4 CHAPTER Sequences and Series of Functions

In this chapter we develop some of the basic properties of power series. In doing so, we will introduce uniform convergence and illustrate its importance. In §26 we prove power series can be differentiated and integrated term-by-term.

§23 Power Series

Given a sequence $(a_n)_{n=0}^{\infty}$ of real numbers, the series $\sum_{n=0}^{\infty} a_n x^n$ is called a *power series*. Observe the variable x. Thus the power series is a function of x provided it converges for some or all x. Of course, it converges for $x = 0$; note the convention $0^0 = 1$. Whether it converges for other values of x depends on the choice of *coefficients* (a_n). It turns out that, given any sequence (a_n), one of the following holds for its power series:

(a) The power series converges for all $x \in \mathbb{R}$;
(b) The power series converges only for $x = 0$;
(c) The power series converges for all x in some bounded interval centered at 0; the interval may be open, half-open or closed.

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These remarks are consequences of the following important theorem.

23.1 Theorem.
For the power series $\sum a_n x^n$, *let*

$$\beta = \limsup |a_n|^{1/n} \quad \text{and} \quad R = \frac{1}{\beta}.$$

[If $\beta = 0$ *we set* $R = +\infty$, *and if* $\beta = +\infty$ *we set* $R = 0$.*] Then*

(i) *The power series converges for* $|x| < R$;

(ii) *The power series diverges for* $|x| > R$.

R is called the *radius of convergence* for the power series. Note that (i) is a vacuous statement if $R = 0$ and that (ii) is a vacuous statement if $R = +\infty$. Note also that (a) above corresponds to the case $R = +\infty$, (b) above corresponds to the case $R = 0$, and (c) above corresponds to the case $0 < R < +\infty$.

Proof of Theorem 23.1
The proof follows quite easily from the Root Test 14.9. Here are the details. We want to apply the Root Test to the series $\sum a_n x^n$. So for each $x \in \mathbb{R}$, let α_x be the number or symbol defined in 14.9 for the series $\sum a_n x^n$. Since the nth term of the series is $a_n x^n$, we have

$$\alpha_x = \limsup |a_n x^n|^{1/n} = \limsup |x||a_n|^{1/n} = |x| \cdot \limsup |a_n|^{1/n} = \beta|x|.$$

The third equality is justified by Exercise 12.6(a). Now we consider cases.

Case 1. Suppose $0 < R < +\infty$. In this case $\alpha_x = \beta|x| = \frac{|x|}{R}$. If $|x| < R$ then $\alpha_x < 1$, so the series converges by the Root Test. Likewise, if $|x| > R$, then $\alpha_x > 1$ and the series diverges.

Case 2. Suppose $R = +\infty$. Then $\beta = 0$ and $\alpha_x = 0$ no matter what x is. Hence the power series converges for all x by the Root Test.

Case 3. Suppose $R = 0$. Then $\beta = +\infty$ and $\alpha_x = +\infty$ for $x \neq 0$. Thus by the Root Test the series diverges for $x \neq 0$. ∎

Recall that if $\lim |\frac{a_{n+1}}{a_n}|$ exists, then this limit equals β of the last theorem by Corollary 12.3. This limit is often easier to calculate than $\limsup |a_n|^{1/n}$; see the examples below.

Example 1
Consider $\sum_{n=0}^{\infty} \frac{1}{n!} x^n$. If $a_n = \frac{1}{n!}$, then $\frac{a_{n+1}}{a_n} = \frac{1}{n+1}$, so $\lim |\frac{a_{n+1}}{a_n}| = 0$. Therefore $\beta = 0$, $R = +\infty$ and this series has radius of convergence $+\infty$. That is, it converges for all x in $\mathbb{R}$. In fact, it converges to e^x for all x, but that is another story; see Example 1 in §31, page 252, and also §37. □

Example 2
Consider $\sum_{n=0}^{\infty} x^n$. Then $\beta = 1$ and $R = 1$. Note this series does not converge for $x = 1$ or $x = -1$, so the interval of convergence is exactly $(-1, 1)$. [By *interval of convergence* we mean the set of x for which the power series converges.] The series converges to $\frac{1}{1-x}$ by formula (2) of Example 1 in §14, page 96. □

Example 3
Consider $\sum_{n=0}^{\infty} \frac{1}{n} x^n$. Since $\lim \frac{\frac{1}{n+1}}{\frac{1}{n}} = 1$, we again have $\beta = 1$ and $R = 1$. This series diverges for $x = 1$ [see Example 1 of §15], but it converges for $x = -1$ by the Alternating Series theorem 15.3 on page 108. Hence the interval of convergence is exactly $[-1, 1)$. □

Example 4
Consider $\sum_{n=0}^{\infty} \frac{1}{n^2} x^n$. Once again $\beta = 1$ and $R = 1$. This series converges at both $x = 1$ and $x = -1$, so its interval of convergence is exactly $[-1, 1]$. □

Example 5
The series $\sum_{n=0}^{\infty} n! x^n$ has radius of convergence $R = 0$ because we have $\lim |\frac{(n+1)!}{n!}| = +\infty$. It diverges for every $x \neq 0$. □

Examples 1–5 illustrate all the possibilities discussed in (a)–(c) prior to Theorem 23.1.

Example 6
Consider $\sum_{n=0}^{\infty} 2^{-n}x^{3n}$. This is deceptive, and it is tempting to calculate $\beta = \limsup(2^{-n})^{1/n} = \frac{1}{2}$ and conclude $R = 2$. *This is wrong* because 2^{-n} is the coefficient of x^{3n} *not* x^n, and the calculation of β must involve the coefficients a_n of x^n. We need to handle this series more carefully. The series can be written $\sum_{n=0}^{\infty} a_n x^n$ where $a_{3k} = 2^{-k}$ and $a_n = 0$ if n is not a multiple of 3. We calculate β by using the subsequence of all nonzero terms, i.e., the subsequence given by $\sigma(k) = 3k$. This yields

$$\beta = \limsup |a_n|^{1/n} = \lim_{k\to\infty} |a_{3k}|^{1/3k} = \lim_{k\to\infty} (2^{-k})^{1/3k} = 2^{-1/3}.$$

Therefore the radius of convergence is $R = \frac{1}{\beta} = 2^{1/3}$. □

One may consider more general power series of the form

$$\sum_{n=0}^{\infty} a_n(x - x_0)^n, \tag{*}$$

where x_0 is a fixed real number, but they reduce to series of the form $\sum_{n=0}^{\infty} a_n y^n$ by the change of variable $y = x - x_0$. The interval of convergence for the series (*) will be an interval centered at x_0.

Example 7
Consider the series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}(x - 1)^n. \tag{1}$$

The radius of convergence for the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} y^n$ is $R = 1$, so the interval of convergence for the series (1) is the interval $(0, 2)$ plus perhaps an endpoint or two. Direct substitution shows the series (1) converges at $x = 2$ [it's an alternating series] and diverges to $-\infty$ at $x = 0$. So the exact interval of convergence is $(0, 2]$. It turns out that the series (1) represents the function $\log_e x$ on $(0, 2]$. See Examples 1 and 2 in §26. □

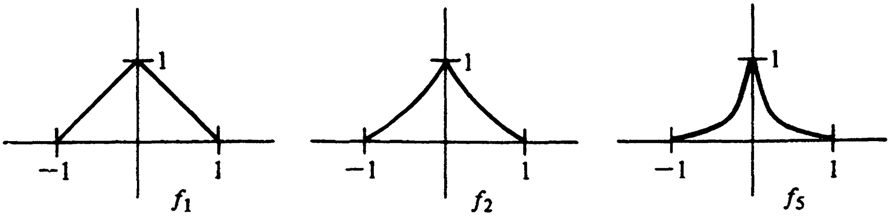

FIGURE 23.1

On of our major goals is to understand the function given by a power series

$$f(x) = \sum_{k=0}^{\infty} a_k x^k \quad \text{for} \quad |x| < R.$$

We are interested in questions like: Is f continuous? Is f differentiable? If so, can one differentiate f term-by-term:

$$f'(x) = \sum_{k=1}^{\infty} k a_k x^{k-1} \,?$$

Can one integrate f term-by-term?

Returning to the question of continuity, what reason is there to believe f is continuous? Its partial sums $f_n = \sum_{k=0}^{n} a_k x^k$ are continuous, since they are polynomials. Moreover, we have $\lim_{n\to\infty} f_n(x) = f(x)$ for $|x| < R$. Therefore f would be continuous *if* a result like the following were true: If (f_n) is a sequence of continuous functions on (a, b) and if $\lim_{n\to\infty} f_n(x) = f(x)$ for all $x \in (a, b)$, then f is continuous on (a, b). However, this fine sounding result is false!

Example 8

Let $f_n(x) = (1 - |x|)^n$ for $x \in (-1, 1)$; see Fig. 23.1. Let $f(x) = 0$ for $x \neq 0$ and let $f(0) = 1$. Then we have $\lim_{n\to\infty} f_n(x) = f(x)$ for all $x \in (-1, 1)$, since $\lim_{n\to\infty} a^n = 0$ if $|a| < 1$. Each f_n is a continuous function, but the limit function f is clearly discontinuous at $x = 0$. □

This example, as well as Exercises 23.7–23.9, may be discouraging, but it turns out that power series do converge to continuous functions. This is because

$$\lim_{n\to\infty}\sum_{k=0}^{n} a_k x^k \quad \textit{converges uniformly} \text{ to } \quad \sum_{k=0}^{\infty} a_k x^k$$

on sets $[-R_1, R_1]$ such that $R_1 < R$. The definition of uniform convergence is given in the next section, and the next two sections will be devoted to this important notion. We return to power series in §26, and again in §31.

Exercises

23.1 For each of the following power series, find the radius of convergence and determine the exact interval of convergence.

(a) $\sum n^2 x^n$
(b) $\sum (\frac{x}{n})^n$
(c) $\sum (\frac{2^n}{n^2})x^n$
(d) $\sum (\frac{n^3}{3^n})x^n$
(e) $\sum (\frac{2^n}{n!})x^n$
(f) $\sum (\frac{1}{(n+1)^2 2^n})x^n$
(g) $\sum (\frac{3^n}{n \cdot 4^n})x^n$
(h) $\sum (\frac{(-1)^n}{n^2 \cdot 4^n})x^n$

23.2 Repeat Exercise 23.1 for the following:

(a) $\sum \sqrt{n} x^n$
(b) $\sum \frac{1}{n\sqrt{n}} x^n$
(c) $\sum x^{n!}$
(d) $\sum \frac{3^n}{\sqrt{n}} x^{2n+1}$

23.3 Find the exact interval of convergence for the series in Example 6.

23.4 For $n = 0, 1, 2, 3, \ldots$, let $a_n = [\frac{4+2(-1)^n}{5}]^n$.

(a) Find $\limsup (a_n)^{1/n}$, $\liminf (a_n)^{1/n}$, $\limsup |\frac{a_{n+1}}{a_n}|$ and $\liminf |\frac{a_{n+1}}{a_n}|$.

(b) Do the series $\sum a_n$ and $\sum (-1)^n a_n$ converge? Explain briefly.

(c) Now consider the power series $\sum a_n x^n$ with the coefficients a_n as above. Find the radius of convergence and determine the exact interval of convergence for the series.

23.5 Consider a power series $\sum a_n x^n$ with radius of convergence R.

(a) Prove that if all the coefficients a_n are integers and if infinitely many of them are nonzero, then $R \leq 1$.

(b) Prove that if $\limsup |a_n| > 0$, then $R \leq 1$.

23.6 **(a)** Suppose $\sum a_n x^n$ has finite radius of convergence R and $a_n \geq 0$ for all n. Show that if the series converges at R, then it also converges at $-R$.

(b) Give an example of a power series whose interval of convergence is exactly $(-1, 1]$.

The next three exercises are designed to show that the notion of convergence of functions discussed prior to Example 8 has many defects.

23.7 For each $n \in \mathbb{N}$, let $f_n(x) = (\cos x)^n$. Each f_n is a continuous function. Nevertheless, show

(a) $\lim f_n(x) = 0$ unless x is a multiple of π,

(b) $\lim f_n(x) = 1$ if x is an even multiple of π,

(c) $\lim f_n(x)$ does not exist if x is an odd multiple of π.

23.8 For each $n \in \mathbb{N}$, let $f_n(x) = \frac{1}{n} \sin nx$. Each f_n is a differentiable function. Show

(a) $\lim f_n(x) = 0$ for all $x \in \mathbb{R}$,

(b) But $\lim f_n'(x)$ need not exist [at $x = \pi$ for instance].

23.9 Let $f_n(x) = nx^n$ for $x \in [0, 1]$ and $n \in \mathbb{N}$. Show

(a) $\lim f_n(x) = 0$ for $x \in [0, 1)$. *Hint*: Use Exercise 9.12.

(b) However, $\lim_{n\to\infty} \int_0^1 f_n(x)\, dx = 1$.

§24 Uniform Convergence

We first formalize the notion of convergence discussed prior to Example 8 in the preceding section.

24.1 Definition.
Let (f_n) be a sequence of real-valued functions defined on a set $S \subseteq \mathbb{R}$. The sequence (f_n) *converges pointwise* [i.e., at each point] to a function f defined on S if

$$\lim_{n\to\infty} f_n(x) = f(x) \quad \text{for all} \quad x \in S.$$

We often write $\lim f_n = f$ *pointwise* [*on* S] or $f_n \to f$ *pointwise* [*on* S].

Example 1
All the functions f obtained in the last section as a limit of a sequence of functions were pointwise limits. See Example 8 of §23 and Exercises 23.7–23.9. In Exercise 23.8 we have $f_n \to 0$ pointwise on $\mathbb{R}$, and in Exercise 23.9 we have $f_n \to 0$ pointwise on $[0, 1)$. □

Example 2
Let $f_n(x) = x^n$ for $x \in [0, 1]$. Then $f_n \to f$ pointwise on $[0, 1]$ where $f(x) = 0$ for $x \in [0, 1)$ and $f(1) = 1$. □

Now observe $f_n \to f$ pointwise on S means exactly the following:

$$\begin{gathered}\text{for each } \epsilon > 0 \text{ and } x \text{ in } S \text{ there exists } N \text{ such that} \\ |f_n(x) - f(x)| < \epsilon \text{ for } n > N.\end{gathered} \tag{1}$$

Note the value of N depends on both $\epsilon > 0$ and x in S. If for each $\epsilon > 0$ we could find N so that

$$|f_n(x) - f(x)| < \epsilon \quad \text{for } \textit{all} \quad x \in S \quad \text{and} \quad n > N,$$

then the values $f_n(x)$ would be "uniformly" close to the values $f(x)$. Here N would depend on ϵ but not on x. This concept is extremely useful.

24.2 Definition.
Let (f_n) be a sequence of real-valued functions defined on a set $S \subseteq \mathbb{R}$. The sequence (f_n) *converges uniformly on* S to a function f defined on S if

$$\begin{gathered}\text{for each } \epsilon > 0 \text{ there exists a number } N \text{ such that} \\ |f_n(x) - f(x)| < \epsilon \text{ for all } x \in S \text{ and all } n > N.\end{gathered} \tag{1}$$

We write $\lim f_n = f$ *uniformly on* S or $f_n \to f$ *uniformly on* S.

Note that if $f_n \to f$ uniformly on S and if $\epsilon > 0$, then there exists N such that $f(x) - \epsilon < f_n(x) < f(x) + \epsilon$ for *all* $x \in S$ and $n > N$. In other words, for $n > N$ the graph of f_n lies in the strip between the graphs of $f - \epsilon$ and $f + \epsilon$. In Fig. 24.1 the graphs of f_n for $n > N$ would all lie between the dotted curves.

We return to our earlier examples.

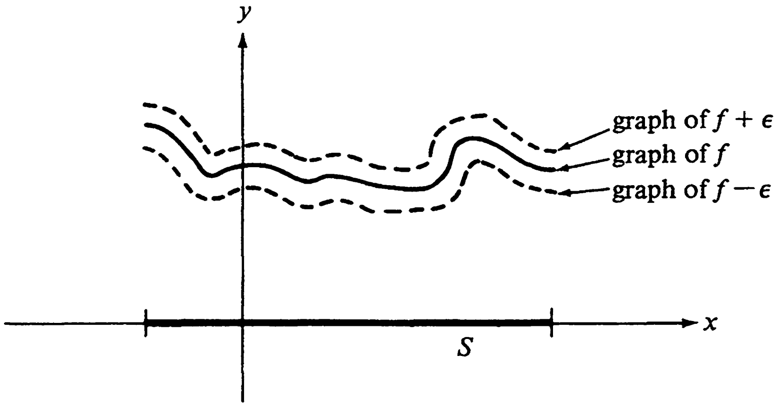

FIGURE 24.1

Example 3
Let $f_n(x) = (1-|x|)^n$ for $x \in (-1, 1)$. Also, let $f(x) = 0$ for $x \neq 0$ and $f(0) = 1$. As noted in Example 8 of §23, $f_n \to f$ pointwise on $(-1, 1)$. It turns out that the sequence (f_n) does not converge uniformly to f on $(-1, 1)$ in view of the next theorem. This can also be shown directly, as follows. Assume $f_n \to f$ uniformly on $(-1, 1)$. Then [with $\epsilon = \frac{1}{2}$ in mind] we see there exists N in $\mathbb{N}$ so that $|f(x) - f_n(x)| < \frac{1}{2}$ for *all* $x \in (-1, 1)$ and $n > N$. Hence

$$x \in (0,1) \quad \text{and} \quad n > N \quad \text{imply} \quad |(1-x)^n| < \frac{1}{2}.$$

In particular,

$$x \in (0,1) \quad \text{implies} \quad (1-x)^{N+1} < \frac{1}{2}.$$

However, this fails for sufficiently small x; for example, if we set $x = 1-2^{-1/(N+1)}$, then $1-x = 2^{-1/(N+1)}$ and $(1-x)^{N+1} = 2^{-1} = \frac{1}{2}$. This contradiction shows (f_n) does not converge uniformly to f on $(-1, 1)$ as had been assumed. □

Example 4
Let $f_n(x) = \frac{1}{n} \sin nx$ for $x \in \mathbb{R}$. Then $f_n \to 0$ pointwise on $\mathbb{R}$ as shown in Exercise 23.8. In fact, $f_n \to 0$ uniformly on $\mathbb{R}$. To see this,

let $\epsilon > 0$ and let $N = \frac{1}{\epsilon}$. Then for $n > N$ and *all* $x \in \mathbb{R}$ we have

$$|f_n(x) - 0| = \left|\frac{1}{n}\sin nx\right| \le \frac{1}{n} < \frac{1}{N} = \epsilon.$$

□

Example 5

Let $f_n(x) = nx^n$ for $x \in [0,1)$. Since $\lim_{n\to\infty} f_n(1) = \lim_{n\to\infty} n = +\infty$, we have dropped the number 1 from the domain under consideration. Then $f_n \to 0$ pointwise on $[0,1)$, as shown in Exercise 23.9. We show the convergence is *not* uniform. If it were, there would exist N in $\mathbb{N}$ such that

$$|nx^n - 0| < 1 \quad \text{for } all \quad x \in [0,1) \quad \text{and} \quad n > N.$$

In particular, we would have $(N+1)x^{N+1} < 1$ for all $x \in [0,1)$. But this fails for x sufficiently close to 1. Consider, for example, the reciprocal x of $(N+1)^{1/(N+1)}$. □

Example 6

As in Example 2, let $f_n(x) = x^n$ for $x \in [0,1]$, $f(x) = 0$ for $x \in [0,1)$ and $f(1) = 1$. Then $f_n \to f$ pointwise on $[0,1]$, but (f_n) does not converge uniformly to f on $[0,1]$, as can be seen directly or by applying the next theorem. □

24.3 Theorem.

The uniform limit of continuous functions is continuous. More precisely, let (f_n) be a sequence of functions on a set $S \subseteq \mathbb{R}$, suppose $f_n \to f$ uniformly on S, and suppose $S = \mathrm{dom}(f)$. If each f_n is continuous at x_0 in S, then f is continuous at x_0. [So if each f_n is continuous on S, then f is continuous on S.]

Proof

This involves the famous "$\frac{\epsilon}{3}$ argument." The critical inequality is

$$|f(x) - f(x_0)| \le |f(x) - f_n(x)| + |f_n(x) - f_n(x_0)| + |f_n(x_0) - f(x_0)|. \tag{1}$$

If n is large enough, the first and third terms on the right side of (1) will be small, since $f_n \to f$ uniformly. *Once such n is selected*, the continuity of f_n implies that the middle term will be small provided x is close to x_0.

For the formal proof, let $\epsilon > 0$. There exists N in $\mathbb{N}$ such that

$$n > N \quad \text{implies} \quad |f_n(x) - f(x)| < \frac{\epsilon}{3} \quad \text{for all} \quad x \in S.$$

In particular,

$$|f_{N+1}(x) - f(x)| < \frac{\epsilon}{3} \quad \text{for all} \quad x \in S. \tag{2}$$

Since f_{N+1} is continuous at x_0 there is a $\delta > 0$ such that

$$x \in S \quad \text{and} \quad |x - x_0| < \delta \quad \text{imply} \quad |f_{N+1}(x) - f_{N+1}(x_0)| < \frac{\epsilon}{3}; \tag{3}$$

see Theorem 17.2. Now we apply (1) with $n = N+1$, (2) twice [once for x and once for x_0] and (3) to conclude

$$x \in S \quad \text{and} \quad |x - x_0| < \delta \quad \text{imply} \quad |f(x) - f(x_0)| < 3 \cdot \frac{\epsilon}{3} = \epsilon.$$

This proves that f is continuous at x_0. ■

One might think this theorem would be useless in practice, since it should be easier to show a single function is continuous than to show a sequence (f_n) consists of continuous functions and the sequence converges to f uniformly. This would no doubt be true if f were given by a simple formula. But consider, for example,

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{n^2} x^n \quad \text{for} \quad x \in [-1, 1]$$

or

$$J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n (\frac{1}{2}x)^{2n}}{(n!)^2} \quad \text{for} \quad x \in \mathbb{R}.$$

The partial sums are clearly continuous, but neither f nor J_0 is given by a simple formula. Moreover, many functions that arise in mathematics and elsewhere, such as the Bessel function J_0, are defined by power series. It would be very useful to know when and where power series converge uniformly; an answer is given in §26.

24.4 Remark.
Uniform convergence can be reformulated as follows. *A sequence (f_n) of functions on a set $S \subseteq \mathbb{R}$ converges uniformly to a function f on*

S if and only if

$$\lim_{n\to\infty} \sup\{|f(x) - f_n(x)| : x \in S\} = 0. \tag{1}$$

We leave the straightforward proof to Exercise 24.12.

According to (1) we can decide whether a sequence (f_n) converges uniformly to f by calculating $\sup\{|f(x) - f_n(x)| : x \in S\}$ for each n. If $f - f_n$ is differentiable, we may use calculus to find these suprema. □

Example 7

Let $f_n(x) = \frac{x}{1+nx^2}$ for $x \in \mathbb{R}$. Clearly we have $\lim_{n\to\infty} f_n(0) = 0$. If $x \neq 0$, then $\lim_{n\to\infty}(1 + nx^2) = +\infty$, so $\lim_{n\to\infty} f_n(x) = 0$. Thus $f_n \to 0$ pointwise on $\mathbb{R}$. To find the maximum and minimum of f_n, we calculate $f_n'(x)$ and set it equal to 0. This leads to $(1 + nx^2) \cdot 1 - x(2nx) = 0$ or $1 - nx^2 = 0$. Thus $f_n'(x) = 0$ if and only if $x = \frac{\pm 1}{\sqrt{n}}$. Further analysis or a sketch of f_n leads one to conclude f_n takes its maximum at $\frac{1}{\sqrt{n}}$ and its minimum at $-\frac{1}{\sqrt{n}}$. Since $f_n\left(\pm\frac{1}{\sqrt{n}}\right) = \pm\frac{1}{2\sqrt{n}}$, we conclude

$$\lim_{n\to\infty} \sup\{|f_n(x)| : x \in S\} = \lim_{n\to\infty} \frac{1}{2\sqrt{n}} = 0.$$

Therefore $f_n \to 0$ uniformly on $\mathbb{R}$ by Remark 24.4. □

Example 8

Let $f_n(x) = n^2x^n(1-x)$ for $x \in [0, 1]$. Then we have $\lim_{n\to\infty} f_n(1) = 0$. For $x \in [0, 1)$ we have $\lim_{n\to\infty} n^2x^n = 0$ by applying Exercise 9.12, since for $x \neq 0$,

$$\frac{(n+1)^2x^{n+1}}{n^2x^n} = \left(\frac{n+1}{n}\right)^2 x \to x.$$

Hence $\lim_{n\to\infty} f_n(x) = 0$. Thus $f_n \to 0$ pointwise on $[0, 1]$. Again, to find the maximum and minimum of f_n we set its derivative equal to 0. We obtain $x^n(-1) + (1 - x)nx^{n-1} = 0$ or $x^{n-1}[n - (n + 1)x] = 0$. Since f_n takes the value 0 at both endpoints of the interval $[0, 1]$, it follows that f_n takes it maximum at $\frac{n}{n+1}$. We have

$$f_n\left(\frac{n}{n+1}\right) = n^2\left(\frac{n}{n+1}\right)^n\left(1 - \frac{n}{n+1}\right) = \frac{n^2}{n+1}\left(\frac{n}{n+1}\right)^n. \tag{1}$$

The reciprocal of $(\frac{n}{n+1})^n$ is $(1+\frac{1}{n})^n$, the nth term of a sequence which has limit e. This was mentioned, but not proved, in Example 3 of §7; a proof is given in Theorem 37.11. Therefore we have $\lim(\frac{n}{n+1})^n = \frac{1}{e}$. Since $\lim[\frac{n^2}{n+1}] = +\infty$, we conclude from (1) that $\lim f_n(\frac{n}{n+1}) = +\infty$; see Exercise 12.9(a). In particular, $\limsup\{|f_n(x)| : x \in [0,1]\} = +\infty$, so (f_n) does *not* converge uniformly to 0 on $[0,1]$. □

Exercises

24.1 Let $f_n(x) = \frac{1+2\cos^2 nx}{\sqrt{n}}$. Prove carefully that (f_n) converges uniformly to 0 on $\mathbb{R}$.

24.2 For $x \in [0,\infty)$, let $f_n(x) = \frac{x}{n}$.

(a) Find $f(x) = \lim f_n(x)$.

(b) Determine whether $f_n \to f$ uniformly on $[0,1]$.

(c) Determine whether $f_n \to f$ uniformly on $[0,\infty)$.

24.3 Repeat Exercise 24.2 for $f_n(x) = \frac{1}{1+x^n}$.

24.4 Repeat Exercise 24.2 for $f_n(x) = \frac{x^n}{1+x^n}$.

24.5 Repeat Exercise 24.2 for $f_n(x) = \frac{x^n}{n+x^n}$.

24.6 Let $f_n(x) = (x - \frac{1}{n})^2$ for $x \in [0,1]$.

(a) Does the sequence (f_n) converge pointwise on the set $[0,1]$? If so, give the limit function.

(b) Does (f_n) converge uniformly on $[0,1]$? Prove your assertion.

24.7 Repeat Exercise 24.6 for $f_n(x) = x - x^n$.

24.8 Repeat Exercise 24.6 for $f_n(x) = \sum_{k=0}^{n} x^k$.

24.9 Consider $f_n(x) = nx^n(1-x)$ for $x \in [0,1]$.

(a) Find $f(x) = \lim f_n(x)$.

(b) Does $f_n \to f$ uniformly on $[0,1]$? Justify.

(c) Does $\int_0^1 f_n(x)\,dx$ converge to $\int_0^1 f(x)\,dx$? Justify.

24.10 **(a)** Prove that if $f_n \to f$ uniformly on a set S, and if $g_n \to g$ uniformly on S, then $f_n + g_n \to f + g$ uniformly on S.

(b) Do you believe the analogue of (a) holds for products? If so, see the next exercise.

24.11 Let $f_n(x) = x$ and $g_n(x) = \frac{1}{n}$ for all $x \in \mathbb{R}$. Let $f(x) = x$ and $g(x) = 0$ for $x \in \mathbb{R}$.

(a) Observe $f_n \to f$ uniformly on $\mathbb{R}$ [obvious!] and $g_n \to g$ uniformly on $\mathbb{R}$ [almost obvious].

(b) Observe the sequence $(f_n g_n)$ does not converge uniformly to fg on $\mathbb{R}$. Compare Exercise 24.2.

24.12 Prove the assertion in Remark 24.4.

24.13 Prove that if (f_n) is a sequence of uniformly continuous functions on an interval (a, b), and if $f_n \to f$ uniformly on (a, b), then f is also uniformly continuous on (a, b). *Hint*: Try an $\frac{\epsilon}{3}$ argument as in the proof of Theorem 24.3.

24.14 Let $f_n(x) = \frac{nx}{1+n^2x^2}$ and $f(x) = 0$ for $x \in \mathbb{R}$.

(a) Show $f_n \to f$ pointwise on $\mathbb{R}$.

(b) Does $f_n \to f$ uniformly on $[0, 1]$? Justify.

(c) Does $f_n \to f$ uniformly on $[1, \infty)$? Justify.

24.15 Let $f_n(x) = \frac{nx}{1+nx}$ for $x \in [0, \infty)$.

(a) Find $f(x) = \lim f_n(x)$.

(b) Does $f_n \to f$ uniformly on $[0, 1]$? Justify.

(c) Does $f_n \to f$ uniformly on $[1, \infty)$? Justify.

24.16 Repeat Exercise 24.15 for $f_n(x) = \frac{nx}{1+nx^2}$.

24.17 Let (f_n) be a sequence of continuous functions on $[a, b]$ that converges uniformly to f on $[a, b]$. Show that if (x_n) is a sequence in $[a, b]$ and if $x_n \to x$, then $\lim_{n\to\infty} f_n(x_n) = f(x)$.

§25 More on Uniform Convergence

Our next theorem shows one can interchange integrals and *uniform* limits. The adjective "uniform" here is important; compare Exercise 23.9.

25.1 Discussion.
To prove Theorem 25.2 below we merely use some basic facts about integration which should be familiar [or believable] even if your calculus is rusty. Specifically, we use:

(a) *If* g *and* h *are integrable on* $[a, b]$ *and if* $g(x) \le h(x)$ *for all* $x \in [a, b]$, *then* $\int_a^b g(x)\,dx \le \int_a^b h(x)\,dx$. See Theorem 33.4(i).

We also use the following corollary:

(b) *If* g *is integrable on* $[a, b]$, *then*

$$\left| \int_a^b g(x)\,dx \right| \le \int_a^b |g(x)|\,dx.$$

Continuous functions on closed intervals are integrable, as noted in Discussion 19.3 and proved in Theorem 33.2. □

25.2 Theorem.
Let (f_n) *be a sequence of continuous functions on* $[a, b]$, *and suppose* $f_n \to f$ *uniformly on* $[a, b]$. *Then*

$$\lim_{n\to\infty} \int_a^b f_n(x)\,dx = \int_a^b f(x)\,dx. \tag{1}$$

Proof
By Theorem 24.3 f is continuous, so the functions $f_n - f$ are all integrable on $[a, b]$. Let $\epsilon > 0$. Since $f_n \to f$ uniformly on $[a, b]$, there exists a number N such that $|f_n(x) - f(x)| < \frac{\epsilon}{b-a}$ for all $x \in [a, b]$ and all $n > N$. Consequently $n > N$ implies

$$\begin{aligned} \left| \int_a^b f_n(x)\,dx - \int_a^b f(x)\,dx \right| &= \left| \int_a^b [f_n(x) - f(x)]\,dx \right| \\ &\le \int_a^b |f_n(x) - f(x)|\,dx \le \int_a^b \frac{\epsilon}{b-a}\,dx = \epsilon. \end{aligned}$$

The first $\le$ follows from Discussion 25.1(b) applied to $g = f_n - f$ and the second $\le$ follows from Discussion 25.1(a) applied to $g = |f_n - f|$ and $h = \frac{\epsilon}{b-a}$; h happens to be a constant function, but this does no harm.

The last paragraph shows that given $\epsilon > 0$, there exists N such that $|\int_a^b f_n(x)\,dx - \int_a^b f(x)\,dx| \le \epsilon$ for $n > N$. Therefore (1) holds. ■

Recall one of the advantages of the notion of Cauchy sequence: A sequence (s_n) of real numbers can be shown to converge *without knowing its limit* by simply verifying that it is a Cauchy sequence. Clearly a similar result for sequences of functions would be valuable, since it is likely that we will not know the limit function in advance. What we need is the idea of "uniformly Cauchy."

25.3 Definition.
A sequence (f_n) of functions defined on a set $S \subseteq \mathbb{R}$ is *uniformly Cauchy on* S if

$$\begin{array}{c}\text{for each } \epsilon > 0 \text{ there exists a number } N \text{ such that} \\ |f_n(x) - f_m(x)| < \epsilon \text{ for all } x \in S \text{ and all } m, n > N.\end{array} \tag{1}$$

Compare this definition with that of a Cauchy sequence of real numbers [Definition 10.8] and that of uniform convergence [Definition 24.2]. It is an easy exercise to show uniformly convergent sequences of functions are uniformly Cauchy; see Exercise 25.4. The interesting and useful result is the converse, just as in the case of sequences of real numbers.

25.4 Theorem.
Let (f_n) be a sequence of functions defined and uniformly Cauchy on a set $S \subseteq \mathbb{R}$. Then there exists a function f on S such that $f_n \to f$ uniformly on S.

Proof
First we have to "find" f. We begin by showing

$$\begin{array}{l}\text{for each } x_0 \in S \text{ the sequence } (f_n(x_0)) \text{ is a Cauchy} \\ \text{sequence of real numbers.}\end{array} \tag{1}$$

For each $\epsilon > 0$, there exists N such that

$$|f_n(x) - f_m(x)| < \epsilon \quad \text{for} \quad x \in S \quad \text{and} \quad m, n > N.$$

In particular, we have

$$|f_n(x_0) - f_m(x_0)| < \epsilon \quad \text{for} \quad m, n > N.$$

This shows $(f_n(x_0))$ is a Cauchy sequence, so (1) holds.

Now for each x in S, assertion (1) implies $\lim_{n\to\infty} f_n(x)$ exists; this is proved in Theorem 10.11 which in the end depends on the Completeness Axiom 4.4. Hence we *define* $f(x) = \lim_{n\to\infty} f_n(x)$. This defines a function f on S such that $f_n \to f$ *pointwise* on S.

Now that we have "found" f, we need to prove $f_n \to f$ uniformly on S. Let $\epsilon > 0$. There is a number N such that

$$|f_n(x) - f_m(x)| < \frac{\epsilon}{2} \quad \text{for all} \quad x \in S \quad \text{and all} \quad m, n > N. \tag{2}$$

Consider $m > N$ and $x \in S$. Assertion (2) tells us that $f_n(x)$ lies in the open interval $(f_m(x) - \frac{\epsilon}{2}, f_m(x) + \frac{\epsilon}{2})$ for all $n > N$. Therefore, as noted in Exercise 8.9, the limit $f(x) = \lim_{n\to\infty} f_n(x)$ lies in the closed interval $[f_m(x) - \frac{\epsilon}{2}, f_m(x) + \frac{\epsilon}{2}]$. In other words,

$$|f(x) - f_m(x)| \le \frac{\epsilon}{2} \quad \text{for all} \quad x \in S \quad \text{and} \quad m > N.$$

Then of course

$$|f(x) - f_m(x)| < \epsilon \quad \text{for all} \quad x \in S \quad \text{and} \quad m > N.$$

This shows $f_m \to f$ uniformly on S, as desired. ■

Theorem 25.4 is especially useful for "series of functions." Let us recall what $\sum_{k=1}^{\infty} a_k$ signifies when the a_k's are real numbers. This signifies $\lim_{n\to\infty} \sum_{k=1}^{n} a_k$ provided this limit exists [as a real number, $+\infty$ or $-\infty$]. Otherwise the symbol $\sum_{k=1}^{\infty} a_k$ has no meaning. Thus the infinite series is the limit of the sequence of partial sums $\sum_{k=1}^{n} a_k$. Similar remarks apply to series of functions. A *series of functions* is an expression $\sum_{k=0}^{\infty} g_k$ or $\sum_{k=0}^{\infty} g_k(x)$ which makes sense provided the sequence of partial sums $\sum_{k=0}^{n} g_k$ converges, or diverges to $+\infty$ or $-\infty$ pointwise. If the sequence of partial sums converges uniformly on a set S to $\sum_{k=0}^{\infty} g_k$, then we say the *series is uniformly convergent on S*.

Example 1

Any power series is a series of functions, since $\sum_{k=0}^{\infty} a_k x^k$ has the form $\sum_{k=0}^{\infty} g_k$ where $g_k(x) = a_k x^k$ for all x. □

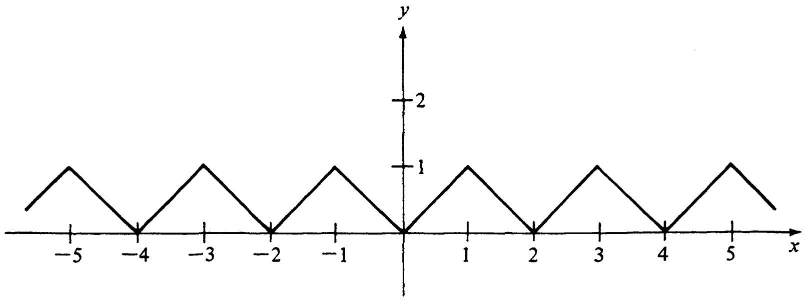

FIGURE 25.1

Example 2

$\sum_{k=0}^{\infty} \frac{x^k}{1+x^k}$ is a series of functions, but is not a power series, at least not in its present form. This is a series $\sum_{k=0}^{\infty} g_k$ where $g_0(x) = \frac{1}{2}$ for all x, $g_1(x) = \frac{x}{1+x}$ for all x, $g_2(x) = \frac{x^2}{1+x^2}$ for all x, etc. □

Example 3

Let g be the function drawn in Fig. 25.1, and let $g_n(x) = g(4^n x)$ for all $x \in \mathbb{R}$. Then $\sum_{n=0}^{\infty} (\frac{3}{4})^n g_n(x)$ is a series of functions. The limit function f is continuous on $\mathbb{R}$, but has the amazing property that it is not differentiable at any point! The proof of the nondifferentiability of f is somewhat delicate; see [62, 7.18]. A similar example is given in Example 38.1 on page 348. □

Theorems for sequences of functions translate easily into theorems for series of functions. Here is an example.

25.5 Theorem.

Consider a series $\sum_{k=0}^{\infty} g_k$ of functions on a set $S \subseteq \mathbb{R}$. Suppose each g_k is continuous on S and the series converges uniformly on S. Then the series $\sum_{k=0}^{\infty} g_k$ represents a continuous function on S.

Proof

Each partial sum $f_n = \sum_{k=1}^{n} g_k$ is continuous and the sequence (f_n) converges uniformly on S. Hence the limit function is continuous by Theorem 24.3. ■

Recall the Cauchy criterion for series $\sum a_k$ given in Definition 14.3:

$$\text{for each } \epsilon > 0 \text{ there exists a number } N \text{ such that} \\ n \geq m > N \text{ implies } |\textstyle\sum_{k=m}^{n} a_k| < \epsilon. \tag{*}$$

The analogue for series of functions is also useful. The sequence of partial sums of a series $\sum_{k=0}^{\infty} g_k$ of functions is uniformly Cauchy on a set S if and only if the series satisfies the *Cauchy criterion* [*uniformly on* S]:

$$\text{for each } \epsilon > 0 \text{ there exists a number } N \text{ such that} \\ n \geq m > N \text{ implies } |\textstyle\sum_{k=m}^{n} g_k(x)| < \epsilon \text{ for all } x \in S. \tag{**}$$

25.6 Theorem.
If a series $\sum_{k=0}^{\infty} g_k$ of functions satisfies the Cauchy criterion uniformly on a set S, then the series converges uniformly on S.

Proof
Let $f_n = \sum_{k=0}^{n} g_k$. The sequence (f_n) of partial sums is uniformly Cauchy on S, so (f_n) converges uniformly on S by Theorem 25.4. ■

Here is a useful corollary.

25.7 Weierstrass M-test.
Let (M_k) be a sequence of nonnegative real numbers where $\sum M_k < \infty$. If $|g_k(x)| \leq M_k$ for all x in a set S, then $\sum g_k$ converges uniformly on S.

Proof
To verify the Cauchy criterion on S, let $\epsilon > 0$. Since the series $\sum M_k$ converges, it satisfies the Cauchy criterion in Definition 14.3. So there exists a number N such that

$$n \geq m > N \quad \text{implies} \quad \sum_{k=m}^{n} M_k < \epsilon.$$

Hence if $n \geq m > N$ and x is in S, then

$$\left|\sum_{k=m}^{n} g_k(x)\right| \leq \sum_{k=m}^{n} |g_k(x)| \leq \sum_{k=m}^{n} M_k < \epsilon.$$

Thus the series $\sum g_k$ satisfies the Cauchy criterion uniformly on S, and Theorem 25.6 shows it converges uniformly on S. ■

Example 4
Show $\sum_{n=1}^{\infty} 2^{-n}x^n$ represents a continuous function f on $(-2,2)$, but the convergence is not uniform. □

Solution
This is a power series with radius of convergence 2. Clearly the series does not converge at $x = 2$ or at $x = -2$, so its interval of convergence is $(-2,2)$.

Consider $0 < a < 2$ and note $\sum_{n=1}^{\infty} 2^{-n}a^n = \sum_{n=1}^{\infty}(\frac{a}{2})^n$ converges. Since $|2^{-n}x^n| \le 2^{-n}a^n = (\frac{a}{2})^n$ for $x \in [-a,a]$, the Weierstrass M-test 25.7 shows the series $\sum_{n=1}^{\infty} 2^{-n}x^n$ converges uniformly to a function on $[-a,a]$. By Theorem 25.5 the limit function f is continuous at each point of the set $[-a,a]$. Since a can be any number less than 2, we conclude f represents a continuous function on $(-2,2)$.

Since we have $\sup\{|2^{-n}x^n| : x \in (-2,2)\} = 1$ for each n, the convergence of the series cannot be uniform on $(-2,2)$ in view of the next example. □

Example 5
Show that if the series $\sum g_n$ converges uniformly on a set S, then

$$\lim_{n\to\infty} \sup\{|g_n(x)| : x \in S\} = 0. \tag{1}$$

□

Solution
Let $\epsilon > 0$. Since the series $\sum g_n$ satisfies the Cauchy criterion, there exists N such that

$$n \ge m > N \quad \text{implies} \quad \left|\sum_{k=m}^{n} g_k(x)\right| < \epsilon \quad \text{for all} \quad x \in S.$$

In particular,

$$n > N \quad \text{implies} \quad |g_n(x)| < \epsilon \quad \text{for all} \quad x \in S.$$

Therefore

$$n > N \quad \text{implies} \quad \sup\{|g_n(x)| : x \in S\} \le \epsilon.$$

This establishes (1). □

Exercises

25.1 Derive Discussions 25.1(b) from 25.1(a). *Hint*: Apply (a) twice, once to g and $|g|$ and once to $-|g|$ and g.

25.2 Let $f_n(x) = \frac{x^n}{n}$. Show (f_n) is uniformly convergent on $[-1, 1]$ and specify the limit function.

25.3 Let $f_n(x) = \frac{n+\cos x}{2n+\sin^2 x}$ for all real numbers x.

(a) Show (f_n) converges uniformly on $\mathbb{R}$. *Hint*: First decide what the limit function is; then show (f_n) converges uniformly to it.

(b) Calculate $\lim_{n\to\infty} \int_2^7 f_n(x)\,dx$. *Hint*: Don't integrate f_n.

25.4 Let (f_n) be a sequence of functions on a set $S \subseteq \mathbb{R}$, and suppose $f_n \to f$ uniformly on S. Prove (f_n) is uniformly Cauchy on S. *Hint*: Use the proof of Lemma 10.9 on page 63 as a model, but be careful.

25.5 Let (f_n) be a sequence of bounded functions on a set S, and suppose $f_n \to f$ uniformly on S. Prove f is a bounded function on S.

25.6 **(a)** Show that if $\sum |a_k| < \infty$, then $\sum a_k x^k$ converges uniformly on $[-1, 1]$ to a continuous function.

(b) Does $\sum_{n=1}^{\infty} \frac{1}{n^2} x^n$ represent a continuous function on $[-1, 1]$?

25.7 Show $\sum_{n=1}^{\infty} \frac{1}{n^2} \cos nx$ converges uniformly on $\mathbb{R}$ to a continuous function.

25.8 Show $\sum_{n=1}^{\infty} \frac{x^n}{n^2 2^n}$ has radius of convergence 2 and the series converges uniformly to a continuous function on $[-2, 2]$.

25.9 **(a)** Let $0 < a < 1$. Show the series $\sum_{n=0}^{\infty} x^n$ converges uniformly on $[-a, a]$ to $\frac{1}{1-x}$.

(b) Does the series $\sum_{n=0}^{\infty} x^n$ converge uniformly on $(-1, 1)$ to $\frac{1}{1-x}$? Explain.

25.10 **(a)** Show $\sum \frac{x^n}{1+x^n}$ converges for $x \in [0, 1)$.

(b) Show that the series converges uniformly on $[0, a]$ for each a, $0 < a < 1$.

(c) Does the series converge uniformly on $[0, 1)$? Explain.

25.11 **(a)** Sketch the functions g_0, g_1, g_2 and g_3 in Example 3.

(b) Prove the function f in Example 3 is continuous.

25.12 Suppose $\sum_{k=1}^{\infty} g_k$ is a series of continuous functions g_k on $[a,b]$ that converges uniformly to g on $[a,b]$. Prove

$$\int_a^b g(x)\,dx = \sum_{k=1}^{\infty} \int_a^b g_k(x)\,dx.$$

25.13 Suppose $\sum_{k=1}^{\infty} g_k$ and $\sum_{k=1}^{\infty} h_k$ converge uniformly on a set S. Show $\sum_{k=1}^{\infty}(g_k + h_k)$ converges uniformly on S.

25.14 Prove that if $\sum g_k$ converges uniformly on a set S and if h is a bounded function on S, then $\sum hg_k$ converges uniformly on S.

25.15 Let (f_n) be a sequence of continuous functions on $[a,b]$.

(a) Suppose that, for each x in $[a,b]$, $(f_n(x))$ is a decreasing sequence of real numbers. Prove that if $f_n \to 0$ pointwise on $[a,b]$, then $f_n \to 0$ uniformly on $[a,b]$. *Hint*: If not, there exists $\epsilon > 0$ and a sequence (x_n) in $[a,b]$ such that $f_n(x_n) \geq \epsilon$ for all n. Obtain a contradiction.

(b) Suppose that, for each x in $[a,b]$, $(f_n(x))$ is an increasing sequence of real numbers. Prove that if $f_n \to f$ pointwise on $[a,b]$ and if f is continuous on $[a,b]$, then $f_n \to f$ uniformly on $[a,b]$. This is *Dini's theorem*.

§26 Differentiation and Integration of Power Series

The following result was mentioned in §23 after Example 8.

26.1 Theorem.
Let $\sum_{n=0}^{\infty} a_n x^n$ be a power series with radius of convergence $R > 0$ [possibly $R = +\infty$]. If $0 < R_1 < R$, then the power series converges uniformly on $[-R_1, R_1]$ to a continuous function.

Proof
Consider $0 < R_1 < R$. A glance at Theorem 23.1 shows the series $\sum a_n x^n$ and $\sum |a_n| x^n$ have the same radius of convergence, since β and R are defined in terms of $|a_n|$. Since $|R_1| < R$, we have $\sum |a_n| R_1^n < \infty$. Clearly we have $|a_n x^n| \leq |a_n| R_1^n$ for all x in

$[-R_1, R_1]$, so the series $\sum a_n x^n$ converges uniformly on $[-R_1, R_1]$ by the Weierstrass M-test 25.7. The limit function is continuous at each point of $[-R_1, R_1]$ by Theorem 25.5. ■

26.2 Corollary.
The power series $\sum a_n x^n$ converges to a continuous function on the open interval $(-R, R)$.

Proof
If $x_0 \in (-R, R)$, then $x_0 \in (-R_1, R_1)$ for some $R_1 < R$. The theorem shows the limit of the series is continuous at x_0. ■

We emphasize that a power series need *not* converge uniformly on its interval of convergence though it might; see Example 4 of §25 and Exercise 25.8.

We are going to differentiate and integrate power series term-by-term, so clearly it would be useful to know where the new series converge. The next lemma tells us.

26.3 Lemma.
If the power series $\sum_{n=0}^{\infty} a_n x^n$ has radius of convergence R, then the power series

$$\sum_{n=1}^{\infty} n a_n x^{n-1} \quad \textit{and} \quad \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}$$

also have radius of convergence R.

Proof
First observe the series $\sum n a_n x^{n-1}$ and $\sum n a_n x^n$ have the same radius of convergence: since the second series is x times the first series, they converge for exactly the same values of x. Likewise $\sum \frac{a_n}{n+1} x^{n+1}$ and $\sum \frac{a_n}{n+1} x^n$ have the same radius of convergence.

Next recall $R = \frac{1}{\beta}$ where $\beta = \limsup |a_n|^{1/n}$. For the series $\sum n a_n x^n$, we consider $\limsup (n|a_n|)^{1/n} = \limsup n^{1/n} |a_n|^{1/n}$. By Theorem 9.7(c) on page 48, we have $\lim n^{1/n} = 1$, so $\limsup (n|a_n|)^{1/n} = \beta$ by Theorem 12.1 on page 78. Hence the series $\sum n a_n x^n$ has radius of convergence R.

For the series $\sum \frac{a_n}{n+1} x^n$, we consider $\limsup(\frac{|a_n|}{n+1})^{1/n}$. It is easy to show $\lim(n+1)^{1/n} = 1$; therefore $\lim(\frac{1}{n+1})^{1/n} = 1$. Hence by Theorem 12.1 we have $\limsup(\frac{|a_n|}{n+1})^{1/n} = \beta$, so the series $\sum \frac{a_n}{n+1} x^n$ has radius of convergence R. ■

26.4 Theorem.
Suppose $f(x) = \sum_{n=0}^{\infty} a_n x^n$ has radius of convergence $R > 0$. Then

$$\int_0^x f(t)\, dt = \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1} \quad \textit{for} \quad |x| < R. \tag{1}$$

Proof
We fix x and assume $x < 0$; the case $x > 0$ is similar [Exercise 26.1]. On the interval $[x, 0]$, the sequence of partial sums $\sum_{k=0}^{n} a_k t^k$ converges uniformly to $f(t)$ by Theorem 26.1. Consequently, by Theorem 25.2 we have

$$\int_x^0 f(t)\, dt = \lim_{n\to\infty} \int_x^0 \left(\sum_{k=0}^{n} a_k t^k \right) dt = \lim_{n\to\infty} \sum_{k=0}^{n} a_k \int_x^0 t^k\, dt$$
$$= \lim_{n\to\infty} \sum_{k=0}^{n} a_k \left[\frac{0^{k+1} - x^{k+1}}{k+1} \right] = -\sum_{k=0}^{\infty} \frac{a_k}{k+1} x^{k+1}. \tag{2}$$

The second equality is valid because we can interchange integrals and *finite* sums; this is a basic property of integrals [Theorem 33.3]. Since $\int_0^x f(t)\, dt = -\int_x^0 f(t)\, dt$, Eq. (2) implies Eq. (1). ■

The theorem just proved shows that a power series can be integrated term-by-term inside its interval of convergence. Term-by-term differentiation is also legal.

26.5 Theorem.
Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$ have radius of convergence $R > 0$. Then f is differentiable on $(-R, R)$ and

$$f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} \quad \textit{for} \quad |x| < R. \tag{1}$$

The proof of Theorem 26.4 was a straightforward application of Theorem 25.2, but the direct analogue of Theorem 25.2 for derivatives is not true [see Exercise 23.8 and Example 4 of §24]. So we give a devious indirect proof of the theorem.

Proof

We begin with the series $g(x) = \sum_{n=1}^{\infty} na_n x^{n-1}$ and observe this series converges for $|x| < R$ by Lemma 26.3. Theorem 26.4 shows that we can *integrate* g term-by-term:

$$\int_0^x g(t)\,dt = \sum_{n=1}^{\infty} a_n x^n = f(x) - a_0 \quad \text{for} \quad |x| < R.$$

Thus if $0 < R_1 < R$, then

$$f(x) = \int_{-R_1}^{x} g(t)\,dt + k \quad \text{for} \quad |x| \le R_1,$$

where k is a constant; in fact, $k = a_0 - \int_{-R_1}^{0} g(t)\,dt$. Since g is continuous, one of the versions of the Fundamental Theorem of Calculus [Theorem 34.3] shows f is differentiable and $f'(x) = g(x)$. Thus

$$f'(x) = g(x) = \sum_{n=1}^{\infty} na_n x^{n-1} \quad \text{for} \quad |x| < R.$$

■

Example 1

Recall

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \quad \text{for} \quad |x| < 1. \tag{1}$$

Differentiating term-by-term, we obtain

$$\sum_{n=1}^{\infty} nx^{n-1} = \frac{1}{(1-x)^2} \quad \text{for} \quad |x| < 1.$$

Integrating (1) term-by-term, we get

$$\sum_{n=0}^{\infty} \frac{1}{n+1} x^{n+1} = \int_0^x \frac{1}{1-t}\,dt = -\log_e(1-x)$$

or

$$\log_e(1-x) = -\sum_{n=1}^{\infty} \frac{1}{n} x^n \quad \text{for} \quad |x| < 1. \tag{2}$$

Replacing x by $-x$, we find

$$\log_e(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \quad \text{for} \quad |x| < 1. \tag{3}$$

It turns out that this equality is also valid for $x = 1$ [see Example 2], so we have the interesting identity

$$\log_e 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots. \tag{4}$$

In Eq. (2) set $x = \frac{m-1}{m}$. Then

$$\sum_{n=1}^{\infty} \frac{1}{n} \left(\frac{m-1}{m}\right)^n = -\log_e\left(1 - \frac{m-1}{m}\right) = -\log_e\left(\frac{1}{m}\right) = \log_e m.$$

Hence we have

$$\sum_{n=1}^{\infty} \frac{1}{n} \geq \sum_{n=1}^{\infty} \frac{1}{n} \left(\frac{m-1}{m}\right)^n = \log_e m \quad \text{for all} \quad m.$$

Here is yet another proof that $\sum_{n=1}^{\infty} \frac{1}{n} = +\infty$. □

To establish (4) we need a relatively difficult theorem about convergence of a power series at the endpoints of its interval of convergence.

26.6 Abel's Theorem.
Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$ be a power series with finite positive radius of convergence R. If the series converges at $x = R$, then f is continuous at $x = R$. If the series converges at $x = -R$, then f is continuous at $x = -R$.

Example 2
As promised, we return to (3) in Example 1:

$$\log_e(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \quad \text{for} \quad |x| < 1.$$

For $x = 1$ the series converges by the Alternating Series Theorem 15.3. Thus the series represents a function f on $(-1, 1]$ that is continuous at $x = 1$ by Abel's theorem. The function $\log_e(1+x)$ is also continuous at $x = 1$, so the functions agree at $x = 1$. [In detail, if (x_n) is a sequence in $(-1, 1)$ converging to 1, then $f(1) = \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} \log_e(1+x_n) = \log_e 2$.] Therefore we have

$$\log_e 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots.$$

Another proof of this identity is given in Example 2 of §31. □

Example 3

Recall $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$ for $|x| < 1$. Note that at $x = -1$ the function $\frac{1}{1-x}$ is continuous and takes the value $\frac{1}{2}$. However, the series does *not* converge for $x = -1$, so Abel's theorem does not apply. □

Proof of Abel's Theorem

The heart of the proof is in Case 1.

Case 1. Suppose $f(x) = \sum_{n=0}^{\infty} a_n x^n$ has radius of convergence 1 and the series converges at $x = 1$. We will prove f is continuous on $[0, 1]$. By subtracting a constant from f, we may assume $f(1) = \sum_{n=0}^{\infty} a_n = 0$. Let $f_n(x) = \sum_{k=0}^{n} a_k x^k$ and $s_n = \sum_{k=0}^{n} a_k = f_n(1)$ for $n = 0, 1, 2, \ldots$. Since $f_n(x) \to f(x)$ pointwise on $[0, 1]$ and each f_n is continuous, Theorem 24.3 on page 196 shows it suffices to show $f_n \to f$ uniformly on $[0, 1]$. Theorem 25.4 on page 202 shows it suffices to show the convergence is uniformly Cauchy.

For $m < n$, we have

$$\begin{aligned} f_n(x) - f_m(x) &= \sum_{k=m+1}^{n} a_k x^k = \sum_{k=m+1}^{n} (s_k - s_{k-1}) x^k \\ &= \sum_{k=m+1}^{n} s_k x^k - x \sum_{k=m+1}^{n} s_{k-1} x^{k-1} \\ &= \sum_{k=m+1}^{n} s_k x^k - x \sum_{k=m}^{n-1} s_k x^k, \end{aligned}$$

and therefore

$$f_n(x) - f_m(x) = s_n x^n - s_m x^{m+1} + (1-x)\sum_{k=m+1}^{n-1} s_k x^k. \qquad (1)$$

Since $\lim s_n = \sum_{k=0}^{\infty} a_k = f(1) = 0$, given $\epsilon > 0$, there is an integer N so that $|s_n| < \frac{\epsilon}{3}$ for all $n \geq N$. Then for $n > m \geq N$ and x in $[0,1)$, we have

$$\begin{aligned}\left|(1-x)\sum_{k=m+1}^{n-1} s_k x^k\right| &\leq \frac{\epsilon}{3}(1-x)\sum_{k=m+1}^{n-1} x^k \\ &= \frac{\epsilon}{3}(1-x)x^{m+1}\frac{1-x^{n-m-1}}{1-x} < \frac{\epsilon}{3}. \qquad (2)\end{aligned}$$

The first term in inequality (2) is also less than $\frac{\epsilon}{3}$ for $x = 1$. Therefore, for $n > m \geq N$ and x in $[0,1]$, (1) and (2) show

$$|f_n(x) - f_m(x)| \leq |s_n|x^n + |s_m|x^{m+1} + \frac{\epsilon}{3} < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.$$

Thus the sequence (f_n) is uniformly Cauchy on $[0,1]$, and its limit f is continuous.

Case 2. Suppose $f(x) = \sum_{n=0}^{\infty} a_n x^n$ has radius of convergence R, $0 < R < \infty$, and the series converges at $x = R$. Let $g(x) = f(Rx)$ and note that

$$g(x) = \sum_{n=0}^{\infty} a_n R^n x^n \quad \text{for} \quad |x| < 1.$$

This series has radius of convergence 1, and it converges at $x = 1$. By Case 1, g is continuous at $x = 1$. Since $f(x) = g(\frac{x}{R})$, it follows that f is continuous at $x = R$.

Case 3. Suppose $f(x) = \sum_{n=0}^{\infty} a_n x^n$ has radius of convergence R, $0 < R < \infty$, and the series converges at $x = -R$. Let $h(x) = f(-x)$ and note that

$$h(x) = \sum_{n=0}^{\infty} (-1)^n a_n x^n \quad \text{for} \quad |x| < R.$$

The series for h converges at $x = R$, so h is continuous at $x = R$ by Case 2. It follows that $f(x) = h(-x)$ is continuous at $x = -R$. ■

The point of view in our extremely brief introduction to power series has been: For a given power series $\sum a_n x^n$, what can one say about the function $f(x) = \sum a_n x^n$? This point of view was misleading. Often, in real life, one begins with a function f and seeks a power series that represents the function for some or all values of x. This is because power series, being limits of polynomials, are in some sense basic objects.

If we have $f(x) = \sum_{n=0}^{\infty} a_n x^n$ for $|x| < R$, then we can differentiate f term-by-term forever. At each step, we may calculate the kth derivative of f at 0, written $f^{(k)}(0)$. It is easy to show $f^{(k)}(0) = k! a_k$ for $k \geq 0$. This tells us that *if* f can be represented by a power series, then that power series must be $\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k$. This is the *Taylor series* for f about 0. Frequently, but not always, the Taylor series will agree with f on the interval of convergence. This turns out to be true for many familiar functions. Thus the following relations can be proved:

$$e^x = \sum_{k=0}^{\infty} \frac{1}{k!} x^k, \quad \cos x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k}, \quad \sin x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1}$$

for all x in $\mathbb{R}$. A detailed study of Taylor series is given in §31.

Exercises

26.1 Prove Theorem 26.4 for $x > 0$.

26.2 **(a)** Observe $\sum_{n=1}^{\infty} n x^n = \frac{x}{(1-x)^2}$ for $|x| < 1$; see Example 1.

(b) Evaluate $\sum_{n=1}^{\infty} \frac{n}{2^n}$. Compare with Exercise 14.13(d).

(c) Evaluate $\sum_{n=1}^{\infty} \frac{n}{3^n}$ and $\sum_{n=1}^{\infty} \frac{(-1)^n n}{3^n}$.

26.3 **(a)** Use Exercise 26.2 to derive an explicit formula for $\sum_{n=1}^{\infty} n^2 x^n$.

(b) Evaluate $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$ and $\sum_{n=1}^{\infty} \frac{n^2}{3^n}$.

26.4 **(a)** Observe $e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^{2n}$ for $x \in \mathbb{R}$, since we have $e^x = \sum_{n=1}^{\infty} \frac{1}{n!} x^n$ for $x \in \mathbb{R}$.

(b) Express $F(x) = \int_0^x e^{-t^2}\, dt$ as a power series.

26.5 Let $f(x) = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$ for $x \in \mathbb{R}$. Show $f' = f$. Do *not* use the fact that $f(x) = e^x$; this is true but has not been established at this point in the text.

26.6 Let $s(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$ and $c(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$ for $x \in \mathbb{R}$.

(a) Prove $s' = c$ and $c' = -s$.

(b) Prove $(s^2 + c^2)' = 0$.

(c) Prove $s^2 + c^2 = 1$.

Actually $s(x) = \sin x$ and $c(x) = \cos x$, but you do *not* need these facts.

26.7 Let $f(x) = |x|$ for $x \in \mathbb{R}$. Is there a power series $\sum a_n x^n$ such that $f(x) = \sum_{n=0}^{\infty} a_n x^n$ for all x? Discuss.

26.8 **(a)** Show $\sum_{n=0}^{\infty} (1)^n x^{2n} = \frac{1}{1+x^2}$ for $x \in (-1, 1)$. *Hint*: $\sum_{n=0}^{\infty} y^n = \frac{1}{1-y}$. Let $y = -x^2$.

(b) Show $\arctan x = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1}$ for $x \in (-1, 1)$.

(c) Show the equality in (b) also holds for $x = 1$. Use this to find a nice formula for π.

(d) What happens at $x = -1$?

§27 * Weierstrass's Approximation Theorem

Suppose a power series has radius of convergence greater than 1, and let f denote the function given by the power series. Theorem 26.1 tells us that the partial sums of the power series get uniformly close to f on $[-1, 1]$. In other words, f can be approximated uniformly on $[-1, 1]$ by polynomials. Weierstrass's approximation theorem is a generalization of this last observation, for it tells us that *any* continuous function on $[-1, 1]$ can be uniformly approximated by polynomials on $[-1, 1]$. This result is quite different because such a function need not be given by a power series; see Exercise 26.7. The approximation theorem is valid for any closed interval $[a, b]$ and can be deduced easily from the case $[0, 1]$; see Exercise 27.1.

We give the beautiful proof due to S. N. Bernstein. Bernstein was motivated by probabilistic considerations, but we will not use any probability here. One of the attractive features of Bernstein's proof is that the approximating polynomials will be given explicitly. There are more abstract proofs in which this is not the case. On the other hand, the abstract proofs lead to far-reaching and important generalizations. See the treatment in [31] or [62].

We need some preliminary facts about polynomials involving binomial coefficients.

27.1 Lemma.
For every $x \in \mathbb{R}$ and $n \geq 0$, we have

$$\sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} = 1.$$

Proof
This is just the binomial theorem [Exercise 1.12] applied to $a = x$ and $b = 1 - x$, since in this case $(a+b)^n = 1^n = 1$. ■

27.2 Lemma.
For $x \in \mathbb{R}$ and $n \geq 0$, we have

$$\sum_{k=0}^{n} (nx-k)^2 \binom{n}{k} x^k (1-x)^{n-k} = nx(1-x) \leq \frac{n}{4}. \tag{1}$$

Proof
Since $k\binom{n}{k} = n\binom{n-1}{k-1}$ for $k \geq 1$, we have

$$\begin{aligned} \sum_{k=0}^{n} k\binom{n}{k} x^k (1-x)^{n-k} &= n \sum_{k=1}^{n} \binom{n-1}{k-1} x^k (1-x)^{n-k} \\ &= nx \sum_{j=0}^{n-1} \binom{n-1}{j} x^j (1-x)^{n-1-j} \\ &= nx. \end{aligned} \tag{2}$$

Since $k(k-1)\binom{n}{k} = n(n-1)\binom{n-2}{k-2}$ for $k \geq 2$, we have

$$\begin{aligned} \sum_{k=0}^{n} k(k-1)\binom{n}{k} x^k (1-x)^{n-k} &= n(n-1)x^2 \sum_{j=0}^{n-2} \binom{n-2}{j} x^j (1-x)^{n-2-j} \\ &= n(n-1)x^2. \end{aligned} \tag{3}$$

Adding the results in (2) and (3), we find

$$\sum_{k=0}^{n} k^2 \binom{n}{k} x^k (1-x)^{n-k} = n(n-1)x^2 + nx = n^2x^2 + nx(1-x). \quad (4)$$

Since $(nx-k)^2 = n^2x^2 - 2nx \cdot k + k^2$, we use Lemma 27.1, (2) and (4) to obtain

$$\sum_{k=0}^{n} (nx-k)^2 \binom{n}{k} x^k (1-x)^{n-k} = n^2x^2 - 2nx(nx) + [n^2x^2 + nx(1-x)]$$
$$= nx(1-x).$$

This establishes the equality in (1). The inequality in (1) simply reflects the inequality $x(1-x) \le \frac{1}{4}$, which is equivalent to $4x^2 - 4x + 1 \ge 0$ or $(2x-1)^2 \ge 0$. ■

27.3 Definition.
Let f be a function defined on $[0,1]$. The polynomials $B_n f$ defined by

$$B_n f(x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) \cdot \binom{n}{k} x^k (1-x)^{n-k}$$

are called *Bernstein polynomials* for the function f.

Here is Bernstein's version of the Weierstrass approximation theorem.

27.4 Theorem.
For every continuous function f on $[0,1]$, we have

$$B_n f \to f \quad \textit{uniformly on} \quad [0,1].$$

Proof
We assume f is not identically zero, and we let

$$M = \sup\{|f(x)| : x \in [0,1]\}.$$

Consider $\epsilon > 0$. Since f is uniformly continuous by Theorem 19.2, there exists $\delta > 0$ such that

$$x, y \in [0,1] \quad \text{and} \quad |x-y| < \delta \quad \text{imply} \quad |f(x) - f(y)| < \frac{\epsilon}{2}. \quad (1)$$

Let $N = \frac{M}{\epsilon\delta^2}$. This choice of N is unmotivated at this point, but we make it here to emphasize that it does not depend on the choice of x. We will show

$$|B_n f(x) - f(x)| < \epsilon \quad \text{for all} \quad x \in [0,1] \quad \text{and all} \quad n > N, \tag{2}$$

completing the proof of the theorem.

To prove (2), consider a fixed $x \in [0,1]$ and $n > N$. In view of Lemma 27.1, we have

$$f(x) = \sum_{k=0}^{n} f(x) \binom{n}{k} x^k (1-x)^{n-k},$$

so

$$|B_n f(x) - f(x)| \leq \sum_{k=0}^{n} \left| f\left(\frac{k}{n}\right) - f(x) \right| \cdot \binom{n}{k} x^k (1-x)^{n-k}. \tag{3}$$

To estimate this sum, we divide the set $\{0, 1, 2, \ldots, n\}$ into two sets:

$$k \in A \ \text{ if } \ \left|\frac{k}{n} - x\right| < \delta \quad \text{while} \quad k \in B \ \text{ if } \ \left|\frac{k}{n} - x\right| \geq \delta.$$

For $k \in A$ we have $|f(\frac{k}{n}) - f(x)| < \frac{\epsilon}{2}$ by (1), so

$$\begin{aligned} &\sum_{k\in A} \left| f\left(\frac{k}{n}\right) - f(x) \right| \cdot \binom{n}{k} x^k (1-x)^{n-k} \\ &\qquad \leq \sum_{k\in A} \frac{\epsilon}{2} \binom{n}{k} x^k (1-x)^{n-k} \leq \frac{\epsilon}{2} \end{aligned} \tag{4}$$

using Lemma 27.1. For $k \in B$, we have $|\frac{k-nx}{n}| \geq \delta$ or $(k - nx)^2 \geq n^2\delta^2$, so

$$\begin{aligned} \sum_{k\in B} \left| f\left(\frac{k}{n}\right) - f(x) \right| \cdot \binom{n}{k} x^k (1-x)^{n-k} &\leq 2M \sum_{k\in B} \binom{n}{k} x^k (1-x)^{n-k} \\ &\leq \frac{2M}{n^2\delta^2} \sum_{k\in B} (k-nx)^2 \binom{n}{k} x^k (1-x)^{n-k}. \end{aligned}$$

By Lemma 27.2, this is bounded by

$$\frac{2M}{n^2\delta^2} \cdot \frac{n}{4} = \frac{M}{2n\delta^2} < \frac{M}{2N\delta^2} = \frac{\epsilon}{2}.$$

This observation, (4) and (3) show

$$|B_n f(x) - f(x)| < \epsilon.$$

That is, (2) holds. ■

27.5 Weierstrass's Approximation Theorem.
Every continuous function on a closed interval $[a, b]$ *can be uniformly approximated by polynomials on* $[a, b]$*.*

In other words, given a continuous function f on $[a, b]$, there exists a sequence (p_n) of polynomials such that $p_n \to f$ uniformly on $[a, b]$.

27.6 Corollary.
Given a continuous function f *on* $[a, b]$*, there exists a sequence* (p_n) *of polynomials such that* $p_n \to f$ *uniformly on* $[a, b]$*, and for each* n*,* $p_n(a) = f(a)$ *and* $p_n(b) = f(b)$*.*

Proof
See Exercise 27.4. ■

Exercises

27.1 Prove Theorem 27.5 from Theorem 27.4. *Hint*: Let $\phi(x) = (b-a)x+a$ so that ϕ maps $[0, 1]$ onto $[a, b]$. If f is continuous on $[a, b]$, then $f \circ \phi$ is continuous on $[0, 1]$.

27.2 Show that if f is continuous on $\mathbb{R}$, then there exists a sequence (p_n) of polynomials such that $p_n \to f$ uniformly on each bounded subset of $\mathbb{R}$. *Hint*: Arrange for $|f(x) - p_n(x)| < \frac{1}{n}$ for $|x| \leq n$.

27.3 Show there does not exist a sequence of polynomials converging uniformly on $\mathbb{R}$ to f if
(a) $f(x) = \sin x$, **(b)** $f(x) = e^x$.

27.4 Prove Corollary 27.6. *Hint*: Select a sequence (q_n) of polynomials such that $q_n \to f$ uniformly on $[a, b]$. For each n, let s_n be the function on $\mathbb{R}$ whose graph is the straight line passing through the points $(a, f(a) - q_n(a))$ and $(b, f(b) - q_n(b))$. Set $p_n = q_n + s_n$.

27.5 Find the sequence $(B_n f)$ of Bernstein polynomials in case
(a) $f(x) = x$, **(b)** $f(x) = x^2$.

27.6 The Bernstein polynomials were defined for any function f on $[0, 1]$. Show that if $B_n f \to f$ uniformly on $[0, 1]$, then f is continuous on $[0, 1]$.

27.7 Let f be a bounded function on $[0, 1]$, say $|f(x)| \leq M$ for all $x \in [0, 1]$. Show that all the Bernstein polynomials $B_n f$ are bounded by M.