

Our version of the Dominated Convergence Theorem is a special case of a much more general theorem, which is usually stated and proved for the family of all "Lebesgue integrable functions," not just for Riemann integrable functions. There is also a Monotone Convergence Theorem for Lebesgue integrable functions, but in that

generality it does not follow immediately from the Dominated Convergence Theorem, because in that setting integrable functions need not be bounded. An elementary proof of the Monotone Convergence Theorem is proved for Riemann integrable functions, without resort to Lebesgue theory, by Brian S. Thomson [67].

Exercises

33.1 Complete the proof of Theorem 33.1 by showing that a decreasing function on $[a, b]$ is integrable.

33.2 This exercise could have appeared just as easily in §4. Let S be a nonempty bounded subset of $\mathbb{R}$. For fixed $c > 0$, let $cS = \{cs : s \in S\}$. Show $\sup(cS) = c \cdot \sup(S)$ and $\inf(cS) = c \cdot \inf(S)$.

33.3 A function f on $[a, b]$ is called a *step function* if there exists a partition $P = \{a = u_0 < u_1 < \cdots < c_m = b\}$ of $[a, b]$ such that f is constant on each interval (u_{j-1}, u_j), say $f(x) = c_j$ for x in (u_{j-1}, u_j).

   (a) Show that a step function f is integrable and evaluate $\int_a^b f$.

   (b) Evaluate the integral $\int_0^4 P(x)\, dx$ for the postage-stamp function P in Exercise 17.16.

33.4 Give an example of a function f on $[0, 1]$ that is *not* integrable for which $|f|$ is integrable. *Hint*: Modify Example 2 in §32.

33.5 Show $\left| \int_{-2\pi}^{2\pi} x^2 \sin^8(e^x)\, dx \right| \leq \frac{16\pi^3}{3}$.

33.6 Prove (2) in the proof of Theorem 33.5. *Hint*: For $x_0, y_0 \in S$, we have $|f(x_0)| - |f(y_0)| \leq |f(x_0) - f(y_0)| \leq M(f, S) - m(f, S)$.

33.7 Let f be a bounded function on $[a, b]$, so that there exists $B > 0$ such that $|f(x)| \leq B$ for all $x \in [a, b]$.

   (a) Show

$$U(f^2, P) - L(f^2, P) \leq 2B[U(f, P) - L(f, P)]$$

   for all partitions P of $[a, b]$. *Hint*: $f(x)^2 - f(y)^2 = [f(x) + f(y)] \cdot [f(x) - f(y)]$.

   (b) Show that if f is integrable on $[a, b]$, then f^2 also is integrable on $[a, b]$.

33.8 Let f and g be integrable functions on $[a, b]$.

    (a) Show fg is integrable on $[a, b]$. *Hint*: Use Exercise 33.7 and $4fg = (f + g)^2 - (f - g)^2$.

    (b) Show $\max(f, g)$ and $\min(f, g)$ are integrable on $[a, b]$. *Hint*: Exercise 17.8.

33.9 Let (f_n) be a sequence of integrable functions on $[a, b]$, and suppose $f_n \to f$ uniformly on $[a, b]$. Prove f is integrable and

$$\int_a^b f = \lim_{n \to \infty} \int_a^b f_n.$$

Compare this result with Theorems 25.2 and 33.11.

33.10 Let $f(x) = \sin \frac{1}{x}$ for $x \neq 0$ and $f(0) = 0$. Show f is integrable on $[-1, 1]$. *Hint*: See the answer to Exercise 33.11(c).

33.11 Let $f(x) = x \, \text{sgn}(\sin \frac{1}{x})$ for $x \neq 0$ and $f(0) = 0$.

    (a) Show f is not piecewise continuous on $[-1, 1]$.

    (b) Show f is not piecewise monotonic on $[-1, 1]$.

    (c) Show f is integrable on $[-1, 1]$.

33.12 Let f be the function described in Exercise 17.14.

    (a) Show f is not piecewise continuous or piecewise monotonic on any interval $[a, b]$.

    (b) Show f is integrable on every interval $[a, b]$ and $\int_a^b f = 0$.

33.13 Suppose f and g are continuous functions on $[a, b]$ such that $\int_a^b f = \int_a^b g$. Prove there exists x in (a, b) such that $f(x) = g(x)$.

33.14 **(a)** Prove the following generalization of the Intermediate Value Theorem for Integrals. If f and g are continuous functions on $[a, b]$ and $g(t) \geq 0$ for all $t \in [a, b]$, then there exists x in (a, b) such that

$$\int_a^b f(t)g(t) \, dt = f(x) \int_a^b g(t) \, dt.$$

    (b) Show Theorem 33.9 is a special case of part (a).

    (c) Does the conclusion in part (a) hold if $[a, b] = [-1, 1]$ and $f(t) = g(t) = t$ for all t?

33.15 For integers $n \geq 3$, define the function f_n on $[0, 1]$ by the rules:

$$f_n(0) = f_n\left(\frac{2}{n}\right) = f_n(1) = 0 \quad \text{and} \quad f_n\left(\frac{1}{n}\right) = n,$$

and so that its graph is a straight line from $(0, 0)$ to $(\frac{1}{n}, n)$, from $(\frac{1}{n}, n)$ to $(\frac{2}{n}, 0)$, and from $(\frac{2}{n}, 0)$ to $(1, 0)$.

(a) Graph f_3, f_4 and f_5.

(b) Show $f_n \to 0$ pointwise on $[0, 1]$.

(c) Show $\lim_n \int_0^1 f_n(x)\, dx \neq \int_0^1 0\, dx$. Why doesn't this contradict the Dominated Convergence Theorem?

§34   Fundamental Theorem of Calculus

There are two versions of the Fundamental Theorem of Calculus. Each says, roughly speaking, that differentiation and integration are inverse operations. In fact, our first version [Theorem 34.1] says "the integral of the derivative of a function is given by the function," and our second version [Theorem 34.3] says "the derivative of the integral of a continuous function is the function." It is somewhat traditional for books to prove our second version first and use it to prove our first version, although some books do avoid this approach. F. Cunningham, Jr. [18] offers some good reasons for avoiding the traditional approach:

(a) Theorem 34.3 implies Theorem 34.1 only for functions g whose derivative g' is continuous; see Exercise 34.1.
(b) Making Theorem 34.1 depend on Theorem 34.3 obscures the fact that the two theorems say different things, have different applications, and may leave the impression Theorem 34.3 is *the* fundamental theorem.
(c) The need for Theorem 34.1 in calculus is immediate and easily motivated.

In what follows, we say a function h defined on (a, b) is *integrable* on $[a, b]$ if every extension of h to $[a, b]$ is integrable. In view of

Exercise 32.7, the value $\int_a^b h$ will not depend on the values of the extensions at a or b.

34.1 Fundamental Theorem of Calculus I.
If g is a continuous function on $[a, b]$ that is differentiable on (a, b), and if g' is integrable on $[a, b]$, then

$$\int_a^b g' = g(b) - g(a). \tag{1}$$

Proof
Let $\epsilon > 0$. By Theorem 32.5, there exists a partition $P = \{a = t_0 < t_1 < \cdots < t_n = b\}$ of $[a, b]$ such that

$$U(g', P) - L(g', P) < \epsilon. \tag{2}$$

We apply the Mean Value Theorem 29.3 to each interval $[t_{k-1}, t_k]$ to obtain x_k in (t_{k-1}, t_k) for which

$$(t_k - t_{k-1})g'(x_k) = g(t_k) - g(t_{k-1}).$$

Hence we have

$$g(b) - g(a) = \sum_{k=1}^{n}[g(t_k) - g(t_{k-1})] = \sum_{k=1}^{n} g'(x_k)(t_k - t_{k-1}).$$

It follows that

$$L(g', P) \leq g(b) - g(a) \leq U(g', P); \tag{3}$$

see Definition 32.1. Since

$$L(g', P) \leq \int_a^b g' \leq U(g', P),$$

inequalities (2) and (3) imply

$$\left| \int_a^b g' - [g(b) - g(a)] \right| < \epsilon.$$

Since ϵ is arbitary, (1) holds. ∎

The integration formulas in calculus all rely in the end on Theorem 34.1.

Example 1
If $g(x) = \frac{x^{n+1}}{n+1}$, then $g'(x) = x^n$, so

$$\int_a^b x^n \, dx = \frac{b^{n+1}}{n+1} - \frac{a^{n+1}}{n+1} = \frac{b^{n+1} - a^{n+1}}{n+1}. \tag{1}$$

In particular,

$$\int_a^b x^2 \, dx = \frac{b^3 - a^3}{3}.$$

Formula (1) is valid for any powers n for which $g(x) = \frac{x^{n+1}}{n+1}$ is defined on $[a, b]$. See Examples 3 and 4 in §28 and Exercises 29.15 and 37.5. For example,

$$\int_a^b \sqrt{x} \, dx = \int_a^b x^{1/2} \, dx = \frac{2}{3}[b^{3/2} - a^{3/2}] \quad \text{for} \quad 0 \le a < b.$$

$\square$

34.2 Theorem [Integration by Parts].
If u and v are continuous functions on $[a, b]$ that are differentiable on (a, b), and if u' and v' are integrable on $[a, b]$, then

$$\int_a^b u(x)v'(x) \, dx + \int_a^b u'(x)v(x) \, dx = u(b)v(b) - u(a)v(a). \tag{1}$$

Proof
Let $g = uv$; then $g' = uv' + u'v$ by Theorem 28.3. Exercise 33.8 shows g' is integrable. Now Theorem 34.1 shows

$$\int_a^b g'(x) \, dx = g(b) - g(a) = u(b)v(b) - u(a)v(a),$$

so (1) holds. ■

Note the use of Exercise 33.8 above can be avoided if u' and v' are continuous, which is normally the case.

Example 2
Here is a simple application of integration by parts. To calculate $\int_0^\pi x \cos x \, dx$, we note the integrand has the form $u(x)v'(x)$ where $u(x) = x$ and $v(x) = \sin x$. Hence

$$\int_0^\pi x \cos x \, dx = u(\pi)v(\pi) - u(0)v(0) - \int_0^\pi 1 \cdot \sin x \, dx = -\int_0^\pi \sin x \, dx = -2.$$

$\square$

In what follows we use the convention $\int_a^b f = -\int_b^a f$ for $a > b$.

34.3 Fundamental Theorem of Calculus II.
Let f be an integrable function on $[a,b]$. For x in $[a,b]$, let

$$F(x) = \int_a^x f(t) \, dt.$$

Then F is continuous on $[a,b]$. If f is continuous at x_0 in (a,b), then F is differentiable at x_0 and

$$F'(x_0) = f(x_0).$$

Proof
Select $B > 0$ so that $|f(x)| \le B$ for all $x \in [a,b]$. If $x, y \in [a,b]$ and $|x - y| < \frac{\epsilon}{B}$ where $x < y$, say, then

$$|F(y) - F(x)| = \left| \int_x^y f(t) \, dt \right| \le \int_x^y |f(t)| \, dt \le \int_x^y B \, dt = B(y - x) < \epsilon.$$

This shows F is [uniformly] continuous on $[a,b]$.

Suppose f is continuous at x_0 in (a,b). Observe

$$\frac{F(x) - F(x_0)}{x - x_0} = \frac{1}{x - x_0} \int_{x_0}^x f(t) \, dt$$

for $x \ne x_0$. The trick is to observe

$$f(x_0) = \frac{1}{x - x_0} \int_{x_0}^x f(x_0) \, dt,$$

and therefore

$$\frac{F(x) - f(x_0)}{x - x_0} - f(x_0) = \frac{1}{x - x_0} \int_{x_0}^x [f(t) - f(x_0)] \, dt. \qquad (1)$$

Let $\epsilon > 0$. Since f is continuous at x_0, there exists $\delta > 0$ such that

$$t \in (a,b) \quad \text{and} \quad |t - x_0| < \delta \quad \text{imply} \quad |f(t) - f(x_0)| < \epsilon;$$

see Theorem 17.2. It follows from (1) that

$$\left| \frac{F(x) - F(x_0)}{x - x_0} - f(x_0) \right| \le \epsilon$$

for x in (a, b) satisfying $|x - x_0| < \delta$; the cases $x > x_0$ and $x < x_0$ require separate arguments. We have just shown

$$\lim_{x \to x_0} \frac{F(x) - F(x_0)}{x - x_0} = f(x_0).$$

In other words, $F'(x_0) = f(x_0)$. ■

A useful technique of integration is known as "substitution." A more accurate description of the process is "change of variable." The technique is the reverse of the chain rule.

34.4 Theorem [Change of Variable].
Let u be a differentiable function on an open interval J such that u' is continuous, and let I be an open interval such that $u(x) \in I$ for all $x \in J$. If f is continuous on I, then $f \circ u$ is continuous on J and

$$\int_a^b f \circ u(x) u'(x) \, dx = \int_{u(a)}^{u(b)} f(u) \, du \tag{1}$$

for a, b in J.

Note $u(a)$ need not be less than $u(b)$, even if $a < b$.

Proof
The continuity of $f \circ u$ follows from Theorem 17.5. Fix c in I and let $F(u) = \int_c^u f(t) \, dt$. Then $F'(u) = f(u)$ for all $u \in I$ by Theorem 34.3. Let $g = F \circ u$. By the Chain Rule 28.4, we have

$$g'(x) = F'(u(x)) \cdot u'(x) = f(u(x)) \cdot u'(x),$$

so by Theorem 34.1

$$\int_a^b f \circ u(x) u'(x) \, dx = \int_a^b g'(x) \, dx = g(b) - g(a) = F(u(b)) - F(u(a))$$

$$= \int_c^{u(b)} f(t) \, dt - \int_c^{u(a)} f(t) \, dt = \int_{u(a)}^{u(b)} f(t) \, dt.$$

This proves (1). ■

Example 3

Let g be a one-to-one differentiable function on an open interval I. Then $J = g(I)$ is an open interval, and the inverse function g^{-1} is differentiable on J by Theorem 29.9. We show

$$\int_a^b g(x)\,dx + \int_{g(a)}^{g(b)} g^{-1}(u)\,du = b \cdot g(b) - a \cdot g(a) \qquad (1)$$

for a, b in I.

We put $f = g^{-1}$ and $u = g$ in the change of variable formula to obtain

$$\int_a^b g^{-1} \circ g(x) g'(x)\,dx = \int_{g(a)}^{g(b)} g^{-1}(u)\,du.$$

Since $g^{-1} \circ g(x) = x$ for x in I, we obtain

$$\int_{g(a)}^{g(b)} g^{-1}(u)\,du = \int_a^b x g'(x)\,dx.$$

Now integrate by parts with $u(x) = x$ and $v(x) = g(x)$:

$$\int_{g(a)}^{g(b)} g^{-1}(u)\,du = b \cdot g(b) - a \cdot g(a) - \int_a^b g(x)\,dx.$$

This is formula (1). □

Exercises

34.1 Use Theorem 34.3 to prove Theorem 34.1 for the case g' is continuous. *Hint*: Let $F(x) = \int_a^x g'$; then $F' = g'$. Apply Corollary 29.5.

34.2 Calculate
(a) $\lim_{x \to 0} \frac{1}{x} \int_0^x e^{t^2}\,dt$ (b) $\lim_{h \to 0} \frac{1}{h} \int_3^{3+h} e^{t^2}\,dt$.

34.3 Let f be defined as follows: $f(t) = 0$ for $t < 0$; $f(t) = t$ for $0 \le t \le 1$; $f(t) = 4$ for $t > 1$.

(a) Determine the function $F(x) = \int_0^x f(t)\,dt$.

(b) Sketch F. Where is F continuous?

(c) Where is F differentiable? Calculate F' at the points of differentiability.

34.4 Repeat Exercise 34.3 for f where $f(t) = t$ for $t < 0$; $f(t) = t^2 + 1$ for $0 \le t \le 2$; $f(t) = 0$ for $t > 2$.

34.5 Let f be a continuous function on $\mathbb{R}$ and define

$$F(x) = \int_{x-1}^{x+1} f(t)\, dt \quad \text{for} \quad x \in \mathbb{R}.$$

Show F is differentiable on $\mathbb{R}$ and compute F'.

34.6 Let f be a continuous function on $\mathbb{R}$ and define

$$G(x) = \int_0^{\sin x} f(t)\, dt \quad \text{for} \quad x \in \mathbb{R}.$$

Show G is differentiable on $\mathbb{R}$ and compute G'.

34.7 Use change of variables to integrate $\int_0^1 x\sqrt{1 - x^2}\, dx$.

34.8 **(a)** Use integration by parts to evaluate

$$\int_0^1 x \arctan x\, dx.$$

Hint: Let $u(x) = \arctan x$, so that $u'(x) = \frac{1}{1+x^2}$.

(b) If you used $v(x) = \frac{x^2}{2}$ in part (a), do the computation again with $v(x) = \frac{x^2+1}{2}$. This interesting example is taken from J.L. Borman [10].

34.9 Use Example 3 to show $\int_0^{1/2} \arcsin x\, dx = \frac{\pi}{12} + \frac{\sqrt{3}}{2} - 1$.

34.10 Let g be a strictly increasing continuous function mapping $[0,1]$ onto $[0,1]$. Give a geometric argument showing $\int_0^1 g(x)dx + \int_0^1 g^{-1}(u)du = 1$.

34.11 Suppose f is a continuous function on $[a,b]$. Show that if $\int_a^b f(x)^2 dx = 0$, then $f(x) = 0$ for all x in $[a,b]$. *Hint*: See Theorem 33.4.

34.12 Show that if f is a continuous real-valued function on $[a,b]$ satisfying $\int_a^b f(x)g(x)\, dx = 0$ for every continuous function g on $[a,b]$, then $f(x) = 0$ for all x in $[a,b]$.

§35 * Riemann-Stieltjes Integrals

In this long section we introduce a useful generalization of the Riemann integral. In the Riemann integral, all intervals of the same length are given the same weight. For example, in our definition of upper sums

$$U(f, P) = \sum_{k=1}^{n} M(f, [t_{k-1}, t_k]) \cdot (t_k - t_{k-1}), \qquad (*)$$

the factors $(t_k - t_{k-1})$ are the lengths of the intervals involved. In applications such as probability and statistics, it is desirable to modify the definition so as to weight the intervals according to some increasing function F. In other words, the idea is to replace the factors $(t_k - t_{k-1})$ in (*) by $[F(t_k) - F(t_{k-1})]$. The Riemann integral is, then, the special case where $F(t) = t$ for all t.

It is also desirable to allow some *points* to have positive weight. This corresponds to the situation where F has jumps, i.e., where the left-hand and right-hand limits of F differ. In fact, if (c_k) is a sequence in $\mathbb{R}$, then the sums

$$\sum_{k=1}^{\infty} c_k f(u_k)$$

can be viewed as a generalized integral for a suitable F [see Examples 1 and 3 on pages 301 and 309]. In this case, F has a jump at each u_k.

The traditional treatment, in all books that I am aware of, replaces the factors $(t_k - t_{k-1})$ in (*) by $[F(t_k) - F(t_{k-1})]$ and develops the theory from there, though some authors emphasize upper and lower sums while others stress generalized Riemann sums. In this section, we offer a slightly different treatment, so

Warning. Theorems in this section do not necessarily correspond to theorems in other texts.

We deviate from tradition because:

(a) Our treatment is more general. Functions that are Riemann-Stieltjes integrable in the traditional sense are integrable in our sense [Theorem 35.20].

(b) In the traditional theory, if f and F have a common discontinuity, then f is not integrable using F. Such unfortunate results disappear in our approach. We will show piecewise continuous and piecewise monotonic functions are always integrable using F [Theorem 35.17]. We also will observe that if F is a step function, then *all* bounded functions are integrable; see Example 1.

(c) We will give a definition involving Riemann-Stieltjes sums that is equivalent to our definition involving upper and lower sums [Theorem 35.25]. The corresponding standard definitions are not equivalent.

As just explained, our development of Riemann-Stieltjes integrals has several positive features. However, this section is long with lots of technical details. Therefore, we recommend readers omit the proofs on first reading, and then decide whether to go through the details.

Many of the results in this section are straightforward generalizations of results in §§32 and 33. Accordingly, many proofs will be brief or omitted.

35.1 Notation.

We assume throughout this section that F is an increasing function on a closed interval $[a, b]$. To avoid trivialities we assume $F(a) < F(b)$. All left-hand and right-hand limits exist; see Definition 20.3 and Exercise 35.1. We use the notation

$$F(t^-) = \lim_{x \to t^-} F(x) \quad \text{and} \quad F(t^+) = \lim_{x \to t^+} F(x).$$

For the endpoints we decree

$$F(a^-) = F(a) \quad \text{and} \quad F(b^+) = F(b).$$

Note that $F(t^-) \leq F(t^+)$ for all $t \in [a, b]$. If F is continuous at t, then $F(t^-) = F(t) = F(t^+)$. Otherwise $F(t^-) < F(t^+)$ and the difference $F(t^+) - F(t^-)$ is called the *jump* of F at t. The actual value of $F(t)$ *at* jumps t will play no role in what follows. □

In the next definition we employ some of the notation established in Definition 32.1.

35.2 Definition.
For a bounded function f on $[a, b]$ and a partition $P = \{a = t_0 < t_1 < \cdots < t_n = b\}$ of $[a, b]$, we write

$$J_F(f, P) = \sum_{k=0}^{n} f(t_k) \cdot [F(t_k^+) - F(t_k^-)]. \tag{1}$$

The *upper Darboux-Stieltjes sum* is

$$U_F(f, P) = J_F(f, P) + \sum_{k=1}^{n} M(f, (t_{k-1}, t_k)) \cdot [F(t_k^-) - F(t_{k-1}^+)] \tag{2}$$

and the *lower Darboux-Stieltjes sum* is

$$L_F(f, P) = J_F(f, P) + \sum_{k=1}^{n} m(f, (t_{k-1}, t_k)) \cdot [F(t_k^-) - F(t_{k-1}^+)]. \tag{3}$$

These definitions explicitly take into account the possible jump effects of F at the points t_k, though F may have jumps at other points. The other terms focus on the effect of F on the open intervals (t_{k-1}, t_k). Observe

$$[a, b] = \bigcup_{k=0}^{n} \{t_k\} \cup \bigcup_{k=1}^{n} (t_{k-1}, t_k)$$

represents $[a, b]$ as a disjoint union.[2] See Remarks 35.26 for a discussion about the choice of open intervals versus closed intervals in the definitions of Riemann integrals and Riemann-Stieltjes integrals.

Note

$$U_F(f, P) - L_F(f, P)$$
$$= \sum_{k=1}^{n} [M(f, (t_{k-1}, t_k)) - m(f, (t_{k-1}, t_k))][F(t_k^-) - F(t_{k-1}^+)] \tag{4}$$

and

$$m(f, [a, b]) \cdot [F(b) - F(a)] \leq L_F(f, P) \leq U_F(f, P)$$
$$\leq M(f, [a, b]) \cdot [F(b) - F(a)]. \tag{5}$$

---

[2]In measure theory this is an example of a measurable partition, that is to say, a family of measurable sets with disjoint union $[a, b]$.

In checking (5), note

$$\sum_{k=0}^{n}[F(t_k^+) - F(t_k^-)] + \sum_{k=1}^{n}[F(t_k^-) - F(t_{k-1}^+)]$$
$$= F(t_0^+) - F(t_0^-) + \sum_{k=1}^{n}[F(t_k^+) - F(t_{k-1}^+)]$$
$$= F(a^+) - F(a^-) + F(b^+) - F(a^+) = F(b) - F(a), \tag{6}$$

since the last sum is a telescoping sum. The *upper Darboux-Stieltjes integral* is

$$U_F(f) = \inf\{U_F(f, P) : P \text{ is a partition of } [a, b]\} \tag{7}$$

and the *lower Darboux-Stieltjes integral* is

$$L_F(f) = \sup\{L_F(f, P) : P \text{ is a partition of } [a, b]\}. \tag{8}$$

Theorem 35.5 below states that $L_F(f) \leq U_F(f)$. Accordingly, we say f is *Darboux-Stieltjes integrable on* $[a, b]$ with respect to F or, more briefly, *F-integrable on* $[a, b]$, provided $L_F(f) = U_F(f)$; in this case we write

$$\int_a^b f\, dF = \int_a^b f(x)\, dF(x) = L_F(f) = U_F(f).$$

Example 1
For each u in $[a, b]$, let

$$J_u(t) = \begin{cases} 0 & \text{for} \quad t < u, \\ 1 & \text{for} \quad t \geq u, \end{cases}$$

for $u > a$, and let

$$J_a(t) = \begin{cases} 0 & \text{for} \quad t = a, \\ 1 & \text{for} \quad t > a. \end{cases}$$

Then J_u is an increasing step function with jump 1 at u. Also, every bounded function f on $[a, b]$ is J_u-integrable and
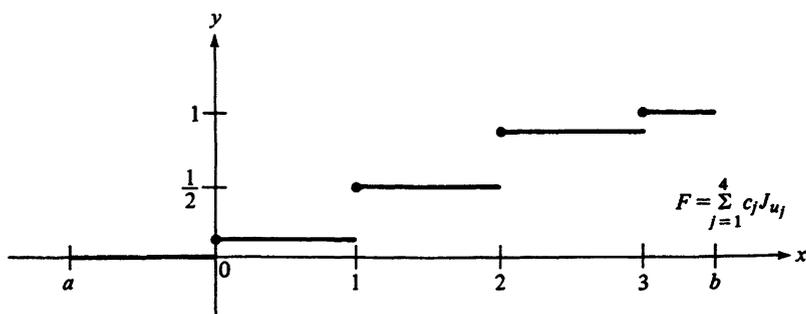
$$\int_a^b f\, dJ_u = f(u).$$

FIGURE 35.1

More generally, if $u_1, u_2, \ldots, u_m$ are distinct points in $[a, b]$ and if $c_1, c_2, \ldots, c_m$ are positive numbers, then

$$F = \sum_{j=1}^{m} c_j J_{u_j}$$

is an increasing step function with jumps c_j at u_j. See Fig. 35.1 for a special case. Every bounded function f on $[a, b]$ is F-integrable and

$$\int_a^b f \, dF = \sum_{j=1}^{m} c_j \cdot f(u_j). \tag{1}$$

To check (1), first note the validity of (1) will not change if some of the u_j are not jump points so that $c_j = 0$. So we may assume a and b are in $\{u_j : 1 \le j \le m\}$, and $a = u_1 < u_2 < \cdots < u_m = b$. Let P be the partition $\{a = u_1 < u_2 < \cdots < u_m = b\}$. Then we have $F(u_j^+) - F(u_j^-) = c_j$ for $j = 1, 2, \ldots, m$ and $F(u_j^-) - F(u_{j-1}^+) = 0$ for $j = 2, 3, \ldots, m$. Therefore

$$U_F(f, P) = L_F(f, P) = J_F(f, P) = \sum_{j=1}^{m} f(u_j) \cdot c_j$$

for any bounded function f on $[a, b]$. From (7) and (8) in Definition 35.2 and $L_F(f) \leq U_F(f)$, it follows that

$$U_F(f) = L_F(f) = \sum_{j=1}^{m} f(u_j) \cdot c_j.$$

Hence f is F-integrable and (1) holds. □

Example 2
We specialize Example 1 to the case $u_1 = 0$, $u_2 = 1$, $u_3 = 2$, $u_4 = 3$; $c_1 = c_4 = \frac{1}{8}$, $c_2 = c_3 = \frac{3}{8}$. Thus we must have $a \leq 0$ and $b \geq 3$; see Fig. 35.1. For any bounded function f on $[a, b]$, we have

$$\int_a^b f \, dF = \frac{1}{8} f(0) + \frac{3}{8} f(1) + \frac{3}{8} f(2) + \frac{1}{8} f(3).$$

□

35.3 Lemma.
Let f be a bounded function on $[a, b]$, and let P and Q be partitions of $[a, b]$ such that $P \subseteq Q$. Then

$$L_F(f, P) \leq L_F(f, Q) \leq U_F(f, Q) \leq U_F(f, P). \tag{1}$$

Proof
We imitate the proof of Lemma 32.2 on page 273 down to, but not including, formula (3). In the present case, the difference $L_F(f, Q) - L_F(f, P)$ equals

$$
\begin{aligned}
f(u) \cdot [F(u^+) - F(u^-)] &+ m(f, (t_{k-1}, u)) \cdot [F(u^-) - F(t_{k-1}^+)] \\
&+ m(f, (u, t_k)) \cdot [F(t_k^-) - F(u^+)] \\
&- m(f, (t_{k-1}, t_k)) \cdot [F(t_k^-) - F(t_{k-1}^+)],
\end{aligned}
\tag{3}
$$

and this is nonnegative because

$$
\begin{aligned}
m(f, (t_{k-1}, t_k)) &\cdot [F(t_k^-) - F(t_{k-1}^+)] \\
= m(f, (t_{k-1}, t_k)) &\cdot [F(t_k^-) - F(u^+) + F(u^+) - F(u^-) \\
&\quad + F(u^-) - F(t_{k-1}^+)] \\
\leq m(f, (u, t_k)) &\cdot [F(t_k^-) - F(u^+)] + f(u) \cdot [F(u^+) - F(u^-)] \\
&+ m(f, (t_{k-1}, u)) \cdot [F(u^-) - F(t_{k-1}^+)].
\end{aligned}
$$

■

35.4 Lemma.
If f is a bounded function on $[a, b]$ and if P and Q are partitions of $[a, b]$, then $L_F(f, P) \leq U_F(f, Q)$.

Proof
Imitates the proof of Lemma 32.3 on page 273. ∎

35.5 Theorem.
For every bounded function f on $[a, b]$, we have $L_F(f) \leq U_F(f)$.

Proof
Imitates the proof of Theorem 32.4 on page 274. ∎

35.6 Theorem.
A bounded function f on $[a, b]$ is F-integrable if and only if for each $\epsilon > 0$ there exists a partition P such that

$$U_F(f, P) - L_F(f, P) < \epsilon.$$

Proof
Imitates the proof of Theorem 32.5 on page 274. ∎

We next develop analogues of results in §33; we return later to generalizations of items Definition 32.6 through Theorem 32.9. We begin with the analogue of Theorem 33.2. The analogue of Theorem 33.1 is true, but its proof requires some preparation, so we defer it to Theorem 35.16.

35.7 Theorem.
Every continuous function f on $[a, b]$ is F-integrable.

Proof
To apply Theorem 35.6, let $\epsilon > 0$. Since f is uniformly continuous, there exists $\delta > 0$ such that

$$x, y \in [a, b] \quad \text{and} \quad |x - y| < \delta \quad \text{imply} \quad |f(x) - f(y)| < \frac{\epsilon}{F(b) - F(a)}.$$

Just as in the proof of Theorem 33.2, there is a partition P of $[a, b]$ such that

$$M(f, (t_{k-1}, t_k)) - m(f, (t_{k-1}, t_k)) < \frac{\epsilon}{F(b) - F(a)}$$

for each k. Hence by (4) of Definition 35.2 we have

$$U_F(f, P) - L_F(f, P) \leq \sum_{k=1}^{n} \frac{\epsilon}{F(b) - F(a)} [F(t_k^-) - F(t_{k-1}^+)] \leq \epsilon.$$

Theorem 35.6 now shows f is F-integrable.   ∎

35.8 Theorem.
Let f and g be F-integrable functions on $[a, b]$, and let c be a real number. Then
  (i) *cf is F-integrable and $\int_a^b (cf) \, dF = c \int_a^b f \, dF$;*
  (ii) *$f + g$ is F-integrable and $\int_a^b (f + g) \, dF = \int_a^b f \, dF + \int_a^b g \, dF$.*

Proof
Imitates the proof of Theorem 33.3, using Theorem 35.6 instead of Theorem 32.5.   ∎

35.9 Theorem.
If f and g are F-integrable on $[a, b]$ and if $f(x) \leq g(x)$ for $x \in [a, b]$, then $\int_a^b f \, dF \leq \int_a^b g \, dF$.

Proof
Imitates the proof of Theorem 33.4(i).   ∎

35.10 Theorem.
If f is F-integrable on $[a, b]$, then $|f|$ is F-integrable and

$$\left| \int_a^b f \, dF \right| \leq \int_a^b |f| \, dF.$$

Proof
Imitates the proof of Theorem 33.5 and uses formula (4) of Definition 35.2.   ∎

35.11 Theorem.
Let f be a function defined on $[a, b]$. If $a < c < b$ and f is F-integrable on $[a, c]$ and on $[c, b]$, then f is F-integrable on $[a, b]$ and

$$\int_a^b f \, dF = \int_a^c f \, dF + \int_c^b f \, dF. \tag{1}$$

Proof
Imitates the proof of Theorem 33.6. Note that an upper or lower sum on $[a, c]$ will include the term $f(c)[F(c) - F(c^-)]$ while an upper or lower sum on $[c, b]$ will include the term $f(c)[F(c^+) - F(c)]$. ∎

The next result clearly has no analogue in §32 or §33.

35.12 Theorem.
(a) *Let F_1 and F_2 be increasing functions on $[a, b]$. If f is F_1-integrable and F_2-integrable on $[a, b]$ and if $c > 0$, then f is cF_1-integrable, f is $(F_1 + F_2)$-integrable,*

$$\int_a^b f \, d(cF_1) = c \int_a^b f \, dF_1, \tag{1}$$

and

$$\int_a^b f \, d(F_1 + F_2) = \int_a^b f \, dF_1 + \int_a^b f \, dF_2. \tag{2}$$

(b) *Let (F_j) be a sequence of increasing functions on $[a, b]$ such that $F = \sum_{j=1}^{\infty} F_j$ defines an (automatically increasing) function on $[a, b]$. Thus the series converges on the entire interval $[a, b]$, and $F(a)$ and $F(b)$ are finite. If a bounded function f on $[a, b]$ is F_j-integrable for each j, then f is F-integrable on $[a, b]$ and*

$$\int_a^b f \, dF = \sum_{j=1}^{\infty} \int_a^b f \, dF_j. \tag{3}$$

Proof
From Theorem 20.4 we see

$$(F_1 + F_2)(t^+) = \lim_{x \to t^+} [F_1(x) + F_2(x)] = \lim_{x \to t^+} F_1(x) + \lim_{x \to t^+} F_2(x)$$
$$= F_1(t^+) + F_2(t^+)$$

with similar identities for $(F_1 + F_2)(t^-)$, $(cF_1)(t^+)$ and $(cF_1)(t^-)$. Hence for any partition P of $[a, b]$, we have

$$\begin{aligned} U_{F_1 + F_2}(f, P) &= U_{F_1}(f, P) + U_{F_2}(f, P) \\ L_{F_1 + F_2}(f, P) &= L_{F_1}(f, P) + L_{F_2}(f, P), \end{aligned} \tag{4}$$

$U_{cF_1}(f, P) = cU_{F_1}(f, P)$ and $L_{cF_1}(f, P) = cL_{F_1}(f, P)$. It is now clear that f is cF_1-integrable and that (1) holds. To check (2), let $\epsilon > 0$. By Theorem 35.6 and Lemma 35.3, there is a single partition P of $[a, b]$ so that both

$$U_{F_1}(f, P) - L_{F_1}(f, P) < \frac{\epsilon}{2} \quad \text{and} \quad U_{F_2}(f, P) - L_{F_2}(f, P) < \frac{\epsilon}{2}.$$

Hence by (4) we have

$$U_{F_1+F_2}(f, P) - L_{F_1+F_2}(f, P) < \epsilon.$$

This and Theorem 35.6 imply f is $(F_1 + F_2)$-integrable. The identity (2) follows from

$$\int_a^b f\, d(F_1 + F_2) \leq U_{F_1+F_2}(f, P) < L_{F_1+F_2}(f, P) + \epsilon$$

$$= L_{F_1}(f, P) + L_{F_2}(f, P) + \epsilon \leq \int_a^b f\,dF_1 + \int_a^b f\,dF_2 + \epsilon$$

and the similar inequality

$$\int_a^b f\, d(F_1 + F_2) > \int_a^b f\, dF_1 + \int_a^b f\, dF_2 - \epsilon.$$

Now we prove part (b). We write S_m for $\sum_{j=1}^m F_j$ and T_m for the difference (or tail sum) $F - S_m = \sum_{j=m+1}^\infty F_j$. Let B be a positive bound for $|f|$ on $[a, b]$, and consider $\epsilon > 0$. Select a positive integer m_0 so that

$$|T_m(b) - T_m(a)| < \frac{\epsilon}{5B} \quad \text{for} \quad m \geq m_0. \tag{5}$$

We will show

$$U_F(f) - L_F(f) < \frac{3\epsilon}{5}. \tag{6}$$

Since $\epsilon > 0$ is arbitrary, this will imply $U_F(f) = L_F(f)$, so that f is F-integrable. We will also show

$$\left| \int_a^b f\,dF - \int_a^b f\,dS_m \right| < \epsilon \quad \text{for} \quad m \geq m_0. \tag{7}$$

Since $\epsilon > 0$ is arbitrary, and $\int_a^b f\, dS_m = \sum_{j=1}^m \int_a^b f\, dF_j$ by (2), this will confirm (3).

We now prove (6). First we claim

$$|U_F(f, P) - U_{S_m}(f, P)| < \frac{\epsilon}{5} \text{ for all partitions } P \text{ of } [a, b] \text{ and } m \geq m_0,$$
(8)

$$|L_F(f, P) - L_{S_m}(f, P)| < \frac{\epsilon}{5} \text{ for all partitions } P \text{ of } [a, b] \text{ and } m \geq m_0.$$
(9)

Let $P = \{a = t_0 < t_1 < \cdots < t_n = b\}$. Note $T_m = \sum_{j=m+1}^{\infty} F_j$ is an increasing function on $[a, b]$, so using Eq. (2) in Definition 35.2 on page 300, the left-hand side of inequality (8) is bounded by

$$\sum_{k=0}^{n} B[T_m(t_k^+) - T_m(t_k^-)] + \sum_{k=1}^{n} B[T_m(t_k^-) - T_m(t_{k-1}^+)].$$

This sum is equal to $B[T_m(b^+) - T_m(a^-)] = B[T_m(b) - T_m(a)]$ which, by (5), is less than $\frac{\epsilon}{5}$, so (8) holds. The verification of (9) is similar.

In view of (2), f is S_m-integrable for each m, so for each m there is a partition P_m of $[a, b]$ satisfying

$$U_{S_m}(f, P_m) - L_{S_m}(f, P_m) < \frac{\epsilon}{5}.$$
(10)

Now we apply the triangle inequality to (8) and (9) [with $P = P_m$] and (10) to obtain

$$U_F(f, P_m) - L_F(f, P_m) < \frac{3\epsilon}{5} \quad \text{for} \quad m \geq m_0.$$
(11)

This implies (6); therefore f is F-integrable.

Now (11) and (10) imply

$$U_F(f, P_m) - \int_a^b f \, dF < \frac{3\epsilon}{5} \quad \text{and} \quad U_{S_m}(f, P_m) - \int_a^b f \, dS_m < \frac{\epsilon}{5}$$

for $m \geq m_0$. Applying these inequalities and (8), we obtain

$$\left| \int_a^b f \, dF - \int_a^b f \, dS_m \right| < \epsilon \quad \text{for} \quad m \geq m_0,$$

i.e., (7) holds. ∎

Example 3

(a) Let (u_j) be a sequence of distinct points in $[a, b]$, and let (c_j) be a sequence of nonnegative numbers[3] such that $\sum c_j < \infty$. Using the notation of Example 1 on page 301, we define

$$F = \sum_{j=1}^{\infty} c_j J_{u_j}.$$

Then F is an increasing function on $[a, b]$; note $F(a) = 0$ and $F(b) = \sum_{j=1}^{\infty} c_j < \infty$. Every bounded function f on $[a, b]$ is F-integrable and

$$\int_a^b f \, dF = \sum_{j=1}^{\infty} c_j f(u_j). \tag{1}$$

This follows from Theorem 35.12(b) with $F_j = c_j J_{u_j}$, since every bounded function is F_j-integrable for every j, as shown in Example 1.

(b) The function F in part (a) satisfies

$$F(t) = \sum \{c_j : u_j \leq t\} \quad \text{for} \quad t \in (a, b], \tag{2}$$

since $J_{u_j}(t) = 1$ if and only if $u_j \leq t$. Also $F(a) = 0$. Moreover,

$$F(t^-) = \sum \{c_j : u_j < t\} \quad \text{for} \quad t \in (a, b], \tag{3}$$

since $F(t^-) = \lim_{x \to t-} \sum \{c_j : u_j \leq x\}$. We also have

$$F(t^+) = F(t) \quad \text{for} \quad t \in (a, b], \tag{4}$$

because $F(b^+) = F(b)$ by decree, and for $t \in (a, b)$ and $x > t$, we have

$$F(x) - F(t) = \sum \{c_j : t < u_j \leq x\}, \tag{5}$$

so that $F(t^+) = \lim_{x \to t+} F(x) = F(t)$. [Details: Given $\epsilon > 0$, there is N so that $\sum_{j=N+1}^{\infty} c_j < \epsilon$. For $x > t$ and x sufficiently close to t, the sets $\{u_j : t < u_j \leq x\}$ and $\{u_1, \ldots, u_N\}$ are disjoint, so the sum in (5) is less than ϵ.]

---

[3] Allowing some c_j's to be 0 won't change the value of F, of course, but allowing c_j's to be 0 is sometimes convenient.

(c) A function F is said to be right continuous at t if $F(t) = F(t^+)$. Equation (4) shows F is right continuous except possibly at a. By (4), (2) and (3), we have

$$F(t^+) - F(t^-) = \sum \{c_j : u_j = t\}. \tag{6}$$

Equation (6) holds for all t in $[a, b]$, even $t = a$. Therefore, F is continuous at t unless $t = u_j$ for some j. Finally, F is clearly discontinuous at each u_j unless $c_j = 0$. □

The next theorem shows F-integrals can often be calculated using ordinary Riemann integrals. In fact, most F-integrals encountered in practice are either covered by Example 3 or this theorem.

35.13 Theorem.
Suppose F is differentiable on $[a, b]$ and F' is continuous on $[a, b]$. If f is continuous on $[a, b]$, then

$$\int_a^b f \, dF = \int_a^b f(x) F'(x) \, dx. \tag{1}$$

Proof
Note fF' is Riemann integrable by Theorem 33.2, and f is F-integrable by Theorem 35.7. By Theorems 32.5 and 35.6, there is a partition $P = \{a = t_0 < t_1 < \cdots < t_n = b\}$ such that

$$U(fF', P) - L(fF', P) < \frac{\epsilon}{2} \quad \text{and} \quad U_F(f, P) - L_F(f, P) < \frac{\epsilon}{2}. \tag{2}$$

By the Mean Value Theorem 29.3 on page 233 applied to F on each interval $[t_{k-1}, t_k]$, there exist x_k in (t_{k-1}, t_k) so that

$$F(t_k) - F(t_{k-1}) = F'(x_k)(t_k - t_{k-1});$$

hence

$$\sum_{k=1}^n f(x_k) \cdot [F(t_k) - F(t_{k-1})] = \sum_{k=1}^n f(x_k) F'(x_k) \cdot (t_k - t_{k-1}). \tag{3}$$

Since F is continuous, it has no jumps and (3) implies

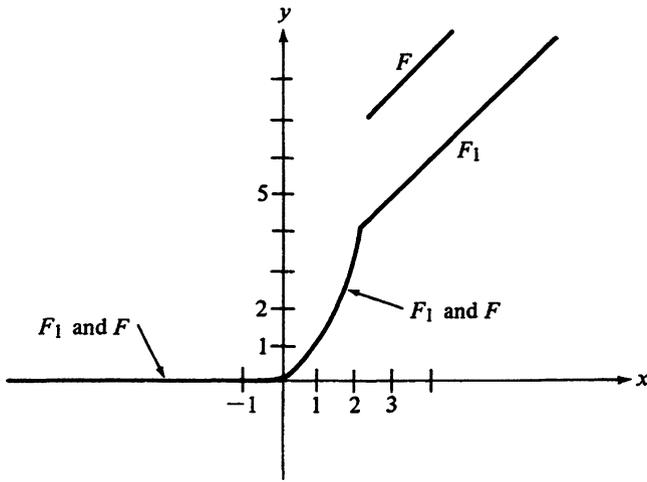$$L_F(f, P) \le U(fF', P) \quad \text{and} \quad L(fF', P) \le U_F(f, P).$$

FIGURE 35.2

Now by (2) we have

$$\int_a^b f\, dF \leq U_F(f, P) < \frac{\epsilon}{2} + L_F(f, P) \leq \frac{\epsilon}{2} + U(fF', P)$$

$$< \frac{\epsilon}{2} + \frac{\epsilon}{2} + L(fF', P) \leq \epsilon + \int_a^b f(x)F'(x)\, dx$$

and similarly $\int_a^b f\, dF > \int_a^b f(x)F'(x)\, dx - \epsilon$. Since $\epsilon > 0$ is arbitrary, (1) holds. ∎

An extension of Theorem 35.13 appears in Theorem 35.29.

Example 4
Let $F(t) = 0$ for $t < 0$, $F(t) = t^2$ for $0 \leq t < 2$, and $F(t) = t + 5$ for $t \geq 2$; see Fig. 35.2. We can write $F = F_1 + 3J_2$ where F_1 is continuous and J_2 is the jump function at 2. The function F_1 is differentiable except at $t = 2$; the differentiability of F_1 at $t = 0$ is shown in Exercise 28.7. Let f be continuous on $[-3, 3]$, say. Clearly $\int_{-3}^0 f\, dF_1 = 0$. Since F_1 agrees with the differentiable function t^2 on $[0, 2]$, we can apply Theorem 35.13 to obtain

$$\int_0^2 f\, dF_1 = \int_0^2 f(x) \cdot 2x\, dx = 2\int_0^2 x f(x)\, dx.$$

Similarly we have

$$\int_2^3 f \, dF_1 = \int_2^3 f(x) \cdot 1 \, dx = \int_2^3 f(x) \, dx.$$

Theorem 35.11 now shows

$$\int_{-3}^3 f \, dF_1 = 2 \int_0^2 x f(x) \, dx + \int_2^3 f(x) \, dx,$$

and then Theorem 35.12(a) shows

$$\int_{-3}^3 f \, dF = \int_{-3}^3 f \, dF_1 + 3 \int_{-3}^3 f \, dJ_2 = 2 \int_0^2 x f(x) \, dx + \int_2^3 f(x) \, dx + 3f(2).$$

As a specific example, if $f(x) = x^3$, then

$$\int_{-3}^3 f \, dF = 2 \int_0^2 x^4 \, dx + \int_2^3 x^3 \, dx + 3 \cdot 8 = \frac{1061}{20} = 53.05.$$

$\square$

For the proof of Lemma 35.15, we will use the next theorem, which is interesting in its own right.

35.14 Theorem.
Let F be a right-continuous increasing function on $[a, b]$. Then we have $F = F_c + F_d$, where F_c is a continuous increasing function on $[a, b]$, and $F_d = \sum c_j J_{u_j}$ where $\{u_1, u_2, \ldots\}$ are the jump points of F and c_j is the jump at u_j, i.e., $c_j = F(u_j^+) - F(u_j^-)$ for each j. (If there are no jump points, then F is continuous, $F = F_c$ and $F_d = 0$.) In general, there might be finitely many or infinitely many such u_j, so we will not decorate the sums until we need to.

See Examples 1 and 3 on pages 301 and 309 for the notation. The function F_c is called the "continuous part" of F, and F_d is called the "discrete part" of F.

Proof

Let $F_c = F - F_d$. To show F_c is an increasing function, it suffices to show $s < t$ implies $F_c(s) \leq F_c(t)$, i.e.,

$$s < t \quad \text{implies} \quad F_d(t) - F_d(s) \leq F(t) - F(s).$$

By (2) in Example 3(b) on page 309, it suffices to show

$$s < t \quad \text{implies} \quad \sum \{c_j : s < u_j \leq t\} \leq F(t) - F(s). \qquad (1)$$

To show this, it suffices to consider a finite subset E of $\{u_j : s < u_j \leq t\}$, which we write as $\{t_0 < t_1 < \cdots < t_n\}$ where $s < t_0$ and $t_n \leq t$. Then

$$\sum \{c_j : u_j \in E\} = \sum_{k=0}^{n} [F(t_k^+) - F(t_k^-)] \leq F(t^+) - F(s) = F(t) - F(s)$$

by (6) in Definition 35.2 on page 300 with $a = s$ and $b = t$. Since E is an arbitrary finite subset of $\{u_j : s < u_j \leq t\}$, (1) holds.

To show F_c is continuous, first note that if $t \neq u_j$ for all j, then F is continuous at t, and F_d is continuous at t as shown in Example 3(c). So $F_c = F - F_d$ is continuous at t in this case. Otherwise, $t = u_j$ for some j. Then $F(u_j^+) - F(u_j^-) = c_j$ by the definition of c_j, and $F_d(u_j^+) - F_d(u_j^-) = c_j$ by (6) of Example 3(b). Therefore

$$F_c(u_j^+) - F_c(u_j^-) = F(u_j^+) - F_d(u_j^+) - F(u_j^-) + F_d(u_j^-)$$

$$= [F(u_j^+) - F(u_j^-)] - [F_d(u_j^+) - F_d(u_j^-)] = c_j - c_j = 0,$$

so that F_c is continuous at u_j, as claimed. ∎

It is easy to check that the representation of F as $F_c + F_d$ is unique in the following sense: If $F_c + F_d = G_c + G_d$ where G is another right-continuous increasing function, then $F_c = G_c$ and $F_d = G_d$.

35.15 Lemma.
If F is an increasing function on $[a, b]$ and if $\epsilon > 0$, then there exists a partition

$$P = \{a = t_0 < t_1 < \cdots < t_n = b\}$$

such that

$$F(t_k^-) - F(t_{k-1}^+) < \epsilon \quad for \quad k = 1, 2, \ldots, n. \tag{1}$$

Proof

The conclusion (1) does not depend on the exact values of F at any of its jump points, though of course they satisfy $F(t^-) \leq F(t) \leq F(t^+)$. Hence it suffices to prove (1) after F has been redefined at its jump points. We choose to define $F(t) = F(t^+)$ at all jump points, so that F is right continuous on $[a, b]$.

Now that F is right continuous, we can apply Theorem 35.14: $F = F_c + \sum c_j J_{u_j}$ where F_c is continuous, each c_j is positive, and $\sum c_j < \infty$. If F is continuous, then the sum disappears and the proof below can be simplified.

Given $\epsilon > 0$, select N so that $\sum_{j=N+1}^{\infty} c_j < \frac{\epsilon}{2}$. Again, if there are only finitely many u_j's, this part of the proof simplifies. Since F_c is uniformly continuous on $[a, b]$ [by Theorem 19.2 on page 143], there is $\delta > 0$ so that

$$|s - t| < \delta \quad \text{implies} \quad |F_c(s) - F_c(t)| < \frac{\epsilon}{2}. \tag{2}$$

Let $P = \{a = t_0 < t_1 < \cdots < t_n\}$ be a partition that includes $\{u_1, \ldots, u_N\}$ and satisfies $|t_k - t_{k-1}| < \delta$ for $k = 1, 2, \ldots, n$. Let

$$F_1 = \sum_{j=1}^{N} c_j J_{u_j} \quad \text{and} \quad F_2 = \sum_{j=N+1}^{\infty} c_j J_{u_j},$$

so that $F = F_c + F_1 + F_2$. For each $k = 1, 2, \ldots, n$, we have

$$F_c(t_k^-) - F_c(t_{k-1}^+) = F_c(t_k) - F_c(t_{k-1}) < \frac{\epsilon}{2} \quad \text{by (2)},$$

$$F_1(t_k^-) - F_1(t_{k-1}^+) = 0$$

because F_1 is constant on the open interval (t_{k-1}, t_k), and

$$F_2(t_k^-) - F_2(t_{k-1}^+) \leq F_2(b) - F_2(a) \leq \sum_{j=N+1}^{\infty} c_j < \frac{\epsilon}{2}.$$

Since $F = F_c + F_1 + F_2$, summing yields $F(t_k^-) - F(t_{k-1}^+) < \epsilon$. ∎

35.16 Theorem.

Every monotonic function f on $[a, b]$ is F-integrable.

Proof

We may assume f is increasing and $f(a) < f(b)$. Since $f(a) \leq f(x) \leq f(b)$ for all x in $[a, b]$, f is bounded on $[a, b]$. For $\epsilon > 0$ we apply Lemma 35.15 to obtain $P = \{a = t_0 < t_1 < \cdots < t_n = b\}$ where

$$F(t_k^-) - F(t_{k-1}^+) < \frac{\epsilon}{f(b) - f(a)}$$

for $k = 1, 2, \ldots, n$. Since $M(f, (t_{k-1}, t_k)) = f(t_k^-) \leq f(t_k)$ and $m(f, (t_{k-1}, t_k)) = f(t_{k-1}^+) \geq f(t_{k-1})$, we have

$$U_F(f, P) - L_F(f, P) \leq \sum_{k=1}^{n} [f(t_k) - f(t_{k-1})] \cdot [F(t_k^-) - F(t_{k-1}^+)]$$

$$< \sum_{k=1}^{n} [f(t_k) - f(t_{k-1})] \cdot \frac{\epsilon}{f(b) - f(a)} = \epsilon.$$

Since ϵ is arbitrary, Theorem 35.6 shows f is F-integrable. ∎

35.17 Theorem.

If f is piecewise continuous, or bounded and piecewise monotonic, on $[a, b]$, then f is F-integrable.

Proof

Just as in the proof of Theorem 33.8, this follows from Theorems 35.7, 35.16 and 35.11, provided we have the following generalization of Exercise 32.7. ∎

35.18 Proposition.

If f is F-integrable on $[a, b]$ and $g(x) = f(x)$ except for finitely many points, then g is F-integrable. We do *not* claim $\int_a^b f \, dF = \int_a^b g \, dF$.

Proof

It suffices to show $g - f$ is F-integrable, because then the sum $(g - f) + f$ would be F-integrable by Theorem 35.8(ii). Thus it suffices to assume $g(x) = 0$ except for finitely many points. Consider any partition $P = \{a = t_0 < t_1 < \cdots < t_n = b\}$ satisfying $g(x) = 0$ for all x in $[a, b] \setminus \{t_0, t_1, \ldots, t_n\}$. Then from Eqs. (2) and (3) in Definition 35.2 on page 300 we have $U_F(g, P) = J_F(g, P) = L_F(g, P)$, so g is clearly F-integrable; see Theorem 35.6, for example. ∎

If F_1 and F_2 are increasing functions with continuous derivatives, then Theorem 35.13 allows the formula on integration by parts [Theorem 34.2] to be recast as

$$\int_a^b F_1 \, dF_2 + \int_a^b F_2 \, dF_1 = F_1(b)F_2(b) - F_1(a)F_2(a).$$

There is no hope to prove this in general because if $F(t) = 0$ for $t < 0$ and $F(t) = 1$ for $t \geq 0$, then

$$\int_{-1}^1 F \, dF + \int_{-1}^1 F \, dF = 2 \neq 1 = F(1)F(1) - F(-1)F(-1).$$

The generalization does hold provided the functions in the integrands take the middle values at each of their jumps, as we next prove. The result is a special case of a theorem given by Edwin Hewitt [29].

35.19 Theorem [Integration by Parts].
Suppose F_1 and F_2 are increasing functions on $[a, b]$ and define

$$F_1^*(t) = \frac{1}{2}[F_1(t^-) + F_1(t^+)] \quad and \quad F_2^*(t) = \frac{1}{2}[F_2(t^-) + F_2(t^+)]$$

for all $t \in [a, b]$. Then

$$\int_a^b F_1^* \, dF_2 + \int_a^b F_2^* \, dF_1 = F_1(b)F_2(b) - F_1(a)F_2(a). \qquad (1)$$

As usual, we decree $F_1(b^+) = F_1(b)$, $F_1(a^-) = F_1(a)$, etc.

Proof
Both integrals in (1) exist in view of Theorem 35.16. For an $\epsilon > 0$, there exists a partition $P = \{a = t_0 < t_1 < \cdots < t_n = b\}$ such that

$$U_{F_1}(F_2^*, P) - L_{F_1}(F_2^*, P) < \epsilon.$$

Some algebraic manipulation [discussed in the next paragraph] shows

$$U_{F_2}(F_1^*, P) + L_{F_1}(F_2^*, P) = F_1(b)F_2(b) - F_1(a)F_2(a), \qquad (2)$$

so that also

$$U_{F_1}(F_2^*, P) + L_{F_2}(F_1^*, P) = F_1(b)F_2(b) - F_1(a)F_2(a). \qquad (3)$$

It follows from (2) that

$$\int_a^b F_1^* \, dF_2 + \int_a^b F_2^* \, dF_1 \leq U_{F_2}(F_1^*, P) + U_{F_1}(F_2^*, P)$$

$$< U_{F_2}(F_1^*, P) + L_{F_1}(F_2^*, P) + \epsilon$$

$$= F_1(b)F_2(b) - F_1(a)F_2(a) + \epsilon,$$

while (3) leads to

$$\int_a^b F_1^* \, dF_2 + \int_a^b F_2^* \, dF_1 > F_1(b)F_2(b) - F_1(a)F_2(a) - \epsilon.$$

Since ϵ is arbitrary, (1) holds.

It remains to verify (2). Observe

$$U_{F_2}(F_1^*, P) + L_{F_1}(F_2^*, P) = \sum_{k=0}^n F_1^*(t_k) \cdot [F_2(t_k^+) - F_2(t_k^-)]$$

$$+ \sum_{k=1}^n M(F_1^*, (t_{k-1}, t_k)) \cdot [F_2(t_k^-) - F_2(t_{k-1}^+)]$$

$$+ \sum_{k=0}^n F_2^*(t_k) \cdot [F_1(t_k^+) - F_1(t_k^-)]$$

$$+ \sum_{k=1}^n m(F_2^*, (t_{k-1}, t_k)) \cdot [F_1(t_k^-) - F_1(t_{k-1}^+)]$$

$$= \sum_{k=0}^n \frac{1}{2}[F_1(t_k^-) + F_1(t_k^+)] \cdot [F_2(t_k^+) - F_2(t_k^-)]$$

$$+ \sum_{k=1}^n F_1(t_k^-) \cdot [F_2(t_k^-) - F_2(t_{k-1}^+)]$$

$$+ \sum_{k=0}^n \frac{1}{2}[F_2(t_k^-) + F_2(t_k^+)] \cdot [F_1(t_k^+) - F_1(t_k^-)]$$

$$+ \sum_{k=1}^n F_2(t_{k-1}^+) \cdot [F_1(t_k^-) - F_1(t_{k-1}^+)].$$

The first and third sums add to

$$\sum_{k=0}^n [F_1(t_k^+)F_2(t_k^+) - F_1(t_k^-)F_2(t_k^-)], \tag{4}$$

while the second and fourth sums add to

$$\sum_{k=1}^{n} [F_1(t_k^-)F_2(t_k^-) - F_1(t_{k-1}^+)F_2(t_{k-1}^+)]. \tag{5}$$

Since the sums in (4) and (5) add to $F_1(b)F_2(b) - F_1(a)F_2(a)$, by Eq. (6) in Definition 35.2 [on page 300] applied to F_1F_2, the equality (2) holds. The preceding algebra simplifies considerably if F_1 and F_2 are continuous. ∎

We next compare our approach to Riemann-Stieltjes integration to the usual approach. For a bounded function f on $[a, b]$, the usual Darboux-Stieltjes integral is defined via the upper sums

$$\tilde{U}_F(f, P) = \sum_{k=1}^{n} M(f, [t_{k-1}, t_k]) \cdot [F(t_k) - F(t_{k-1})]$$

and the lower sums

$$\tilde{L}_F(f, P) = \sum_{k=1}^{n} m(f, [t_{k-1}, t_k]) \cdot [F(t_k) - F(t_{k-1})].$$

The expressions $\tilde{U}_F(f)$, $\tilde{L}_F(f)$ and $\tilde{\int}_a^b f \, dF$ are defined in analogy to those in Definition 35.2 on page 300. The usual Riemann-Stieltjes integral is defined via the sums

$$\tilde{S}_F(f, P) = \sum_{k=1}^{n} f(x_k)[F(t_k) - F(t_{k-1})],$$

where x_k is in $[t_{k-1}, t_k]$, and the mesh is defined in Definition 32.6; compare Definition 35.24.

The usual Riemann-Stieltjes integrability criterion implies the usual Darboux-Stieltjes integrability criterion; these criteria are not equivalent in general, but they are equivalent if F is continuous. See, for example, [53, §12.2], [55, Chap. 8], or [62, Chap. 6], the most complete treatment being in [55].

35.20 Theorem.
If f is Darboux-Stieltjes integrable on $[a, b]$ with respect to F in the usual sense, then f is F-integrable and the integrals agree.

Proof

For any partition P, $\tilde{L}_F(f, P)$ equals

$$\sum_{k=1}^{n} m(f, [t_{k-1}, t_k]) \cdot [F(t_k) - F(t_k^-) + F(t_k^-) - F(t_{k-1}^+)$$

$$+ F(t_{k-1}^+) - F(t_{k-1})]$$

$$\leq \sum_{k=1}^{n} f(t_k)[F(t_k) - F(t_k^-)]$$

$$+ \sum_{k=1}^{n} m(f, (t_{k-1}, t_k)) \cdot [F(t_k^-) - F(t_{k-1}^+)]$$

$$+ \sum_{k=1}^{n} f(t_{k-1})[F(t_{k-1}^+) - F(t_{k-1})].$$

The first and third sums add to

$$\sum_{k=1}^{n} f(t_k)[F(t_k) - F(t_k^-)] + \sum_{k=0}^{n-1} f(t_k)[F(t_k^+) - F(t_k)]$$

$$= f(t_n)[F(t_n) - F(t_n^-)] + \sum_{k=1}^{n-1} f(t_k)[F(t_k^+) - F(t_k^-)]$$

$$+ f(t_0)[F(t_0^+) - F(t_0)]$$

$$= \sum_{k=0}^{n} f(t_k)[F(t_k^+) - F(t_k^-)] = J_F(f, P).$$

These observations and a glance at the definition of $L_F(f, P)$ now show $\tilde{L}_F(f, P) \leq L_F(f, P)$. Likewise we have $\tilde{U}_F(f, P) \geq U_F(f, P)$, so

$$U_F(f, P) - L_F(f, P) \leq \tilde{U}_F(f, P) - \tilde{L}_F(f, P). \tag{1}$$

If $\epsilon > 0$, the usual theory [Theorem 32.5] shows there exists a partition P such that $\tilde{U}_F(f, P) - \tilde{L}_F(f, P) < \epsilon$. By (1) we see that we also have $U_F(f, P) - L_F(f, P) < \epsilon$, so f is F-integrable by Theorem 35.6.

To see equality of the integrals, simply observe

$$\tilde{\int_a^b} f\, dF \leq \tilde{U}_F(f, P) < \tilde{L}_F(f, P) + \epsilon \leq L_F(f, P) + \epsilon \leq \int_a^b f\, dF + \epsilon$$

and similarly

$$\int_a^{\tilde{b}} f \, dF > \int_a^b f \, dF - \epsilon.$$ ∎

We will define Riemann-Stieltjes integrals using a mesh defined in terms of F instead of the usual mesh in Definition 32.6.

35.21 Definition.
The *F-mesh* of a partition P is

$$F\text{-mesh}(P) = \max\{F(t_k^-) - F(t_{k-1}^+) : k = 1, 2, \ldots, n\}.$$

It is convenient to restate Lemma 35.15:

35.22 Lemma.
If $\delta > 0$, there exists a partition P such that F-mesh$(P) < \delta$.

35.23 Theorem.
A bounded function f on $[a, b]$ is F-integrable if and only if for each $\epsilon > 0$ there exists $\delta > 0$ such that

$$F\text{-mesh}(P) < \delta \quad implies \quad U_F(f, P) - L_F(f, P) < \epsilon \qquad (1)$$

for all partitions P of $[a, b]$.

Proof
Suppose the ϵ–δ condition stated in the theorem holds. If we have $\epsilon > 0$, then (1) applies to some partition P by Lemma 35.22 and hence $U_F(f, P) - L_F(f, P) < \epsilon$. Since this remark applies to all $\epsilon > 0$, Theorem 35.6 implies f is F-integrable.

The converse is proved just as in Theorem 32.7 with "mesh" replaced by "F-mesh" and references to Lemma 32.2 replaced by references to Lemma 35.3. ∎

35.24 Definition.
Let f be bounded on $[a, b]$, and let

$$P = \{a = t_0 < t_1 < \cdots < t_n = b\}.$$

A *Riemann-Stieltjes sum* of f associated with P and F is a sum of the form

$$J_F(f, P) + \sum_{k=1}^{n} f(x_k)[F(t_k^-) - F(t_{k-1}^+)]$$

where x_k is in (t_{k-1}, t_k) for $k = 1, 2, \ldots, n$.

The function f is *Riemann-Stieltjes integrable on* $[a, b]$ if there exists r in $\mathbb{R}$ with the following property. For each $\epsilon > 0$ there exists $\delta > 0$ such that

$$|S - r| < \epsilon \tag{1}$$

for every Riemann-Stieltjes sum S of f associated with a partition P having F-mesh$(P) < \delta$. We call r the *Riemann-Stieltjes integral* of f and temporarily write it as

$$\mathcal{RS} \int_a^b f \, dF.$$

35.25 Theorem.
A bounded function f on $[a, b]$ is F-integrable if and only if it is Riemann-Stieltjes integrable, in which case the integrals are equal.

Proof
The proof that F-integrability implies Riemann-Stieltjes integrability imitates the corresponding proof in Theorem 32.9. The proof of the converse also imitates the corresponding proof, but a little care is needed, so we give it.

Let f be a Riemann-Stieltjes integrable function, and let r be as in Definition 35.24. Consider $\epsilon > 0$, and let $\delta > 0$ be as provided in Definition 35.24. By Lemma 35.22 there exists a partition $P = \{a = t_0 < t_1 < \cdots < t_n = b\}$ with F-mesh$(P) < \delta$. For each $k = 1, 2, \ldots, n$, select x_k in (t_{k-1}, t_k) so that $f(x_k) < m(f, (t_{k-1}, t_k)) + \epsilon$. The Riemann-Stieltjes sum S for this choice of x_k's satisfies

$$S \le L_F(f, P) + \epsilon[F(b) - F(a)]$$

and also

$$|S - r| < \epsilon;$$

hence $L_F(f) \geq L_F(f, P) > r - \epsilon - \epsilon[F(b) - F(a)]$. Since $\epsilon > 0$ is arbitrary, it follows that $L_F(f) \geq r$ and similarly $U_F(f) \leq r$. Therefore $L_F(f) = U_F(f) = r$. Thus f is F-integrable and

$$\int_a^b f \, dF = r = \mathcal{RS} \int_a^b f \, dF.$$

∎

35.26 Remarks.

In the definition of upper and lower Darboux sums, we used closed intervals $[t_{k-1}, t_k]$ for Riemann integrals, in §32, and we used open intervals (t_{k-1}, t_k) in this section. The reason we used closed intervals in §32 is because that is completely standard. The reason we used open intervals in this section is because our development was motivated by the desire to have Riemann-Stieltjes integrals more compatible with their measure-theoretic generalization, Lebesgue-Stieltjes integrals, where the "measurable partition"

$$[a, b] = \bigcup_{k=0}^n \{t_k\} \cup \bigcup_{k=1}^n (t_{k-1}, t_k)$$

is a natural starting point. Moreover, the arguments in Examples 1 and 3 and in Proposition 35.18 would be somewhat more complicated if we had used closed intervals.

The reader may wonder whether the theories would have changed if we had used closed intervals in this section or open intervals in §32. It turns out there would be no essential changes, as we now explain. We do so using increasing functions F, as in this section; see Corollary 35.28 for the case of Riemann integrals. We write $\overline{L}_F(f, P)$ and $\overline{U}_F(f, P)$ for the lower and upper Darboux-Stieltjes sums using closed intervals $[t_{k-1}, t_k]$ instead of open intervals (t_{k-1}, t_k). These will sometimes be different from the corresponding Darboux-Stieltjes sums defined in Definition 35.2 on page 300. But if we define

$$\overline{U}_F(f) = \inf\{\overline{U}_F(f, P) : P \text{ partitions } [a, b]\}$$

and we define $\overline{L}_F(f)$ similarly, we have the following fact.

35.27 Proposition.
If f is bounded on $[a, b]$, then $\overline{U}_F(f) = U_F(f)$ and $\overline{L}_F(f) = L_F(f)$. Thus f is integrable using closed intervals in the Darboux-Stieltjes definitions if and only if it is integrable using open intervals, and in this case the integrals are equal.

Proof
We prove $\overline{U}_F(f) = U_F(f)$; the proof of $\overline{L}_F(f) = L_F(f)$ is similar. For each partition P of $[a, b]$, it is clear that $U_F(f, P) \leq \overline{U}_F(f, P)$. Therefore $U_F(f) \leq \overline{U}_F(f)$ and it suffices to show

$$\overline{U}_F(f) \leq U_F(f). \tag{1}$$

Let $\epsilon > 0$. There is a partition P of $[a, b]$ such that

$$U_F(f, P) < U_F(f) + \frac{\epsilon}{4}. \tag{2}$$

Note (2) holds for any refinement of P by Lemma 35.3.

As before, if there are any jumps we list them as $u_1, u_2, \dots$ and define $c_j = F(u_j^+) - F(u_j^-)$. Since $\sum c_j$ is finite, there is an integer N so that $\sum_{j=N+1}^{\infty} c_j < \frac{\epsilon}{4}$. Of course, this step is trivial if F has only finitely many jumps. We first refine P so that for every $j \leq N$, the jump u_j is equal to some t_k in P. Then if P_1 and P_2 are such refinements of P, we have

$$|J_F(f, P_1) - J_F(f, P_2)| \leq \sum_{j=N+1}^{\infty} c_j < \frac{\epsilon}{4}. \tag{3}$$

Now we further refine P as follows. Using the definitions of $F(t_k^-)$ and $F(t_k^+)$, we see there are points s_k and u_k for $k = 0, 1, \dots, n$ so that

$$a = s_0 = t_0 < u_0 < s_1 < t_1 < u_1 < s_2 < t_2 < u_2 < s_3 < \cdots$$

$$\cdots < s_{n-1} < t_{n-1} < u_{n-1} < s_n < t_n = u_n = b,$$

and all $F(u_k^-) - F(t_k^+)$ and $F(t_k^-) - F(s_k^+)$ are less than $\frac{\epsilon}{4(n+1)B}$, where B is a positive bound for the absolute value $|f|$. We estimate $\overline{U}_F(f, P^\sharp)$ for the new partition $P^\sharp$ consisting of all s_k, t_k and u_k.

Observe:

$$\sum_{k=0}^{n} M(f, [s_k, t_k]) \cdot [F(t_k^-) - F(s_k^+)] \le \sum_{k=0}^{n} B \cdot [F(t_k^-) - F(s_k^+)]) < \frac{\epsilon}{4};$$

$$\sum_{k=0}^{n} M(f, [t_k, u_k]) \cdot [F(u_k^-) - F(t_k^+)] \le \sum_{k=0}^{n} B \cdot [F(u_k^-) - F(t_k^+)] < \frac{\epsilon}{4};$$

$$\text{and} \quad \sum_{k=1}^{n} M(f, [u_{k-1}, s_k]) \cdot [F(s_k^-) - F(u_{k-1}^+)]$$

$$\le \sum_{k=1}^{n} M(f, (t_{k-1}, t_k)) \cdot [F(t_k^-) - F(t_{k-1}^+)] = U_F(f, P) - J_F(f, P).$$

Summing these three inequalities, we see

$$\overline{U}_F(f, P^\sharp) - \overline{J}_F(f, P^\sharp) < U_F(f, P) - J_F(f, P) + \frac{2\epsilon}{4}.$$

Now we invoke (3) to conclude

$$\overline{U}_F(f, P^\sharp) < U_F(f, P) - J_F(f, P) + \overline{J}_F(f, P^\sharp) + \frac{2\epsilon}{4} < U_F(f, P) + \frac{3\epsilon}{4}.$$

Finally, using (2), we obtain $\overline{U}_F(f, P^\sharp) < U_F(f) + \epsilon$; hence

$$\overline{U}_F(f) \le \overline{U}_F(f, P^\sharp) < U_F(f) + \epsilon.$$

Since ϵ is arbitrary, (1) is established. ∎

Proposition 35.27 is true for $F(t) = t$, so it is true for Riemann integrals. However, for this important case, we use closed intervals as is traditional. Thus our notation $U(f, P)$, $U(f)$ refers to the theory using closed intervals, and we do not want to confuse the situation here by switching to $\overline{U}(f, P, \overline{U}(f)$, etc. To distinguish the theory using open intervals, we introduce new notation $U^\flat(f, P)$, $U^\flat(f)$, etc.

35.28 Corollary.
In §32, on the Riemann integral, the definitions of $L(f, P)$ and $U(f, P)$ for upper and lower Darboux sums could have used open intervals (t_{k-1}, t_k) instead of closed intervals $[t_{k-1}, t_k]$. The resulting

Darboux sums $L^\flat(f, P)$ and $U^\flat(f, P)$ might be different. But if we define

$$U^\flat(f) = \inf\{U^\flat(f, P) : P \text{ partitions } [a, b]\}$$

and we define $L^\flat(f)$ similarly, then we have $U^\flat(f) = U(f)$ and $L^\flat(f) = L(f)$. Thus the two approaches to Riemann integration give the same integrals.

Proof
This is Proposition 35.27 for $F(t) = t$; only the notation here is different. ∎

We next give a generalization of Theorem 35.13. The proof is somewhat complicated and uses Corollary 35.28. It was originally worked out by me and my colleague Theodore W. Palmer.

35.29 Theorem.
Suppose F is differentiable on $[a, b]$ and F' is Riemann integrable on $[a, b]$. A bounded function f on $[a, b]$ is F-integrable if and only if fF' is Riemann integrable, in which case

$$\int_a^b f \, dF = \int_a^b f(x)F'(x) \, dx.$$

We will give the proof after Lemma 35.30 and its corollary. First look at Definition 35.2 on page 300. Being differentiable, F is continuous. So the J_F terms in the definition are 0, and we have

$$U_F(f, P) = \sum_{k=1}^n M(f, (t_{k-1}, t_k)) \cdot [F(t_k) - F(t_{k-1})]$$

and

$$L_F(f, P) = \sum_{k=1}^n m(f, (t_{k-1}, t_k)) \cdot [F(t_k) - F(t_{k-1})].$$

We will use these formulas in the proof of Lemma 35.30 below. We also use the notation $U^\flat(f)$, etc., from Corollary 35.28.

35.30 Lemma.
Assume F is differentiable on $[a, b]$, F' is Riemann integrable on $[a, b]$, and f is a bounded function on $[a, b]$. Let B be a positive bound for $|f|$. If we have $U^\flat(F', P) - L^\flat(F', P) < \epsilon/B$, then

$$|U_F(f, P) - U^\flat(fF', P)| \le \epsilon \qquad (1)$$

and

$$|L_F(f, P) - L^\flat(fF', P)| \le \epsilon. \qquad (2)$$

Proof
Let $P = \{a = t_0 < t_1 < \cdots < t_n = b\}$. By the Mean Value Theorem 29.3, for each k there exists x_k in (t_{k-1}, t_k) so that

$$F(t_k) - F(t_{k-1}) = F'(x_k) \cdot (t_k - t_{k-1}). \qquad (3)$$

Now consider an arbitrary y_k in (t_{k-1}, t_k) and observe

$$\sum_{k=1}^{n} |F'(x_k) - F'(y_k)| \cdot (t_k - t_{k-1}) \le U^\flat(F', P) - L^\flat(F', P) < \frac{\epsilon}{B} \quad (4)$$

since the sum is bounded by $\sum_{k=1}^{n}[M(F', (t_{k-1}, t_k)) - m(F', (t_{k-1}, t_k))] \cdot (t_k - t_{k-1})$. Applying this and (3), we obtain

$$\begin{aligned}
&|\textstyle\sum_{k=1}^{n} f(y_k)[F(t_k) - F(t_{k-1})] - \sum_{k=1}^{n} f(y_k) \cdot F'(y_k) \cdot (t_k - t_{k-1})| \\
&= |\textstyle\sum_{k=1}^{n} f(y_k) \cdot F'(x_k) \cdot (t_k - t_{k-1}) - \sum_{k=1}^{n} f(y_k) \cdot F'(y_k) \cdot (t_k - t_{k-1})| \\
&\qquad \le \textstyle\sum_{k=1}^{n} |f(y_k)| \cdot |F'(x_k) - F'(y_k)| \cdot (t_k - t_{k-1}) \\
&\qquad \le B \cdot \textstyle\sum_{k=1}^{n} |F'(x_k) - F'(y_k)| \cdot (t_k - t_{k-1}) < \epsilon.
\end{aligned}$$

Therefore for y_k in (t_{k-1}, t_k), we have

$$\begin{aligned}
\textstyle\sum_{k=1}^{n} f(y_k)[F(t_k) - F(t_{k-1})] &< \epsilon + \textstyle\sum_{k=1}^{n} f(y_k) \cdot F'(y_k) \cdot (t_k - t_{k-1}) \\
&\le \epsilon + \textstyle\sum_{k=1}^{n} M(fF', (t_{k-1}, t_k)) \cdot (t_k - t_{k-1}) = \epsilon + U^\flat(fF', P)
\end{aligned}$$

$$(5)$$

and

$$\begin{aligned}
\textstyle\sum_{k=1}^{n} f(y_k) \cdot F'(y_k) \cdot (t_k - t_{k-1}) &< \epsilon + \textstyle\sum_{k=1}^{n} f(y_k)[F(t_k) - F(t_{k-1})] \\
&\le \epsilon + \textstyle\sum_{k=1}^{n} M(f, (t_{k-1}, t_k))[F(t_k) - F(t_{k-1})] = \epsilon + U_F(f, P).
\end{aligned}$$

$$(6)$$

Since (5) and (6) hold for arbitrary y_k in (t_{k-1}, t_k), for each k, we conclude[4]

$$U_F(f, P) = \sum_{k=1}^{n} M(f, (t_{k-1}, t_k)) \cdot [F(t_k) - F(t_{k-1})] \leq \epsilon + U^\flat(fF', P).$$
(7)

and

$$U^\flat(fF', P) = \sum_{k=1}^{n} M(fF', (t_{k-1}, t_k)) \cdot (t_k - t_{k-1}) \leq \epsilon + U_F(f, P). \quad (8)$$

Now (1) follows from (7) and (8); (2) has a similar proof. ∎

35.31 Corollary.
With F and f as in Lemma 35.30, we have $U_F(f) = U(fF')$ and $L_F(f) = L(fF')$.

Proof
Consider $\epsilon > 0$. Select partitions P_1 and P_2 of $[a, b]$ so that

$$|U_F(f, P_1) - U_F(f)| < \epsilon \qquad \text{and} \qquad |U(fF') - U^\flat(fF', P_2)| < \epsilon.$$

Such a partition P_2 exists because $U(fF') = U^\flat(fF')$ by Corollary 35.28. Since F' is Riemann integrable by hypothesis, $U^\flat(F') = U(F') = L(F') = L^\flat(F')$. So there exists a partition P_3 so that

$$U^\flat(F', P_3) - L^\flat(F', P_3) < \epsilon/B.$$

The last three inequalities still hold if we replace P_1, P_2, P_3 by a common refinement P. Now by Lemma 35.30, we have

$$|U_F(f, P) - U^\flat(fF', P)| \leq \epsilon,$$

so by the triangle inequality, we have

$$|U_F(f) - U(fF')| < 3\epsilon.$$

Since ϵ is arbitrary, $U_F(f) = U(fF')$. Likewise $L_F(f) = L(fF')$. ∎

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[4]Some readers may appreciate more explanation. To avoid breaking the flow of the proofs, this will be provided after the proof of Theorem 35.29 is completed.

Proof of Theorem 35.29
Suppose f is F-integrable. Then $U_F(f) = L_F(f)$. Hence $U(fF') = L(fF')$ by Corollary 35.31. Thus fF' is Riemann integrable and

$$\int_a^b f \, dF = U_F(f) = U(fF') = \int_a^b f(x)F'(x)dx.$$

Now suppose fF' is Riemann integrable. Then $U(fF') = L(fF')$; hence $U_F(f) = L_F(f)$. Thus f is F-integrable and

$$\int_a^b f(x)F'(x)dx = U(fF') = U_F(f) = \int_a^b f \, dF.$$

∎

How did we so quickly conclude inequalities (5) and (6) in Lemma 35.30? Intuitively, we knew we could select each y_k so that $f(y_k)$ is very close to $M(f, (t_{k-1}, t_k))$, which appears in the definition of $U_F(f, P)$ given after the statement of Theorem 35.29. Similarly, we concluded inequality (6) because we could select each y_k so that $(fF')(y_k)$ is very close to $M(fF', (t_{k-1}, t_k))$, which appears in the definition of $U^\flat(fF', P)$. In fact, these conclusions follow from the following lemma.

35.32 Lemma.
Suppose $a_1, \ldots, a_n$ are (fixed) nonnegative numbers and $B_1, \ldots, B_n$ are nonempty bounded subsets of $\mathbb{R}$. If for some constant K, we have

$$\sum_{k=1}^n a_k b_k \le K$$

for all choices of b_k in B_k, $k = 1, 2, \ldots, n$, then

$$\sum_{k=1}^n a_k \sup B_k \le K.$$

Proof
We prove the following by induction on $m = 1, 2, \ldots, n$:

$$P_m : \text{``} a_m b_m \le K - \sum_{k=1}^{m-1} a_k \sup B_k - \sum_{k=m+1}^n a_k b_k,$$

for all choices of b_k in B_k, $k = m, \ldots, n$." As always, sums $\sum_{k=1}^{0}$ and $\sum_{k=n+1}^{n}$ are taken to be 0. Assertion P_1 holds by the hypothesis of the lemma. Assume P_m holds for some $m = 1, \ldots, n-1$. Since b_m in B_m is arbitrary, P_m implies

$$Q_m : \text{``} a_m \sup B_m \leq K - \sum_{k=1}^{m-1} a_k \sup B_k - \sum_{k=m+1}^{n} a_k b_k,$$

for all choices of b_k in B_k, $k = m+1, \ldots, n$," but this is exactly assertion P_{m+1}. By induction, all P_n and Q_n hold. Since Q_n is the conclusion of the lemma, the proof is complete. ■

The nontrivial part of Exercise 4.14, $\sup A + \sup B \leq \sup(A+B)$, is a special case of Lemma 35.32 where $n = 2$, $a_1 = a_2 = 1$, $B_1 = A$, $B_2 = B$, and $K = \sup(A+B)$.

Exercise 35.6 asserts $\int_a^b f \, dF = \lim_{n \to \infty} \int_a^b f_n \, dF$ provided the sequence (f_n) of F-integrable functions converges uniformly to f on $[a, b]$. Moreover, all of the comments in Discussion 33.10 carry over to this setting.

Exercises

35.1 Let F be an increasing function on $[a, b]$.

   (a) Show $\lim_{x \to t^-} F(x)$ exists for t in $(a, b]$ and is equal to $\sup\{F(x) : x \in (a, t)\}$.

   (b) Show $\lim_{x \to t^+} F(x)$ exists for t in $[a, b)$ and is equal to $\inf\{F(x) : x \in (t, b)\}$.

35.2 Calculate $\int_0^3 x^2 \, dF(x)$ for the function F in Example 4.

35.3 Let F be the step function such that $F(t) = n$ for $t \in [n, n+1)$, n an integer. Calculate
   (a) $\int_0^6 x \, dF(x)$,   **(b)** $\int_0^3 x^2 \, dF(x)$,
   (c) $\int_{1/4}^{\pi/4} x^2 \, dF(x)$.

35.4 Let $F(t) = \sin t$ for $t \in [-\frac{\pi}{2}, \frac{\pi}{2}]$. Calculate
   (a) $\int_0^{\pi/2} x \, dF(x)$ **(b)** $\int_{-\pi/2}^{\pi/2} x \, dF(x)$.

35.5 Let $f(x) = 1$ for rational x and $f(x) = 0$ for irrational x.

    (a) Show that if F is continuous on $[a, b]$ and $F(a) < F(b)$, then f is not F-integrable on $[a, b]$.

    (b) Observe f is F-integrable if F is as in Example 1 or 3.

35.6 Let (f_n) be a sequence of F-integrable functions on $[a, b]$, and suppose $f_n \to f$ uniformly on $[a, b]$. Show f is F-integrable and

$$\int_a^b f \, dF = \lim_{n \to \infty} \int_a^b f_n \, dF.$$

35.7 Let f and g be F-integrable functions on $[a, b]$. Show

    (a) f^2 is F-integrable.

    (b) fg is F-integrable.

    (c) $\max(f, g)$ and $\min(f, g)$ are F-integrable.

35.8 Let g be continuous on $[a, b]$ where $g(x) \geq 0$ for all $x \in [a, b]$ and define $F(t) = \int_a^t g(x) \, dx$ for $t \in [a, b]$. Show that if f is continuous, then

$$\int_a^b f \, dF = \int_a^b f(x)g(x) \, dx.$$

35.9 Let f be continuous on $[a, b]$.

    (a) Show $\int_a^b f \, dF = f(x)[F(b) - F(a)]$ for some x in $[a, b]$.

    (b) Show Exercise 33.14 is a special case of part (a).

35.10 Let $F(x) = f(x) = x^n$ for $x \in [a, b]$ and some positive odd integer n. Show $\int_a^b f \, dF = \frac{1}{2}[b^{2n} - a^{2n}]$.

35.11 Here is a "change of variable" formula. Let f be F-integrable on $[a, b]$. Let ϕ be a continuous, strictly increasing function on an interval $[c, d]$ such that $\phi(c) = a$ and $\phi(d) = b$. Define

$$g(u) = f(\phi(u)) \quad \text{and} \quad G(u) = F(\phi(u)) \quad \text{for} \quad u \in [c, d].$$

Show g is G-integrable and $\int_c^d g \, dG = \int_a^b f \, dF$.

35.12 Let (u_j) be an enumeration of the rationals in $[a, b]$ and let (c_j) be a sequence of positive integers such that $\sum c_j < \infty$.

    (a) Show $F = \sum c_j J_{u_j}$ defines a strictly increasing function on $[a, b]$.

    (b) At what points is F continuous?

§36   * Improper Integrals

The Riemann integral in §32 has been defined only for functions that are bounded on a closed interval $[a, b]$. It is convenient to be able to integrate some functions that are unbounded or are defined on an unbounded interval.

36.1 Definition.
Consider an interval $[a, b)$ where b is finite or $+\infty$. Suppose f is a function on $[a, b)$ that is integrable on each $[a, d]$ for $a < d < b$, and suppose the limit

$$\lim_{d \to b^-} \int_a^d f(x)\, dx$$

exists either as a finite number, $+\infty$ or $-\infty$. Then we define

$$\int_a^b f(x)\, dx = \lim_{d \to b^-} \int_a^d f(x)\, dx. \tag{1}$$

If b is finite and f is integrable on $[a, b]$, this definition agrees with that in Definition 32.1 [Exercise 36.1]. If $b = +\infty$ or if f is not integrable on $[a, b]$, but the limit in (1) exists, then (1) defines an *improper integral*.

An analogous definition applies if f is defined on $(a, b]$ where a is finite or $-\infty$ and if f is integrable on each $[c, b]$ for $a < c < b$. Then we define

$$\int_a^b f(x)\, dx = \lim_{c \to a^+} \int_c^b f(x)\, dx \tag{2}$$

whenever the limit exists.

If f is defined on (a, b) and integrable on all closed subintervals $[c, d]$, then we fix α in (a, b) and define

$$\int_a^b f(x)\, dx = \int_a^\alpha f(x)\, dx + \int_\alpha^b f(x)\, dx \tag{3}$$

provided the integrals on the right exist and the sum is not of the form $+\infty + (-\infty)$. Here we agree $\infty + L = \infty$ if $L \neq -\infty$ and $(-\infty) + L = -\infty$ if $L \neq \infty$. It is easy [Exercise 36.2] to see this definition does not depend on the choice of α.

Whenever the improper integrals defined above exist and are finite, the integrals are said to *converge*. Otherwise they *diverge* to $+\infty$ or to $-\infty$.

Example 1

Consider $f(x) = \frac{1}{x}$ for $x \in (0, \infty)$. For $d > 1$, we have $\int_1^d \frac{1}{x}\, dx = \log_e d$, so

$$\int_1^\infty \frac{1}{x}\, dx = \lim_{d \to \infty} \log_e d = +\infty.$$

This improper integral diverges to $+\infty$. For $0 < c < 1$, we have $\int_c^1 \frac{1}{x}\, dx = -\log_e c$, so

$$\int_0^1 \frac{1}{x}\, dx = \lim_{c \to 0^+} [-\log_e c] = +\infty.$$

Also we have

$$\int_0^\infty \frac{1}{x} = +\infty.$$

□

Example 2

Consider $f(x) = x^{-p}$ for $x \in [1, \infty)$ and a fixed positive number $p \neq 1$. For $d > 1$,

$$\int_1^d x^{-p}\, dx = \frac{1}{1-p}[d^{1-p} - 1].$$

It follows that

$$\int_1^\infty x^{-p}\, dx = \frac{1}{1-p}[0 - 1] = \frac{1}{p-1} \quad \text{if} \quad p > 1$$

and

$$\int_1^\infty x^{-p}\, dx = +\infty \quad \text{if} \quad 0 < p < 1.$$

□

Example 3

We have $\int_0^d \sin x\, dx = 1 - \cos d$ for all d. The value $(1 - \cos d)$ oscillates between 0 and 2, as $d \to \infty$, and therefore the limit

$$\lim_{d \to \infty} \int_0^d \sin x\, dx \quad \text{does not exist.}$$

Thus the symbol $\int_0^\infty \sin x \, dx$ has no meaning and is not an improper integral. Similarly, $\int_{-\infty}^0 \sin x \, dx$ and $\int_{-\infty}^\infty \sin x \, dx$ have no meaning.

Note the limit

$$\lim_{a \to \infty} \int_{-a}^a \sin x \, dx$$

clearly exists and equals 0. When such a "symmetric" limit exists even though the improper integral $\int_{-\infty}^\infty$ does not, we have what is called a *Cauchy principal value* of $\int_{-\infty}^\infty$. Thus 0 is the Cauchy principal value of $\int_{-\infty}^\infty \sin x \, dx$, but this is not an improper integral. ☐

It is especially valuable to extend Riemann-Stieltjes integrals to infinite intervals; see the discussion after Theorem 36.4 below. Let F be a bounded increasing function on some interval I. The function F can be extended to all of $\mathbb{R}$ by a simple device: if I is bounded below, define

$$F(t) = \inf\{F(u) : u \in I\} \quad \text{for} \quad t \le \inf I;$$

if I is bounded above, define

$$F(t) = \sup\{F(u) : u \in I\} \quad \text{for} \quad t \ge \sup I.$$

For this reason, we will henceforth assume F is an increasing function on all of $\mathbb{R}$. We will use the notations

$$F(-\infty) = \lim_{t \to -\infty} F(t) \quad \text{and} \quad F(\infty) = \lim_{t \to \infty} F(t).$$

Improper Riemann-Stieltjes integrals are defined in analogy to improper Riemann integrals.

36.2 Definition.
Suppose f is F-integrable on each interval $[a, b]$ in $\mathbb{R}$. We make the following definitions whenever the limits exist:

$$\int_0^\infty f \, dF = \lim_{b \to \infty} \int_0^b f \, dF; \qquad \int_{-\infty}^0 f \, dF = \lim_{a \to -\infty} \int_a^0 f \, dF.$$

If both limits exist and their sum does not have the form $\infty + (-\infty)$, we define

$$\int_{-\infty}^{\infty} f \, dF = \int_{-\infty}^{0} f \, dF + \int_{0}^{\infty} f \, dF.$$

If this sum is finite, we say f is F-integrable on $\mathbb{R}$. If f is F-integrable on $\mathbb{R}$ for $F(t) = t$ [i.e., the integrals are Riemann integrals], we say f is *integrable* on $\mathbb{R}$.

36.3 Theorem.
If f is F-integrable on each interval $[a, b]$ and if $f(x) \geq 0$ for all $x \in \mathbb{R}$, then f is F-integrable on $\mathbb{R}$ or else $\int_{-\infty}^{\infty} f \, dF = +\infty$.

Proof
We indicate why $\lim_{a \to -\infty} \int_{a}^{0} f \, dF$ exists, and leave the case of $\lim_{b \to \infty} \int_{0}^{b} f \, dF$ to the reader. Let $h(a) = \int_{a}^{0} f \, dF$ for $a < 0$, and note $a' < a < 0$ implies $h(a') \geq h(a)$. This property implies $\lim_{a \to -\infty} h(a)$ exists and

$$\lim_{a \to -\infty} h(a) = \sup\{h(a) : a \in (-\infty, 0)\}.$$

We omit the simple argument. ∎

36.4 Theorem.
Suppose $-\infty < F(-\infty) < F(\infty) < \infty$. Let f be a bounded function on $\mathbb{R}$ that is F-integrable on each interval $[a, b]$. Then f is F-integrable on $\mathbb{R}$.

Proof
Select a constant B such that $|f(x)| \leq B$ for all $x \in \mathbb{R}$. Since we have $F(\infty) - F(-\infty) < \infty$, constant functions are F-integrable. Since $0 \leq f + B \leq 2B$, Theorem 36.3 shows $f + B$ is F-integrable. It follows [Exercise 36.10] that $f = (f + B) + (-B)$ is also F-integrable. ∎

Increasing functions F defined on $\mathbb{R}$ come up naturally in probability and statistics. In these disciplines, F is called a *distribution function* if we also have $F(-\infty) = 0$ and $F(\infty) = 1$. Of course, the function $F(t) = t$ that corresponds to the Riemann integral is not a distribution function. Here is how a distribution function

comes up in probability. Consider a random experiment with numerical outcomes; then $F(t)$ can represent the probability the numerical value will be $\leq t$. Thus F will be right continuous. As a very simple example, suppose the experiment involves tossing three fair coins and counting the number of heads. The numerical values 0, 1, 2, and 3 will result with probabilities $\frac{1}{8}$, $\frac{3}{8}$, $\frac{3}{8}$, and $\frac{1}{8}$, respectively. The corresponding distribution function is defined in Example 2 of §35, page 303, and sketched in Fig. 35.1.

Frequently a distribution function F has the form

$$F(t) = \int_{-\infty}^{t} g(x)\, dx$$

for an integrable function g satisfying $g(x) \geq 0$ for all $x \in \mathbb{R}$. Then g is called a *density* for F. Note we must have

$$\int_{-\infty}^{\infty} g(x)\, dx = 1.$$

If g is continuous, then $g(t) = F'(t)$ for all t by Theorem 34.3.

Example 4
It turns out that $\int_{-\infty}^{\infty} e^{-x^2}\, dx = \sqrt{\pi}$ [Exercise 36.7] and hence

$$\int_{-\infty}^{\infty} e^{-x^2/2}\, dx = \sqrt{2\pi}.$$

The most important density in probability is the *normal density*

$$g(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

which gives rise to the *normal distribution*

$$F(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{t} e^{-x^2/2}\, dx;$$

see Fig. 36.1. ☐

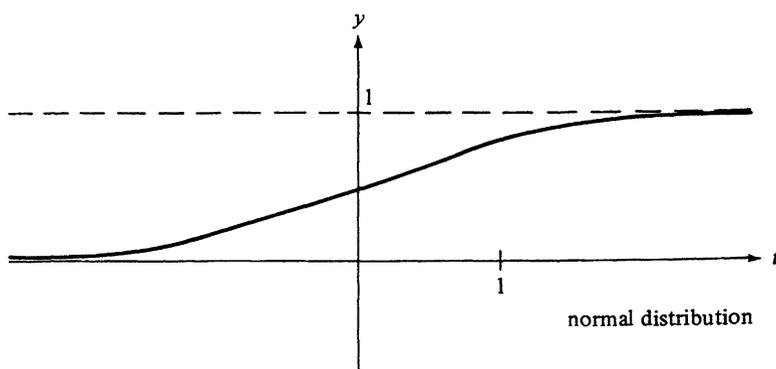Exercises 36.1–36.8 below deal only with Riemann integrals.

FIGURE 36.1

Exercises

36.1 Show that if f is integrable on $[a, b]$ as in Definition 32.1, then

$$\lim_{d \to b^-} \int_a^d f(x) \, dx = \int_a^b f(x) \, dx.$$

36.2 Show the definition (3) in Definition 36.1 does not depend on the choice of α.

36.3 **(a)** Show

$$\int_0^1 x^{-p} \, dx = \frac{1}{1 - p} \quad \text{if} \quad 0 < p < 1 \quad \text{and} \quad \int_0^1 x^{-p} \, dx = +\infty \quad \text{if} \quad p > 1.$$

(b) Show $\int_0^\infty x^{-p} \, dx = +\infty$ for all $p > 0$.

36.4 Calculate
(a) $\int_0^1 \log_e x \, dx$,   **(b)** $\int_2^\infty \frac{\log_e x}{x} \, dx$,
(c) $\int_0^\infty \frac{1}{1+x^2} \, dx$.

36.5 Let f be a continuous function on (a, b) such that $f(x) \geq 0$ for all $x \in (a, b)$; a can be $-\infty$, b can be $+\infty$. Show the improper integral $\int_a^b f(x) \, dx$ exists and equals

$$\sup \left\{ \int_c^d f(x) \, dx : [c, d] \subseteq (a, b) \right\}.$$

36.6 Prove the following *comparison tests*. Let f and g be continuous functions on (a, b) such that $0 \leq f(x) \leq g(x)$ for all x in (a, b); a can be $-\infty$, b can be $+\infty$.

(a) If $\int_a^b g(x)\,dx < \infty$, then $\int_a^b f(x)\,dx < \infty$.

(b) If $\int_a^b f(x)\,dx = +\infty$, then $\int_a^b g(x)\,dx = +\infty$.

36.7 **(a)** Use Exercise 36.6 to show $\int_{-\infty}^\infty e^{-x^2}\,dx < \infty$.

(b) Show this integral equals $\sqrt{\pi}$. *Hint*: Calculate the double integral $\int_{-\infty}^\infty \int_{-\infty}^\infty e^{-x^2} e^{-y^2}\,dx\,dy$ using polar coordinates.

36.8 Suppose f is continuous on (a, b) and $\int_a^b |f(x)|\,dx < \infty$; again a can be $-\infty$, b can be $+\infty$. Show the integral $\int_a^b f(x)\,dx$ exists and is finite.

36.9 Let F be the normal distribution function in Example 4.

(a) Show that if f is continuous on $\mathbb{R}$ and if the improper integral $\int_{-\infty}^\infty f(x)e^{-x^2/2}\,dx$ exists, then the improper integral $\int_{-\infty}^\infty f\,dF$ exists and

$$\int_{-\infty}^\infty f\,dF = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty f(x)e^{-x^2/2}\,dx.$$

Calculate
(b) $\int_{-\infty}^\infty x^2\,dF(x)$,   **(c)** $\int_{-\infty}^\infty e^{x^2}\,dF(x)$,
(d) $\int_{-\infty}^\infty |x|\,dF(x)$,   **(e)** $\int_{-\infty}^\infty x\,dF(x)$.

36.10 Let f and g be F-integrable functions on $\mathbb{R}$. Show $f+g$ is F-integrable on $\mathbb{R}$ and

$$\int_{-\infty}^\infty (f+g)\,dF = \int_{-\infty}^\infty f\,dF + \int_{-\infty}^\infty g\,dF.$$

36.11 Show that if f and g are F-integrable on $\mathbb{R}$ and if $f(x) \le g(x)$ for x in $\mathbb{R}$, then $\int_{-\infty}^\infty f\,dF \le \int_{-\infty}^\infty g\,dF$.

36.12 Generalize Exercise 36.6 to F-integrals on $\mathbb{R}$.

36.13 Generalize Exercise 36.8 to F-integrals on $\mathbb{R}$.

36.14 Let (u_j) be a sequence of distinct points in $\mathbb{R}$, and let (c_j) be a sequence of nonnegative numbers such that $\sum c_j < \infty$.

(a) Observe $F = \sum_{j=1}^\infty c_j J_{u_j}$ is an increasing function on $\mathbb{R}$. See Example 3 on page 309.

(b) Show every bounded function f on $\mathbb{R}$ is F-integrable and

$$\int_{-\infty}^\infty f\,dF = \sum_{j=1}^\infty c_j f(u_j).$$

    (c) When will F be a distribution function?

    (d) As in Exercise 35.12, if (u_j) is an enumeration of the rationals, then F is strictly increasing on $\mathbb{R}$.

36.15 **(a)** Give an example of a sequence (f_n) of integrable functions on $\mathbb{R}$ where $\int_{-\infty}^{\infty} f_n(x)\,dx = 1$ for all n and yet $f_n \to 0$ uniformly on $\mathbb{R}$.

    (b) Suppose F is a distribution function on $\mathbb{R}$. Show that if (f_n) is a sequence of F-integrable functions on $\mathbb{R}$ and if $f_n \to f$ uniformly on $\mathbb{R}$, then f is F-integrable on $\mathbb{R}$ and

$$\int_{-\infty}^{\infty} f\,dF = \lim_{n\to\infty} \int_{-\infty}^{\infty} f_n\,dF.$$