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CHAPTER

Sequences

§7 Limits of Sequences

A *sequence* is a function whose domain is a set of the form $\{n \in \mathbb{Z} : n \geq m\}$; m is usually 1 or 0. Thus a sequence is a function that has a specified value for each integer $n \geq m$. It is customary to denote a sequence by a letter such as s and to denote its value at n as s_n rather than $s(n)$. It is often convenient to write the sequence as $(s_n)_{n=m}^{\infty}$ or $(s_m, s_{m+1}, s_{m+2}, \ldots)$. If $m = 1$ we may write $(s_n)_{n\in\mathbb{N}}$ or of course $(s_1, s_2, s_3, \ldots)$. Sometimes we will write (s_n) when the domain is understood or when the results under discussion do not depend on the specific value of m. In this chapter, we will be interested in sequences whose range values are real numbers, i.e., each s_n represents a real number.

Example 1

(a) Consider the sequence $(s_n)_{n\in\mathbb{N}}$ where $s_n = \frac{1}{n^2}$. This is the sequence $(1, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}, \frac{1}{25}, \ldots)$. Formally, of course, this is the function with domain $\mathbb{N}$ whose value at each n is $\frac{1}{n^2}$. The *set* of values is $\{1, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}, \frac{1}{25}, \ldots\}$.

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Undergraduate Texts in Mathematics, DOI 10.1007/978-1-4614-6271-2_2,

(b) Consider the sequence given by $a_n = (-1)^n$ for $n \geq 0$, i.e., $(a_n)_{n=0}^{\infty}$ where $a_n = (-1)^n$. Note that the first term of the sequence is $a_0 = 1$ and the sequence is $(1, -1, 1, -1, 1, -1, 1, \ldots)$. Formally, this is a function whose domain is $\{0, 1, 2, \ldots\}$ and whose *set* of values is $\{-1, 1\}$.

It is important to distinguish between a sequence and its set of values, since the validity of many results in this book depends on whether we are working with a sequence or a set. We will always use parentheses ( ) to signify a sequence and braces { } to signify a set. The sequence given by $a_n = (-1)^n$ has an infinite number of terms even though their values are repeated over and over. On the other hand, the *set* $\{(-1)^n : n = 0, 1, 2, \ldots\}$ is exactly the set $\{-1, 1\}$ consisting of two numbers.

(c) Consider the sequence $\cos(\frac{n\pi}{3})$, $n \in \mathbb{N}$. The first term of this sequence is $\cos(\frac{\pi}{3}) = \cos 60^\circ = \frac{1}{2}$ and the sequence looks like

$$(\tfrac{1}{2}, -\tfrac{1}{2}, -1, -\tfrac{1}{2}, \tfrac{1}{2}, 1, \tfrac{1}{2}, -\tfrac{1}{2}, -1, -\tfrac{1}{2}, \tfrac{1}{2}, 1, \tfrac{1}{2}, -\tfrac{1}{2}, -1, \ldots).$$

The set of values is $\{\cos(\frac{n\pi}{3}) : n \in \mathbb{N}\} = \{\frac{1}{2}, -\frac{1}{2}, -1, 1\}$.

(d) If $a_n = \sqrt[n]{n}$, $n \in \mathbb{N}$, the sequence is $(1, \sqrt{2}, \sqrt[3]{3}, \sqrt[4]{4}, \ldots)$. If we approximate values to four decimal places, the sequence looks like

$$(1, 1.4142, 1.4422, 1.4142, 1.3797, 1.3480, 1.3205, 1.2968, \ldots).$$

It turns out that a_{100} is approximately 1.0471 and $a_{1,000}$ is approximately 1.0069.

(e) Consider the sequence $b_n = (1 + \frac{1}{n})^n$, $n \in \mathbb{N}$. This is the sequence $(2, (\frac{3}{2})^2, (\frac{4}{3})^3, (\frac{5}{4})^4, \ldots)$. If we approximate the values to four decimal places, we obtain

$$(2, 2.25, 2.3704, 2.4414, 2.4883, 2.5216, 2.5465, 2.5658, \ldots).$$

Also b_{100} is approximately 2.7048 and $b_{1,000}$ is approximately 2.7169. □

The "limit" of a sequence (s_n) is a real number that the values s_n are close to for large values of n. For instance, the values of the sequence in Example 1(a) are close to 0 for large n and the values of the sequence in Example 1(d) appear to be close to 1 for large n.

The sequence (a_n) given by $a_n = (-1)^n$ requires some thought. We might say 1 is a limit because in fact $a_n = 1$ for the large values of n that are even. On the other hand, $a_n = -1$ [which is quite a distance from 1] for other large values of n. We need a precise definition in order to decide whether 1 is a limit of $a_n = (-1)^n$. It turns out that our definition will require the values to be close to the limit value for *all* large n, so 1 will *not* be a limit of the sequence $a_n = (-1)^n$.

7.1 Definition.
A sequence (s_n) of real numbers is said to *converge* to the real number s provided that

$$\text{for each} \quad \epsilon > 0 \quad \text{there exists a number} \quad N \quad \text{such that} \\ n > N \quad \text{implies} \quad |s_n - s| < \epsilon. \tag{1}$$

If (s_n) converges to s, we will write $\lim_{n\to\infty} s_n = s$, or $s_n \to s$. The number s is called the *limit* of the sequence (s_n). A sequence that does not converge to some real number is said to *diverge*.

Several comments are in order. First, in view of the Archimedean property, the number N in Definition 7.1 can be taken to be a positive integer if we wish. Second, the symbol ϵ [lower case Greek epsilon] in this definition represents a positive number, not some new exotic number. However, it is traditional in mathematics to use ϵ and δ [lower case Greek delta] in situations where the interesting or challenging values are the small positive values. Third, condition (1) is an infinite number of statements, one for each positive value of ϵ. The condition states that to each $\epsilon > 0$ there corresponds a number N with a certain property, namely $n > N$ implies $|s_n - s| < \epsilon$. The value N depends on the value ϵ, and normally N will have to be large if ϵ is small. We illustrate these remarks in the next example.

Example 2
Consider the sequence $s_n = \frac{3n+1}{7n-4}$. If we write s_n as $\frac{3+\frac{1}{n}}{7-\frac{4}{n}}$ and note $\frac{1}{n}$ and $\frac{4}{n}$ are very small for large n, it seems reasonable to conclude $\lim s_n = \frac{3}{7}$. In fact, this reasoning will be completely valid after we

have the limit theorems in §9:

$$\lim s_n = \lim \left[\frac{3+\frac{1}{n}}{7-\frac{4}{n}}\right] = \frac{\lim 3 + \lim(\frac{1}{n})}{\lim 7 - 4\lim(\frac{1}{n})} = \frac{3+0}{7-4\cdot 0} = \frac{3}{7}.$$

However, for now we are interested in analyzing exactly what we mean by $\lim s_n = \frac{3}{7}$. By Definition 7.1, $\lim s_n = \frac{3}{7}$ means

$$\text{for each} \quad \epsilon > 0 \quad \text{there exists a number} \quad N \quad \text{such that}$$
$$n > N \quad \text{implies} \quad \left|\frac{3n+1}{7n-4} - \frac{3}{7}\right| < \epsilon. \tag{1}$$

As ϵ varies, N varies. In Example 2 of the next section we will show that, for this particular sequence, N can be taken to be $\frac{19}{49\epsilon} + \frac{4}{7}$. Using this observation, we find that for ϵ equal to 1, 0.1, 0.01, 0.001, and 0.000001, respectively, N can be taken to be approximately 0.96, 4.45, 39.35, 388.33, and 387,755.67, respectively. Since we are interested only in integer values of n, we may as well drop the fractional part of N. Then we see five of the infinitely many statements given by (1) are:

$$n > 0 \quad \text{implies} \quad \left|\frac{3n+1}{7n-4} - \frac{3}{7}\right| < 1; \tag{2}$$

$$n > 4 \quad \text{implies} \quad \left|\frac{3n+1}{7n-4} - \frac{3}{7}\right| < 0.1; \tag{3}$$

$$n > 39 \quad \text{implies} \quad \left|\frac{3n+1}{7n-4} - \frac{3}{7}\right| < 0.01; \tag{4}$$

$$n > 388 \quad \text{implies} \quad \left|\frac{3n+1}{7n-4} - \frac{3}{7}\right| < 0.001; \tag{5}$$

$$n > 387{,}755 \quad \text{implies} \quad \left|\frac{3n+1}{7n-4} - \frac{3}{7}\right| < 0.000001. \tag{6}$$

Table 7.1 partially confirms assertions (2) through (6). We could go on and on with these numerical illustrations, but it should be clear we need a more theoretical approach if we are going to *prove* results about limits. □

Example 3

We return to the examples in Example 1.

TABLE 7.1

n	$s_n = \frac{3n+1}{7n-4}$ is approximately	$\lvert s_n - \frac{3}{7}\rvert$ is approximately
2	0.7000	0.2714
3	0.5882	0.1597
4	0.5417	0.1131
5	0.5161	0.0876
6	0.5000	0.0714
40	0.4384	0.0098
400	0.4295	0.0010

(a) $\lim \frac{1}{n^2} = 0$. This will be proved in Example 1 of the next section.

(b) The sequence (a_n) where $a_n = (-1)^n$ does not converge. Thus the expression "$\lim a_n$" is meaningless in this case. We will discuss this example again in Example 4 of the next section.

(c) The sequence $\cos(\frac{n\pi}{3})$ does not converge. See Exercise 8.7.

(d) The sequence $n^{1/n}$ appears to converge to 1. We will prove $\lim n^{1/n} = 1$ in Theorem 9.7(c) on page 48.

(e) The sequence (b_n) where $b_n = (1+\frac{1}{n})^n$ converges to the number e that should be familiar from calculus. The limit $\lim b_n$ and the number e will be discussed further in Example 6 in §16 and in §37. Recall e is approximately 2.7182818. □

We conclude this section by showing that limits are unique. That is, if $\lim s_n = s$ and $\lim s_n = t$, then we must have $s = t$. In short, the values s_n cannot be getting arbitrarily close to different values for large n. To prove this, consider $\epsilon > 0$. By the definition of limit there exists N_1 so that

$$n > N_1 \quad \text{implies} \quad |s_n - s| < \frac{\epsilon}{2}$$

and there exists N_2 so that

$$n > N_2 \quad \text{implies} \quad |s_n - t| < \frac{\epsilon}{2}.$$

For $n > \max\{N_1, N_2\}$, the Triangle Inequality 3.7 shows

$$|s - t| = |(s - s_n) + (s_n - t)| \le |s - s_n| + |s_n - t| \le \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

This shows $|s-t| < \epsilon$ for all $\epsilon > 0$. It follows that $|s-t| = 0$; hence $s = t$.

Exercises

7.1 Write out the first five terms of the following sequences.
(a) $s_n = \frac{1}{3n+1}$ **(b)** $b_n = \frac{3n+1}{4n-1}$
(c) $c_n = \frac{n}{3^n}$ **(d)** $\sin(\frac{n\pi}{4})$

7.2 For each sequence in Exercise 7.1, determine whether it converges. If it converges, give its limit. No proofs are required.

7.3 For each sequence below, determine whether it converges and, if it converges, give its limit. No proofs are required.
(a) $a_n = \frac{n}{n+1}$ **(b)** $b_n = \frac{n^2+3}{n^2-3}$
(c) $c_n = 2^{-n}$ **(d)** $t_n = 1 + \frac{2}{n}$
(e) $x_n = 73 + (-1)^n$ **(f)** $s_n = (2)^{1/n}$
(g) $y_n = n!$ **(h)** $d_n = (-1)^n n$
(i) $\frac{(-1)^n}{n}$ **(j)** $\frac{7n^3+8n}{2n^3-3}$
(k) $\frac{9n^2-18}{6n+18}$ **(l)** $\sin(\frac{n\pi}{2})$
(m) $\sin(n\pi)$ **(n)** $\sin(\frac{2n\pi}{3})$
(o) $\frac{1}{n}\sin n$ **(p)** $\frac{2^{n+1}+5}{2^n-7}$
(q) $\frac{3^n}{n!}$ **(r)** $(1+\frac{1}{n})^2$
(s) $\frac{4n^2+3}{3n^2-2}$ **(t)** $\frac{6n+4}{9n^2+7}$

7.4 Give examples of

(a) A sequence (x_n) of irrational numbers having a limit $\lim x_n$ that is a rational number.

(b) A sequence (r_n) of rational numbers having a limit $\lim r_n$ that is an irrational number.

7.5 Determine the following limits. No proofs are required, but show any relevant algebra.

(a) $\lim s_n$ where $s_n = \sqrt{n^2+1} - n$,

(b) $\lim(\sqrt{n^2+n} - n)$,

(c) $\lim(\sqrt{4n^2+n} - 2n)$.
Hint for (a): First show $s_n = \frac{1}{\sqrt{n^2+1}+n}$.

§8 A Discussion about Proofs

In this section we give several examples of proofs using the definition of the limit of a sequence. With a little study and practice, students should be able to do proofs of this sort themselves. We will sometimes refer to a proof as a *formal proof* to emphasize it is a rigorous mathematical proof.

Example 1
Prove $\lim \frac{1}{n^2} = 0$.

Discussion. Our task is to consider an arbitrary $\epsilon > 0$ and show there exists a number N [which will depend on ϵ] such that $n > N$ implies $|\frac{1}{n^2} - 0| < \epsilon$. So we expect our formal proof to begin with "Let $\epsilon > 0$" and to end with something like "Hence $n > N$ implies $|\frac{1}{n^2} - 0| < \epsilon$." In between the proof should specify an N and then verify N has the desired property, namely $n > N$ does indeed imply $|\frac{1}{n^2} - 0| < \epsilon$.

As is often the case with trigonometric identities, we will initially work backward from our desired conclusion, but in the formal proof we will have to be sure our steps are reversible. In the present example, we want $|\frac{1}{n^2} - 0| < \epsilon$ and we want to know how big n must be. So we will operate on this inequality algebraically and try to "solve" for n. Thus we want $\frac{1}{n^2} < \epsilon$. By multiplying both sides by n^2 and dividing both sides by ϵ, we find we want $\frac{1}{\epsilon} < n^2$ or $\frac{1}{\sqrt{\epsilon}} < n$. If our steps are reversible, we see $n > \frac{1}{\sqrt{\epsilon}}$ implies $|\frac{1}{n^2} - 0| < \epsilon$. This suggests we put $N = \frac{1}{\sqrt{\epsilon}}$.

Formal Proof
Let $\epsilon > 0$. Let $N = \frac{1}{\sqrt{\epsilon}}$. Then $n > N$ implies $n > \frac{1}{\sqrt{\epsilon}}$ which implies $n^2 > \frac{1}{\epsilon}$ and hence $\epsilon > \frac{1}{n^2}$. Thus $n > N$ implies $|\frac{1}{n^2} - 0| < \epsilon$. This proves $\lim \frac{1}{n^2} = 0$. ■

Example 2
Prove $\lim \frac{3n+1}{7n-4} = \frac{3}{7}$.

Discussion. For each $\epsilon > 0$, we need to to decide how big n must be to guarantee $|\frac{3n+1}{7n-4} - \frac{3}{7}| < \epsilon$. Thus we want

$$\left|\frac{21n+7-21n+12}{7(4n-4)}\right| < \epsilon \quad \text{or} \quad \left|\frac{19}{7(7n-4)}\right| < \epsilon.$$

Since $7n - 4 > 0$, we can drop the absolute value and manipulate the inequality further to "solve" for n:

$$\frac{19}{7\epsilon} < 7n - 4 \quad \text{or} \quad \frac{19}{7\epsilon} + 4 < 7n \quad \text{or} \quad \frac{19}{49\epsilon} + \frac{4}{7} < n.$$

Our steps are reversible, so we will put $N = \frac{19}{49\epsilon} + \frac{4}{7}$. Incidentally, we could have chosen N to be any number larger than $\frac{19}{49\epsilon} + \frac{4}{7}$.

Formal Proof

Let $\epsilon > 0$ and let $N = \frac{19}{49\epsilon} + \frac{4}{7}$. Then $n > N$ implies $n > \frac{19}{49\epsilon} + \frac{4}{7}$, hence $7n > \frac{19}{7\epsilon} + 4$, hence $7n - 4 > \frac{19}{7\epsilon}$, hence $\frac{19}{7(7n-4)} < \epsilon$, and hence $|\frac{3n+1}{7n-4} - \frac{3}{7}| < \epsilon$. This proves $\lim \frac{3n+1}{7n-4} = \frac{3}{7}$. ■

Example 3

Prove $\lim \frac{4n^3+3n}{n^3-6} = 4$.

Discussion. For each $\epsilon > 0$, we need to determine how large n has to be to imply

$$\left|\frac{4n^3+3n}{n^3-6} - 4\right| < \epsilon \quad \text{or} \quad \left|\frac{3n+24}{n^3-6}\right| < \epsilon.$$

By considering $n > 1$, we may drop the absolute values; thus we need to find how big n must be to give $\frac{3n+24}{n^3-6} < \epsilon$. This time it would be very difficult to "solve" for or isolate n. Recall we need to find some N such that $n > N$ implies $\frac{3n+24}{n^3-6} < \epsilon$, but we do not need to find the least such N. So we will simplify matters by making estimates. The idea is that $\frac{3n+24}{n^3-6}$ is bounded by some constant times $\frac{n}{n^3} = \frac{1}{n^2}$ for sufficiently large n. To find such a bound we will find an upper bound for the numerator and a lower bound for the denominator. For example, since $3n + 24 \leq 27n$, it suffices for us to get $\frac{27n}{n^3-6} < \epsilon$. To make the denominator smaller and yet a constant multiple of n^3, we note $n^3 - 6 \geq \frac{n^3}{2}$ provided n is sufficiently large; in fact, all we need is $\frac{n^3}{2} \geq 6$ or $n^3 \geq 12$ or $n > 2$. So it suffices to get $\frac{27n}{n^3/2} < \epsilon$ or $\frac{54}{n^2} < \epsilon$ or $n > \sqrt{\frac{54}{\epsilon}}$, provided $n > 2$.

Formal Proof
Let $\epsilon > 0$ and let $N = \max\{2, \sqrt{\frac{54}{\epsilon}}\}$. Then $n > N$ implies $n > \sqrt{\frac{54}{\epsilon}}$, hence $\frac{54}{n^2} < \epsilon$, hence $\frac{27n}{n^3/2} < \epsilon$. Since $n > 2$, we have $\frac{n^3}{2} \leq n^3 - 6$ and also $27n \geq 3n + 24$. Thus $n > N$ implies

$$\frac{3n+24}{n^3-6} \leq \frac{27n}{\frac{1}{2}n^3} = \frac{54}{n^2} < \epsilon,$$

and hence

$$\left| \frac{4n^3+3n}{n^3-6} - 4 \right| < \epsilon,$$

as desired. ■

Example 3 illustrates direct proofs of even rather simple limits can get complicated. With the limit theorems of §9 we would just write

$$\lim \left[\frac{4n^3+3n}{n^3-6} \right] = \lim \left[\frac{4 + \frac{3}{n^2}}{1 - \frac{6}{n^3}} \right] = \frac{\lim 4 + 3 \cdot \lim(\frac{1}{n^2})}{\lim 1 - 6 \cdot \lim(\frac{1}{n^3})} = 4.$$

Example 4
Show that the sequence $a_n = (-1)^n$ does not converge.

Discussion. We will assume $\lim(-1)^n = a$ and obtain a contradiction. No matter what a is, either 1 or -1 will have distance at least 1 from a. Thus the inequality $|(-1)^n - a| < 1$ will not hold for all large n.

Formal Proof
Assume $\lim(-1)^n = a$ for some $a \in \mathbb{R}$. Letting $\epsilon = 1$ in the definition of the limit, we see that there exists N such that

$$n > N \quad \text{implies} \quad |(-1)^n - a| < 1.$$

By considering both an even and an odd $n > N$, we see that

$$|1 - a| < 1 \quad \text{and} \quad |-1 - a| < 1.$$

Now by the Triangle Inequality 3.7

$$2 = |1-(-1)| = |1-a+a-(-1)| \leq |1-a| + |a-(-1)| < 1+1 = 2.$$

This absurdity shows our assumption $\lim(-1)^n = a$ must be wrong, so the sequence $(-1)^n$ does not converge. ■

Example 5

Let (s_n) be a sequence of nonnegative real numbers and suppose $s = \lim s_n$. Note $s \geq 0$; see Exercise 8.9(a). Prove $\lim \sqrt{s_n} = \sqrt{s}$.

Discussion. We need to consider $\epsilon > 0$ and show there exists N such that

$$n > N \quad \text{implies} \quad |\sqrt{s_n} - \sqrt{s}| < \epsilon.$$

This time we cannot expect to obtain N explicitly in terms of ϵ because of the general nature of the problem. But we can hope to show such N exists. The trick here is to violate our training in algebra and "irrationalize the denominator":

$$\sqrt{s_n} - \sqrt{s} = \frac{(\sqrt{s_n} - \sqrt{s})(\sqrt{s_n} + \sqrt{s})}{\sqrt{s_n} + \sqrt{s}} = \frac{s_n - s}{\sqrt{s_n} + \sqrt{s}}.$$

Since $s_n \to s$ we will be able to make the numerator small [for large n]. Unfortunately, if $s = 0$ the denominator will also be small. So we consider two cases. If $s > 0$, the denominator is bounded below by $\sqrt{s}$ and our trick will work:

$$|\sqrt{s_n} - \sqrt{s}| \leq \frac{|s_n - s|}{\sqrt{s}},$$

so we will select N so that $|s_n - s| < \sqrt{s}\epsilon$ for $n > N$. Note that N exists, since we can apply the definition of limit to $\sqrt{s}\epsilon$ just as well as to ϵ. For $s = 0$, it can be shown directly that $\lim s_n = 0$ implies $\lim \sqrt{s_n} = 0$; the trick of "irrationalizing the denominator" is not needed in this case.

Formal Proof

Case I: $s > 0$. Let $\epsilon > 0$. Since $\lim s_n = s$, there exists N such that

$$n > N \quad \text{implies} \quad |s_n - s| < \sqrt{s}\epsilon.$$

Now $n > N$ implies

$$|\sqrt{s_n} - \sqrt{s}| = \frac{|s_n - s|}{\sqrt{s_n} + \sqrt{s}} \leq \frac{|s_n - s|}{\sqrt{s}} < \frac{\sqrt{s}\epsilon}{\sqrt{s}} = \epsilon.$$

Case II: $s = 0$. This case is left to Exercise 8.3. ■

Example 6

Let (s_n) be a convergent sequence of real numbers such that $s_n \neq 0$ for all $n \in \mathbb{N}$ and $\lim s_n = s \neq 0$. Prove $\inf\{|s_n| : n \in \mathbb{N}\} > 0$.

Discussion. The idea is that "most" of the terms s_n are close to s and hence not close to 0. More explicitly, "most" of the terms s_n are within $\frac{1}{2}|s|$ of s, hence most s_n satisfy $|s_n| \geq \frac{1}{2}|s|$. This seems clear from Fig. 8.1, but a formal proof will use the triangle inequality.

Formal Proof

Let $\epsilon = \frac{1}{2}|s| > 0$. Since $\lim s_n = s$, there exists N in $\mathbb{N}$ so that

$$n > N \quad \text{implies} \quad |s_n - s| < \frac{|s|}{2}.$$

Now

$$n > N \quad \text{implies} \quad |s_n| \geq \frac{|s|}{2}, \tag{1}$$

since otherwise the triangle inequality would imply

$$|s| = |s - s_n + s_n| \leq |s - s_n| + |s_n| < \frac{|s|}{2} + \frac{|s|}{2} = |s|$$

which is absurd. If we set

$$m = \min\left\{\frac{|s|}{2}, |s_1|, |s_2|, \ldots, |s_N|\right\},$$

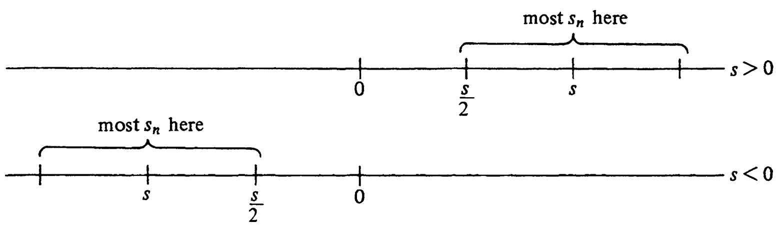

FIGURE 8.1

then we clearly have $m > 0$ and $|s_n| \geq m$ for *all* $n \in \mathbb{N}$ in view of (1). Thus $\inf\{|s_n| : n \in \mathbb{N}\} \geq m > 0$, as desired. ■

Formal proofs are required in the following exercises.

Exercises

8.1 Prove the following:
(a) $\lim \frac{(-1)^n}{n} = 0$ **(b)** $\lim \frac{1}{n^{1/3}} = 0$
(c) $\lim \frac{2n-1}{3n+2} = \frac{2}{3}$ **(d)** $\lim \frac{n+6}{n^2-6} = 0$

8.2 Determine the limits of the following sequences, and then prove your claims.
(a) $a_n = \frac{n}{n^2+1}$ **(b)** $b_n = \frac{7n-19}{3n+7}$
(c) $c_n = \frac{4n+3}{7n-5}$ **(d)** $d_n = \frac{2n+4}{5n+2}$
(e) $s_n = \frac{1}{n} \sin n$

8.3 Let (s_n) be a sequence of nonnegative real numbers, and suppose $\lim s_n = 0$. Prove $\lim \sqrt{s_n} = 0$. This will complete the proof for Example 5.

8.4 Let (t_n) be a bounded sequence, i.e., there exists M such that $|t_n| \leq M$ for all n, and let (s_n) be a sequence such that $\lim s_n = 0$. Prove $\lim(s_n t_n) = 0$.

8.5 ★[1]

(a) Consider three sequences (a_n), (b_n) and (s_n) such that $a_n \leq s_n \leq b_n$ for all n and $\lim a_n = \lim b_n = s$. Prove $\lim s_n = s$. This is called the "squeeze lemma."

(b) Suppose (s_n) and (t_n) are sequences such that $|s_n| \leq t_n$ for all n and $\lim t_n = 0$. Prove $\lim s_n = 0$.

8.6 Let (s_n) be a sequence in $\mathbb{R}$.

(a) Prove $\lim s_n = 0$ if and only if $\lim |s_n| = 0$.

(b) Observe that if $s_n = (-1)^n$, then $\lim |s_n|$ exists, but $\lim s_n$ does not exist.

8.7 Show the following sequences do not converge.
(a) $\cos(\frac{n\pi}{3})$ **(b)** $s_n = (-1)^n n$
(c) $\sin(\frac{n\pi}{3})$

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[1]This exercise is referred to in several places.

8.8 Prove the following [see Exercise 7.5]:
(a) $\lim[\sqrt{n^2+1}-n]=0$ **(b)** $\lim[\sqrt{n^2+n}-n]=\frac{1}{2}$
(c) $\lim[\sqrt{4n^2+n}-2n]=\frac{1}{4}$

8.9 ★[2] Let (s_n) be a sequence that converges.

(a) Show that if $s_n \geq a$ for all but finitely many n, then $\lim s_n \geq a$.

(b) Show that if $s_n \leq b$ for all but finitely many n, then $\lim s_n \leq b$.

(c) Conclude that if all but finitely many s_n belong to $[a,b]$, then $\lim s_n$ belongs to $[a,b]$.

8.10 Let (s_n) be a convergent sequence, and suppose $\lim s_n > a$. Prove there exists a number N such that $n > N$ implies $s_n > a$.

§9 Limit Theorems for Sequences

In this section we prove some basic results that are probably already familiar to the reader. First we prove convergent sequences are bounded. A sequence (s_n) of real numbers is said to be *bounded* if the set $\{s_n : n \in \mathbb{N}\}$ is a bounded set, i.e., if there exists a constant M such that $|s_n| \leq M$ for all n.

9.1 Theorem.
Convergent sequences are bounded.

Proof
Let (s_n) be a convergent sequence, and let $s = \lim s_n$. Applying Definition 7.1 with $\epsilon = 1$ we obtain N in $\mathbb{N}$ so that

$$n > N \quad \text{implies} \quad |s_n - s| < 1.$$

From the triangle inequality we see $n > N$ implies $|s_n| < |s| + 1$. Define $M = \max\{|s|+1, |s_1|, |s_2|, \ldots, |s_N|\}$. Then we have $|s_n| \leq M$ for all $n \in \mathbb{N}$, so (s_n) is a bounded sequence. ■

In the proof of Theorem 9.1 we only needed to use property 7.1(1) for a single value of ϵ. Our choice of $\epsilon = 1$ was quite arbitrary.

[2] This exercise is referred to in several places.

9.2 Theorem.
If the sequence (s_n) *converges to* s *and* k *is in* $\mathbb{R}$*, then the sequence* (ks_n) *converges to* ks*. That is,* $\lim(ks_n) = k \cdot \lim s_n$*.*

Proof
We assume $k \neq 0$, since this result is trivial for $k = 0$. Let $\epsilon > 0$ and note we need to show $|ks_n - ks| < \epsilon$ for large n. Since $\lim s_n = s$, there exists N such that

$$n > N \quad \text{implies} \quad |s_n - s| < \frac{\epsilon}{|k|}.$$

Then

$$n > N \quad \text{implies} \quad |ks_n - ks| < \epsilon.$$

■

9.3 Theorem.
If (s_n) *converges to* s *and* (t_n) *converges to* t*, then* (s_n+t_n) *converges to* $s+t$*. That is,*

$$\lim(s_n + t_n) = \lim s_n + \lim t_n.$$

Proof
Let $\epsilon > 0$; we need to show

$$|s_n + t_n - (s+t)| < \epsilon \quad \text{for large} \quad n.$$

We note $|s_n + t_n - (s+t)| \leq |s_n - s| + |t_n - t|$. Since $\lim s_n = s$, there exists N_1 such that

$$n > N_1 \quad \text{implies} \quad |s_n - s| < \frac{\epsilon}{2}.$$

Likewise, there exists N_2 such that

$$n > N_2 \quad \text{implies} \quad |t_n - t| < \frac{\epsilon}{2}.$$

Let $N = \max\{N_1, N_2\}$. Then clearly

$$n>N \quad \text{implies} \quad |s_n+t_n-(s+t)| \leq |s_n-s|+|t_n-t| < \frac{\epsilon}{2}+\frac{\epsilon}{2} = \epsilon.$$

■

9.4 Theorem.
If (s_n) *converges to* s *and* (t_n) *converges to* t*, then* $(s_n t_n)$ *converges to* st*. That is,*

$$\lim(s_n t_n) = (\lim s_n)(\lim t_n).$$

Discussion. The trick here is to look at the inequality

$$|s_n t_n - st| = |s_n t_n - s_n t + s_n t - st| \\ \leq |s_n t_n - s_n t| + |s_n t - st| = |s_n| \cdot |t_n - t| + |t| \cdot |s_n - s|.$$

For large n, $|t_n - t|$ and $|s_n - s|$ are small and $|t|$ is, of course, constant. Fortunately, Theorem 9.1 shows $|s_n|$ is bounded, so we will be able to show $|s_n t_n - s_n t|$ is small.

Proof

Let $\epsilon > 0$. By Theorem 9.1 there is a constant $M > 0$ such that $|s_n| \leq M$ for all n. Since $\lim t_n = t$ there exists N_1 such that

$$n > N_1 \quad \text{implies} \quad |t_n - t| < \frac{\epsilon}{2M}.$$

Also, since $\lim s_n = s$ there exists N_2 such that

$$n > N_2 \quad \text{implies} \quad |s_n - s| < \frac{\epsilon}{2(|t| + 1)}.$$

[We used $\frac{\epsilon}{2(|t|+1)}$ instead of $\frac{\epsilon}{2|t|}$, because t could be 0.] Now if $N = \max\{N_1, N_2\}$, then $n > N$ implies

$$|s_n t_n - st| \leq |s_n| \cdot |t_n - t| + |t| \cdot |s_n - s| \\ \leq M \cdot \frac{\epsilon}{2M} + |t| \cdot \frac{\epsilon}{2(|t| + 1)} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

■

To handle quotients of sequences, we first deal with reciprocals.

9.5 Lemma.

If (s_n) converges to s, if $s_n \neq 0$ for all n, and if $s \neq 0$, then $(1/s_n)$ converges to $1/s$.

Discussion. We begin by considering the equality

$$\left| \frac{1}{s_n} - \frac{1}{s} \right| = \left| \frac{s - s_n}{s_n s} \right|.$$

For large n, the numerator is small. The only possible difficulty would be if the denominator were also small for large n. This difficulty was solved in Example 6 of §8 where we proved $m = \inf\{|s_n| : n \in \mathbb{N}\} > 0$. Thus

$$\left| \frac{1}{s_n} - \frac{1}{s} \right| \leq \frac{|s - s_n|}{m|s|},$$

and it is clear how our proof should proceed.

Proof

Let $\epsilon > 0$. By Example 6 of §8, there exists $m > 0$ such that $|s_n| \geq m$ for all n. Since $\lim s_n = s$ there exists N such that

$$n > N \quad \text{implies} \quad |s - s_n| < \epsilon \cdot m|s|.$$

Then $n > N$ implies

$$\left|\frac{1}{s_n} - \frac{1}{s}\right| = \frac{|s - s_n|}{|s_n s|} \leq \frac{|s - s_n|}{m|s|} < \epsilon.$$

■

9.6 Theorem.

Suppose (s_n) converges to s and (t_n) converges to t. If $s \neq 0$ and $s_n \neq 0$ for all n, then (t_n/s_n) converges to t/s.

Proof

By Lemma 9.5, the sequence $(1/s_n)$ converges to $1/s$, so

$$\lim \frac{t_n}{s_n} = \lim \frac{1}{s_n} \cdot t_n = \frac{1}{s} \cdot t = \frac{t}{s}$$

by Theorem 9.4. ■

The preceding limit theorems and a few standard examples allow one to easily calculate many limits.

9.7 Theorem (Basic Examples).

(a) $\lim_{n\to\infty}(\frac{1}{n^p}) = 0$ for $p > 0$.
(b) $\lim_{n\to\infty} a^n = 0$ if $|a| < 1$.
(c) $\lim(n^{1/n}) = 1$.
(d) $\lim_{n\to\infty}(a^{1/n}) = 1$ for $a > 0$. □

Proof

(a) Let $\epsilon > 0$ and let $N = (\frac{1}{\epsilon})^{1/p}$. Then $n > N$ implies $n^p > \frac{1}{\epsilon}$ and hence $\epsilon > \frac{1}{n^p}$. Since $\frac{1}{n^p} > 0$, this shows $n > N$ implies $|\frac{1}{n^p} - 0| < \epsilon$. [The meaning of n^p when p is not an integer will be discussed in §37.]

(b) We may suppose $a \neq 0$, because $\lim_{n\to\infty} a^n = 0$ is obvious for $a = 0$. Since $|a| < 1$, we can write $|a| = \frac{1}{1+b}$ where $b > 0$. By the binomial theorem [Exercise 1.12], $(1+b)^n \geq 1+nb > nb$, so

$$|a^n - 0| = |a^n| = \frac{1}{(1+b)^n} < \frac{1}{nb}.$$

Now consider $\epsilon > 0$ and let $N = \frac{1}{\epsilon b}$. Then $n > N$ implies $n > \frac{1}{\epsilon b}$ and hence $|a^n - 0| < \frac{1}{nb} < \epsilon$.

(c) Let $s_n = (n^{1/n}) - 1$ and note $s_n \geq 0$ for all n. By Theorem 9.3 it suffices to show $\lim s_n = 0$. Since $1 + s_n = (n^{1/n})$, we have $n = (1+s_n)^n$. For $n \geq 2$ we use the binomial expansion of $(1+s_n)^n$ to conclude

$$n = (1+s_n)^n \geq 1 + ns_n + \frac{1}{2}n(n-1)s_n^2 > \frac{1}{2}n(n-1)s_n^2.$$

Thus $n > \frac{1}{2}n(n-1)s_n^2$, so $s_n^2 < \frac{2}{n-1}$. Consequently, we have $s_n < \sqrt{\frac{2}{n-1}}$ for $n \geq 2$. A standard argument now shows $\lim s_n = 0$; see Exercise 9.7.

(d) First suppose $a \geq 1$. Then for $n \geq a$ we have $1 \leq a^{1/n} \leq n^{1/n}$. Since $\lim n^{1/n} = 1$, it follows easily that $\lim(a^{1/n}) = 1$; compare Exercise 8.5(a). Suppose $0 < a < 1$. Then $\frac{1}{a} > 1$, so $\lim(\frac{1}{a})^{1/n} = 1$ from above. Lemma 9.5 now shows $\lim(a^{1/n}) = 1$. ■

Example 1

Prove $\lim s_n = \frac{1}{4}$, where

$$s_n = \frac{n^3 + 6n^2 + 7}{4n^3 + 3n - 4}.$$

□

Solution

We have

$$s_n = \frac{1 + \frac{6}{n} + \frac{7}{n^3}}{4 + \frac{3}{n^2} - \frac{4}{n^3}}.$$

By Theorem 9.7(a) we have $\lim \frac{1}{n} = 0$ and $\lim \frac{1}{n^3} = 0$. Hence by Theorems 9.3 and 9.2 we have

$$\lim\left(1 + \frac{6}{n} + \frac{7}{n^3}\right) = \lim(1) + 6 \cdot \lim\left(\frac{1}{n}\right) + 7 \cdot \lim\left(\frac{1}{n^3}\right) = 1.$$

Similarly, we have

$$\lim\left(4+\frac{3}{n^2}-\frac{4}{n^3}\right)=4.$$

Hence Theorem 9.6 implies $\lim s_n = \frac{1}{4}$. □

Example 2
Find $\lim \frac{n-5}{n^2+7}$. □

Solution
Let $s_n = \frac{n-5}{n^2+7}$. We can write s_n as $\frac{1-\frac{5}{n}}{n+\frac{7}{n}}$, but then the denominator does not converge. So we write

$$s_n = \frac{\frac{1}{n}-\frac{5}{n^2}}{1+\frac{7}{n^2}}.$$

Now $\lim(\frac{1}{n}-\frac{5}{n^2}) = 0$ by Theorems 9.7(a), 9.3 and 9.2. Likewise $\lim(1+\frac{7}{n^2}) = 1$, so Theorem 9.6 implies $\lim s_n = \frac{0}{1} = 0$. □

Example 3
Find $\lim \frac{n^2+3}{n+1}$. □

Solution
We can write $\frac{n^2+3}{n+1}$ as

$$\frac{n+\frac{3}{n}}{1+\frac{1}{n}} \quad \text{or} \quad \frac{1+\frac{3}{n^2}}{\frac{1}{n}+\frac{1}{n^2}}.$$

Both fractions lead to problems: either the numerator does not converge or else the denominator converges to 0. It turns out $\frac{n^2+3}{n+1}$ does not converge and the symbol $\lim \frac{n^2+3}{n+1}$ is undefined, at least for the present; see Example 6. The reader may have the urge to use the symbol $+\infty$ here. Our next task is to make such use of the symbol $+\infty$ legitimate. For a sequence (s_n), $\lim s_n = +\infty$ will signify that the terms s_n are eventually all large. Here is the precise definition. □

9.8 Definition.
For a sequence (s_n), we write $\lim s_n = +\infty$ provided
  for each $M > 0$ there is a number N such that
  $n > N$ implies $s_n > M$.

In this case we say the sequence *diverges to* $+\infty$.

Similarly, we write $\lim s_n = -\infty$ provided

for each $M < 0$ there is a number N such that
$n > N$ implies $s_n < M$.

Henceforth we will say (s_n) has a *limit* or the *limit exists* provided (s_n) converges or diverges to $+\infty$ or diverges to $-\infty$. In the definition of $\lim s_n = +\infty$ the challenging values of M are large positive numbers: the larger M is the larger N will need to be. In the definition of $\lim s_n = -\infty$ the challenging values of M are "large" negative numbers like $-10{,}000{,}000{,}000$.

Example 4

We have $\lim n^2 = +\infty$, $\lim(-n) = -\infty$, $\lim 2^n = +\infty$ and $\lim(\sqrt{n} + 7) = +\infty$. Of course, many sequences do not have limits $+\infty$ or $-\infty$ even if they are unbounded. For example, the sequences defined by $s_n = (-1)^n n$ and $t_n = n\cos^2(\frac{n\pi}{2})$ are unbounded, but they do not diverge to $+\infty$ or $-\infty$, so the expressions $\lim[(-1)^n n]$ and $\lim[n\cos^2(\frac{n\pi}{2})]$ are meaningless. Note $t_n = n$ when n is even and $t_n = 0$ when n is odd. □

The strategy for proofs involving infinite limits is very much the same as for finite limits. We give some examples.

Example 5

Give a formal proof that $\lim(\sqrt{n} + 7) = +\infty$. □

Discussion. We need to consider an arbitrary $M > 0$ and show there exists N [which will depend on M] such that

$$n > N \quad \text{implies} \quad \sqrt{n} + 7 > M.$$

To see how big N must be we "solve" for n in the inequality $\sqrt{n}+7 > M$. This inequality holds provided $\sqrt{n} > M - 7$ or $n > (M - 7)^2$. Thus we will take $N = (M - 7)^2$.

Formal Proof

Let $M > 0$ and let $N = (M-7)^2$. Then $n > N$ implies $n > (M-7)^2$, hence $\sqrt{n} > M - 7$, hence $\sqrt{n} + 7 > M$. This shows $\lim(\sqrt{n} + 7) = +\infty$. ■

Example 6
Give a formal proof that $\lim \frac{n^2+3}{n+1} = +\infty$; see Example 3. □

Discussion. Consider $M > 0$. We need to determine how large n must be to guarantee $\frac{n^2+3}{n+1} > M$. The idea is to bound the fraction $\frac{n^2+3}{n+1}$ below by some multiple of $\frac{n^2}{n} = n$; compare Example 3 of §8. Since $n^2 + 3 > n^2$ and $n + 1 \leq 2n$, we have $\frac{n^2+3}{n+1} > \frac{n^2}{2n} = \frac{1}{2}n$, and it suffices to arrange for $\frac{1}{2}n > M$.

Formal Proof
Let $M > 0$ and let $N = 2M$. Then $n > N$ implies $\frac{1}{2}n > M$, which implies

$$\frac{n^2+3}{n+1} > \frac{n^2}{2n} = \frac{1}{2}n > M.$$

Hence $\lim \frac{n^2+3}{n+1} = +\infty$. ■

The limit in Example 6 would be easier to handle if we could apply a limit theorem. But the limit Theorems 9.2–9.6 do not apply.

WARNING. Do not attempt to apply the limit Theorems 9.2–9.6 to infinite limits. Use Theorem 9.9 or 9.10 below or Exercises 9.9–9.12.

9.9 Theorem.
Let (s_n) and (t_n) be sequences such that $\lim s_n = +\infty$ and $\lim t_n > 0$ [$\lim t_n$ can be finite or $+\infty$]. Then $\lim s_n t_n = +\infty$.

Discussion. Let $M > 0$. We need to show $s_n t_n > M$ for large n. We have $\lim s_n = +\infty$, and we need to be sure the t_n's are bounded away from 0 for large n. We will choose a real number m so that $0 < m < \lim t_n$ and observe $t_n > m$ for large n. Then all we need is $s_n > \frac{M}{m}$ for large n.

Proof
Let $M > 0$. Select a real number m so that $0 < m < \lim t_n$. Whether $\lim t_n = +\infty$ or not, it is clear there exists N_1 such that

$$n > N_1 \quad \text{implies} \quad t_n > m;$$

see Exercise 8.10. Since $\lim s_n = +\infty$, there exists N_2 so that

$$n > N_2 \quad \text{implies} \quad s_n > \frac{M}{m}.$$

Put $N = \max\{N_1, N_2\}$. Then $n > N$ implies $s_n t_n > \frac{M}{m} \cdot m = M$. ■

Example 7
Use Theorem 9.9 to prove $\lim \frac{n^2+3}{n+1} = +\infty$; see Example 6. □

Solution
We observe $\frac{n^2+3}{n+1} = \frac{n+\frac{3}{n}}{1+\frac{1}{n}} = s_n t_n$ where $s_n = n + \frac{3}{n}$ and $t_n = \frac{1}{1+\frac{1}{n}}$. It is easy to show $\lim s_n = +\infty$ and $\lim t_n = 1$. So by Theorem 9.9, we have $\lim s_n t_n = +\infty$. □

Here is another useful theorem.

9.10 Theorem.
For a sequence (s_n) of positive real numbers, we have $\lim s_n = +\infty$ if and only if $\lim(\frac{1}{s_n}) = 0$.

Proof
Let (s_n) be a sequence of positive real numbers. We have to show

$$\lim s_n = +\infty \quad \text{implies} \quad \lim\left(\frac{1}{s_n}\right) = 0 \tag{1}$$

and

$$\lim\left(\frac{1}{s_n}\right) = 0 \quad \text{implies} \quad \lim s_n = +\infty. \tag{2}$$

In this case the proofs will appear very similar, but the thought processes will be quite different.

To prove (1), suppose $\lim s_n = +\infty$. Let $\epsilon > 0$ and let $M = \frac{1}{\epsilon}$. Since $\lim s_n = +\infty$, there exists N such that $n > N$ implies $s_n > M = \frac{1}{\epsilon}$. Therefore $n > N$ implies $\epsilon > \frac{1}{s_n} > 0$, so

$$n > N \quad \text{implies} \quad \left|\frac{1}{s_n} - 0\right| < \epsilon.$$

That is, $\lim(\frac{1}{s_n}) = 0$. This proves (1).

To prove (2), we abandon the notation of the last paragraph and begin anew. Suppose $\lim(\frac{1}{s_n}) = 0$. Let $M > 0$ and let $\epsilon = \frac{1}{M}$. Then

$\epsilon > 0$, so there exists N such that $n > N$ implies $|\frac{1}{s_n} - 0| < \epsilon = \frac{1}{M}$. Since $s_n > 0$, we can write

$$n > N \quad \text{implies} \quad 0 < \frac{1}{s_n} < \frac{1}{M}$$

and hence

$$n > N \quad \text{implies} \quad M < s_n.$$

That is, $\lim s_n = +\infty$ and (2) holds. ■

Exercises

9.1 Using the limit Theorems 9.2–9.7, prove the following. Justify all steps.
(a) $\lim \frac{n+1}{n} = 1$ **(b)** $\lim \frac{3n+7}{6n-5} = \frac{1}{2}$
(c) $\lim \frac{17n^5+73n^4-18n^2+3}{23n^5+13n^3} = \frac{17}{23}$

9.2 Suppose $\lim x_n = 3$, $\lim y_n = 7$ and all y_n are nonzero. Determine the following limits:
(a) $\lim(x_n + y_n)$ **(b)** $\lim \frac{3y_n - x_n}{y_n^2}$

9.3 Suppose $\lim a_n = a$, $\lim b_n = b$, and $s_n = \frac{a_n^3+4a_n}{b_n^2+1}$. Prove $\lim s_n = \frac{a^3+4a}{b^2+1}$ carefully, using the limit theorems.

9.4 Let $s_1 = 1$ and for $n \geq 1$ let $s_{n+1} = \sqrt{s_n + 1}$.

(a) List the first four terms of (s_n).

(b) It turns out that (s_n) converges. Assume this fact and prove the limit is $\frac{1}{2}(1 + \sqrt{5})$.

9.5 Let $t_1 = 1$ and $t_{n+1} = \frac{t_n^2+2}{2t_n}$ for $n \geq 1$. Assume (t_n) converges and find the limit.

9.6 Let $x_1 = 1$ and $x_{n+1} = 3x_n^2$ for $n \geq 1$.

(a) Show if $a = \lim x_n$, then $a = \frac{1}{3}$ or $a = 0$.

(b) Does $\lim x_n$ exist? Explain.

(c) Discuss the apparent contradiction between parts (a) and (b).

9.7 Complete the proof of Theorem 9.7(c), i.e., give the standard argument needed to show $\lim s_n = 0$.

9.8 Give the following when they exist. Otherwise assert "NOT EXIST."

(a) $\lim n^3$ **(b)** $\lim(-n^3)$
(c) $\lim(-n)^n$ **(d)** $\lim(1.01)^n$
(e) $\lim n^n$

9.9 Suppose there exists N_0 such that $s_n \le t_n$ for all $n > N_0$.

(a) Prove that if $\lim s_n = +\infty$, then $\lim t_n = +\infty$.

(b) Prove that if $\lim t_n = -\infty$, then $\lim s_n = -\infty$.

(c) Prove that if $\lim s_n$ and $\lim t_n$ exist, then $\lim s_n \le \lim t_n$.

9.10 **(a)** Show that if $\lim s_n = +\infty$ and $k > 0$, then $\lim(ks_n) = +\infty$.

(b) Show $\lim s_n = +\infty$ if and only if $\lim(-s_n) = -\infty$.

(c) Show that if $\lim s_n = +\infty$ and $k < 0$, then $\lim(ks_n) = -\infty$.

9.11 **(a)** Show that if $\lim s_n = +\infty$ and $\inf\{t_n : n \in \mathbb{N}\} > -\infty$, then $\lim(s_n + t_n) = +\infty$.

(b) Show that if $\lim s_n = +\infty$ and $\lim t_n > -\infty$, then $\lim(s_n + t_n) = +\infty$.

(c) Show that if $\lim s_n = +\infty$ and if (t_n) is a bounded sequence, then $\lim(s_n + t_n) = +\infty$.

9.12 ★[3] Assume all $s_n \neq 0$ and that the limit $L = \lim |\frac{s_{n+1}}{s_n}|$ exists.

(a) Show that if $L < 1$, then $\lim s_n = 0$. *Hint*: Select a so that $L < a < 1$ and obtain N so that $|s_{n+1}| < a|s_n|$ for $n \ge N$. Then show $|s_n| < a^{n-N}|s_N|$ for $n > N$.

(b) Show that if $L > 1$, then $\lim |s_n| = +\infty$. *Hint*: Apply (a) to the sequence $t_n = \frac{1}{|s_n|}$; see Theorem 9.10.

9.13 Show

$$\lim_{n\to\infty} a^n = \begin{cases} 0 & \text{if} \quad |a| < 1 \\ 1 & \text{if} \quad a = 1 \\ +\infty & \text{if} \quad a > 1 \\ \text{does not exist} & \text{if} \quad a \le -1. \end{cases}$$

9.14 Let $p > 0$. Use Exercise 9.12 to show

$$\lim_{n\to\infty} \frac{a^n}{n^p} = \begin{cases} 0 & \text{if} \quad |a| \le 1 \\ +\infty & \text{if} \quad a > 1 \\ \text{does not exist} & \text{if} \quad a < -1. \end{cases}$$

Hint: For the $a > 1$ case, use Exercise 9.12(b).

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[3]This exercise is referred to in several places.

9.15 Show $\lim_{n\to\infty} \frac{a^n}{n!} = 0$ for all $a \in \mathbb{R}$.

9.16 Use Theorems 9.9 and 9.10 or Exercises 9.9–9.15 to prove the following:

(a) $\lim \frac{n^4+8n}{n^2+9} = +\infty$

(b) $\lim[\frac{2^n}{n^2} + (-1)^n] = +\infty$

(c) $\lim[\frac{3^n}{n^3} - \frac{3^n}{n!}] = +\infty$

9.17 Give a formal proof that $\lim n^2 = +\infty$ using only Definition 9.8.

9.18 **(a)** Verify $1 + a + a^2 + \cdots + a^n = \frac{1-a^{n+1}}{1-a}$ for $a \neq 1$.

(b) Find $\lim_{n\to\infty}(1 + a + a^2 + \cdots + a^n)$ for $|a| < 1$.

(c) Calculate $\lim_{n\to\infty}(1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \cdots + \frac{1}{3^n})$.

(d) What is $\lim_{n\to\infty}(1 + a + a^2 + \cdots + a^n)$ for $a \geq 1$?

§10 Monotone Sequences and Cauchy Sequences

In this section we obtain two theorems [Theorems 10.2 and 10.11] that will allow us to conclude certain sequences converge *without* knowing the limit in advance. These theorems are important because in practice the limits are not usually known in advance.

10.1 Definition.
A sequence (s_n) of real numbers is called an *increasing sequence* if $s_n \leq s_{n+1}$ for all n, and (s_n) is called a *decreasing sequence* if $s_n \geq s_{n+1}$ for all n. Note that if (s_n) is increasing, then $s_n \leq s_m$ whenever $n < m$. A sequence that is increasing or decreasing[4] will be called a *monotone sequence* or a *monotonic sequence*.

Example 1
The sequences defined by $a_n = 1 - \frac{1}{n}$, $b_n = n^3$ and $c_n = (1+\frac{1}{n})^n$ are increasing sequences, although this is not obvious for the

[4]In the First Edition of this book, increasing and decreasing sequences were referred to as "nondecreasing" and "nonincreasing" sequences, respectively.

sequence (c_n). The sequence $d_n = \frac{1}{n^2}$ is decreasing. The sequences $s_n = (-1)^n$, $t_n = \cos(\frac{n\pi}{3})$, $u_n = (-1)^n n$ and $v_n = \frac{(-1)^n}{n}$ are not monotonic sequences. Also $x_n = n^{1/n}$ is not monotonic, as can be seen by examining the first four values; see Example 1(d) on page 33 in §7.

Of the sequences above, (a_n), (c_n), (d_n), (s_n), (t_n), (v_n) and (x_n) are bounded sequences. The remaining sequences, (b_n) and (u_n), are unbounded sequences. □

10.2 Theorem.
All bounded monotone sequences converge.

Proof
Let (s_n) be a bounded increasing sequence. Let S denote the set $\{s_n : n \in \mathbb{N}\}$, and let $u = \sup S$. Since S is bounded, u represents a real number. We show $\lim s_n = u$. Let $\epsilon > 0$. Since $u - \epsilon$ is not an upper bound for S, there exists N such that $s_N > u - \epsilon$. Since (s_n) is increasing, we have $s_N \leq s_n$ for all $n \geq N$. Of course, $s_n \leq u$ for all n, so $n > N$ implies $u - \epsilon < s_n \leq u$, which implies $|s_n - u| < \epsilon$. This shows $\lim s_n = u$.

The proof for bounded decreasing sequences is left to Exercise 10.2. ■

Note the Completeness Axiom 4.4 is a vital ingredient in the proof of Theorem 10.2.

Example 2
Consider the sequence (s_n) defined *recursively* by

$$s_1 = 5 \quad \text{and} \quad s_n = \frac{s_{n-1}^2 + 5}{2s_{n-1}} \quad \text{for} \quad n \geq 2. \tag{1}$$

Thus $s_2 = 3$ and $s_3 = \frac{7}{3} \approx 2.333$. First, note a simple induction argument shows $s_n > 0$ for all n. We will show $\lim_n s_n$ exists by showing the sequence is decreasing and bounded; see Theorem 10.2. In fact, we will prove the following by induction:

$$\sqrt{5} < s_{n+1} < s_n \leq 5 \quad \text{for} \quad n \geq 1. \tag{2}$$

Since $\sqrt{5} \approx 2.236$, our computations show (2) holds for $n \le 2$. For the induction step, assume (2) holds for some $n \ge 2$. To show $s_{n+2} < s_{n+1}$, we need

$$\frac{s_{n+1}^2 + 5}{2s_{n+1}} < s_{n+1} \quad \text{or} \quad s_{n+1}^2 + 5 < 2s_{n+1}^2 \quad \text{or} \quad 5 < s_{n+1}^2,$$

but this holds because $s_{n+1} > \sqrt{5}$ by the assumption (2) for n. To show $s_{n+2} > \sqrt{5}$, we need

$$\frac{s_{n+1}^2 + 5}{2s_{n+1}} > \sqrt{5} \quad \text{or} \quad s_{n+1}^2 + 5 > 2\sqrt{5}s_{n+1}$$

or $s_{n+1}^2 - 2\sqrt{5}s_{n+1} + 5 > 0$, which is true because $s_{n+1}^2 - 2\sqrt{5}s_{n+1} + 5 = (s_{n+1} - \sqrt{5})^2 > 0$. Thus (2) holds for $n+1$ whenever (2) holds for n. Hence (2) holds for all n by induction. Thus $s = \lim_n s_n$ exists.

If one looks at $s_4 = \frac{47}{21} \approx 2.238095$ and compares with $\sqrt{5} \approx 2.236068$, one might suspect $s = \sqrt{5}$. To verify this, we apply the limit Theorems 9.2–9.4 and the fact $s = \lim_n s_{n+1}$ to the equation $2 \cdot s_{n+1}s_n = s_n^2 + 5$ to obtain $2s^2 = s^2 + 5$. Thus $s^2 = 5$ and $s = \sqrt{5}$, since the limit is certainly not $-\sqrt{5}$. □

10.3 Discussion of Decimals.
We have not given much attention to the notion that real numbers are simply decimal expansions. This notion is substantially correct, but there are subtleties to be faced. For example, different decimal expansions can represent the same real number. The somewhat more abstract developments of the set $\mathbb{R}$ of real numbers discussed in §6 turn out to be more satisfactory.

We restrict our attention to nonnegative decimal expansions and nonnegative real numbers. From our point of view, every nonnegative decimal expansion is shorthand for the limit of a bounded increasing sequence of real numbers. Suppose we are given a decimal expansion $K.d_1d_2d_3d_4\cdots$, where K is a nonnegative integer and each d_j belongs to $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$. Let

$$s_n = K + \frac{d_1}{10} + \frac{d_2}{10^2} + \cdots + \frac{d_n}{10^n}. \tag{1}$$

Then (s_n) is an increasing sequence of real numbers, and (s_n) is bounded [by $K + 1$, in fact]. So by Theorem 10.2, (s_n) converges to a real number we traditionally write as $K.d_1d_2d_3d_4\cdots$. For example, $3.3333\cdots$ represents

$$\lim_{n\to\infty}\left(3+\frac{3}{10}+\frac{3}{10^2}+\cdots+\frac{3}{10^n}\right).$$

To calculate this limit, we borrow the following fact about geometric series from Example 1 on page 96 in §14:

$$\lim_{n\to\infty} a(1+r+r^2+\cdots+r^n)=\frac{a}{1-r}\quad\text{for}\quad |r|<1; \tag{2}$$

see also Exercise 9.18. In our case, $a=3$ and $r=\frac{1}{10}$, so $3.3333\cdots$ represents $\frac{3}{1-\frac{1}{10}}=\frac{10}{3}$, as expected. Similarly, $0.9999\cdots$ represents

$$\lim_{n\to\infty}\left(\frac{9}{10}+\frac{9}{10^2}+\cdots+\frac{9}{10^n}\right)=\frac{\frac{9}{10}}{1-\frac{1}{10}}=1.$$

Thus $0.9999\cdots$ and $1.0000\cdots$ are different decimal expansions that represent the same real number!

The converse of the preceding discussion also holds. That is, every nonnegative real number x has at least one decimal expansion. This will be proved, along with some related results, in §16. □

Unbounded monotone sequences also have limits.

10.4 Theorem.

(i) *If (s_n) is an unbounded increasing sequence, then* $\lim s_n = +\infty$.

(ii) *If (s_n) is an unbounded decreasing sequence, then* $\lim s_n = -\infty$.

Proof

(i) Let (s_n) be an unbounded increasing sequence. Let $M>0$. Since the set $\{s_n : n\in\mathbb{N}\}$ is unbounded and it is bounded below by s_1, it must be unbounded above. Hence for some N in $\mathbb{N}$ we have $s_N>M$. Clearly $n>N$ implies $s_n\ge s_N>M$, so $\lim s_n=+\infty$.

(ii) The proof is similar and is left to Exercise 10.5. ■

10.5 Corollary.
If (s_n) is a monotone sequence, then the sequence either converges, diverges to $+\infty$, or diverges to $-\infty$. Thus $\lim s_n$ is always meaningful for monotone sequences.

Proof
Apply Theorems 10.2 and 10.4. ■

Let (s_n) be a bounded sequence in $\mathbb{R}$; it may or may not converge. It is apparent from the definition of limit in 7.1 that the limiting behavior of (s_n) depends only on sets of the form $\{s_n : n > N\}$. For example, if $\lim s_n$ exists, clearly it lies in the interval $[u_N, v_N]$ where

$$u_N = \inf\{s_n : n > N\} \quad \text{and} \quad v_N = \sup\{s_n : n > N\};$$

see Exercise 8.9. As N increases, the sets $\{s_n : n > N\}$ get smaller, so we have

$$u_1 \leq u_2 \leq u_3 \leq \cdots \quad \text{and} \quad v_1 \geq v_2 \geq v_3 \geq \cdots;$$

see Exercise 4.7(a). By Theorem 10.2 the limits $u = \lim_{N\to\infty} u_N$ and $v = \lim_{N\to\infty} v_N$ both exist, and $u \leq v$ since $u_N \leq v_N$ for all N. If $\lim s_n$ exists then, as noted above, $u_N \leq \lim s_n \leq v_N$ for all N, so we must have $u \leq \lim s_n \leq v$. The numbers u and v are useful whether $\lim s_n$ exists or not and are denoted $\liminf s_n$ and $\limsup s_n$, respectively.

10.6 Definition.
Let (s_n) be a sequence in $\mathbb{R}$. We define

$$\limsup s_n = \lim_{N\to\infty} \sup\{s_n : n > N\} \tag{1}$$

and

$$\liminf s_n = \lim_{N\to\infty} \inf\{s_n : n > N\}. \tag{2}$$

Note that in this definition we do not restrict (s_n) to be bounded. However, we adopt the following conventions. If (s_n) is not bounded above, $\sup\{s_n : n > N\} = +\infty$ for all N and we decree $\limsup s_n = +\infty$. Likewise, if (s_n) is not bounded below, $\inf\{s_n : n > N\} = -\infty$ for all N and we decree $\liminf s_n = -\infty$.

We emphasize $\limsup s_n$ need not equal $\sup\{s_n : n \in \mathbb{N}\}$, but $\limsup s_n \le \sup\{s_n : n \in \mathbb{N}\}$. Some of the values s_n may be much larger than $\limsup s_n$; $\limsup s_n$ is the largest value that *infinitely many* s_n's can get close to. Similar remarks apply to $\liminf s_n$. These remarks will be clarified in Theorem 11.8 and §12, where we will give a thorough treatment of lim inf's and lim sup's. For now, we need a theorem that shows (s_n) has a limit if and only if $\liminf s_n = \limsup s_n$.

10.7 Theorem.
Let (s_n) be a sequence in $\mathbb{R}$.
(i) *If $\lim s_n$ is defined [as a real number, $+\infty$ or $-\infty$], then $\liminf s_n = \lim s_n = \limsup s_n$.*
(ii) *If $\liminf s_n = \limsup s_n$, then $\lim s_n$ is defined and $\lim s_n = \liminf s_n = \limsup s_n$.*

Proof
We use the notation $u_N = \inf\{s_n : n > N\}$, $v_N = \sup\{s_n : n > N\}$, $u = \lim u_N = \liminf s_n$ and $v = \lim v_N = \limsup s_n$.

(i) Suppose $\lim s_n = +\infty$. Let M be a positive real number. Then there is a positive integer N so that

$$n > N \quad \text{implies} \quad s_n > M.$$

Then $u_N = \inf\{s_n : n > N\} \ge M$. It follows that $m > N$ implies $u_m \ge M$. In other words, the sequence (u_N) satisfies the condition defining $\lim u_N = +\infty$, i.e., $\liminf s_n = +\infty$. Likewise $\limsup s_n = +\infty$.

The case $\lim s_n = -\infty$ is handled in a similar manner.

Now suppose $\lim s_n = s$ where s is a real number. Consider $\epsilon > 0$. There exists a positive integer N such that $|s_n - s| < \epsilon$ for $n > N$. Thus $s_n < s + \epsilon$ for $n > N$, so

$$v_N = \sup\{s_n : n > N\} \le s + \epsilon.$$

Also, $m > N$ implies $v_m \le s+\epsilon$, so $\limsup s_n = \lim v_m \le s+\epsilon$. Since $\limsup s_n \le s + \epsilon$ for all $\epsilon > 0$, no matter how small, we conclude $\limsup s_n \le s = \lim s_n$. A similar argument shows $\lim s_n \le \liminf s_n$. Since $\liminf s_n \le \limsup s_n$, we infer all three numbers are equal:

$$\liminf s_n = \lim s_n = \limsup s_n.$$

(ii) If $\liminf s_n = \limsup s_n = +\infty$ it is easy to show $\lim s_n = +\infty$. And if $\liminf s_n = \limsup s_n = -\infty$ it is easy to show $\lim s_n = -\infty$. We leave these two special cases to the reader.

Suppose, finally, that $\liminf s_n = \limsup s_n = s$ where s is a real number. We need to prove $\lim s_n = s$. Let $\epsilon > 0$. Since $s = \lim v_N$ there exists a positive integer N_0 such that

$$|s - \sup\{s_n : n > N_0\}| < \epsilon.$$

Thus $\sup\{s_n : n > N_0\} < s + \epsilon$, so

$$s_n < s + \epsilon \quad \text{for all} \quad n > N_0. \tag{1}$$

Similarly, there exists N_1 such that $|s - \inf\{s_n : n > N_1\}| < \epsilon$, hence $\inf\{s_n : n > N_1\} > s - \epsilon$, hence

$$s_n > s - \epsilon \quad \text{for all} \quad n > N_1. \tag{2}$$

From (1) and (2) we conclude

$$s - \epsilon < s_n < s + \epsilon \quad \text{for} \quad n > \max\{N_0, N_1\},$$

equivalently

$$|s_n - s| < \epsilon \quad \text{for} \quad n > \max\{N_0, N_1\}.$$

This proves $\lim s_n = s$ as desired. ■

If (s_n) converges, then $\liminf s_n = \limsup s_n$ by the theorem just proved, so for large N the numbers $\sup\{s_n : n > N\}$ and $\inf\{s_n : n > N\}$ are close together. This implies that all the numbers in the set $\{s_n : n > N\}$ are close to each other. This leads us to a concept of great theoretical importance that will be used throughout the book.

10.8 Definition.
A sequence (s_n) of real numbers is called a *Cauchy sequence* if

$$\text{for each } \epsilon > 0 \text{ there exists a number } N \text{ such that}$$
$$m, n > N \text{ implies } |s_n - s_m| < \epsilon. \tag{1}$$

Compare this definition with Definition 7.1.

10.9 Lemma.
Convergent sequences are Cauchy sequences.

Proof
Suppose $\lim s_n = s$. The idea is that, since the terms s_n are close to s for large n, they also must be close to each other; indeed

$$|s_n - s_m| = |s_n - s + s - s_m| \le |s_n - s| + |s - s_m|.$$

To be precise, let $\epsilon > 0$. Then there exists N such that

$$n > N \quad \text{implies} \quad |s_n - s| < \frac{\epsilon}{2}.$$

Clearly we may also write

$$m > N \quad \text{implies} \quad |s_m - s| < \frac{\epsilon}{2},$$

so

$$m, n > N \quad \text{implies} \quad |s_n - s_m| \le |s_n - s| + |s - s_m| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Thus (s_n) is a Cauchy sequence. ■

10.10 Lemma.
Cauchy sequences are bounded.

Proof
The proof is similar to that of Theorem 9.1. Applying Definition 10.8 with $\epsilon = 1$ we obtain N in $\mathbb{N}$ so that

$$m, n > N \quad \text{implies} \quad |s_n - s_m| < 1.$$

In particular, $|s_n - s_{N+1}| < 1$ for $n > N$, so $|s_n| < |s_{N+1}| + 1$ for $n > N$. If $M = \max\{|s_{N+1}| + 1, |s_1|, |s_2|, \ldots, |s_N|\}$, then $|s_n| \le M$ for all $n \in \mathbb{N}$. ■

The next theorem is very important because it shows that to verify that a sequence converges it suffices to check it is a Cauchy sequence, a property that does not involve the limit itself.

10.11 Theorem.
A sequence is a convergent sequence if and only if it is a Cauchy sequence.

Proof

The expression "if and only if" indicates that we have two assertions to verify: (i) convergent sequences are Cauchy sequences, and (ii) Cauchy sequences are convergent sequences. We already verified (i) in Lemma 10.9. To check (ii), consider a Cauchy sequence (s_n) and note (s_n) is bounded by Lemma 10.10. By Theorem 10.7 we need only show

$$\liminf s_n = \limsup s_n. \qquad (1)$$

Let $\epsilon > 0$. Since (s_n) is a Cauchy sequence, there exists N so that

$$m, n > N \quad \text{implies} \quad |s_n - s_m| < \epsilon.$$

In particular, $s_n < s_m + \epsilon$ for all $m, n > N$. This shows $s_m + \epsilon$ is an upper bound for $\{s_n : n > N\}$, so $v_N = \sup\{s_n : n > N\} \le s_m + \epsilon$ for $m > N$. This, in turn, shows $v_N - \epsilon$ is a lower bound for $\{s_m : m > N\}$, so $v_N - \epsilon \le \inf\{s_m : m > N\} = u_N$. Thus

$$\limsup s_n \le v_N \le u_N + \epsilon \le \liminf s_n + \epsilon.$$

Since this holds for all $\epsilon > 0$, we have $\limsup s_n \le \liminf s_n$. The opposite inequality always holds, so we have established (1). ■

The proof of Theorem 10.11 uses Theorem 10.7, and Theorem 10.7 relies implicitly on the Completeness Axiom 4.4, since without the completeness axiom it is not clear that $\liminf s_n$ and $\limsup s_n$ are meaningful. The completeness axiom assures us that the expressions $\sup\{s_n : n > N\}$ and $\inf\{s_n : n > N\}$ in Definition 10.6 are meaningful, and Theorem 10.2 [which itself relies on the completeness axiom] assures us that the limits in Definition 10.6 also are meaningful.

Exercises on lim sup's and lim inf's appear in §§11 and 12.

Exercises

10.1 Which of the following sequences are increasing? decreasing? bounded?

(a) $\frac{1}{n}$ **(b)** $\frac{(-1)^n}{n^2}$

(c) n^5 **(d)** $\sin(\frac{n\pi}{7})$

(e) $(-2)^n$ **(f)** $\frac{n}{3^n}$

10.2 Prove Theorem 10.2 for bounded decreasing sequences.

10.3 For a decimal expansion $K.d_1d_2d_3d_4\cdots$, let (s_n) be defined as in Discussion 10.3. Prove $s_n < K+1$ for all $n \in \mathbb{N}$. *Hint*: $\frac{9}{10} + \frac{9}{10^2} + \cdots + \frac{9}{10^n} = 1 - \frac{1}{10^n}$ for all n.

10.4 Discuss why Theorems 10.2 and 10.11 would fail if we restricted our world of numbers to the set $\mathbb{Q}$ of rational numbers.

10.5 Prove Theorem 10.4(ii).

10.6 **(a)** Let (s_n) be a sequence such that

$$|s_{n+1} - s_n| < 2^{-n} \quad \text{for all} \quad n \in \mathbb{N}.$$

Prove (s_n) is a Cauchy sequence and hence a convergent sequence.

(b) Is the result in (a) true if we only assume $|s_{n+1} - s_n| < \frac{1}{n}$ for all $n \in \mathbb{N}$?

10.7 Let S be a bounded nonempty subset of $\mathbb{R}$ such that $\sup S$ is not in S. Prove there is a sequence (s_n) of points in S such that $\lim s_n = \sup S$. See also Exercise 11.11.

10.8 Let (s_n) be an increasing sequence of positive numbers and define $\sigma_n = \frac{1}{n}(s_1 + s_2 + \cdots + s_n)$. Prove (σ_n) is an increasing sequence.

10.9 Let $s_1 = 1$ and $s_{n+1} = (\frac{n}{n+1})s_n^2$ for $n \geq 1$.

(a) Find s_2, s_3 and s_4.

(b) Show $\lim s_n$ exists.

(c) Prove $\lim s_n = 0$.

10.10 Let $s_1 = 1$ and $s_{n+1} = \frac{1}{3}(s_n + 1)$ for $n \geq 1$.

(a) Find s_2, s_3 and s_4.

(b) Use induction to show $s_n > \frac{1}{2}$ for all n.

(c) Show (s_n) is a decreasing sequence.

(d) Show $\lim s_n$ exists and find $\lim s_n$.

10.11 Let $t_1 = 1$ and $t_{n+1} = [1 - \frac{1}{4n^2}] \cdot t_n$ for $n \geq 1$.

(a) Show $\lim t_n$ exists.

(b) What do you think $\lim t_n$ is?

10.12 Let $t_1 = 1$ and $t_{n+1} = [1 - \frac{1}{(n+1)^2}] \cdot t_n$ for $n \geq 1$.

(a) Show $\lim t_n$ exists.

(b) What do you think $\lim t_n$ is?

(c) Use induction to show $t_n = \frac{n+1}{2n}$.

(d) Repeat part (b).

§11 Subsequences

11.1 Definition.
Suppose $(s_n)_{n \in \mathbb{N}}$ is a sequence. A *subsequence* of this sequence is a sequence of the form $(t_k)_{k \in \mathbb{N}}$ where for each k there is a positive integer n_k such that

$$n_1 < n_2 < \cdots < n_k < n_{k+1} < \cdots \tag{1}$$

and

$$t_k = s_{n_k}. \tag{2}$$

Thus (t_k) is just a selection of some [possibly all] of the s_n's taken in order.

Here are some alternative ways to approach this concept. Note that (1) defines an infinite subset of $\mathbb{N}$, namely $\{n_1, n_2, n_3, \ldots\}$. Conversely, every infinite subset of $\mathbb{N}$ can be described by (1). Thus a subsequence of (s_n) is a sequence obtained by selecting, in order, an infinite subset of the terms.

For a more precise definition, recall we can view the sequence $(s_n)_{n \in \mathbb{N}}$ as a function s with domain $\mathbb{N}$; see §7. For the subset $\{n_1, n_2, n_3, \ldots\}$, there is a natural function σ [lower case Greek sigma] given by $\sigma(k) = n_k$ for $k \in \mathbb{N}$. The function σ "selects" an infinite subset of $\mathbb{N}$, in order. The subsequence of s corresponding to σ is simply the composite function $t = s \circ \sigma$. That is,

$$t_k = t(k) = s \circ \sigma(k) = s(\sigma(k)) = s(n_k) = s_{n_k} \quad \text{for} \quad k \in \mathbb{N}. \tag{3}$$

Thus a sequence t is a subsequence of a sequence s if and only if $t = s \circ \sigma$ for some increasing function σ mapping $\mathbb{N}$ into $\mathbb{N}$. We will usually suppress the notation σ and often suppress the notation t

also. Thus the phrase "a subsequence (s_{n_k}) of (s_n)" will refer to the subsequence defined by (1) and (2) or by (3), depending upon your point of view.

Example 1
Let (s_n) be the sequence defined by $s_n = n^2(-1)^n$. The positive terms of this sequence comprise a subsequence. In this case, the sequence (s_n) is

$$(-1, 4, -9, 16, -25, 36, -49, 64, \ldots)$$

and the subsequence is

$$(4, 16, 36, 64, 100, 144, \ldots).$$

More precisely, the subsequence is $(s_{n_k})_{k\in\mathbb{N}}$ where $n_k{=}2k$ so that $s_{n_k}{=}(2k)^2(-1)^{2k}{=}4k^2$. The selection function σ is given by $\sigma(k){=}2k$. □

Example 2
Consider the sequence $a_n = \sin(\frac{n\pi}{3})$ and its subsequence (a_{n_k}) of nonnegative terms. The sequence $(a_n)_{n\in\mathbb{N}}$ is

$$(\frac{1}{2}\sqrt{3}, \frac{1}{2}\sqrt{3}, 0, -\frac{1}{2}\sqrt{3}, -\frac{1}{2}\sqrt{3}, 0, \frac{1}{2}\sqrt{3}, \frac{1}{2}\sqrt{3}, 0, -\frac{1}{2}\sqrt{3}, -\frac{1}{2}\sqrt{3}, 0, \ldots)$$

and the desired subsequence is

$$(\frac{1}{2}\sqrt{3}, \frac{1}{2}\sqrt{3}, 0, 0, \frac{1}{2}\sqrt{3}, \frac{1}{2}\sqrt{3}, 0, 0, \ldots).$$

It is evident that $n_1 = 1$, $n_2 = 2$, $n_3 = 3$, $n_4 = 6$, $n_5 = 7$, $n_6 = 8$, $n_7 = 9$, $n_8 = 12$, $n_9 = 13$, etc. We won't need a formula for n_k, but here is one: $n_k = k + 2\lfloor \frac{k}{4} \rfloor$ for $k \geq 1$, where $\lfloor x \rfloor$ is the "floor function," i.e., $\lfloor x \rfloor$ is the largest integer less than or equal to x, for $x \in \mathbb{R}$. □

After some reflection, the next theorem will seem obvious, but it is good to have a complete proof that covers all situations. The proof is a little bit complicated, but we will apply the theorem several times rather than having to recreate a similar proof several times.[5] Thus it is important to understand the proof.

[5]In the first edition of this book, we did create similar proofs instead.

11.2 Theorem.
Let (s_n) *be a sequence.*

(i) *If* t *is in* $\mathbb{R}$*, then there is a subsequence of* (s_n) *converging to* t *if and only if the set* $\{n \in \mathbb{N} : |s_n - t| < \epsilon\}$ *is infinite for all* $\epsilon > 0$*.*

(ii) *If the sequence* (s_n) *is unbounded above, it has a subsequence with limit* $+\infty$*.*

(iii) *Similarly, if* (s_n) *is unbounded below, a subsequence has limit* $-\infty$*.*

In each case, the subsequence can be taken to be monotonic.[6]

Proof
The forward implications $\Longrightarrow$ in (i)–(iii) are all easy to check. For example, if $\lim_k s_{n_k} = t$ and $\epsilon > 0$, then all but finitely many of the n_ks are in $\{n \in \mathbb{N} : |s_n - t| < \epsilon\}$. We focus on the other implications.

(i) First suppose the set $\{n \in \mathbb{N} : s_n = t\}$ is infinite. Then there are subsequences $(s_{n_k})_{k \in \mathbb{N}}$ such that $s_{n_k} = t$ for all k. Such subsequences of (s_n) are boring monotonic sequences converging to t.

Henceforth, we assume $\{n \in \mathbb{N} : s_n = t\}$ is finite. Then

$$\{n \in \mathbb{N} : 0 < |s_n - t| < \epsilon\} \quad \text{is infinite for all} \quad \epsilon > 0.$$

Since these sets equal

$$\{n \in \mathbb{N} : t - \epsilon < s_n < t\} \cup \{n \in \mathbb{N} : t < s_n < t + \epsilon\},$$

and these sets get smaller as $\epsilon \to 0$, we have

$$\{n \in \mathbb{N} : t - \epsilon < s_n < t\} \quad \text{is infinite for all} \quad \epsilon > 0, \tag{1}$$

or

$$\{n \in \mathbb{N} : t < s_n < t + \epsilon\} \quad \text{is infinite for all} \quad \epsilon > 0; \tag{2}$$

otherwise, for sufficiently small $\epsilon > 0$, the sets in both (1) and (2) would be finite.

We assume (1) holds, and leave the case that (2) holds to the reader. We will show how to define or construct

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[6]This will be proved easily here, but is also a consequence of the more general Theorem 11.4.

step-by-step a subsequence $(s_{n_k})_{k\in\mathbb{N}}$ satisfying $t-1 < s_{n_1} < t$ and

$$\max\left\{s_{n_{k-1}}, t-\frac{1}{k}\right\} \le s_{n_k} < t \quad \text{for} \quad k \ge 2. \tag{3}$$

Specifically, we will assume $n_1, n_2, \ldots, n_{k-1}$ have been selected satisfying (3) and show how to select n_k. This will give us an infinite increasing sequence $(n_k)_{k\in\mathbb{N}}$ and hence a subsequence (s_{n_k}) of (s_n) satisfying (3). Since we will have $s_{n_{k-1}} \le s_{n_k}$ for all k, this subsequence will be monotonically increasing. Since (3) also will imply $t - \frac{1}{k} \le s_{n_k} < t$ for all k, we will have $\lim_k s_{n_k} = t$; compare Exercise 8.5(a) on page 44.

A construction like the one described above, and executed below, is called an "inductive definition" or "definition by induction," even though the validity of the process is not a direct consequence of Peano's axiom N5 in §1.[7]

Here is the construction. Select n_1 so that $t-1 < s_{n_1} < t$; this is possible by (1). Suppose $n_1, n_2, \ldots, n_{k-1}$ have been selected so that

$$n_1 < n_2 < \cdots < n_{k-1} \tag{4}$$

and

$$\max\left\{s_{n_{j-1}}, t-\frac{1}{j}\right\} \le s_{n_j} < t \quad \text{for} \quad j = 2, \ldots, k-1. \tag{5}$$

Using (1) with $\epsilon = \max\{s_{n_{k-1}}, t-\frac{1}{k}\}$, we can select $n_k > n_{k-1}$ satisfying (5) for $j = k$, so that (3) holds for k. The procedure defines the sequence $(n_k)_{k\in\mathbb{N}}$. This completes the proof of (i), and is the crux of the full proof.

(ii) Let $n_1 = 1$, say. Given $n_1 < \cdots < n_{k-1}$, select n_k so that $s_{n_k} > \max\{s_{n_{k-1}}, k\}$. This is possible, since (s_n) is unbounded above. The sequence so obtained will be monotonic and have limit $+\infty$. A similar proof verifies (iii). ■

[7] Recursive definitions of sequences, which first appear in Exercises 9.4–9.6, can be viewed as simple examples of definitions by induction.

Example 3

It can be shown that the set $\mathbb{Q}$ of rational numbers can be listed as a sequence (r_n), though it is tedious to specify an exact formula. Figure 11.1 suggests such a listing [with repetitions] where $r_1 = 0$, $r_2 = 1$, $r_3 = \frac{1}{2}$, $r_4 = -\frac{1}{2}$, $r_5 = -1$, $r_6 = -2$, $r_7 = -1$, etc. Readers familiar with some set theory will recognize this assertion as "$\mathbb{Q}$ is countable." This sequence has an amazing property: given any real number a there exists a subsequence (r_{n_k}) of (r_n) converging to a. Since there are infinitely many rational numbers in every interval $(a-\epsilon, a+\epsilon)$ by Exercise 4.11, Theorem 11.2 shows that a subsequence of (r_n) converges to a. □

Example 4

Suppose (s_n) is a sequence of positive numbers such that $\inf\{s_n : n \in \mathbb{N}\} = 0$. The sequence (s_n) need not converge or even be bounded, but it has a subsequence converging monotonically to 0. By Theorem 11.2, it suffices to show $\{n \in \mathbb{N} : s_n < \epsilon\}$ is infinite for each $\epsilon > 0$. Otherwise, this set would be finite for some $\epsilon_0 > 0$. If the set is nonempty, then $\inf\{s_n : n \in \mathbb{N}\} = \min\{s_n : s_n < \epsilon_0\} > 0$, because each s_n is positive and the set $\{s_n : s_n < \epsilon_0\}$ is finite. This contradicts our assumption $\inf\{s_n : n \in \mathbb{N}\} = 0$. If the set is empty, then $\inf\{s_n : n \in \mathbb{N}\} \geq \epsilon_0 > 0$, again contrary to our assumption. □

The next theorem is almost obvious.

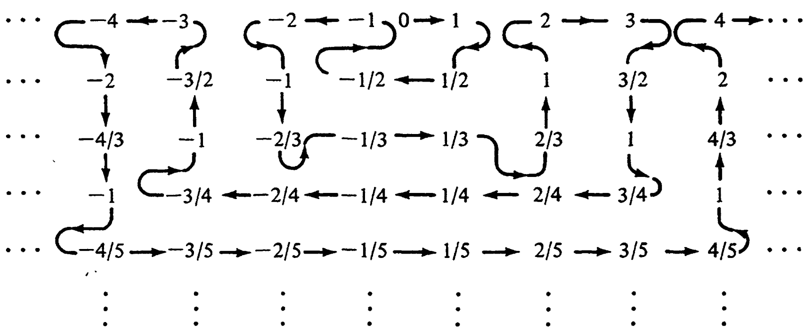

FIGURE 11.1

11.3 Theorem.
If the sequence (s_n) converges, then every subsequence converges to the same limit.

Proof
Let (s_{n_k}) denote a subsequence of (s_n). Note that $n_k \geq k$ for all k. This is easy to prove by induction; in fact, $n_1 \geq 1$ and $n_{k-1} \geq k-1$ implies $n_k > n_{k-1} \geq k-1$ and hence $n_k \geq k$.

Let $s = \lim s_n$ and let $\epsilon > 0$. There exists N so that $n > N$ implies $|s_n - s| < \epsilon$. Now $k > N$ implies $n_k > N$, which implies $|s_{n_k} - s| < \epsilon$. Thus

$$\lim_{k\to\infty} s_{n_k} = s.$$

■

Our immediate goal is to prove the Bolzano-Weierstrass theorem which asserts that every bounded sequence has a convergent subsequence. First we prove a theorem about monotonic subsequences.

11.4 Theorem.
Every sequence (s_n) has a monotonic subsequence.

Proof
Let's say that the n-th term is *dominant* if it is greater than every term which follows it:

$$s_m < s_n \quad \text{for all} \quad m > n. \tag{1}$$

Case 1. Suppose there are infinitely many dominant terms, and let (s_{n_k}) be any subsequence consisting solely of dominant terms. Then $s_{n_{k+1}} < s_{n_k}$ for all k by (1), so (s_{n_k}) is a decreasing sequence.

Case 2. Suppose there are only finitely many dominant terms. Select n_1 so that s_{n_1} is beyond all the dominant terms of the sequence. Then

$$\text{given } N \geq n_1 \text{ there exists } m > N \text{ such that} \quad s_m \geq s_N. \tag{2}$$

Applying (2) with $N = n_1$ we select $n_2 > n_1$ such that $s_{n_2} \geq s_{n_1}$. Suppose $n_1, n_2, \ldots, n_{k-1}$ have been selected so that

$$n_1 < n_2 < \cdots < n_{k-1} \tag{3}$$

and

$$s_{n_1} \le s_{n_2} \le \cdots \le s_{n_{k-1}}. \tag{4}$$

Applying (2) with $N = n_{k-1}$ we select $n_k > n_{k-1}$ such that $s_{n_k} \ge s_{n_{k-1}}$. Then (3) and (4) hold with k in place of $k-1$, the procedure continues by induction, and we obtain an increasing subsequence (s_{n_k}). ■

The elegant proof of Theorem 11.4 was brought to our attention by David M. Bloom and is based on a solution in D. J. Newman's beautiful book [48].

11.5 Bolzano-Weierstrass Theorem.
Every bounded sequence has a convergent subsequence.

Proof
If (s_n) is a bounded sequence, it has a monotonic subsequence by Theorem 11.4, which converges by Theorem 10.2. ■

The Bolzano-Weierstrass theorem is very important and will be used at critical points in Chap. 3. Our proof, based on Theorem 11.4, is somewhat nonstandard for reasons we now discuss. Many of the notions introduced in this chapter make equally good sense in more general settings. For example, the ideas of convergent sequence, Cauchy sequence and bounded sequence all make sense for a sequence $(\boldsymbol{s}_n)$ where each $\boldsymbol{s}_n$ belongs to the plane. But the idea of a monotonic sequence does not carry over. It turns out that the Bolzano-Weierstrass theorem also holds in the plane and in many other settings [see Theorem 13.5], but clearly it would no longer be possible to prove it directly from an analogue of Theorem 11.4. Since the Bolzano-Weierstrass Theorem 11.5 generalizes to settings where Theorem 11.4 makes little sense, in applications we will emphasize the Bolzano-Weierstrass Theorem 11.5 rather than Theorem 11.4.

We need one more notion, and then we will be able to tie our various concepts together in Theorem 11.8.

11.6 Definition.
Let (s_n) be a sequence in $\mathbb{R}$. A *subsequential limit* is any real number or symbol $+\infty$ or $-\infty$ that is the limit of some subsequence of (s_n).

When a sequence has a limit s, then all subsequences have limit s, so $\{s\}$ is the set of subsequential limits. The interesting case is when the original sequence does not have a limit. We return to some of the examples discussed after Definition 11.1.

Example 5
Consider (s_n) where $s_n = n^2(-1)^n$. The subsequence of even terms diverges to $+\infty$, and the subsequence of odd terms diverges to $-\infty$. All subsequences that have a limit diverge to $+\infty$ or $-\infty$, so that $\{-\infty, +\infty\}$ is exactly the set of subsequential limits of (s_n). □

Example 6
Consider the sequence $a_n = \sin(\frac{n\pi}{3})$ in Example 2. This sequence takes each of the values $\frac{1}{2}\sqrt{3}$, 0 and $-\frac{1}{2}\sqrt{3}$ an infinite number of times. The only convergent subsequences are constant from some term on, and $\{-\frac{1}{2}\sqrt{3}, 0, \frac{1}{2}\sqrt{3}\}$ is the set of subsequential limits of (a_n). If $n_k = 3k$, then $a_{n_k} = 0$ for all $k \in \mathbb{N}$ and obviously $\lim_{k\to\infty} a_{n_k} = 0$. If $n_k = 6k+1$, then $a_{n_k} = \frac{1}{2}\sqrt{3}$ for all k and $\lim_{k\to\infty} a_{n_k} = \frac{1}{2}\sqrt{3}$. And if $n_k = 6k+4$, then $\lim_{k\to\infty} a_{n_k} = -\frac{1}{2}\sqrt{3}$. □

Example 7
Let (r_n) be a list of all rational numbers. It was shown in Example 3 that every real number is a subsequential limit of (r_n). Also, $+\infty$ and $-\infty$ are subsequential limits; see Exercise 11.7. Consequently, $\mathbb{R} \cup \{-\infty, +\infty\}$ is the set of subsequential limits of (r_n). □

Example 8
Let $b_n = n[1 + (-1)^n]$ for $n \in \mathbb{N}$. Then $b_n = 2n$ for even n and $b_n = 0$ for odd n. Thus the 2-element set $\{0, +\infty\}$ is the set of subsequential limits of (b_n). □

We now turn to the connection between subsequential limits and lim sup's and lim inf's.

11.7 Theorem.
Let (s_n) be any sequence. There exists a monotonic subsequence whose limit is $\limsup s_n$, and there exists a monotonic subsequence whose limit is $\liminf s_n$.

Proof
If (s_n) is not bounded above, then by Theorem 11.2(ii), a monotonic subsequence of (s_n) has limit $+\infty = \limsup s_n$. Similarly, if (s_n) is not bounded below, a monotonic subsequence has limit $-\infty = \liminf s_n$.

The remaining cases are that (s_n) is bounded above or is bounded below. These cases are similar, so we only consider the case that (s_n) is bounded above, so that $\limsup s_n$ is finite. Let $t = \limsup s_n$, and consider $\epsilon > 0$. There exists N_0 so that

$$\sup\{s_n : n > N\} < t + \epsilon \quad \text{for} \quad N \geq N_0.$$

In particular, $s_n < t + \epsilon$ for all $n > N_0$. We now claim

$$\{n \in \mathbb{N} : t - \epsilon < s_n < t + \epsilon\} \quad \text{is infinite.} \tag{1}$$

Otherwise, there exists $N_1 > N_0$ so that $s_n \leq t - \epsilon$ for $n > N_1$. Then $\sup\{s_n : n > N\} \leq t - \epsilon$ for $N \geq N_1$, so that $\limsup s_n < t$, a contradiction. Since (1) holds for each $\epsilon > 0$, Theorem 11.2(i) shows that a monotonic subsequence of (s_n) converges to $t = \limsup s_n$. ■

11.8 Theorem.
Let (s_n) be any sequence in $\mathbb{R}$, and let S denote the set of subsequential limits of (s_n).

- **(i)** *S is nonempty.*
- **(ii)** $\sup S = \limsup s_n$ *and* $\inf S = \liminf s_n$.
- **(iii)** $\lim s_n$ *exists if and only if S has exactly one element, namely* $\lim s_n$.

Proof
(i) is an immediate consequence of Theorem 11.7.

To prove (ii), consider any limit t of a subsequence (s_{n_k}) of (s_n). By Theorem 10.7 we have $t = \liminf_k s_{n_k} = \limsup_k s_{n_k}$. Since $n_k \geq k$ for all k, we have $\{s_{n_k} : k > N\} \subseteq \{s_n : n > N\}$ for each N in $\mathbb{N}$. Therefore

$$\liminf_n s_n \leq \liminf_k s_{n_k} = t = \limsup_k s_{n_k} \leq \limsup_n s_n.$$

This inequality holds for all t in S; therefore

$$\liminf s_n \leq \inf S \leq \sup S \leq \limsup s_n.$$

Theorem 11.7 shows that $\liminf s_n$ and $\limsup s_n$ both belong to S. Therefore (ii) holds.

Assertion (iii) is simply a reformulation of Theorem 10.7. ■

Theorems 11.7 and 11.8 show that $\limsup s_n$ is exactly the largest subsequential limit of (s_n), and $\liminf s_n$ is exactly the smallest subsequential limit of (s_n). This makes it easy to calculate lim sup's and lim inf's.

We return to the examples given before Theorem 11.7.

Example 9

If $s_n = n^2(-1)^n$, then $S = \{-\infty, +\infty\}$ as noted in Example 5. Therefore $\limsup s_n = \sup S = +\infty$ and $\liminf s_n = \inf S = -\infty$. □

Example 10

If $a_n = \sin(\frac{n\pi}{3})$, then $S = \{-\frac{1}{2}\sqrt{3}, 0, \frac{1}{2}\sqrt{3}\}$ as observed in Example 6. Hence $\limsup a_n = \sup S = \frac{1}{2}\sqrt{3}$ and $\liminf a_n = \inf S = -\frac{1}{2}\sqrt{3}$. □

Example 11

If (r_n) denotes a list of all rational numbers, then the set $\mathbb{R} \cup \{-\infty, +\infty\}$ is the set of subsequential limits of (r_n). Consequently we have $\limsup r_n = +\infty$ and $\liminf r_n = -\infty$. □

Example 12

If $b_n = n[1 + (-1)^n]$, then $\limsup b_n = +\infty$ and $\liminf b_n = 0$; see Example 8. □

The next result shows that the set S of subsequential limits always contains all limits of sequences *from* S. Such sets are called *closed sets*. Sets of this sort will be discussed further in the enrichment §13.

11.9 Theorem.

Let S denote the set of subsequential limits of a sequence (s_n). Suppose (t_n) is a sequence in $S \cap \mathbb{R}$ and that $t = \lim t_n$. Then t belongs to S.

Proof
Suppose t is finite. Consider the interval $(t-\epsilon, t+\epsilon)$. Then some t_n is in this interval. Let $\delta = \min\{t+\epsilon-t_n, t_n-t+\epsilon\}$, so that

$$(t_n-\delta, t_n+\delta) \subseteq (t-\epsilon, t+\epsilon).$$

Since t_n is a subsequential limit, the set $\{n \in \mathbb{N} : s_n \in (t_n-\delta, t_n+\delta)\}$ is infinite, so the set $\{n \in \mathbb{N} : s_n \in (t-\epsilon, t+\epsilon)\}$ is also infinite. Thus, by Theorem 11.2(i), t itself is a subsequential limit of (s_n).

If $t = +\infty$, then clearly the sequence (s_n) is unbounded above, so a subsequence of (s_n) has limit $+\infty$ by Theorem 11.2(ii). Thus $+\infty$ is also in S. A similar argument applies if $t = -\infty$. ■

Exercises

11.1 Let $a_n = 3 + 2(-1)^n$ for $n \in \mathbb{N}$.

- **(a)** List the first eight terms of the sequence (a_n).
- **(b)** Give a subsequence that is constant [takes a single value]. Specify the selection function σ.

11.2 Consider the sequences defined as follows:

$$a_n = (-1)^n, \qquad b_n = \frac{1}{n}, \qquad c_n = n^2, \qquad d_n = \frac{6n+4}{7n-3}.$$

- **(a)** For each sequence, give an example of a monotone subsequence.
- **(b)** For each sequence, give its set of subsequential limits.
- **(c)** For each sequence, give its lim sup and lim inf.
- **(d)** Which of the sequences converges? diverges to $+\infty$? diverges to $-\infty$?
- **(e)** Which of the sequences is bounded?

11.3 Repeat Exercise 11.2 for the sequences:

$$s_n = \cos(\frac{n\pi}{3}), \quad t_n = \frac{3}{4n+1}, \quad u_n = \left(-\frac{1}{2}\right)^n, \quad v_n = (-1)^n + \frac{1}{n}.$$

11.4 Repeat Exercise 11.2 for the sequences:

$$w_n = (-2)^n, \quad x_n = 5^{(-1)^n}, \quad y_n = 1 + (-1)^n, \quad z_n = n\cos\left(\frac{n\pi}{4}\right).$$

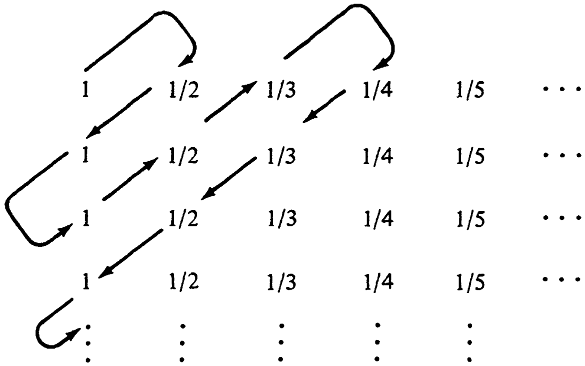

FIGURE 11.2

11.5 Let (q_n) be an enumeration of all the rationals in the interval $(0, 1]$.

(a) Give the set of subsequential limits for (q_n).

(b) Give the values of $\limsup q_n$ and $\liminf q_n$.

11.6 Show every subsequence of a subsequence of a given sequence is itself a subsequence of the given sequence. *Hint*: Define subsequences as in (3) of Definition 11.1.

11.7 Let (r_n) be an enumeration of the set $\mathbb{Q}$ of all rational numbers. Show there exists a subsequence (r_{n_k}) such that $\lim_{k\to\infty} r_{n_k} = +\infty$.

11.8 ★[8] Use Definition 10.6 and Exercise 5.4 to prove $\liminf s_n = -\limsup(-s_n)$ for every sequence (s_n).

11.9 **(a)** Show the closed interval $[a, b]$ is a closed set.

(b) Is there a sequence (s_n) such that $(0, 1)$ is its set of subsequential limits?

11.10 Let (s_n) be the sequence of numbers in Fig. 11.2 listed in the indicated order.

(a) Find the set S of subsequential limits of (s_n).

(b) Determine $\limsup s_n$ and $\liminf s_n$.

[8] This exercise is referred to in several places.

11.11 Let S be a bounded set. Prove there is an increasing sequence (s_n) of points in S such that $\lim s_n = \sup S$. Compare Exercise 10.7. *Note*: If $\sup S$ is in S, it's sufficient to define $s_n = \sup S$ for all n.

§12 lim sup's and lim inf's

Let (s_n) be any sequence of real numbers, and let S be the set of subsequential limits of (s_n). Recall

$$\limsup s_n = \lim_{N\to\infty} \sup\{s_n : n > N\} = \sup S \tag{*}$$

and

$$\liminf s_n = \lim_{N\to\infty} \inf\{s_n : n > N\} = \inf S. \tag{**}$$

The first equalities in (*) and (**) are from Definition 10.6, and the second equalities are proved in Theorem 11.8. This section is designed to increase the students' familiarity with these concepts. Most of the material is given in the exercises. We illustrate the techniques by proving some results that will be needed later in the text.

12.1 Theorem.
If (s_n) converges to a positive real number s and (t_n) is any sequence, then

$$\limsup s_n t_n = s \cdot \limsup t_n.$$

Here we allow the conventions $s \cdot (+\infty) = +\infty$ and $s \cdot (-\infty) = -\infty$ for $s > 0$.

Proof
We first show

$$\limsup s_n t_n \geq s \cdot \limsup t_n. \tag{1}$$

We have three cases. Let $\beta = \limsup t_n$.

Case 1. Suppose β is finite.

By Theorem 11.7, there exists a subsequence (t_{n_k}) of (t_n) such that $\lim_{k\to\infty} t_{n_k} = \beta$. We also have $\lim_{k\to\infty} s_{n_k} = s$ [by Theorem 11.3], so $\lim_{k\to\infty} s_{n_k} t_{n_k} = s\beta$. Thus $(s_{n_k} t_{n_k})$ is a subsequence of $(s_n t_n)$ converging to $s\beta$, and therefore $s\beta \leq \limsup s_n t_n$. [Recall that

$\limsup s_n t_n$ is the largest possible limit of a subsequence of $(s_n t_n)$.] Thus (1) holds.

Case 2. Suppose $\beta = +\infty$.

There exists a subsequence (t_{n_k}) of (t_n) such that $\lim_{k\to\infty} t_{n_k} = +\infty$. Since $\lim_{k\to\infty} s_{n_k} = s > 0$, Theorem 9.9 shows $\lim_{k\to\infty} s_{n_k} t_{n_k} = +\infty$. Hence $\limsup s_n t_n = +\infty$, so (1) clearly holds.

Case 3. Suppose $\beta = -\infty$.

Since $s > 0$, the right-hand side of (1) is equal to $s \cdot (-\infty) = -\infty$. Hence (1) is obvious in this case.

We have now established (1) in all cases. For the reversed inequality, we resort to a little trick. First note that we may ignore the first few terms of (s_n) and assume all $s_n \neq 0$. Then we can write $\lim \frac{1}{s_n} = \frac{1}{s}$ by Lemma 9.5. Now we apply (1) with s_n replaced by $\frac{1}{s_n}$ and t_n replaced by $s_n t_n$:

$$\limsup t_n = \limsup \left(\frac{1}{s_n}\right)(s_n t_n) \geq \left(\frac{1}{s}\right) \limsup s_n t_n,$$

i.e.,

$$\limsup s_n t_n \leq s \cdot \limsup t_n.$$

This inequality and (1) prove the theorem. ■

Example 1

The hypothesis s be positive in Theorem 12.1 cannot be relaxed to allow $s = 0$. To see this, consider (s_n) and (t_n), where $s_n = -\frac{1}{n}$ and $t_n = -n^2$ for all n. In this setting, we don't define $0 \cdot (-\infty)$, but even if we did, we wouldn't define this product to be $+\infty$. □

The next theorem will be useful in dealing with infinite series; see the proof of the Ratio Test 14.8.

12.2 Theorem.

Let (s_n) be any sequence of nonzero real numbers. Then we have

$$\liminf \left|\frac{s_{n+1}}{s_n}\right| \leq \liminf |s_n|^{1/n} \leq \limsup |s_n|^{1/n} \leq \limsup \left|\frac{s_{n+1}}{s_n}\right|.$$

Proof

The middle inequality is obvious. The first and third inequalities have similar proofs. We will prove the third inequality and leave the first inequality to Exercise 12.11.

Let $\alpha = \limsup |s_n|^{1/n}$ and $L = \limsup |\frac{s_{n+1}}{s_n}|$. We need to prove $\alpha \le L$. This is obvious if $L = +\infty$, so we assume $L < +\infty$. To prove $\alpha \le L$ it suffices to show

$$\alpha \le L_1 \quad \text{for any} \quad L_1 > L. \tag{1}$$

Since

$$L = \limsup \left|\frac{s_{n+1}}{s_n}\right| = \lim_{N\to\infty} \sup\left\{\left|\frac{s_{n+1}}{s_n}\right| : n > N\right\} < L_1,$$

there exists a positive integer N such that

$$\sup\left\{\left|\frac{s_{n+1}}{s_n}\right| : n \ge N\right\} < L_1.$$

Thus

$$\left|\frac{s_{n+1}}{s_n}\right| < L_1 \quad \text{for} \quad n \ge N. \tag{2}$$

Now for $n > N$ we can write

$$|s_n| = \left|\frac{s_n}{s_{n-1}}\right| \cdot \left|\frac{s_{n-1}}{s_{n-2}}\right| \cdots \left|\frac{s_{N+1}}{s_N}\right| \cdot |s_N|.$$

There are $n - (N + 1) + 1 = n - N$ fractions here, so applying (2) we see that

$$|s_n| < L_1^{n-N}|s_N| \quad \text{for} \quad n > N.$$

Since L_1 and N are fixed in this argument, $a = L_1^{-N}|s_N|$ is a positive constant and we may write

$$|s_n| < L_1^n a \quad \text{for} \quad n > N.$$

Therefore we have

$$|s_n|^{1/n} < L_1 a^{1/n} \quad \text{for} \quad n > N.$$

Since $\lim_{n\to\infty} a^{1/n} = 1$ by Theorem 9.7(d) on page 48, we conclude $\alpha = \limsup |s_n|^{1/n} \le L_1$; see Exercise 12.1. Consequently (1) holds as desired. ■

12.3 Corollary.
If $\lim |\frac{s_{n+1}}{s_n}|$ *exists* [*and equals* L], *then* $\lim |s_n|^{1/n}$ *exists* [*and equals* L].

Proof
If $\lim |\frac{s_{n+1}}{s_n}| = L$, then all four values in Theorem 12.2 are equal to L. Hence $\lim |s_n|^{1/n} = L$; see Theorem 10.7. ■

Example 2

(a) If $L = \limsup s_n \neq \infty$, then for every $\alpha > L$, the set $\{n : s_n > \alpha\}$ is finite. If $L \neq -\infty$, then for every $\beta < L$, the set $\{n : s_n > \beta\}$ is infinite.

If $L \neq \infty$ and $\alpha > L$, the set $\{n : s_n > \alpha\}$ is finite; otherwise $\sup\{s_n : n > N\} > \alpha$ for all N and hence $L = \limsup s_n \geq \alpha > L$, a contradiction; see Definition 10.6. If $L \neq -\infty$ and $\beta < L$, the set $\{n : s_n > \beta\}$ is infinite; otherwise there exists a positive integer N_0 so that $s_n \leq \beta$ for all $n \geq N_0$, and therefore $\sup\{s_n : n > N\} \leq \beta$ for all $N \geq N_0$. Then $\limsup s_n \leq \beta < L$, a contradiction.

(b) The set $\{n : s_n > \limsup s_n\}$ can be infinite. For example, consider (s_n) where $s_n = \frac{1}{n}$. The set $\{n : s_n < \liminf s_n\}$ can also be infinite; use $s_n = -\frac{1}{n}$.

(c) If $L_0 = \liminf t_n \neq -\infty$, then the set $\{n : t_n < \beta_0\}$ is finite for $\beta_0 < L_0$. If $L_0 \neq \infty$, then the set $\{n : t_n < \alpha_0\}$ is infinite for $\alpha_0 > L_0$.

This follows from part (a) and Exercise 11.8:

$$L_0 = \liminf t_n = -\limsup(-t_n) = -L = -\limsup s_n,$$

where $s_n = -t_n$ and L is as defined in part (a). Now $\beta_0 < L_0$ implies $-\beta_0 > -L_0 = L$, so by part (a),

$$\{n : t_n < \beta_0\} = \{n : -t_n > -\beta_0\} = \{n : s_n > -\beta_0\} \quad \text{is finite.}$$

Similarly, $\alpha_0 > L_0$ implies $-\alpha_0 < L$, so

$$\{n : t_n < \alpha_0\} = \{n : -t_n > -\alpha_0\} = \{n : s_n > -\alpha_0\} \quad \text{is infinite.}$$

(d) If $\liminf s_n < \limsup s_n$, the set

$$\{n : \liminf s_n \leq s_n \leq \limsup s_n\}$$

can be empty. Use, for example, $s_n = (-1)^n(1 + \frac{1}{n})$. □

Exercises

12.1 Let (s_n) and (t_n) be sequences and suppose there exists N_0 such that $s_n \le t_n$ for all $n > N_0$. Show $\liminf s_n \le \liminf t_n$ and $\limsup s_n \le \limsup t_n$. *Hint*: Use Definition 10.6 and Exercise 9.9(c).

12.2 Prove $\limsup |s_n| = 0$ if and only if $\lim s_n = 0$.

12.3 Let (s_n) and (t_n) be the following sequences that repeat in cycles of four:

$$(s_n) = (0, 1, 2, 1, 0, 1, 2, 1, 0, 1, 2, 1, 0, 1, 2, 1, 0, \ldots)$$
$$(t_n) = (2, 1, 1, 0, 2, 1, 1, 0, 2, 1, 1, 0, 2, 1, 1, 0, 2, \ldots)$$

Find
(a) $\liminf s_n + \liminf t_n$, **(b)** $\liminf(s_n + t_n)$,
(c) $\liminf s_n + \limsup t_n$, **(d)** $\limsup(s_n + t_n)$,
(e) $\limsup s_n + \limsup t_n$, **(f)** $\liminf(s_n t_n)$,
(g) $\limsup(s_n t_n)$

12.4 Show $\limsup(s_n + t_n) \le \limsup s_n + \limsup t_n$ for bounded sequences (s_n) and (t_n). *Hint*: First show

$$\sup\{s_n + t_n : n > N\} \le \sup\{s_n : n > N\} + \sup\{t_n : n > N\}.$$

Then apply Exercise 9.9(c).

12.5 Use Exercises 11.8 and 12.4 to prove

$$\liminf(s_n + t_n) \ge \liminf s_n + \liminf t_n$$

for bounded sequences (s_n) and (t_n).

12.6 Let (s_n) be a bounded sequence, and let k be a nonnegative real number.

(a) Prove $\limsup(ks_n) = k \cdot \limsup s_n$.

(b) Do the same for $\liminf$. *Hint*: Use Exercise 11.8.

(c) What happens in (a) and (b) if $k < 0$?

12.7 Prove if $\limsup s_n = +\infty$ and $k > 0$, then $\limsup(ks_n) = +\infty$.

12.8 Let (s_n) and (t_n) be bounded sequences of nonnegative numbers. Prove $\limsup s_n t_n \le (\limsup s_n)(\limsup t_n)$.

12.9 **(a)** Prove that if $\lim s_n = +\infty$ and $\liminf t_n > 0$, then $\lim s_n t_n = +\infty$.

(b) Prove that if $\limsup s_n = +\infty$ and $\liminf t_n > 0$, then $\limsup s_n t_n = +\infty$.

(c) Observe that Exercise 12.7 is the special case of (b) where $t_n = k$ for all $n \in \mathbb{N}$.

12.10 Prove (s_n) is bounded if and only if $\limsup |s_n| < +\infty$.

12.11 Prove the first inequality in Theorem 12.2.

12.12 Let (s_n) be a sequence of nonnegative numbers, and for each n define $\sigma_n = \frac{1}{n}(s_1 + s_2 + \cdots + s_n)$.

(a) Show

$$\liminf s_n \le \liminf \sigma_n \le \limsup \sigma_n \le \limsup s_n.$$

Hint: For the last inequality, show first that $M > N$ implies

$$\sup\{\sigma_n : n > M\} \le \frac{1}{M}(s_1 + s_2 + \cdots + s_N) + \sup\{s_n : n > N\}.$$

(b) Show that if $\lim s_n$ exists, then $\lim \sigma_n$ exists and $\lim \sigma_n = \lim s_n$.

(c) Give an example where $\lim \sigma_n$ exists, but $\lim s_n$ does not exist.

12.13 Let (s_n) be a bounded sequence in $\mathbb{R}$. Let A be the set of $a \in \mathbb{R}$ such that $\{n \in \mathbb{N} : s_n < a\}$ is finite, i.e., all but finitely many s_n are $\ge a$. Let B be the set of $b \in \mathbb{R}$ such that $\{n \in \mathbb{N} : s_n > b\}$ is finite. Prove $\sup A = \liminf s_n$ and $\inf B = \limsup s_n$.

12.14 Calculate (a) $\lim (n!)^{1/n}$, (b) $\lim \frac{1}{n}(n!)^{1/n}$.

§13 * Some Topological Concepts in Metric Spaces

In this book we are restricting our attention to analysis on $\mathbb{R}$. Accordingly, we have taken full advantage of the order properties of $\mathbb{R}$ and studied such important notions as lim sup's and lim inf's. In §3 we briefly introduced a distance function on $\mathbb{R}$. Most of our analysis could have been based on the notion of distance, in which case it becomes easy and natural to work in a more general setting. For example, analysis on the k-dimensional Euclidean spaces $\mathbb{R}^k$ is important, but these spaces do not have the useful natural ordering that $\mathbb{R}$ has, unless of course $k = 1$.

13.1 Definition.
Let S be a set, and suppose d is a function defined for all pairs (x, y) of elements from S satisfying

D1. $d(x, x) = 0$ for all $x \in S$ and $d(x, y) > 0$ for distinct x, y in S.
D2. $d(x, y) = d(y, x)$ for all $x, y \in S$.
D3. $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in S$.

Such a function d is called a *distance function* or a *metric* on S. A *metric space* S is a set S together with a metric on it. Properly speaking, the metric space is the pair (S, d) since a set S may well have more than one metric on it; see Exercise 13.1.

Example 1
As in Definition 3.4, let $\text{dist}(a, b) = |a - b|$ for $a, b \in \mathbb{R}$. Then dist is a metric on $\mathbb{R}$. Note Corollary 3.6 gives D3 in this case. As remarked there, the inequality

$$\text{dist}(a, c) \leq \text{dist}(a, b) + \text{dist}(b, c)$$

is called the triangle inequality. In fact, for any metric d, property D3 is called the *triangle inequality*. □

Example 2
The space of all k-tuples

$$\boldsymbol{x} = (x_1, x_2, \ldots, x_k) \quad \text{where} \quad x_j \in \mathbb{R} \quad \text{for} \quad j = 1, 2, \ldots, k,$$

is called *k-dimensional Euclidean space* and written $\mathbb{R}^k$. As noted in Exercise 13.1, $\mathbb{R}^k$ has several metrics on it. The most familiar metric is the one that gives the ordinary distance in the plane $\mathbb{R}^2$ or in 3-space $\mathbb{R}^3$:

$$d(\boldsymbol{x}, \boldsymbol{y}) = \left[\sum_{j=1}^{k} (x_j - y_j)^2\right]^{1/2} \qquad \text{for} \quad \boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^k.$$

[The summation notation $\sum$ is explained in 14.1.] Obviously this function d satisfies properties D1 and D2. The triangle inequality D3 is not so obvious. For a proof, see for example [53, §6.1] or [62, 1.37]. □

13.2 Definition.
A sequence (s_n) in a metric space (S,d) *converges to* s in S if $\lim_{n\to\infty} d(s_n,s) = 0$. A sequence (s_n) in S is a *Cauchy sequence* if for each $\epsilon > 0$ there exists an N such that

$$m,n > N \quad \text{implies} \quad d(s_m,s_n) < \epsilon.$$

The metric space (S,d) is said to be *complete* if every Cauchy sequence in S converges to some element in S.

Since the Completeness Axiom 4.4 deals with least upper bounds, the word "complete" now appears to have two meanings. However, these two uses of the term are very closely related and both reflect the property that the space is complete, i.e., has no gaps. Theorem 10.11 asserts that the metric space $(\mathbb{R}, \text{dist})$ is a complete metric space, and the proof uses the Completeness Axiom 4.4. We could just as well have taken as an axiom the completeness of $(\mathbb{R}, \text{dist})$ as a metric space and proved the least upper bound property in 4.4 as a theorem. We did not do so because the concept of least upper bound in $\mathbb{R}$ seems to us more fundamental than the concept of Cauchy sequence.

We will prove $\mathbb{R}^k$ is complete. But we have a notational problem, since we like subscripts for sequences and for coordinates of points in $\mathbb{R}^k$. When there is a conflict, we will write $(\boldsymbol{x}^{(n)})$ for a sequence instead of $(\boldsymbol{x}_n)$. In this case,

$$\boldsymbol{x}^{(n)} = (x_1^{(n)}, x_2^{(n)}, \ldots, x_k^{(n)}).$$

Unless otherwise specified, *the metric in* $\mathbb{R}^k$ *is always as given in* Example 2.

13.3 Lemma.
A sequence $(\boldsymbol{x}^{(n)})$ *in* $\mathbb{R}^k$ *converges if and only if for each* $j = 1,2,\ldots,k$*, the sequence* $(x_j^{(n)})$ *converges in* $\mathbb{R}$*. A sequence* $(\boldsymbol{x}^{(n)})$ *in* $\mathbb{R}^k$ *is a Cauchy sequence if and only if each sequence* $(x_j^{(n)})$ *is a Cauchy sequence in* $\mathbb{R}$*.*

Proof
The proof of the first assertion is left to Exercise 13.2(b). For the second assertion, we first observe for $\boldsymbol{x}$, $\boldsymbol{y}$ in $\mathbb{R}^k$ and $j = 1,2,\ldots,k$,

$$|x_j - y_j| \le d(\boldsymbol{x},\boldsymbol{y}) \le \sqrt{k}\max\{|x_j - y_j| : j = 1,2,\ldots,k\}. \tag{1}$$

Suppose $(\boldsymbol{x}^{(n)})$ is a Cauchy sequence in $\mathbb{R}^k$, and consider fixed j. If $\epsilon > 0$, there exists N such that

$$m, n > N \quad \text{implies} \quad d(\boldsymbol{x}^{(m)}, \boldsymbol{x}^{(n)}) < \epsilon.$$

From the first inequality in (1) we see that

$$m, n > N \quad \text{implies} \quad |x_j^{(m)} - x_j^{(n)}| < \epsilon,$$

so $(x_j^{(n)})$ is a Cauchy sequence in $\mathbb{R}$.

Now suppose each sequence $(x_j^{(n)})$ is a Cauchy sequence in $\mathbb{R}$. Let $\epsilon > 0$. For $j = 1, 2, \ldots, k$, there exist N_j such that

$$m, n > N_j \quad \text{implies} \quad |x_j^{(m)} - x_j^{(n)}| < \frac{\epsilon}{\sqrt{k}}.$$

If $N = \max\{N_1, N_2, \ldots, N_k\}$, then by the second inequality in (1),

$$m, n > N \quad \text{implies} \quad d(\boldsymbol{x}^{(m)}, \boldsymbol{x}^{(n)}) < \epsilon,$$

i.e., $(\boldsymbol{x}^{(n)})$ is a Cauchy sequence in $\mathbb{R}^k$. ■

13.4 Theorem.
Euclidean k-space $\mathbb{R}^k$ is complete.

Proof
Consider a Cauchy sequence $(\boldsymbol{x}^{(n)})$ in $\mathbb{R}^k$. By Lemma 13.3, $(x_j^{(n)})$ is a Cauchy sequence in $\mathbb{R}$ for each j. Hence by Theorem 10.11, $(x_j^{(n)})$ converges to some real number x_j. By Lemma 13.3 again, the sequence $(\boldsymbol{x}^{(n)})$ converges, in fact to $\boldsymbol{x} = (x_1, x_2, \ldots, x_k)$. ■

We now prove the Bolzano-Weierstrass theorem for $\mathbb{R}^k$; compare Theorem 11.5. A set S in $\mathbb{R}^k$ is *bounded* if there exists $M > 0$ such that

$$\max\{|x_j| : j = 1, 2, \ldots, k\} \leq M \quad \text{for all} \quad \boldsymbol{x} = (x_1, x_2, \ldots, x_k) \in S.$$

13.5 Bolzano-Weierstrass Theorem.
Every bounded sequence in $\mathbb{R}^k$ has a convergent subsequence.

Proof
Let $(\boldsymbol{x}^{(n)})$ be a bounded sequence in $\mathbb{R}^k$. Then each sequence $(x_j^{(n)})$ is bounded in $\mathbb{R}$. By Theorem 11.5, we may replace $(\boldsymbol{x}^{(n)})$ by a

subsequence such that $(x_1^{(n)})$ converges. By the same theorem, we may replace $(\boldsymbol{x}^{(n)})$ by a subsequence of the subsequence such that $(x_2^{(n)})$ converges. Of course, $(x_1^{(n)})$ still converges by Theorem 11.3. Repeating this argument k times, we obtain a sequence $(\boldsymbol{x}^{(n)})$ so that each sequence $(x_j^{(n)})$ converges, $j = 1, 2, \ldots, k$. This sequence represents a subsequence of the original sequence, and it converges in $\mathbb{R}^k$ by Lemma 13.3. ■

13.6 Definition.
Let (S, d) be a metric space. Let E be a subset of S. An element $s_0 \in E$ is *interior to* E if for some $r > 0$ we have

$$\{s \in S : d(s, s_0) < r\} \subseteq E.$$

We write E° for the set of points in E that are interior to E. The set E is *open in* S if every point in E is interior to E, i.e., if $E = E^\circ$.

13.7 Discussion.
One can show [Exercise 13.4]

- **(i)** S is open in S [trivial].
- **(ii)** The empty set $\varnothing$ is open in S [trivial].
- **(iii)** The union of *any* collection of open sets is open.
- **(iv)** The intersection of *finitely many* open sets is again an open set. □

Our study of $\mathbb{R}^k$ and the exercises suggest that metric spaces are fairly general and useful objects. When one is interested in convergence of certain objects [such as points or functions], there is often a metric that assists in the study of the convergence. But sometimes no metric will work and yet there is still some sort of convergence notion. Frequently the appropriate vehicle is what is called a *topology*. This is a set S for which certain subsets are decreed to be *open sets*. In general, all that is required is that the family of open sets satisfies (i)–(iv) above. In particular, the open sets defined by a metric form a topology. We will not pursue this abstract theory. However, because of this abstract theory, concepts that can be defined in terms of open sets [see Definitions 13.8, 13.11, and 22.1] are called *topological*, hence the title of this section.

13.8 Definition.
Let (S,d) be a metric space. A subset E of S is *closed* if its complement $S \setminus E$ is an open set. In other words, E is closed if $E = S \setminus U$ where U is an open set.

Because of (iii) in Discussion 13.7, the intersection of *any* collection of closed sets in closed [Exercise 13.5]. The *closure* E^- of a set E is the intersection of all closed sets containing E. The *boundary* of E is the set $E^- \setminus E^\circ$; points in this set are called *boundary points of* E.

To get a feel for these notions, we state some easy facts and leave the proofs as exercises.

13.9 Proposition.
Let E be a subset of a metric space (S,d).

- **(a)** *The set E is closed if and only if $E = E^-$.*
- **(b)** *The set E is closed if and only if it contains the limit of every convergent sequence of points in E.*
- **(c)** *An element is in E^- if and only if it is the limit of some sequence of points in E.*
- **(d)** *A point is in the boundary of E if and only if it belongs to the closure of both E and its complement.*

Example 3
In $\mathbb{R}$, open intervals (a,b) are open sets. Closed intervals $[a,b]$ are closed sets. The interior of $[a,b]$ is (a,b). The boundary of both (a,b) and $[a,b]$ is the two-element set $\{a,b\}$.

Every open set in $\mathbb{R}$ is the union of a disjoint sequence of open intervals [Exercise 13.7]. A closed set in $\mathbb{R}$ need not be the union of a disjoint sequence of closed intervals and points; such a set appears in Example 5.

No open interval (a,b) or closed interval $[a,b]$, with $a < b$, can be written as the disjoint union of two or more closed intervals, each having more than one point. This is proved in Theorem 21.11. $\square$

Example 4
In $\mathbb{R}^k$, open balls $\{\boldsymbol{x} : d(\boldsymbol{x}, \boldsymbol{x}_0) < r\}$ are open sets, and closed balls $\{\boldsymbol{x} : d(\boldsymbol{x}, \boldsymbol{x}_0) \leq r\}$ are closed sets. The boundary of each of these

sets is $\{\boldsymbol{x} : d(\boldsymbol{x}, \boldsymbol{x}_0) = r\}$. In the plane $\mathbb{R}^2$, the sets

$$\{(x_1, x_2) : x_1 > 0\} \quad \text{and} \quad \{(x_1, x_2) : x_1 > 0 \text{ and } x_2 > 0\}$$

are open. If $>$ is replaced by $\geq$, we obtain closed sets. Many sets are neither open nor closed. For example, $[0, 1)$ is neither open nor closed in $\mathbb{R}$, and $\{(x_1, x_2) : x_1 > 0 \text{ and } x_2 \geq 0\}$ is neither open nor closed in $\mathbb{R}^2$. □

13.10 Theorem.
Let (F_n) be a decreasing sequence [i.e., $F_1 \supseteq F_2 \supseteq \cdots$] of closed bounded nonempty sets in $\mathbb{R}^k$. Then $F = \cap_{n=1}^{\infty} F_n$ is also closed, bounded and nonempty.

Proof
Clearly F is closed and bounded. It is the nonemptiness that needs proving! For each n, select an element $\boldsymbol{x}_n$ in F_n. By the Bolzano-Weierstrass Theorem 13.5, a subsequence $(\boldsymbol{x}_{n_m})_{m=1}^{\infty}$ of $(\boldsymbol{x}_n)$ converges to some element $\boldsymbol{x}_0$ in $\mathbb{R}^k$. To show $\boldsymbol{x}_0 \in F$, it suffices to show $\boldsymbol{x}_0 \in F_{n_0}$ with n_0 fixed. If $m \geq n_0$, then $n_m \geq n_0$, so $\boldsymbol{x}_{n_m} \in F_{n_m} \subseteq F_{n_0}$. Hence the sequence $(\boldsymbol{x}_{n_m})_{m=n_0}^{\infty}$ consists of points in F_{n_0} and converges to $\boldsymbol{x}_0$. Thus $\boldsymbol{x}_0$ belongs to F_{n_0} by (b) of Proposition 13.9. ■

Example 5
Here is a famous nonempty closed set in $\mathbb{R}$ called the *Cantor set.* Pictorially, $F = \cap_{n=1}^{\infty} F_n$ where F_n are sketched in Fig. 13.1. The Cantor set has some remarkable properties. The sum of the lengths of the intervals comprising F_n is $(\frac{2}{3})^{n-1}$ and this tends to 0 as $n \to \infty$. Yet the intersection F is so large that it cannot be written as a sequence; in set-theoretic terms it is “uncountable.” The interior of F is the empty set, so F is equal to its boundary. For more details, see [62, 2.44], or [31, 6.62]. □

13.11 Definition.
Let (S, d) be a metric space. A family $\mathcal{U}$ of open sets is said to be an *open cover for a set* E if each point of E belongs to at least one set in $\mathcal{U}$, i.e.,

$$E \subseteq \bigcup \{U : U \in \mathcal{U}\}.$$

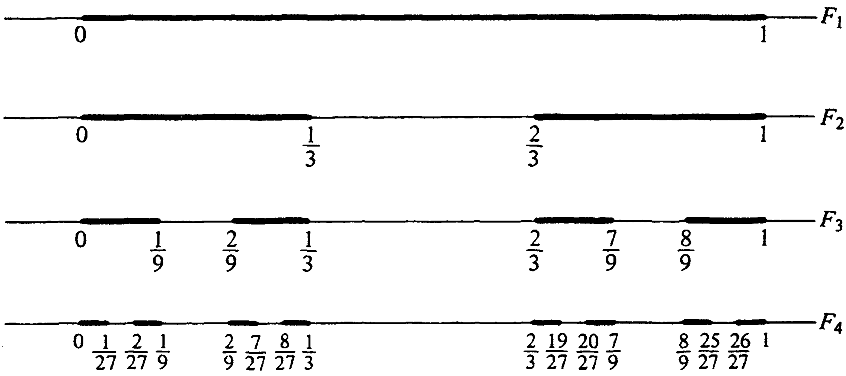

FIGURE 13.1

A *subcover* of $\mathcal{U}$ is any subfamily of $\mathcal{U}$ that also covers E. A cover or subcover is *finite* if it contains only finitely many sets; the sets themselves may be infinite.

A set E is *compact* if every open cover of E has a finite subcover of E.

This rather abstract definition is very important in advanced analysis; see, for example, [30]. In $\mathbb{R}^k$, compact sets are nicely characterized, as follows.

13.12 Heine-Borel Theorem.
A subset E of $\mathbb{R}^k$ is compact if and only if it is closed and bounded.

Proof
Suppose E is compact. For each $m \in \mathbb{N}$, let U_m consist of all $\boldsymbol{x}$ in $\mathbb{R}^k$ such that

$$\max\{|x_j| : j = 1, 2, \ldots, k\} < m.$$

The family $\mathcal{U} = \{U_m : m \in \mathbb{N}\}$ is an open cover of E [it covers $\mathbb{R}^k$!], so a finite subfamily of $\mathcal{U}$ covers E. If U_{m_0} is the largest member of the subfamily, then $E \subseteq U_{m_0}$. It follows that E is bounded. To show E is closed, consider any point $\boldsymbol{x}_0$ in $\mathbb{R}^k \setminus E$. For $m \in \mathbb{N}$, let

$$V_m = \left\{ \boldsymbol{x} \in \mathbb{R}^k : d(\boldsymbol{x}, \boldsymbol{x}_0) > \frac{1}{m} \right\}.$$

Then each V_m is open in $\mathbb{R}^k$ and $\mathcal{V} = \{V_m : m \in \mathbb{N}\}$ covers E since $\cup_{m=1}^{\infty} V_m = \mathbb{R}^k \setminus \{\boldsymbol{x}_0\}$. Since E can be covered by finitely many V_m, for some m_0 we have

$$E \subset \left\{ \boldsymbol{x} \in \mathbb{R}^k : d(\boldsymbol{x}, \boldsymbol{x}_0) > \frac{1}{m_0} \right\}.$$

Thus $\{\boldsymbol{x} \in \mathbb{R}^k : d(\boldsymbol{x}, \boldsymbol{x}_0) < \frac{1}{m_0}\} \subseteq \mathbb{R}^k \setminus E$, so that $\boldsymbol{x}_0$ is interior to $\mathbb{R}^k \setminus E$. Since $\boldsymbol{x}_0$ in $\mathbb{R}^k \setminus E$ was arbitrary, $\mathbb{R}^k \setminus E$ is an open set. Hence E is a closed set.

Now suppose E is closed and bounded. Since E is bounded, E is a subset of some set F having the form

$$F = \{\boldsymbol{x} \in \mathbb{R}^k : |x_j| \leq m \quad \text{for} \quad j = 1, 2, \ldots, k\}.$$

As noted in Exercise 13.12, it suffices to prove F is compact. We do so in the next proposition after some preparation. ■

The set F in the last proof is a k-*cell* because it has the following form. There exist closed intervals $[a_1, b_1]$, $[a_2, b_2]$, ..., $[a_k, b_k]$ so that

$$F = \{\boldsymbol{x} \in \mathbb{R}^k : x_j \in [a_j, b_j] \quad \text{for} \quad j = 1, 2, \ldots, k\},$$

which is sometimes written as

$$F = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_k, b_k],$$

so it is a k-dimensional box in $\mathbb{R}^k$. Thus a 2-cell in $\mathbb{R}^2$ is a closed rectangle. A 3-cell in $\mathbb{R}^3$ is called a "rectangular parallelepiped." The *diameter* of F is

$$\delta = \left[\sum_{j=1}^{k} (b_j - a_j)^2 \right]^{1/2};$$

that is, $\delta = \sup\{d(\boldsymbol{x}, \boldsymbol{y}) : \boldsymbol{x}, \boldsymbol{y} \in F\}$. Using midpoints $c_j = \frac{1}{2}(a_j + b_j)$ of $[a_j, b_j]$, we see that F is a union of 2^k k-cells each having diameter $\frac{\delta}{2}$. If this remark is not clear, consider first the cases $k = 2$ and $k = 3$.

13.13 Proposition.
Every k-cell F in $\mathbb{R}^k$ is compact.

Proof
Assume F is not compact. Then there exists an open cover $\mathcal{U}$ of F, no finite subfamily of which covers F. Let δ denote the diameter of F.

As noted above, F is a union of 2^k k-cells having diameter $\frac{\delta}{2}$. At least one of these 2^k k-cells, which we denote by F_1, cannot be covered by finitely many sets from $\mathcal{U}$. Likewise, F_1 contains a k-cell F_2 of diameter $\frac{\delta}{4}$ which cannot be covered by finitely many sets from $\mathcal{U}$. Continuing in this fashion, we obtain a sequence (F_n) of k-cells such that

$$F_1 \supseteq F_2 \supseteq F_3 \supseteq \cdots; \tag{1}$$

$$F_n \text{ has diameter } \delta \cdot 2^{-n}; \tag{2}$$

$$F_n \text{ cannot be covered by finitely many sets from } \mathcal{U}. \tag{3}$$

By Theorem 13.10, the intersection $\cap_{n=1}^{\infty} F_n$ contains a point $\boldsymbol{x}_0$. This point belongs to some set U_0 in $\mathcal{U}$. Since U_0 is open, there exists $r > 0$ so that

$$\{\boldsymbol{x} \in \mathbb{R}^k : d(\boldsymbol{x}, \boldsymbol{x}_0) < r\} \subseteq U_0.$$

It follows that $F_n \subseteq U_0$ provided $\delta \cdot 2^{-n} < r$, but this contradicts (3) in a dramatic way. ∎

Since $\mathbb{R} = \mathbb{R}^1$, the preceding results apply to $\mathbb{R}$.

Example 6

Let E be a nonempty subset of a metric space (S, d). Consider the function $d(E, x) = \inf\{d(y, x) : y \in E\}$ for $x \in S$. This function satisfies $|d(E, x_1) - d(E, x_2)| \leq d(x_1, x_2)$ for x_1, x_2 in S.

We show that if E is compact and if $E \subseteq U$ for some open subset U of S, then for some $\delta > 0$ we have

$$\{x \in S : d(E, x) < \delta\} \subseteq U. \tag{1}$$

For each $x \in E$, we have

$$\{y \in S : d(y, x) < r_x\} \subseteq U \quad \text{for some} \quad r_x > 0. \tag{2}$$

The open balls $\{y \in S : d(y, x) < r_x/2\}$ cover E, so a finite subfamily also covers E. I.e., there are $x_1, \ldots, x_n$ in E so that

$$E \subseteq \bigcup_{k=1}^{n} \left\{ y \in S : d(y, x_k) < \frac{r_k}{2} \right\},$$

where we write r_k for r_{x_k}. Now let $\delta = \frac{1}{2} \min\{r_1, \ldots, r_n\}$. To prove (1), consider $x \in S$ and suppose $d(E, x) < \delta$. Then for some

$y \in E$, we have $d(y, x) < \delta$. Moreover, $d(y, x_k) < \frac{r_k}{2}$ for some $k \in \{1, 2, \ldots, n\}$. Therefore, for this k we have

$$d(x, x_k) \le d(x, y) + d(y, x_k) < \delta + \frac{r_k}{2} \le \frac{r_k}{2} + \frac{r_k}{2} = r_k.$$

Thus, by (2) applied to $x = x_k$, we see that x belongs to U. Hence (1) holds. □

Exercises

13.1 For points $\boldsymbol{x}$, $\boldsymbol{y}$ in $\mathbb{R}^k$, let

$$d_1(\boldsymbol{x}, \boldsymbol{y}) = \max\{|x_j - y_j| : j = 1, 2, \ldots, k\}$$

and

$$d_2(\boldsymbol{x}, \boldsymbol{y}) = \sum_{j=1}^{k} |x_j - y_j|.$$

(a) Show d_1 and d_2 are metrics for $\mathbb{R}^k$.

(b) Show d_1 and d_2 are complete metrics on $\mathbb{R}^k$.

13.2 **(a)** Prove (1) in Lemma 13.3.

(b) Prove the first assertion in Lemma 13.3.

13.3 Let B be the set of all bounded sequences $\boldsymbol{x} = (x_1, x_2, \ldots)$, and define $d(\boldsymbol{x}, \boldsymbol{y}) = \sup\{|x_j - y_j| : j = 1, 2, \ldots\}$.

(a) Show d is a metric for B.

(b) Does $d^*(\boldsymbol{x}, \boldsymbol{y}) = \sum_{j=1}^{\infty} |x_j - y_j|$ define a metric for B?

13.4 Prove (iii) and (iv) in Discussion 13.7.

13.5 **(a)** Verify one of DeMorgan's Laws for sets:

$$\bigcap\{S \setminus U : U \in \mathcal{U}\} = S \setminus \bigcup\{U : U \in \mathcal{U}\}.$$

(b) Show that the intersection of any collection of closed sets is a closed set.

13.6 Prove Proposition 13.9.

13.7 Show that every open set in $\mathbb{R}$ is the disjoint union of a finite or infinite sequence of open intervals.

13.8 **(a)** Verify the assertions in the first paragraph of Example 3.

(b) Verify the assertions in Example 4.

13.9 Find the closures of the following sets:

(a) $\{\frac{1}{n} : n \in \mathbb{N}\}$,

(b) $\mathbb{Q}$, the set of rational numbers,

(c) $\{r \in \mathbb{Q} : r^2 < 2\}$.

13.10 Show that the interior of each of the following sets is the empty set.

(a) $\{\frac{1}{n} : n \in \mathbb{N}\}$,

(b) $\mathbb{Q}$, the set of rational numbers,

(c) The Cantor set in Example 5.

13.11 Let E be a subset of $\mathbb{R}^k$. Show that E is compact if and only if every sequence in E has a subsequence converging to a point *in* E.

13.12 Let (S, d) be any metric space.

(a) Show that if E is a closed subset of a compact set F, then E is also compact.

(b) Show that the finite union of compact sets in S is compact.

13.13 Let E be a compact nonempty subset of $\mathbb{R}$. Show $\sup E$ and $\inf E$ belong to E.

13.14 Let E be a compact nonempty subset of $\mathbb{R}^k$, and let $\delta = \sup\{d(\boldsymbol{x}, \boldsymbol{y}) : \boldsymbol{x}, \boldsymbol{y} \in E\}$. Show E contains points $\boldsymbol{x}_0$, $\boldsymbol{y}_0$ such that $d(\boldsymbol{x}_0, \boldsymbol{y}_0) = \delta$.

13.15 Let (B, d) be as in Exercise 13.3, and let F consist of all $\boldsymbol{x} \in B$ such that $\sup\{|x_j| : j = 1, 2, \ldots\} \leq 1$.

(a) Show F is closed and bounded. [A set F in a metric space (S, d) is *bounded* if there exist $s_0 \in S$ and $r > 0$ such that $F \subseteq \{s \in S : d(s, s_0) \leq r\}$.]

(b) Show F is not compact. *Hint*: For each $\boldsymbol{x}$ in F, let $U(\boldsymbol{x}) = \{\boldsymbol{y} \in B : d(\boldsymbol{y}, \boldsymbol{x}) < 1\}$, and consider the cover $\mathcal{U}$ of F consisting of all $U(\boldsymbol{x})$. For each $n \in \mathbb{N}$, let $\boldsymbol{x}^{(n)}$ be defined so that $x_n^{(n)} = -1$ and $x_j^{(n)} = 1$ for $j \neq n$. Show that distinct $\boldsymbol{x}^{(n)}$ cannot belong to the same member of $\mathcal{U}$.

§14 Series

Our thorough treatment of sequences allows us to now quickly obtain the basic properties of infinite series.

14.1 Summation Notation.
The notation $\sum_{k=m}^{n} a_k$ is shorthand for the sum $a_m + a_{m+1} + \cdots + a_n$. The symbol "$\sum$" instructs us to sum and the decorations "$k = m$" and "n" tell us to sum the summands obtained by successively substituting $m, m+1, \ldots, n$ for k. For example, $\sum_{k=2}^{5} \frac{1}{k^2+k}$ is shorthand for

$$\frac{1}{2^2+2} + \frac{1}{3^2+3} + \frac{1}{4^2+4} + \frac{1}{5^2+5} = \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \frac{1}{30}$$

and $\sum_{k=0}^{n} 2^{-k}$ is shorthand for $1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^n}$.

The symbol $\sum_{n=m}^{\infty} a_n$ is shorthand for $a_m + a_{m+1} + a_{m+2} + \cdots$, although we have not yet assigned meaning to such an infinite sum. We now do so. □

14.2 Infinite Series.
To assign meaning to $\sum_{n=m}^{\infty} a_n$, we consider the sequences $(s_n)_{n=m}^{\infty}$ of *partial sums*:

$$s_n = a_m + a_{m+1} + \cdots + a_n = \sum_{k=m}^{n} a_k.$$

The infinite series $\sum_{n=m}^{\infty} a_n$ is said to *converge* provided the sequence (s_n) of partial sums converges to a real number S, in which case we define $\sum_{n=m}^{\infty} a_n = S$. Thus

$$\sum_{n=m}^{\infty} a_n = S \quad \text{means} \quad \lim s_n = S \quad \text{or} \quad \lim_{n\to\infty}\left(\sum_{k=m}^{n} a_k\right) = S.$$

A series that does not converge is said to *diverge*. We say that $\sum_{n=m}^{\infty} a_n$ *diverges to* $+\infty$ and we write $\sum_{n=m}^{\infty} a_n = +\infty$ provided $\lim s_n = +\infty$; a similar remark applies to $-\infty$. The symbol $\sum_{n=m}^{\infty} a_n$ has no meaning unless the series converges, or diverges to $+\infty$ or $-\infty$. Often we will be concerned with properties of infinite series but not their exact values or precisely where the summation begins, in which case we may write $\sum a_n$ rather than $\sum_{n=m}^{\infty} a_n$.

If the terms a_n of an infinite series $\sum a_n$ are all nonnegative, then the partial sums (s_n) form an increasing sequence, so Theorems 10.2 and 10.4 show that $\sum a_n$ either converges, or diverges to $+\infty$. In particular, $\sum |a_n|$ is meaningful for any sequence (a_n) whatever. The series $\sum a_n$ is said to *converge absolutely* or to be *absolutely convergent* if $\sum |a_n|$ converges. Absolutely convergent series are convergent, as we shall see in Corollary 14.7. □

Example 1

A series of the form $\sum_{n=0}^{\infty} ar^n$ for constants a and r is called a *geometric series*. These are the easiest series to sum. For $r \neq 1$, the partial sums s_n are given by

$$\sum_{k=0}^{n} ar^k = a\frac{1-r^{n+1}}{1-r}. \tag{1}$$

This identity can be verified by mathematical induction or by multiplying both sides by $1-r$, in which case the right hand side equals $a - ar^{n+1}$ and the left side becomes

$$\begin{aligned}(1-r)\sum_{k=0}^{n} ar^k &= \sum_{k=0}^{n} ar^k - \sum_{k=0}^{n} ar^{k+1} \\ &= a + ar + ar^2 + \cdots + ar^n \\ &\quad -(ar + ar^2 + \cdots + ar^n + ar^{n+1}) \\ &= a - ar^{n+1}.\end{aligned}$$

For $|r| < 1$, we have $\lim_{n\to\infty} r^{n+1} = 0$ by Theorem 9.7(b) on page 48, so from (1) we have $\lim_{n\to\infty} s_n = \frac{a}{1-r}$. This proves

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r} \quad \text{if} \quad |r| < 1. \tag{2}$$

If $a \neq 0$ and $|r| \geq 1$, then the sequence (ar^n) does not converge to 0, so the series $\sum ar^n$ diverges by Corollary 14.5 below. □

Example 2

Formula (2) of Example 1 and the next result are very important and both should be used whenever possible, even though we will not prove (1) below until the next section. Consider a fixed positive real number p. Then

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \quad \text{converges if and only if} \quad p > 1. \tag{1}$$

In particular, for $p \le 1$, we can write $\sum 1/n^p = +\infty$. The exact values of the series for $p > 1$ are not easy to determine. Here are some remarkable formulas that can be shown by techniques [Fourier series or complex variables, to name two possibilities] that will not be covered in this text.

$$\textstyle\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} = 1.6449\cdots, \tag{2}$$

$$\textstyle\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90} = 1.0823\cdots. \tag{3}$$

Similar formulas hold for $\sum_{n=1}^{\infty} \frac{1}{n^p}$ when p is any even integer, but no such elegant formulas are known for p odd. In particular, no such formula is known for $\sum_{n=1}^{\infty} \frac{1}{n^3}$ though of course this series converges and can be approximated as closely as desired. □

It is worth emphasizing that it is often easier to prove limits exist or series converge than to determine their exact values. In the next section we will show without much difficulty that $\sum \frac{1}{n^p}$ converges for all $p > 1$, but it is a lot harder to show the sum is $\frac{\pi^2}{6}$ when $p = 2$ and no one knows exactly what the sum is for $p = 3$.

14.3 Definition.
We say a series $\sum a_n$ satisfies the *Cauchy criterion* if its sequence (s_n) of partial sums is a Cauchy sequence [see Definition 10.8]:

$$\text{for each} \quad \epsilon > 0 \quad \text{there exists a number} \quad N \quad \text{such that}$$
$$m, n > N \quad \text{implies} \quad |s_n - s_m| < \epsilon. \tag{1}$$

Nothing is lost in this definition if we impose the restriction $n > m$. Moreover, it is only a notational matter to work with $m - 1$ where $m \le n$ instead of m where $m < n$. Therefore (1) is equivalent to

$$\text{for each} \quad \epsilon > 0 \quad \text{there exists a number} \quad N \quad \text{such that}$$
$$n \ge m > N \quad \text{implies} \quad |s_n - s_{m-1}| < \epsilon. \tag{2}$$

Since $s_n - s_{m-1} = \sum_{k=m}^{n} a_k$, condition (2) can be rewritten

$$\text{for each} \quad \epsilon > 0 \quad \text{there exists a number} \quad N \quad \text{such that} \\ n \geq m > N \quad \text{implies} \quad \left| \sum_{k=m}^{n} a_k \right| < \epsilon. \tag{3}$$

We will usually use version (3) of the *Cauchy criterion.* Theorem 10.11 implies the following.

14.4 Theorem.
A series converges if and only if it satisfies the Cauchy criterion.

14.5 Corollary.
If a series $\sum a_n$ *converges, then* $\lim a_n = 0$.

Proof
Since the series converges, (3) in Definition 14.3 holds. In particular, (3) in 14.3 holds for $n = m$; i.e., for each $\epsilon > 0$ there exists a number N such that $n > N$ implies $|a_n| < \epsilon$. Thus $\lim a_n = 0$. ∎

The converse of Corollary 14.5 does not hold as the example $\sum 1/n = +\infty$ shows.

We next give several tests to assist us in determining whether a series converges. The first test is elementary but useful.

14.6 Comparison Test.
Let $\sum a_n$ *be a series where* $a_n \geq 0$ *for all* n.

(i) *If* $\sum a_n$ *converges and* $|b_n| \leq a_n$ *for all* n, *then* $\sum b_n$ *converges.*
(ii) *If* $\sum a_n = +\infty$ *and* $b_n \geq a_n$ *for all* n, *then* $\sum b_n = +\infty$.

Proof
(i) For $n \geq m$ we have

$$\left| \sum_{k=m}^{n} b_k \right| \leq \sum_{k=m}^{n} |b_k| \leq \sum_{k=m}^{n} a_k; \tag{1}$$

the first inequality follows from the triangle inequality [Exercise 3.6(b)]. Since $\sum a_n$ converges, it satisfies the Cauchy criterion 14.3(1). It follows from (1) that $\sum b_n$ also satisfies the Cauchy criterion, and hence $\sum b_n$ converges.

(ii) Let (s_n) and (t_n) be the sequences of partial sums for $\sum a_n$ and $\sum b_n$, respectively. Since $b_n \geq a_n$ for all n, we obviously have $t_n \geq s_n$ for all n. Since $\lim s_n = +\infty$, we conclude $\lim t_n = +\infty$, i.e., $\sum b_n = +\infty$. ■

14.7 Corollary.
Absolutely convergent series are convergent.[9]

Proof
Suppose $\sum b_n$ is absolutely convergent. This means $\sum a_n$ converges where $a_n = |b_n|$ for all n. Then $|b_n| \leq a_n$ trivially, so $\sum b_n$ converges by 14.6(i). ■

We next state the Ratio Test which is popular because it is often easy to use. But it has defects: It isn't as general as the Root Test. Moreover, an important result concerning the radius of convergence of a power series uses the Root Test. Finally, the Ratio Test is worthless if some of the a_n's equal 0. To review lim sup's and lim inf's, see Definition 10.6, Theorems 10.7 and 11.8, and §12.

14.8 Ratio Test.
A series $\sum a_n$ of nonzero terms

(i) *converges absolutely if* $\limsup |a_{n+1}/a_n| < 1$,

(ii) *diverges if* $\liminf |a_{n+1}/a_n| > 1$.

(iii) *Otherwise* $\liminf |a_{n+1}/a_n| \leq 1 \leq \limsup |a_{n+1}/a_n|$ *and the test gives no information.*

We give the proof after the proof of the Root Test.

Remember that if $\lim |a_{n+1}/a_n|$ exists, then it is equal to both $\limsup |a_{n+1}/a_n|$ and $\liminf |a_{n+1}/a_n|$ and hence the Ratio Test will give information unless, of course, the limit $\lim |a_{n+1}/a_n|$ equals 1.

14.9 Root Test.
Let $\sum a_n$ be a series and let $\alpha = \limsup |a_n|^{1/n}$. The series $\sum a_n$

(i) *converges absolutely if $\alpha < 1$,*

(ii) *diverges if $\alpha > 1$.*

---

[9] As noted in [35], the proofs of this corollary and the Alternating Series Theorem 15.3 use the completeness of $\mathbb{R}$.

(iii) *Otherwise* $\alpha = 1$ *and the test gives no information.*

Proof

(i) Suppose $\alpha < 1$, and select $\epsilon > 0$ so that $\alpha + \epsilon < 1$. Then by Definition 10.6 there is a positive integer N such that

$$\alpha - \epsilon < \sup\{|a_n|^{1/n} : n > N\} < \alpha + \epsilon.$$

In particular, we have $|a_n|^{1/n} < \alpha + \epsilon$ for $n > N$, so

$$|a_n| < (\alpha + \epsilon)^n \quad \text{for} \quad n > N.$$

Since $0 < \alpha+\epsilon < 1$, the geometric series $\sum_{n=N+1}^{\infty}(\alpha+\epsilon)^n$ converges and the Comparison Test shows the series $\sum_{n=N+1}^{\infty} a_n$ also converges. Then clearly $\sum a_n$ converges; see Exercise 14.9.

(ii) If $\alpha > 1$, then by Theorem 11.7 a subsequence of $|a_n|^{1/n}$ has limit $\alpha > 1$. It follows that $|a_n| > 1$ for infinitely many choices of n. In particular, the sequence (a_n) cannot possibly converge to 0, so the series $\sum a_n$ cannot converge by Corollary 14.5.

(iii) For each of the series $\sum \frac{1}{n}$ and $\sum \frac{1}{n^2}$, α turns out to equal 1 as can be seen by applying Theorem 9.7(c) on page 48. Since $\sum \frac{1}{n}$ diverges and $\sum \frac{1}{n^2}$ converges, the equality $\alpha = 1$ does not guarantee either convergence or divergence of the series. ■

Proof of the Ratio Test

Let $\alpha = \limsup |a_n|^{1/n}$. By Theorem 12.2 we have

$$\liminf \left|\frac{a_{n+1}}{a_n}\right| \leq \alpha \leq \limsup \left|\frac{a_{n+1}}{a_n}\right|. \tag{1}$$

If $\limsup |a_{n+1}/a_n| < 1$, then $\alpha < 1$ and the series converges by the Root Test. If $\liminf |a_{n+1}/a_n| > 1$, then $\alpha > 1$ and the series diverges by the Root Test. Assertion 14.8(iii) is verified by again examining the series $\sum 1/n$ and $\sum 1/n^2$. ■

Inequality (1) in the proof of the Ratio Test shows that the Root Test is superior to the Ratio Test in the following sense: Whenever the Root Test gives no information [i.e., $\alpha = 1$] the Ratio Test will surely also give no information. On the other hand, Example 8 below gives a series for which the Ratio Test gives no information but which converges by the Root Test. Nevertheless, the tests usually fail together as the next remark shows.

14.10 Remark.
If the terms a_n are nonzero and if $\lim |a_{n+1}/a_n| = 1$, then $\alpha = \limsup |a_n|^{1/n} = 1$ by Corollary 12.3, so neither the Ratio Test nor the Root Test gives information concerning the convergence of $\sum a_n$.

We have three tests for convergence of a series [Comparison, Ratio, Root], and we will obtain two more in the next section. There is no clearcut strategy advising us which test to try first. However, if the form of a given series $\sum a_n$ does not suggest a particular strategy, and if the ratios a_{n+1}/a_n are easy to calculate, one may as well try the Ratio Test first.

Example 3
Consider the series

$$\sum_{n=2}^{\infty}\left(-\frac{1}{3}\right)^n = \frac{1}{9} - \frac{1}{27} + \frac{1}{81} - \frac{1}{243} + \cdots. \tag{1}$$

This is a geometric series and has the form $\sum_{n=0}^{\infty} ar^n$ if we write it as $(1/9)\sum_{n=0}^{\infty}(-1/3)^n$. Here $a = 1/9$ and $r = -1/3$, so by (2) of Example 1 the sum is $(1/9)/[1-(-1/3)] = 1/12$.

The series (1) can also be shown to converge by the Comparison Test, since $\sum 1/3^n$ converges by the Ratio Test or by the Root Test. In fact, if $a_n = (-1/3)^n$, then $\lim |a_{n+1}/a_n| = \limsup |a_n|^{1/n} = 1/3$. Of course, none of these tests will give us the exact value of the series (1). □

Example 4
Consider the series

$$\sum \frac{n}{n^2+3}. \tag{1}$$

If $a_n = \dfrac{n}{n^2+3}$, then

$$\frac{a_{n+1}}{a_n} = \frac{n+1}{(n+1)^2+3}\cdot\frac{n^2+3}{n} = \frac{n+1}{n}\cdot\frac{n^2+3}{n^2+2n+4},$$

so $\lim |a_{n+1}/a_n| = 1$. As noted in 14.10, neither the Ratio Test nor the Root Test gives any information in this case. Before trying the Comparison Test we need to decide whether we *believe* the series converges or not. Since a_n is approximately $1/n$ for large n and since $\sum(1/n)$ diverges, we expect the series (1) to diverge. Now

$$\frac{n}{n^2+3} \geq \frac{n}{n^2+3n^2} = \frac{n}{4n^2} = \frac{1}{4n}.$$

Since $\sum(1/n)$ diverges, $\sum(1/4n)$ also diverges [its partial sums are $s_n/4$ where $s_n = \sum_{k=1}^{n}(1/k)$], so (1) diverges by the Comparison Test. □

Example 5

Consider the series

$$\sum \frac{1}{n^2+1}. \tag{1}$$

As the reader should check, neither the Ratio Rest nor the Root Test gives any information. The nth term is approximately $\frac{1}{n^2}$ and in fact $\frac{1}{n^2+1} \leq \frac{1}{n^2}$. Since $\sum \frac{1}{n^2}$ converges, the series (1) converges by the Comparison Test. □

Example 6

Consider the series

$$\sum \frac{n}{3^n}. \tag{1}$$

If $a_n = n/3^n$, then $a_{n+1}/a_n = (n+1)/(3n)$, so $\lim |a_{n+1}/a_n| = 1/3$. Hence the series (1) converges by the Ratio Test. In this case, applying the Root Test is not much more difficult provided we recall $\lim n^{1/n} = 1$. It is also possible to show (1) converges by comparing it with a suitable geometric series. □

Example 7

Consider the series

$$\sum a_n \quad \text{where} \quad a_n = \left[\frac{2}{(-1)^n - 3}\right]^n. \tag{1}$$

The form of a_n suggests the Root Test. Since $|a_n|^{1/n} = 1$ for even n and $|a_n|^{1/n} = 1/2$ for odd n, we have $\alpha = \limsup |a_n|^{1/n} = 1$.

So the Root Test gives no information, and the Ratio Test cannot help either. On the other hand, if we had been alert, we would have observed $a_n = 1$ for even n, so (a_n) cannot converge to 0. Therefore the series (1) diverges by Corollary 14.5. □

Example 8
Consider the series

$$\sum_{n=0}^{\infty} 2^{(-1)^n - n} = 2 + \frac{1}{4} + \frac{1}{2} + \frac{1}{16} + \frac{1}{8} + \frac{1}{64} + \cdots. \qquad (1)$$

Let $a_n = 2^{(-1)^n - n}$. Since $a_n \le \frac{1}{2^{n-1}}$ for all n, we can quickly conclude the series converges by the Comparison Test. But our real interest in this series is that it illustrates the difference between the Ratio Test and the Root Test. Since $a_{n+1}/a_n = 1/8$ for even n and $a_{n+1}/a_n = 2$ for odd n, we have

$$\frac{1}{8} = \liminf \left| \frac{a_{n+1}}{a_n} \right| < 1 < \limsup \left| \frac{a_{n+1}}{a_n} \right| = 2.$$

Hence the Ratio Test gives no information.

Note that $(a_n)^{1/n} = 2^{\frac{1}{n} - 1}$ for even n and $(a_n)^{1/n} = 2^{-\frac{1}{n} - 1}$ for odd n. Since $\lim 2^{\frac{1}{n}} = \lim 2^{-\frac{1}{n}} = 1$ by Theorem 9.7(d) on page 48, we conclude $\lim (a_n)^{1/n} = \frac{1}{2}$. Therefore $\alpha = \limsup (a_n)^{1/n} = \frac{1}{2} < 1$ and the series (1) converges by the Root Test. □

Example 9
Consider the series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n}} = 1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \frac{1}{2} + \frac{1}{\sqrt{5}} - \cdots. \qquad (1)$$

Since $\lim \sqrt{n/(n+1)} = 1$, neither the Ratio Test nor the Root Test gives any information. Since $\sum \frac{1}{\sqrt{n}}$ diverges, we will not be able to use the Comparison Test 14.6(i) to show (1) converges. Since the terms of the series (1) are not all nonnegative, we will not be able to use the Comparison Test 14.6(ii) to show (1) diverges. It turns out that this series converges by the Alternating Series Test 15.3 which we have deferred to the next section. □

Exercises

14.1 Determine which of the following series converge. Justify your answers.

(a) $\sum \frac{n^4}{2^n}$ **(b)** $\sum \frac{2^n}{n!}$

(c) $\sum \frac{n^2}{3^n}$ **(d)** $\sum \frac{n!}{n^4+3}$

(e) $\sum \frac{\cos^2 n}{n^2}$ **(f)** $\sum_{n=2}^{\infty} \frac{1}{\log n}$

14.2 Repeat Exercise 14.1 for the following.

(a) $\sum \frac{n-1}{n^2}$ **(b)** $\sum (-1)^n$

(c) $\sum \frac{3n}{n^3}$ **(d)** $\sum \frac{n^3}{3^n}$

(e) $\sum \frac{n^2}{n!}$ **(f)** $\sum \frac{1}{n^n}$

(g) $\sum \frac{n}{2^n}$

14.3 Repeat Exercise 14.1 for the following.

(a) $\sum \frac{1}{\sqrt{n!}}$ **(b)** $\sum \frac{2+\cos n}{3^n}$

(c) $\sum \frac{1}{2^n+n}$ **(d)** $\sum (\frac{1}{2})^n (50 + \frac{2}{n})$

(e) $\sum \sin(\frac{n\pi}{9})$ **(f)** $\sum \frac{(100)^n}{n!}$

14.4 Repeat Exercise 14.1 for the following.

(a) $\sum_{n=2}^{\infty} \frac{1}{[n+(-1)^n]^2}$ **(b)** $\sum [\sqrt{n+1} - \sqrt{n}]$

(c) $\sum \frac{n!}{n^n}$

14.5 Suppose $\sum a_n = A$ and $\sum b_n = B$ where A and B are real numbers. Use limit theorems from §9 to quickly prove the following.

(a) $\sum (a_n + b_n) = A + B$.

(b) $\sum k a_n = kA$ for $k \in \mathbb{R}$.

(c) Is $\sum a_n b_n = AB$ a reasonable conjecture? Discuss.

14.6 **(a)** Prove that if $\sum |a_n|$ converges and (b_n) is a bounded sequence, then $\sum a_n b_n$ converges. *Hint*: Use Theorem 14.4.

(b) Observe that Corollary 14.7 is a special case of part (a).

14.7 Prove that if $\sum a_n$ is a convergent series of nonnegative numbers and $p > 1$, then $\sum a_n^p$ converges.

14.8 Show that if $\sum a_n$ and $\sum b_n$ are convergent series of nonnegative numbers, then $\sum \sqrt{a_n b_n}$ converges. *Hint*: Show $\sqrt{a_n b_n} \le a_n + b_n$ for all n.

14.9 The convergence of a series does not depend on any finite number of the terms, though of course the value of the limit does. More precisely, consider series $\sum a_n$ and $\sum b_n$ and suppose the set $\{n \in \mathbb{N} : a_n \neq b_n\}$

is finite. Then the series both converge or else they both diverge. Prove this. *Hint*: This is almost obvious from Theorem 14.4.

14.10 Find a series $\sum a_n$ which diverges by the Root Test but for which the Ratio Test gives no information. Compare Example 8.

14.11 Let (a_n) be a sequence of nonzero real numbers such that the sequence $(\frac{a_{n+1}}{a_n})$ of ratios is a constant sequence. Show $\sum a_n$ is a geometric series.

14.12 Let $(a_n)_{n\in\mathbb{N}}$ be a sequence such that $\liminf |a_n| = 0$. Prove there is a subsequence $(a_{n_k})_{k\in\mathbb{N}}$ such that $\sum_{k=1}^{\infty} a_{n_k}$ converges.

14.13 We have seen that it is often a lot harder to find the value of an infinite sum than to show it exists. Here are some sums that can be handled.

- **(a)** Calculate $\sum_{n=1}^{\infty}(\frac{2}{3})^n$ and $\sum_{n=1}^{\infty}(-\frac{2}{3})^n$.
- **(b)** Prove $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$. *Hint*: Note that $\sum_{k=1}^{n} \frac{1}{k(k+1)} = \sum_{k=1}^{n}[\frac{1}{k} - \frac{1}{k+1}]$.
- **(c)** Prove $\sum_{n=1}^{\infty} \frac{n-1}{2^{n+1}} = \frac{1}{2}$. *Hint*: Note $\frac{k-1}{2^{k+1}} = \frac{k}{2^k} - \frac{k+1}{2^{k+1}}$.
- **(d)** Use (c) to calculate $\sum_{n=1}^{\infty} \frac{n}{2^n}$.

14.14 Prove $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges by comparing with the series $\sum_{n=2}^{\infty} a_n$ where (a_n) is the sequence

$$(\frac{1}{2}, \frac{1}{4}, \frac{1}{4}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{16}, \frac{1}{16}, \frac{1}{16}, \frac{1}{16}, \frac{1}{16}, \frac{1}{16}, \frac{1}{16}, \frac{1}{16}, \frac{1}{32}, \frac{1}{32}, \ldots).$$

§15 Alternating Series and Integral Tests

Sometimes one can check convergence or divergence of series by comparing the partial sums with familiar integrals. We illustrate.

Example 1

We show $\sum \frac{1}{n} = +\infty$.

Consider the picture of the function $f(x) = \frac{1}{x}$ in Fig. 15.1. For $n \geq 1$ it is evident that

$$\sum_{k=1}^{n} \frac{1}{k} = \text{Sum of the areas of the first } n \text{ rectangles in Fig. 15.1}$$

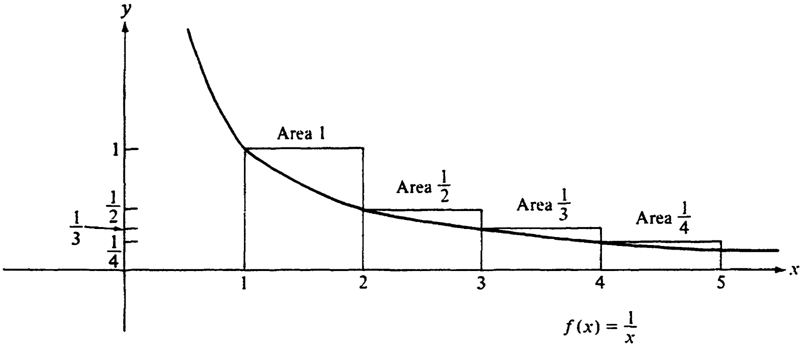

FIGURE 15.1

$$\geq \text{Area under the curve } \frac{1}{x} \text{ between 1 and } n+1$$
$$= \int_1^{n+1} \frac{1}{x}\,dx = \log_e(n+1).$$

Since $\lim_{n\to\infty} \log_e(n+1) = +\infty$, we conclude $\sum_{n=1}^{\infty} \frac{1}{n} = +\infty$.

The series $\sum \frac{1}{n}$ diverges very slowly. In Example 7 on page 120, we observe $\sum_{n=1}^{N} \frac{1}{n}$ is approximately $\log_e N + 0.5772$. Thus for $N = 1,000$ the sum is approximately 7.485, and for $N = 1,000,000$ the sum is approximately 14.393. □

Another proof that $\sum \frac{1}{n}$ diverges was indicated in Exercise 14.14. However, an integral test is useful to establish the next result.

Example 2

We show $\sum \frac{1}{n^2}$ converges.

Consider the graph of $f(x) = \frac{1}{x^2}$ in Fig. 15.2. Then we have

$$\sum_{k=1}^{n} \frac{1}{k^2} = \text{Sum of the areas of the first } n \text{ rectangles}$$
$$\leq 1 + \int_1^n \frac{1}{x^2}\,dx = 2 - \frac{1}{n} < 2$$

for all $n \geq 1$. Thus the partial sums form an increasing sequence that is bounded above by 2. Therefore $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges and its

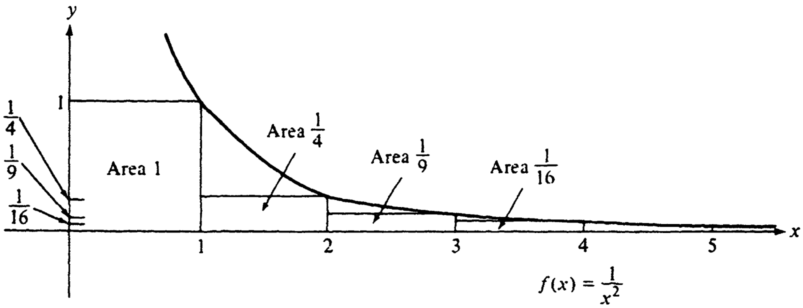

FIGURE 15.2

sum is less than or equal to 2. Actually, we have already mentioned [without proof!] that the sum is $\frac{\pi^2}{6} = 1.6449\cdots$.

Note that in estimating $\sum_{k=1}^{n} \frac{1}{k^2}$ we did not simply write $\sum_{k=1}^{n} \frac{1}{k^2} \le \int_0^n \frac{1}{x^2}\,dx$, even though this is true, because this integral is infinite. We were after a *finite* upper bound for the partial sums. □

The techniques just illustrated can be used to prove the following theorem.

15.1 Theorem.
$\sum \frac{1}{n^p}$ *converges if and only if* $p > 1$.

Proof
Supply your own picture and observe that if $p > 1$, then

$$\sum_{k=1}^{n} \frac{1}{k^p} \le 1 + \int_1^n \frac{1}{x^p}\,dx = 1 + \frac{1}{p-1}\left(1 - \frac{1}{n^{p-1}}\right) < 1 + \frac{1}{p-1} = \frac{p}{p-1}.$$

Consequently $\sum_{n=1}^{\infty} \frac{1}{n^p} \le \frac{p}{p-1} < +\infty$.

Suppose $0 < p \le 1$. Then $\frac{1}{n} \le \frac{1}{n^p}$ for all n. Since $\sum \frac{1}{n}$ diverges, we see that $\sum \frac{1}{n^p}$ diverges by the Comparison Test. ■

15.2 Integral Tests.
Here are the conditions under which an integral test is advisable:

(a) The tests in §14 do not seem to apply.

(b) The terms a_n of the series $\sum a_n$ are nonnegative.

(c) There is a nice decreasing function f on $[1, \infty)$ such that $f(n) = a_n$ for all n [f is *decreasing* if $x < y$ implies $f(x) \geq f(y)$].

(d) The integral of f is easy to calculate or estimate.

If $\lim_{n\to\infty} \int_1^n f(x)\,dx = +\infty$, then the series will diverge just as in Example 1. If $\lim_{n\to\infty} \int_1^n f(x)\,dx < +\infty$, then the series will converge just as in Example 2. The interested reader may formulate and prove the general result [Exercise 15.8]. □

The following result enables us to conclude that series like $\sum \frac{(-1)^n}{\sqrt{n}}$ converge even though they do not converge absolutely. See Example 9 in §14.

15.3 Alternating Series Theorem.
If $a_1 \geq a_2 \geq \cdots \geq a_n \geq \cdots \geq 0$ and $\lim a_n = 0$, then the alternating series $\sum(-1)^{n+1}a_n$ converges. Moreover, the partial sums $s_n = \sum_{k=1}^n (-1)^{k+1} a_k$ satisfy $|s - s_n| \leq a_n$ for all n.

The series $\sum(-1)^n a_n$ is called an *alternating series* because the signs of the terms alternate between $+$ and $-$.

Proof

We need to show that the sequence (s_n) converges. Note that the subsequence (s_{2n}) is increasing because $s_{2n+2} - s_{2n} = -a_{2n+2} + a_{2n+1} \geq 0$. Similarly, the subsequence (s_{2n-1}) is decreasing since $s_{2n+1} - s_{2n-1} = a_{2n+1} - a_{2n} \leq 0$. We claim

$$s_{2m} \leq s_{2n+1} \quad \text{for all} \quad m, n \in \mathbb{N}. \tag{1}$$

First note that $s_{2n} \leq s_{2n+1}$ for all n, because $s_{2n+1} - s_{2n} = a_{2n+1} \geq 0$. If $m \leq n$, then (1) holds because $s_{2m} \leq s_{2n} \leq s_{2n+1}$. If $m \geq n$, then (1) holds because $s_{2n+1} \geq s_{2m+1} \geq s_{2m}$. Thanks to (1), we see that (s_{2n}) is an increasing subsequence of (s_n) bounded above by each odd partial sum, and (s_{2n+1}) is a decreasing subsequence of (s_n) bounded below by each even partial sum. By Theorem 10.2, these subsequences converge, say to s and t. Now

$$t - s = \lim_{n\to\infty} s_{2n+1} - \lim_{n\to\infty} s_{2n} = \lim_{n\to\infty} (s_{2n+1} - s_{2n}) = \lim_{n\to\infty} a_{2n+1} = 0,$$

so $s = t$. It follows that $\lim_n s_n = s$.

To check the last claim, note that $s_{2k} \le s \le s_{2k+1}$, so both $s_{2k+1} - s$ and $s - s_{2k}$ are clearly bounded by $s_{2k+1} - s_{2k} = a_{2k+1} \le a_{2k}$. So, whether n is even or odd, we have $|s - s_n| \le a_n$. ■

Exercises

15.1 Determine which of the following series converge. Justify your answers.
(a) $\sum \frac{(-1)^n}{n}$ **(b)** $\sum \frac{(-1)^n n!}{2^n}$

15.2 Repeat Exercise 15.1 for the following.
(a) $\sum [\sin(\frac{n\pi}{6})]^n$ **(b)** $\sum [\sin(\frac{n\pi}{7})]^n$

15.3 Show $\sum_{n=2}^{\infty} \frac{1}{n(\log n)^p}$ converges if and only if $p > 1$.

15.4 Determine which of the following series converge. Justify your answers.
(a) $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}\log n}$ **(b)** $\sum_{n=2}^{\infty} \frac{\log n}{n}$
(c) $\sum_{n=4}^{\infty} \frac{1}{n(\log n)(\log\log n)}$ **(d)** $\sum_{n=2}^{\infty} \frac{\log n}{n^2}$

15.5 Why didn't we use the Comparison Test to prove Theorem 15.1 for $p > 1$?

15.6 **(a)** Give an example of a divergent series $\sum a_n$ for which $\sum a_n^2$ converges.

(b) Observe that if $\sum a_n$ is a convergent series of nonnegative terms, then $\sum a_n^2$ also converges. See Exercise 14.7.

(c) Give an example of a convergent series $\sum a_n$ for which $\sum a_n^2$ diverges.

15.7 **(a)** Prove if (a_n) is a decreasing sequence of real numbers and if $\sum a_n$ converges, then $\lim na_n = 0$. *Hint*: Consider $|a_{N+1} + a_{N+2} + \cdots + a_n|$ for suitable N.

(b) Use (a) to give another proof that $\sum \frac{1}{n}$ diverges.

15.8 Formulate and prove a general integral test as advised in 15.2.

§16 * Decimal Expansions of Real Numbers

We begin by recalling the brief discussion of decimals in Discussion 10.3. There we considered a decimal expansion $K.d_1d_2d_3\cdots$, where K is a nonnegative integer and each digit d_j belongs to $\{0,1,2,3,4,5,6,7,8,9\}$. This expansion represents the real number

$$K + \sum_{j=1}^{\infty} \frac{d_j}{10^j} = K + \sum_{j=1}^{\infty} d_j \cdot 10^{-j}$$

which we also can write as

$$\lim_{n\to\infty} s_n \quad \text{where} \quad s_n = K + \sum_{j=1}^{n} d_j \cdot 10^{-j}.$$

Thus *every such decimal expansion represents a nonnegative real number*. We will prove the converse after we formalize the process of long division. The development here is based on some suggestions by Karl Stromberg.

16.1 Long Division.
Consider a/b where a and b are positive integers. Since the K term above causes no difficulty, we assume $a < b$. We analyze the familiar long division process which gives a decimal expansion for a/b. Figure 16.1 shows the first few steps where $a = 3$ and $b = 7$. If we name the digits $d_1, d_2, d_3, \ldots$ and the remainders $r_1, r_2, r_3, \ldots$, then so far $d_1 = 4$, $d_2 = 2$ and $r_1 = 2$, $r_2 = 6$. At the next step we divide 7 into $60 = 10 \cdot r_2$ and obtain $60 = 7 \cdot 8 + 4$. The quotient 8 becomes the third digit d_3, we place the product 56 under 60, subtract and obtain a new remainder $4 = r_3$. That is, we are calculating the remainder

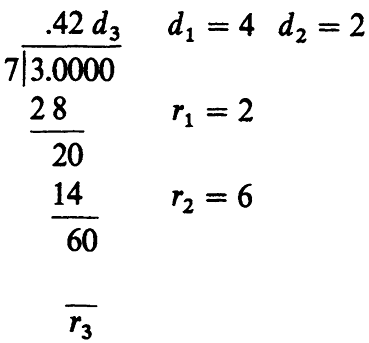

FIGURE 16.1

obtained by dividing 60 by 7. Next we multiply the remainder $r_3 = 4$ by 10 and repeat the process. At each stage

$$d_n \in \{0,1,2,3,4,5,6,7,8,9\}$$
$$r_n = 10 \cdot r_{n-1} - 7 \cdot d_n$$
$$0 \le r_n < 7.$$

These results hold for $n = 1, 2, \ldots$ if we set $r_0 = 3$. In general, we set $r_0 = a$ and obtain

$$d_n \in \{0,1,2,3,4,5,6,7,8,9\} \tag{1}$$
$$r_n = 10 \cdot r_{n-1} - b \cdot d_n \tag{2}$$
$$0 \le r_n < b. \tag{3}$$

As we will show, this construction yields:

$$\frac{a}{b} = \frac{d_1}{10} + \frac{1}{10}\frac{r_1}{b} \tag{4}$$

and

$$\frac{a}{b} = \frac{d_1}{10} + \cdots + \frac{d_n}{10^n} + \frac{1}{10^n}\frac{r_n}{b}. \tag{5}$$

for $n \ge 2$. Since each r_n is less than b, we have $\lim_n \frac{1}{10^n}\frac{r_n}{b} = 0$, so (5) shows

$$\frac{a}{b} = \sum_{j=1}^{\infty} \frac{d_j}{10^j};$$

thus $.d_1 d_2 d_3 \cdots$ is a decimal expansion for $\frac{a}{b}$.

Now Eq. (4) follows from $r_1 = 10a - bd_1$, and we will verify (5) by mathematical induction. From (2), we have

$$r_{n+1} = 10r_n - bd_{n+1}, \quad \text{so} \quad \frac{r_n}{b} = \frac{1}{10}d_{n+1} + \frac{1}{10}\frac{r_{n+1}}{b}.$$

Substituting this into (5) gives

$$\frac{a}{b} = \frac{d_1}{10} + \cdots + \frac{d_n}{10^n} + \frac{d_{n+1}}{10^{n+1}} + \frac{1}{10^{n+1}}\frac{r_{n+1}}{b}, \tag{6}$$

i.e., (5) holds with n replaced by $n+1$. Thus by induction, (5) holds for all n. □

16.2 Theorem.
Every nonnegative real number x has at least one decimal expansion.

Proof
It suffices to consider x in $[0, 1)$. The proof will be similar to that for results in 16.1, starting with Eqs. (4) and (5) in its proof. We will use the "floor function" $\lfloor\ \rfloor$ on $\mathbb{R}$, where $\lfloor y \rfloor$ is defined to be the largest integer less than or equal to y, for each $y \in \mathbb{R}$.

Since $10x < 10$, we can write

$$10x = d_1 + x_1 \quad \text{where} \quad d_1 = \lfloor 10x \rfloor \quad \text{and} \quad x_1 \in [0, 1). \tag{1}$$

Note d_1 is in $\{0, 1, 2, 3, \ldots, 9\}$ since $10x < 10$. Therefore

$$x = \frac{d_1}{10} + \frac{1}{10} x_1. \tag{2}$$

Suppose, for some $n \geq 1$, we have chosen $d_1, \ldots, d_n$ in $\{0, 1, 2, 3, \ldots, 9\}$ and $x_1, \ldots, x_n$ in $[0, 1)$ so that

$$x = \frac{d_1}{10} + \cdots + \frac{d_n}{10^n} + \frac{1}{10^n} x_n. \tag{3}$$

Since $10x_n < 10$, we can write

$$10x_n = d_{n+1} + x_{n+1} \quad \text{where} \quad d_{n+1} = \lfloor 10x_n \rfloor, x_{n+1} \in [0, 1). \tag{4}$$

Solving for x_n in (4) and substituting the value in (3), we get

$$x = \frac{d_1}{10} + \cdots + \frac{d_{n+1}}{10^{n+1}} + \frac{1}{10^{n+1}} x_{n+1}.$$

This completes the induction step. Since $\lim_n \frac{1}{10^{n+1}} x_{n+1} = 0$, we conclude

$$x = \lim_n \sum_{j=1}^{n+1} \frac{d_j}{10^j} = \sum_{j=1}^{\infty} \frac{d_j}{10^j},$$

so that $.d_1 d_2 d_3 \cdots$ is a decimal expansion for x. ■

As noted in Discussion 10.3, $1.000\cdots$ and $0.999\cdots$ are decimal expansions for the same real number. That is, the series

$$1 + \sum_{j=1}^{\infty} 0 \cdot 10^{-j} \qquad \text{and} \qquad \sum_{j=1}^{\infty} 9 \cdot 10^{-j}$$

have the same value, namely 1. Similarly, $2.75000\cdots$ and $2.74999\cdots$ are both decimal expansions for $\frac{11}{4}$ [Exercise 16.1]. The next theorem shows this is essentially the only way a number can have distinct decimal expansions.

16.3 Theorem.
A real number x has exactly one decimal expansion or else x has two decimal expansions, one ending in a sequence of all 0's and the other ending in a sequence of all 9's.

Proof
We assume $x \geq 0$. If x has decimal expansion $K.000\cdots$ with $K > 0$, then it has one other decimal expansion, namely $(K-1).999\cdots$. If x has decimal expansion $K.d_1d_2d_3\cdots d_r000\cdots$ where $d_r \neq 0$, then it has one other decimal expansion $K.d_1d_2d_3\cdots(d_r-1)9999\cdots$. The reader can easily check these claims [Exercise 16.2].

Now suppose x has two distinct decimal expansions $K.d_1d_2d_3\cdots$ and $L.e_1e_2e_3\cdots$. Suppose $K < L$. If any $d_j < 9$, then by Exercise 16.3 we have

$$x < K + \sum_{j=1}^{\infty} 9 \cdot 10^{-j} = K + 1 \leq \text{Ł} \leq x,$$

a contradiction. It follows that $x = K + 1 = L$ and its decimal expansions are $K.999\cdots$ and $(K+1).000\cdots$. In the remaining case, we have $K = L$. Let

$$m = \min\{j : d_j \neq e_j\}.$$

We may assume $d_m < e_m$. If $d_j < 9$ for any $j > m$, then by Exercise 16.3,

$$\begin{aligned} x &< K + \sum_{j=1}^{m} d_j \cdot 10^{-j} + \sum_{j=m+1}^{\infty} 9 \cdot 10^{-j} = K + \sum_{j=1}^{m} d_j \cdot 10^{-j} + 10^{-m} \\ &= K + \sum_{j=1}^{m-1} e_j \cdot 10^{-j} + d_m \cdot 10^{-m} + 10^{-m} \leq K + \sum_{j=1}^{m} e_j \cdot 10^{-j} \leq x, \end{aligned}$$

a contradiction. Thus $d_j = 9$ for $j > m$. Likewise, if $e_j > 0$ for any $j > m$, then

$$\begin{aligned} x > K + \sum_{j=1}^{m} e_j \cdot 10^{-j} &= K + \sum_{j=1}^{m-1} d_j \cdot 10^{-j} + e_m \cdot 10^{-m} \\ &\geq K + \sum_{j=1}^{m-1} d_j \cdot 10^{-j} + d_m \cdot 10^{-m} + 10^{-m} \\ &= K + \sum_{j=1}^{m} d_j \cdot 10^{-j} + \sum_{j=m+1}^{\infty} 9 \cdot 10^{-j} \geq x, \end{aligned}$$

a contradiction. So in this case, $d_j = 9$ for $j > m$, $e_m = d_m + 1$ and $e_j = 0$ for $j > m$. ■

16.4 Definition.
An expression of the form

$$K.d_1 d_2 \cdots d_\ell \overline{d_{\ell+1} \cdots d_{\ell+r}}$$

represents the decimal expansion in which the block $d_{\ell+1} \cdots d_{\ell+r}$ is repeated indefinitely:

$$K.d_1 d_2 \cdots d_\ell d_{\ell+1} \cdots d_{\ell+r} d_{\ell+1} \cdots d_{\ell+r} d_{\ell+1} \cdots d_{\ell+r} d_{\ell+1} \cdots d_{\ell+r} \cdots .$$

We call such an expansion a *repeating decimal*.

Example 1
Every integer is a repeating decimal. For example, $17 = 17.\overline{0} = 17.000\cdots$. Another simple example is

$$.\overline{8} = .888\cdots = \sum_{j=1}^{\infty} 8 \cdot 10^{-j} = \frac{8}{10} \sum_{j=0}^{\infty} 10^{-j} = \frac{8}{10} \cdot \frac{10}{9} = \frac{8}{9}.$$

□

Example 2
The expression $3.9\overline{67}$ represents the repeating decimal $3.9676767\cdots$. We evaluate this as follows:

$$\begin{aligned} 3.9\overline{67} &= 3 + 9 \cdot 10^{-1} + 6 \cdot 10^{-2} + 7 \cdot 10^{-3} + 6 \cdot 10^{-4} + 7 \cdot 10^{-5} + \cdots \\ &= 3 + 9 \cdot 10^{-1} + 67 \cdot 10^{-3} \sum_{j=0}^{\infty} (10^{-2})^j \end{aligned}$$

$$= 3 + 9 \cdot 10^{-1} + 67 \cdot 10^{-3} \left(\frac{100}{99}\right) = 3 + \frac{9}{10} + \frac{67}{990}$$
$$= \frac{3,928}{990} = \frac{1,964}{495}.$$

Thus the repeating decimal $3.9\overline{67}$ represents the rational number $\frac{1,964}{495}$. Any repeating decimal can be evaluated as a rational number in this way, as we'll show in the next theorem. □

Example 3
We find the decimal expansion for $\frac{11}{7}$. By the usual long division process in 16.1, we find

$$\frac{11}{7} = 1.571428571428571428571428571428571\cdots,$$

i.e., $\frac{11}{7} = 1.\overline{571428}$. To check this, observe

$$1.\overline{571428} = 1 + 571,428 \cdot 10^{-6} \sum_{j=0}^{\infty} (10^{-6})^j = 1 + \frac{571,428}{999,999}$$
$$= 1 + \frac{4}{7} = \frac{11}{7}.$$

□

Many books give the next theorem as an exercise, probably to avoid the complicated notation. If the details seem too complicated to you, move on to Examples 4–7.

16.5 Theorem.
A real number x is rational if and only if its decimal expansion is repeating. [*Theorem 16.3 shows that if x has two decimal expansions, they are both repeating.*]

Proof
First assume $x \geq 0$ has a repeating decimal expansion $x = K.d_1 d_2 \cdots d_\ell \overline{d_{\ell+1} \cdots d_{\ell+r}}$. Then

$$x = K + \sum_{j=1}^{\ell} d_j \cdot 10^{-j} + 10^{-\ell} y$$

where

$$y = .\overline{d_{\ell+1} \cdots d_{\ell+r}},$$

so it suffices to show y is rational. To simplify the notation, we write

$$y = .\overline{e_1 e_2 \cdots e_r}.$$

A little computation shows

$$y = \sum_{j=1}^{r} e_j \cdot 10^{-j} \left[\sum_{j=0}^{\infty} (10^{-r})^j \right] = \sum_{j=1}^{r} e_j \cdot 10^{-j} \frac{10^r}{10^r - 1}.$$

Thus y is rational. In fact, if we write $e_1 e_2 \cdots e_r$ for the usual *decimal* $\sum_{j=0}^{r-1} e_j \cdot 10^{r-1-j}$ *not the product*, then $y = \frac{e_1 e_2 \cdots e_r}{10^r - 1}$; see Example 3.

Next consider any positive rational, say $\frac{a}{b}$ where $a, b \in \mathbb{N}$. We may assume $a < b$. As we saw in 16.1, $\frac{a}{b}$ is given by the decimal expansion $.d_1 d_2 d_3 \cdots$ where $r_0 = a$,

$$d_k \in \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\} \tag{1}$$

$$r_k = 10 \cdot r_{k-1} - d_k b \tag{2}$$

$$0 \le r_k < b, \tag{3}$$

for $k \ge 1$. Since a and b are integers, each r_k is an integer. Thus (3) can be written

$$r_k \in \{0, 1, 2, \ldots, b - 1\} \quad \text{for} \quad k \ge 0. \tag{4}$$

This set has b elements, so the first $b + 1$ remainders r_k cannot all be distinct. That is, there exist integers $m \ge 0$ and $p > 0$ so that

$$0 \le m < m + p \le b \quad \text{and} \quad r_m = r_{m+p}.$$

From the construction giving (1)–(3) it is clear that given r_{k-1}, the integers r_k and d_k are uniquely determined. Thus

$$r_j = r_k \quad \text{implies} \quad r_{j+1} = r_{k+1} \quad \text{and} \quad d_{j+1} = d_{k+1}.$$

Since $r_m = r_{m+p}$, we conclude $r_{m+1} = r_{m+1+p}$ and $d_{m+1} = d_{m+1+p}$. A simple induction shows that the statement

$$\text{``}r_k = r_{k+p} \quad \text{and} \quad d_k = d_{k+p}\text{''}$$

holds for all integers $k \ge m + 1$. Thus the decimal expansion of $\frac{a}{b}$ is periodic with period p after the first m digits. That is,

$$\frac{a}{b} = .d_1 d_2 \cdots d_m \overline{d_{m+1} \cdots d_{m+p}}.$$

■

Remark. Given r_{k-1}, the uniqueness of r_k and d_k follows from the so-called "division algorithm," which is actually a theorem that shows the algorithm for division never breaks down. It says that if b is a positive integer and $m \in \mathbb{Z}$, then there are unique integers q and r so that

$$m = bq + r \quad \text{and} \quad 0 \le r < b;$$

q is called the quotient and r is called the remainder. With $m = 10 \cdot r_{k-1}$ in Theorem 16.5, this yields

$$10 \cdot r_{k-1} = bq + r \quad \text{where} \quad 0 \le r < b.$$

If we name $q = d_k$ and $r = r_k$, then we obtain formula (2) in Theorem 16.5. For more details, see for example, [60, §3.5].

Example 4
An expansion such as

$$.1010010001000010000010000001000000010000000010000000001000000000100\cdots$$

represents an irrational number, since it cannot be a repeating decimal: we've arranged for arbitrarily long blocks of 0's. □

Example 5
We do not know the complete decimal expansions of $\sqrt{2}$, $\sqrt{3}$ and many other familiar irrational numbers, but we know that they cannot be repeating by virtue of the last theorem. □

Example 6
We have claimed π and e are irrational. These facts and many others are proved in a fascinating book by Ivan Niven [49].

(a) Here is a proof that

$$e = \sum_{k=0}^{\infty} \frac{1}{k!}$$

is irrational. Assume $e = \frac{a}{b}$ where $a, b \in \mathbb{N}$. Then both $(b+1)! \cdot e$ and $(b+1)! \cdot \sum_{k=0}^{b+1} \frac{1}{k!}$ must be integers, so the difference

$$(b+1)! \sum_{k=b+2}^{\infty} \frac{1}{k!}$$

must be a positive integer. On the other hand, this last number is less than

$$\frac{1}{b+2}+\frac{1}{(b+2)^2}+\frac{1}{(b+2)^3}+\cdots=\frac{\frac{1}{b+2}}{1-\frac{1}{b+2}}=\frac{1}{b+1}<1,$$

a contradiction.

(b) We will prove π^2 is irrational, from which the irrationality of π follows; see Exercise 16.10.

The key to the proof that π^2 is irrational is the sequence of integrals $I_n = \int_0^\pi \frac{(x(\pi-x))^n}{n!} \sin x\,dx$. Thus

$$I_n = \int_0^\pi P_n(x)\sin x\,dx \quad \text{where} \quad P_n(x) = \frac{(x(\pi-x))^n}{n!}.$$

Claim 1. There is a sequence $(Q_n)_{n=0}^\infty$ of polynomials with *integer* coefficients, of degree at most n, satisfying $I_n = Q_n(\pi^2)$ for all n.

Proof. First, we obtain a recursive relation for I_n; see (3). We use integration by parts (Theorem 34.2) twice to show

$$I_n = \int_0^\pi P_n(x)\sin x\,dx = -\int_0^\pi P_n''(x)\sin x\,dx \quad \text{for} \quad n \geq 1. \tag{1}$$

In fact,

$$\begin{aligned}\int_0^\pi P_n(x)\sin x\,dx &= [-P_n(\pi)\cos(\pi) + P_n(0)\cos(0)] + \int_0^\pi P_n'(x)\cos x\,dx \\ &= \int_0^\pi P_n'(x)\cos x\,dx,\end{aligned}$$

since $P_n(x)$ contains a factor of $x(\pi - x)$ which is 0 at both π and 0. Therefore

$$\begin{aligned}I_n &= \int_0^\pi P_n'(x)\cos x\,dx \\ &= [P_n'(\pi)\sin(\pi) - P_n'(0)\sin(0)] - \int_0^\pi P_n''(x)\sin x\,dx \\ &= -\int_0^\pi P_n''(x)\sin x\,dx,\end{aligned}$$

and (1) holds. For $n \geq 2$, we have

$$P_n'(x) = \frac{(x(\pi - x))^{n-1}}{(n-1)!}(\pi - 2x) = P_{n-1}(x)(\pi - 2x),$$

so, using this for $n-1$ and the product rule, we have

$$\begin{aligned} P_n''(x) &= P_{n-2}(x)(\pi - 2x)^2 + P_{n-1}(x)(-2) \\ &= \pi^2 P_{n-2}(x) + P_{n-2}(x)(-4\pi x + 4x^2) - 2P_{n-1}(x). \end{aligned}$$

Since

$$P_{n-2}(x)(-4\pi x + 4x^2) = -4 \cdot \frac{(x(\pi - x))^{n-1}}{(n-2)!} = -4 \cdot P_{n-1}(x)(n-1),$$

we conclude

$$P_n''(x) = \pi^2 P_{n-2}(x) - [4(n-1) + 2]P_{n-1}(x).$$

Therefore

$$P_n''(x) = \pi^2 P_{n-2}(x) - (4n-2)P_{n-1}(x) \quad \text{for} \quad n \geq 2. \tag{2}$$

To prove Claim 1, note that $I_0 = 2$ is clear. Using (1) and $P_1''(x) = -2$, we see that $I_1 = 4$. And from (1) and (2), we see that

$$I_n = -\pi^2 I_{n-2} + (4n-2)I_{n-1} \quad \text{for} \quad n \geq 2. \tag{3}$$

Now Claim 1 holds by a simple induction argument, where $Q_0 = 2$, $Q_1 = 4$, and

$$Q_n(x) = -xQ_{n-2}(x) + (4n-2)Q_{n-1}(x) \quad \text{for} \quad n \geq 2.$$

Claim 2. π^2 is irrational.

Proof. Suppose $\pi^2 = a/b$. Using Claim 1, we see that each $b^n I_n = b^n Q_n(a/b)$ is an integer. Since $x(\pi - x)$ takes its maximum at $\pi/2$, we can write

$$\begin{aligned} 0 < b^n I_n &= b^n \int_0^\pi \frac{(x(\pi - x))^n}{n!} \sin x \, dx \\ &< b^n \int_0^\pi \frac{(\frac{\pi}{2} \cdot \frac{\pi}{2})^n}{n!} \, dx = \frac{\left(\frac{b\pi^2}{4}\right)^n}{n!} \pi. \end{aligned}$$

As noted in Exercise 9.15, the right-hand side converges to 0. So, for large n, the integer $b^n I_n$ lies in the interval $(0,1)$, a contradiction.

This simplification of Ivan Niven's famous short proof (1947) is due to Zhou and Markov [72]. Zhou and Markov use a similar technique to prove $\tan r$ is irrational for nonzero rational r and $\cos r$ is irrational if r^2 is a nonzero rational. Compare with results in Niven's book [49, Chap. 2].

(c) It is even more difficult to prove π and e are not algebraic numbers; see Definition 2.1. These results are proved in Niven's book [49, Theorems 2.12 and 9.11]. □

Example 7

There is a famous number introduced by Euler over 200 years ago that arises in the study of the gamma function. It is known as *Euler's constant* and is defined by

$$\gamma = \lim_{n\to\infty}\left[\sum_{k=1}^{n}\frac{1}{k} - \log_e n\right].$$

Even though

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{k} = +\infty \quad \text{and} \quad \lim_{n\to\infty}\log_e n = +\infty,$$

the limit defining γ exists and is finite [Exercise 16.9]. In fact, γ is approximately 0.577216. The amazing fact is that no one knows whether γ is rational or not. Most mathematicians believe γ is irrational. This is because it is "easier" for a number to be irrational, since repeating decimal expansions are regular. The remark in Exercise 16.8 hints at another reason it is easier for a number to be irrational. □

Exercises

16.1 **(a)** Show $2.74\overline{9}$ and $2.75\overline{0}$ are both decimal expansions for $\frac{11}{4}$.

(b) Which of these expansions arises from the long division process described in 16.1?

16.2 Verify the claims in the first paragraph of the proof of Theorem 16.3.

16.3 Suppose $\sum a_n$ and $\sum b_n$ are convergent series of nonnegative numbers. Show that if $a_n \le b_n$ for all n and if $a_n < b_n$ for at least one n, then $\sum a_n < \sum b_n$.

16.4 Write the following repeating decimals as rationals, i.e., as fractions of integers.

(a) $.2$ **(b)** $.0\overline{2}$
(c) $.\overline{02}$ **(d)** $3.\overline{14}$
(e) $.\overline{10}$ **(f)** $.1\overline{492}$

16.5 Find the decimal expansions of the following rational numbers.

(a) $1/8$ **(b)** $1/16$
(c) $2/3$ **(d)** $7/9$
(e) $6/11$ **(f)** $22/7$

16.6 Find the decimal expansions of $\frac{1}{7}$, $\frac{2}{7}$, $\frac{3}{7}$, $\frac{4}{7}$, $\frac{5}{7}$ and $\frac{6}{7}$. Note the interesting pattern.

16.7 Is $.12345678910111213141516171819202122232425 26\cdots$ rational?

16.8 Let (s_n) be a sequence of numbers in $(0,1)$. Each s_n has a decimal expansion $0.d_1^{(n)}d_2^{(n)}d_3^{(n)}\cdots$. For each n, let $e_n = 6$ if $d_n^{(n)} \neq 6$ and $e_n = 7$ if $d_n^{(n)} = 6$. Show $.e_1e_2e_3\cdots$ is the decimal expansion for some number y in $(0,1)$ and $y \neq s_n$ for all n. *Remark*: This shows the elements of $(0,1)$ cannot be listed as a sequence. In set-theoretic parlance, $(0,1)$ is "uncountable." Since the set $\mathbb{Q} \cap (0,1)$ can be listed as a sequence, there are a lot of irrational numbers in $(0,1)$!

16.9 Let $\gamma_n = (\sum_{k=1}^{n} \frac{1}{k}) - \log_e n = \sum_{k=1}^{n} \frac{1}{k} - \int_1^n \frac{1}{t} dt$.

(a) Show (γ_n) is a decreasing sequence. *Hint*: Look at $\gamma_n - \gamma_{n+1}$.

(b) Show $0 < \gamma_n \le 1$ for all n.

(c) Observe that $\gamma = \lim_n \gamma_n$ exists and is finite.

16.10 In Example 6(b), we showed π^2 is irrational. Use this to show π is irrational. What can you say about $\sqrt{\pi}$ and $\sqrt[3]{\pi}$? π^4?