Chapter 16
Belief Contraction in the Context of the General Theory of Rational Choice

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Introduction

The theory of partial meet contraction and revision was developed by Alchourrón, Gärdenfors and Makinson (henceforth, "AGM") in a paper published in the *Journal of Symbolic Logic* in 1985. That paper is the by now classic reference of the flourishing research programme of *theory change*, or as it is alternatively called, *belief revision* (Fuhrmann and Morreau 1991; Gärdenfors 1988, 1992; Katsuno and Mendelzon 1991). In particular, it has been shown that partial meet contraction is a powerful tool for the reconstruction of other kinds of theory change such as safe contraction, epistemic entrenchment contraction, and base contraction (Alchourrón and Makinson 1986; Nebel 1989, 1992; Rott 1991, 1992a).

The basic idea of partial meet contraction is as follows. In order to eliminate a proposition x from a theory A while obeying the constraint of deductive closure and minimizing the loss of information, it is plausible to look at the maximal subsets B of A that fail to imply x. In an earlier paper, Alchourrón and Makinson had proved that when $A = Cn(A)$ then taking one such B leaves an agent with too many propositions, while taking the intersection of all such B's leaves him with too few. In AGM (1985), AGM investigate the idea of taking the intersection of *a select set* of such B's. The choice which B's to take is made with the help of a *selection function*. A natural question is whether all these selections can be represented as the selections of *preferred B*'s, where the preferences between maximally nonimplying subsets of A are independent of the proposition x to be deleted.

The purpose of the present paper is threefold. First, we put the theory of partial meet contraction in a broader perspective. We decompose it into two layers, each

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of which can be cultivated with the help of methods developed in different research areas. On the one hand, in relating selection functions to preference relations we can draw on existing work in social choice theory (Chernoff 1954; Herzberger 1973; Sen 1982; Suzumura 1983). On the other hand, we shall elaborate on a remark of Grove (1988) and link maximally nonimplying subsets to "possible worlds" or models, thereby making it possible to compare partial meet contraction with semantical approaches to belief revision and nonmonotonic reasoning (Kraus et al. 1990; Katsuno and Mendelzon 1991; Lindström 1991; Lehmann and Magidor 1992). Exaggerating somewhat, we can say that the theory of partial meet contraction emerges from juxtaposing the general theory of rational choice and a tiny bit of model theory. After introducing abstract postulates for contraction functions, we reprove the two main representation theorems of AGM (1985, pp. 521, 530) concerning partial meet contraction and transitively relational partial meet contraction in a slightly more systematic fashion.

Second, we provide a partial solution to a problem left unanswered by AGM and still considered to be an interesting open question in Makinson and Gärdenfors (1991, p. 195). More precisely, we present two new results that lie strictly "between" those of AGM (1985), viz., representation theorems for *relational* and *negatively transitively relational* partial meet contraction. However, these results hold only under certain preconditions. If the theory to be contracted is logically finite, then all these conditions are met. Our decomposition allows for a uniform method of proof using so-called Samuelson preferences. It increases our understanding of the partial meet contraction mechanism by localizing exactly in which parts of the proofs the finiteness assumption is effective.

Third, as an application, we explore the logic of a variant of syntax-based belief change, namely *simple* and *prioritized base contractions* in the finite case. The basic idea here is that real life theories are generated from finite axiomatizations and that the axioms may carry different epistemic weight. Both the syntactical encoding of a theory and the prioritization are held to be relevant for theory change. We achieve a logical characterization of simple and prioritized base contraction by combining one of our representation theorems with observations originating with Lewis (1981) and Nebel (1989, 1992).

Independently from the research for this article, related work has been carried out by Katsuno and Mendelzon (1991) and Lindström (1991). Lindström proves numerous representation theorems, and like us, he heavily draws on results from the general theory of rational choice. However, there are some important differences. He adopts a semantic approach right from the outset, while our reconstruction starts by turning a syntactic approach into an essentially semantic one, with the notable feature that any two "worlds" in our sense are linguistically distinguishable (our models are "injective", to use a term of Michael Freund). Lindström's main concern is nonmonotonic inference relations and the related area of belief *revision*, whereas we shall focus on belief *contraction*. (Everything we might wish to say about the intimate connection between revisions and contractions is contained in AGM 1985 and Gärdenfors 1988). He applies *liberal* maximization where we apply *stringent* maximization. He uses different postulates for choice functions and a

different notion of revealed preference. The main contrast between his approach and ours, however, is that Lindström permits revisions by—possibly infinite—*sets* of propositions whilst we stick to the original AGM model of changing theories by *single* propositions. Lindström's generalization allows him to revise by "worlds" and thus dispense with a finiteness assumption which will prove to be necessary at several places in the AGM framework. In fact, it is a major aim of our paper to make transparent the reasons where and why logical finiteness is needed in the AGM theory. Further important results on purely finitistic belief revision are found in Katsuno and Mendelzon (1991) who incorporate a revision operator into their object language. Both papers are highly recommended and should be read in conjunction with the present one.

Unfortunately, space limitations do not allow a presentation that makes for easy reading. Familiarity with earlier work either in the area of theory change—an excellent overview is Gärdenfors (1988)—or in the general theory of rational choice—of particular relevance is Herzberger (1973)— will greatly facilitate the reader's task. However, we will shortly repeat the basic definitions so that our presentation is in principle self-contained. It may be useful to inform the reader in advance about the fundamental entities that will show up as determinants of theory change. We shall meet contraction functions $\dot{-}$, selection functions γ, preference relations $\leq$ (and $\underline{\ll}$), maximally nonimplying sets M (and N), maximally consistent sets or "worlds" W, as well as simple and prioritized belief bases B and $\langle B_i \rangle$. We shall frequently jump to and fro between these kinds of entities. When we wish to generate a *c*ontraction function, *s*election function, *p*reference relation, *m*aximal nonimplying set, *w*orld, or belief *b*ase from another kind of entity, we shall use metalevel mappings denoted by $\mathscr{C}, \mathscr{S}, \mathscr{P}, \mathscr{M}, \mathscr{W}, \mathscr{B}$ respectively.

Selection Functions and Preference Relations

General Selection Functions

Let X be a set and $\mathscr{X}$ be a nonempty subset of $2^X - \{\emptyset\}$. A *selection function* (or *choice function*) over $\mathscr{X}$ is a function $\gamma : \mathscr{X} \to 2^X$ such that $\gamma(S)$ is a nonempty subset of S, for every $S \in \mathscr{X}$. Intuitively, selection functions are supposed to give us the best elements of each S in $\mathscr{X}$. The requirement that $\gamma(S)$ be nonempty means that a selection function is effective or "decisive": In every case it reaches a decision of which elements are best. As Lindström (1991) points out, the decisiveness of selection functions corresponds to a condition of consistency preservation in belief revision and nonmonotonic reasoning. Here consistency is judged by *some* monotonic background consequence relation, not necessarily by the classical one.

Domain conditions for selection functions are important. A set $\mathscr{X}$ of subsets of X is called *n-covering* ($n = 1, 2, 3, \ldots$) if it contains all subsets of X with exactly n elements, $\mathscr{X}$ is called $n_1 n_2$-*covering* (or $n_1 n_2 n_3$-*covering*) if it is n_1-covering and

n_2-covering (and n_3-covering). $\mathscr{X}$ is called *ω-covering* if it is *n*-covering for all natural numbers $n = 1, 2, 3, \ldots$. The set $\mathscr{X}$ is called *additive* if it is closed under arbitrary unions, and it is called *finitely additive* if it is closed under finite unions; it is called *subtractive* if for every S and S' in $\mathscr{X}$ such that $S \nsubseteq S'$, $S - S'$ is also in $\mathscr{X}$, and it is called *full* if $\mathscr{X} = 2^X - \{\emptyset\}$. Of course, if $\mathscr{X}$ is 1-covering and finitely additive, then it is *ω-covering*. Finally, we say that $\mathscr{X}$ is *compact* if for every T and $S_i, i \in I$, from $\mathscr{X}$, if $T \subseteq \bigcup\{S_i : i \in I\}$ then $T \subseteq \bigcup\{S_i : i \in I_0\}$ for some finite $I_0 \subseteq I$. We say that a *selection function* γ with domain $\mathscr{X}$ is *n*-covering, additive, subtractive, etc., if its domain $\mathscr{X}$ is *n*-covering, additive, subtractive, etc.

Constraints on Selection Functions

We consider the following "coherence postulates" for selection functions. Their motivation, where it is not obvious, is given in Herzberger (1973), Sen (1982), and Suzumura (1983).

(I) For all $S, S' \in \mathscr{X}$, if $S \subseteq S'$ then $S \cap \gamma(S') \subseteq \gamma(S)$ ("Chernoff's axiom", Sen's Property α)

(II) For all $\{S_i : i \in I\} \subseteq \mathscr{X}$ such that $\bigcup\{S_i : i \in I\} \in \mathscr{X}$, $\bigcap\{\gamma(S_i) : i \in I\} \subseteq \gamma(\bigcup\{S_i : i \in I\})$ (Sen's Property γ)

(III) For all $S, S' \in \mathscr{X}$ such that $S \subseteq S'$, if $\gamma(S') \subseteq \gamma(S)$ then $\gamma(S) \subseteq \gamma(S')$ ("Superset Axiom", Sen's Property ε)

(IV) For all $S, S' \in \mathscr{X}$ such that $S \subseteq S'$, if $\gamma(S') \cap S \neq \emptyset$, then $\gamma(S) \subseteq \gamma(S')$ ("Arrow's axiom", Sen's Property $\beta+$)

Condition (III) is independent of condition (II), even for finite X and in the presence of condition (I). Example: Let $X = \{x, y, z\}$, $\mathscr{X}$ be the power set of X minus $\emptyset$, and consider γ_1 and γ_2 defined by $\gamma_1(\{x, y\}) = \gamma_2(\{x, y\}) = \{x, y\}$, $\gamma_1(\{x, z\}) = \{x\}$, $\gamma_1(\{y, z\}) = \{z\}$, $\gamma_1(\{x, y, z\}) = \{x\}$, $\gamma_2(\{x, z\}) = \{x, z\}$, $\gamma_2(\{y, z\}) = \{y, z\}$, and $\gamma_2(\{x, y, z\}) = \{x, y\}$. Evidently, γ_1 satisfies (I) and (II) and violates (III), while γ_2 satisfies (I) and (III) and violates (II). On the other hand, condition (IV) implies condition (II): Let $\{S_i : i \in I\} \subseteq \mathscr{X}$, $\bigcup\{S_i : i \in I\} \in \mathscr{X}$. Now $\emptyset \neq \gamma(\bigcup\{S_i : i \in I\}) \subseteq \bigcup\{S_i : i \in I\}$, so $\gamma(\bigcup\{S_i : i \in I\}) \cap S_{i_0} \neq \emptyset$ for some $i_0 \in I$. So by (IV), $\bigcap\{\gamma(S_i) : i \in I\} \subseteq \gamma(S_{i_0}) \subseteq \gamma(\bigcup\{S_i : i \in I\})$. It is trivial to see that condition (IV) also implies condition (III), since $\emptyset \neq \gamma(S') \subseteq \gamma(S)$ implies $\gamma(S') \cap S \neq \emptyset$.

Rational Choice

In the general theory of choice and preference we often find an idea which can be phrased in the slogan "*Rational choice is relational choice*". That is, rational choice is choice which can be construed as based on an underlying preference relation.

The intended interpretation of the set $\gamma(S)$, called the *choice set* for S, is that its elements are regarded as equally adequate or satisfactory choices for an agent whose values are represented by the function γ, and who faces a decision problem represented by the set S. Following Chernoff (1954) ..., this relativistic concept of equiadequacy for a given decision problem bears sharp distinction from invariant concepts like preferential matching or indifference which for a given agent are not relativized to decision problems, and which may be subject to more stringent constraints, both for rational agents and for agents in general. (Herzberger 1973, p. 189, notation adapted)

Choice sets are taken to be sets of "best" elements. There are basically two ideas to make this precise. The first is based on a nonstrict (reflexive) preference relation $\leq$:

$$\gamma(S) = \{y \in S : y' \leq y \text{ for all } y' \in S\}$$

The second idea is based on a strict (asymmetric) preference relation $<$:

$$\gamma(S) = \{y \in S : y \not< y' \text{ for all } y' \in S\}$$

These suggestions are respectively referred to as *stringent* and *liberal maximization* in Herzberger (1973, p. 197), and *G-rationality* and *M-rationality* in Suzumura (1983, p. 21). (Stringent maximization is often attributed to Condorcet 1785.) If $<$ is the converse complement of $\leq$, then stringent and liberal maximization coincide. If $<$ is the asymmetric part of $\leq$, then every liberal maximizer with respect to $<$ is a stringent maximizer with respect to the *augmentation* $\leq^+$ of $\leq$ which is defined by $x \leq^+ y$ iff $x \leq y$ or $y \not\leq x$, i.e., iff not $y < x$. Clearly, $\leq^+$ is *connected*, i.e., it holds for every x and y that either $x \leq^+ y$ or $y \leq^+ x$. While $\leq$ allows us to keep apart indifferences (both $x \leq y$ and $y \leq x$) from incomparabilities (neither $x \leq y$ nor $y \leq x$), $\leq^+$ blurs just this distinction, for whenever we have neither $x \leq y$ nor $y \leq x$ we have both $x \leq^+ y$ and $y \leq^+ x$. Thus, we may not expect $\leq^+$ to be transitive. If, on the other hand, $\leq$ is already connected then $\leq^+ = \leq$, and stringent maximization with respect to the asymmetric part $<$ of $\leq$ coincides with liberal maximization with respect to $\leq$. For all this, cf. Herzberger (1973, §3).

In accordance with AGM (1985) as well as with the dominant approach in the theory of choice and preference, we shall focus on stringent maximization.[1] From now on, when we say that γ is *relational with respect to $\leq$ over X*, we mean that $\gamma(S) = \{y \in S : y' \leq y \text{ for all } y' \in S\}$ for every $S \in \mathscr{X}$. In this case we write

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[1]Intuitively, however, I think that liberal maximization is preferable. Liberal maximization is based on strict relations which do not allow to distinguish between incomparabilities and indifferences. Nonstrict relations do make this distinction, but stringent maximization tends to require connected relations which often can be had only if incomparabilities are turned into indifferences—i.e., if augmentations are used. The interpretation of nonstrict relations as the converse complements of—more intuitive—strict relations explains the crucial role of *negative* transitivity and *negative* well-foundedness in the following. Also compare the recommendation in Rott (1992b) to regard the nonstrict epistemic entrenchment relation $\leq_E$ of Gärdenfors and Makinson (1988) as the converse complement of a more intuitive strict relation $<_E$.

$\gamma = \mathscr{S}(\leq)$. When we say that γ is *relational*, we mean that there is some relation $\leq$ over X such that $\gamma = \mathscr{S}(\leq)$.

For any nonstrict relation $\leq$, $<$ is to denote the *asymmetric part* of $\leq$, which is defined by $x < y$ iff $x \leq y$ and not $y \leq x$. $\leq$ is called *n-acyclic*, if no n objects $x_1, x_2, \ldots, x_n$ form a cycle under $<$, i.e., if a chain $x_1 < x_2 < \cdots < x_n < x_1$ does not occur. 1-acyclicity is irreflexivity, 2-acyclicity is asymmetry. $\leq$ is called *acyclic*, if it is *n*-acyclic for every $n = 1, 2, 3, \ldots$. $\leq$ is called *negatively acyclic* if there is no cycle under $\not\leq$, i.e., if never $x_1 \not\leq x_2 \not\leq \cdots \not\leq x_n \not\leq x_1$, and it is called *negatively well-founded* if there is no infinite descending chain under $\not\leq$, i.e., if never $\cdots \not\leq x_3 \not\leq x_2 \not\leq x_1$. For connected relations $\leq$, negative well-foundedness coincides with converse well-foundedness where infinite ascending chains $x_1 < x_2 < x_3 < \ldots$ do not occur. Obviously, if $\leq$ is conversely (or negatively) well-founded then it is acyclic (negatively acyclic). $\leq$ is *smooth* with respect to $\mathscr{X}$ if there are no infinite descending $\not\leq$-chains in S, for every $S \in \mathscr{X}$. Smoothness is a restricted form of negative well-foundedness. $\leq$ is called *negatively transitive* (*virtually connected, modular, ranked*) if $x \not\leq y$ and $y \not\leq z$ implies $x \not\leq z$. It is quickly verified that a connected relation $\leq$ is negatively transitive iff it is *quasi-transitive* in the sense that its asymmetric part $<$ is transitive. Quasi-transitivity is a much disputed requirement in social choice theory (Herzberger 1973; Sen 1982; Suzumura 1983). It should be noted that all transitive relations $\leq$ are both acyclic and quasi-transitive.

Two Kinds of Revealed Preferences

In many contexts one can hope to recover the underlying preferences of an agent from observed choice behavior. The two most commonly used types of "revealed preference" relations are the *Samuelson preferences* (Samuelson (1950))

$$\leq_\gamma = \{\langle x, x'\rangle \in X \times X : \exists S \in \mathscr{X} \text{ such that } \{x, x'\} \subseteq S \quad \text{and } x' \in \gamma(S)\} \quad (16.1)$$

$$= \bigcup \{S \times \gamma(S) : S \in \mathscr{X}\} \quad (16.2)$$

and the so-called *base preferences* (Uzawa (1956); Arrow (1959)):

$$\leq_{\gamma,2} = \{\langle x, x'\rangle \in X \times X : \{x, x'\} \in \mathscr{X} \text{ and } x' \in \gamma(\{x, x'\})\} \quad (16.3)$$

$$= \bigcup \{S \times \gamma(S) : S \in \mathscr{X} \text{ and } S \text{ has at most two elements}\} \quad (16.4)$$

The terminology is taken from Herzberger (1973), where many other possibilities of defining notions of revealed preference are discussed. Obviously, $x \leq_{\gamma,2} y$ implies $x \leq_\gamma y$. Notice that neither of these relations is guaranteed to be reflexive, unless γ is 1-covering. $\leq_{\gamma,2}$ is defined for arbitrary γ's, but the definition makes good sense only for 2-covering ones for which $\{x, x'\}$ is always in $\mathscr{X}$. In this case, base

preferences and Samuelson preferences are connected. Given a selection function γ, we shall also denote $\leq_\gamma$ by $\mathscr{P}(\gamma)$ and $\leq_{\gamma,2}$ by $\mathscr{P}_2(\gamma)$.[2]

Some Basic Properties and Reformulations

The following lemmas list a number of important facts which are basically common knowledge in the general theory of rational choice (cf. Herzberger 1973; Sen 1982; Suzumura 1983). For the straightforward proofs, see Appendix 2.

Lemma 1.   (a) *If γ is relational then it satisfies (I) and (II).*
(b) *If γ is 12-covering and satisfies (I) and (II), then $\gamma = \mathscr{S}(\mathscr{P}_2(\gamma))$.*
(c) *If γ is 12-covering and satisfies (I), then $\mathscr{P}(\gamma) = \mathscr{P}_2(\gamma)$.*
(d) *If γ is 12-covering and satisfies (I) and (II), then $\gamma = \mathscr{S}(\mathscr{P}(\gamma))$.*
(e) *If γ is 12-covering and relational, then $\gamma = \mathscr{S}(\mathscr{P}(\gamma)) = \mathscr{S}(\mathscr{P}_2(\gamma))$.*
(f) *If γ is additive and satisfies (I) and (II), then $\gamma = \mathscr{S}(\mathscr{P}(\gamma))$.*

Lemma 2.   (a) *If γ is 12n-covering and satisfies (I), then $\mathscr{P}(\gamma) = \mathscr{P}_2(\gamma)$ is n-acyclic. If γ is ω-covering and satisfies (I), then $\mathscr{P}(\gamma)$ is acyclic.*
(b) *If γ is 123-covering and satisfies (I) and (III), then $\mathscr{P}(\gamma)$ is negatively transitive.*
(c) *If γ is finitely additive and satisfies (IV), then $\mathscr{P}(\gamma)$ is transitive.*

Lemma 3.   (a) *If $\leq$ is smooth with respect to $\mathscr{X}$, then $\mathscr{S}(\leq)$ is a selection function over $\mathscr{X}$ which satisfies (I) and (II).*
(b) *If $\leq$ is negatively transitive and negatively well-founded (or: if $\leq$ is negatively transitive and smooth and $\mathscr{X}$ is subtractive), then $\mathscr{S}(\leq)$ satisfies (III).*
(c) *If $\leq$ is transitive, then $\mathscr{S}(\leq)$ satisfies (IV).*

Now we bring conditions (I)–(IV) into a form which is more suitable for our purposes.

(I′) For all $S, S' \in \mathscr{X}$ such that $S \cup S' \in \mathscr{X}$, $\gamma(S \cup S') \subseteq \gamma(S) \cup \gamma(S')$
(II′) For all $S, S' \in \mathscr{X}$ such that $S \cup S' \in \mathscr{X}$, $\gamma(S) \cap \gamma(S') \subseteq \gamma(S \cup S')$
(III′) For all $S \in \mathscr{X}$ and S' such that $S \cup S' \in \mathscr{X}$, if $\gamma(S \cup S') \cap S' = \emptyset$ then $\gamma(S) \subseteq \gamma(S \cup S')$
(IV′) For all $S \in \mathscr{X}$ and S' such that $S \cup S' \in \mathscr{X}$, if $\gamma(S \cup S') \cap S \neq \emptyset$, then $\gamma(S) \subseteq \gamma(S \cup S')$

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[2]It is worth pointing out that the characteristic definition of a relation of epistemic entrenchment (see Gärdenfors and Makinson 1988; Rott 1992b) between propositions from an observed contraction behavior, viz.

$$x \leq_E y \text{ iff } x \notin A \dotdiv (x \wedge y) \text{ or } y \in A \dotdiv (x \wedge y)$$

can also be interpreted as a base preference (Rott 1992b, p. 61). In that paper it is argued that the instruction "remove $x \wedge y$" should be regarded as an instruction to remove x *or* remove y, where the agent holding theory A has free choice which proposition(s) out of $\{x, y\}$ to remove. [Note added in 2015: This sketch of an idea was turned into a theory in Rott (2003).]

(I&II) For all $S, S_i \in \mathcal{X}, i \in I$, if $S \subseteq \bigcup\{S_i : i \in I\}$ then $S \cap \bigcap\{\gamma(S_i) : i \in I\} \subseteq \gamma(S)$
(I&II′) For all $S, S', S'' \in \mathcal{X}$, if $S \subseteq S' \cup S''$ then $S \cap \gamma(S') \cap \gamma(S'') \subseteq \gamma(S)$

Note that condition (II′) is just a restriction of (II) to index sets with at most two elements, and similarly for (I&II) and (I&II′).

Lemma 4. (a) *If γ is subtractive, then condition (I) is equivalent to (I′).*
(b) *If γ is finitely additive, compact, and satisfies (I) and (II′), then $\gamma = \mathcal{S}(\mathcal{P}(\gamma))$.*
(c) *If γ satisfies (I), then (III) is equivalent to (III′).*
(d) *(IV) is equivalent to (IV′).*
(e) *If γ is additive, then the conjunction of (I) and (II) is equivalent to (I&II).*
(f) *If γ is finitely additive, then the conjunction of (I) and (II′) is equivalent to (I&II′).*

Lemma 4(b) is a substitute for Lemma 1(f) in the absence of infinite additivity. The conditions (I&II) and (I&II′) are generalized forms of the conditions (γ7:∞) and (γ7:2) in AGM (1985). AGM (1985, Observation 4.10) noticed that (I&II′) is the key to relationality in partial meet contraction (because the domain $\mathcal{X}$ encountered there is compact), but as they failed to decompose that condition into the simpler ones, (I) and (II′), it remained undigestible for them. As we shall see, they provided a postulate for contraction functions which corresponds to (I), but nothing in AGM (1985) corresponds to (II′).

The results of the theory of rational choice are quite nice, and we may be optimistic about their applicability to theory change as determined by partial meet contraction s. Still there are at least two problems which should not be underestimated. First, we do not know very much about selection functions which are neither 123-covering nor additive.[3] And second, in choice-and-preference theory we are faced with a bewildering pluralism of revealed preference relations (see Herzberger 1973). It is hard to decide in advance which notion of revealed preference is "the" right notion for a given purpose. It will turn out, happily, that the relation $\mathcal{P}(\gamma)$ defined above, and, to a smaller degree, also the relation $\mathcal{P}_2(\gamma)$ which is equivalent in the finite case are suitable for partial meet contraction s.

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[3]This is no problem for Lindström (1991) whose selection functions are always ω-covering. Consequently, Lindström's constructions can always make use of the base preferences $\mathcal{P}_2(\gamma)$.

Selection Functions and Preference Relations in Partial Meet Contraction

The General Case

We presuppose a propositional language L, with the usual connectives $\neg, \wedge, \vee, \rightarrow$ and $\leftrightarrow$. From now on, small roman letters $a, b, c, \ldots, x, y, z, \ldots$ denote propositions of L, and capital roman letters $A, B, A', B', \ldots, M, N, M', N' \ldots, W, W', \ldots$ denote sets of propositions. Notational conventions: Capital A is to denote a theory, B an arbitrary set of propositions (in general, a nontheory), M and N maximally nonimplying subsets (of some A or B, respectively), and W a maximally consistent set of L. All these metavariables may occur with primes or subscripts.

As in AGM (1985), L is assumed to be governed by some reflexive, monotonic and idempotent logic (consequence operation) Cn which is supraclassical and compact and satisfies the deduction theorem: $B \subseteq Cn(B)$; if $B \subseteq B'$ then $Cn(B) \subseteq Cn(B')$; $Cn(Cn(B)) = Cn(B)$; if Cn_0 is classical tautological implication then $Cn_0(B) \subseteq Cn(B)$; if $y \in Cn(B)$ then $y \in Cn(B_0)$ for some finite subset B_0 of B; and finally, $y \in Cn(B \cup \{x\})$ iff $x \rightarrow y \in Cn(B)$. That is, Cn is a classical logic in the sense of Lindström (1991). Note that in the present context, idempotence is equivalent to the cut rule, and the deduction theorem is equivalent to disjunction in the antecedents.

We also write $B \vdash x$ for $x \in Cn(B)$. A is a called a *theory* if $A = Cn(A)$. By $[\![x]\!]$ and $[\![B]\!]$ we denote the set of all "possible worlds"(valuations, models) in which x (every x in B) is true. Here and throughout this paper, we identify a "world" satisfying x or B with a maximally Cn-consistent set of propositions which contains x (respectively, B). The set of all "worlds" is denoted $\mathbf{W}$.

Let $B \perp x = \{M \subseteq B : x \notin Cn(M) \text{ and } x \in Cn(N) \text{ for all } N \text{ with } M \subset N \subseteq B\}$. Note that the elements of $B \perp x$ are theories if B is a theory. If $\nvdash x$, then $B \perp x$ is nonempty, by the compactness of Cn.[4] We define $B \perp = \{B \perp x : x \in Cn(B) - Cn(\emptyset)\}$ and $U_B = \bigcup (B \perp) = \bigcup \{B \perp x : x \in Cn(B) - Cn(\emptyset)\}$. (Here we slightly deviate from the definitions in AGM (1985) which also include x's from $Cn(\emptyset)$.) If $Cn(B) \neq Cn(\emptyset)$, then $B \perp$ is a nonempty subset of $2^{2^B} - \{\emptyset\}$. The case $Cn(B) = Cn(\emptyset)$ will be handled separately in our reconstruction of partial meet contraction.

Lemma 5. *Let A be a theory and $x \in A$. Then $M \in A \perp x$ iff there is a maximally consistent set ("world") W such that $\neg x \in W$ and $M = A \cap W$.*

For the proof, compare §4 of Grove (1988).

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[4]This marks a difference with Lewis (1981) who identifies propositions with sets of extra-linguistic possible worlds and logical consequence with set-theoretic inclusion. Lacking compactness, Lewis has to ponder the impact of a "Limit Assumption" for premise semantics.

Now let A again be a theory. For $M \in U_A$, put $\mathscr{W}(M) = Cn(M \cup \{\neg x\})$ where x is any proposition in $A - M$, and put $\mathscr{M}(W) = W \cap A$, for any maximally consistent set W such that $A \not\subseteq W$. The reader is invited to verify: $\mathscr{W}$ is well-defined(!), $\mathscr{W}$ is a bijection from U_A to $V_A = \{W \in W : A \not\subseteq W\}$, and $\mathscr{M}$ is the converse of $\mathscr{W}$, i.e., $\mathscr{M}(\mathscr{W}(M)) = M$ and $\mathscr{W}(\mathscr{M}(W)) = W$.

Given this representation of the elements of U_A, it is clear that they satisfy the following *fullness* and *primeness conditions*: If $x \in A - M$ and $y \in A$ then $x \rightarrow y \in M$, and if $x, y \in A - M$ then $x \vee y \in A - M$ (cf. Observations 6.1 and 6.2 in AGM 1985). U_A is just the set of all maximal proper subtheories of the theory A. Moreover, we immediately get

Corollary 1. *Let A be a theory and x, y, y_i be in $A - Cn(\emptyset)$.*

(i) *For $M \in U_A$, $x \notin M$ iff $M \in A \perp x$.*
(ii) $A \perp (x \wedge y) = A \perp x \cup A \perp y$.
(iii) $A \perp (x \vee y) = A \perp x \cap A \perp y$.
(iv) $A \perp (x \vee \neg y) = A \perp x - A \perp y$.
(v) *If $A \perp x \subseteq \bigcup \{A \perp y_i : i \in I\}$, then $A \perp x \subseteq \bigcup \{A \perp y_i : i \in I_0\}$ for some finite $I_0 \subseteq I$.*

Facts (i) and (ii) are contained in AGM (1985, Lemma 2.4 and Lemma 4.1). We see that $A \perp$, the special domain $\mathscr{X}$ of the selection functions which will figure in partial meet contraction s, is closed under finite unions, finite intersections, and differences. We give a direct proof of the compactness property (v). Let $A \perp x \subseteq \bigcup \{A \perp y_i : i \in I\}$. Then $\{y_i : i \in I\} \vdash x$. For otherwise, by Lindenbaum's Lemma and Lemma 5, there would be an $M \in A \perp x$ such that $\{y_i : i \in I\} \subseteq M$, so $M \notin A \perp y_i$ for every i, contradicting our hypothesis. Compactness of Cn gives us $\{y_i : i \in I_0\} \vdash x$ for some finite $I_0 \subseteq I$. Thus, there is, for every $M \in A \perp x$, an $i \in I_0$ such that $y_i \notin M$. Hence, by (i), there is, for every $M \in A \perp x$, an $i \in I_0$ such that $M \in A \perp y_i$. Thus $A \perp x \subseteq \bigcup \{A \perp y_i : i \in I_0\}$, as desired.

Now we can introduce selection functions for belief revision. Let A be a theory. A selection function $\gamma : A \perp \rightarrow 2^{2^A}$ is called a *selection function for A*. It follows from Corollary 1 that every selection function γ for A is finitely additive, subtractive and compact.

Let γ be a selection function for A. The *completion γ^** of γ is defined by $\gamma^*(A \perp x) = \{M \in A \perp x : \bigcap \gamma(A \perp x) \subseteq M\}$, for all $x \in A - Cn(\emptyset)$. Following AGM (1985), we call a selection function γ *complete* if $\gamma = \gamma^*$. If γ is complete, then $M \in \gamma(A \perp x)$ whenever $\bigcap \gamma(A \perp x) \subseteq M \in U_A$ (Proof: Since $\{x \rightarrow y : y \in A\} \in \bigcap \gamma(A \perp x)$, $\bigcap \gamma(A \perp x) \subseteq M$ implies $M \not\vdash x$, so $M \in A \perp x$ by Corollary 1(i).)

A *contraction function* over a theory A is a function $\dot{-}_A : L \rightarrow 2^L$. We write $A \dot{-} x$ for $\dot{-}_A(x)$, and as there will be no danger of confusion, we shall often write just $\dot{-}$ for $\dot{-}_A$. The term '$A \dot{-} x$' should be read as the 'the result of rationally removing x from A'. The idea of a contraction function dictates that it should satisfy at least the postulates $A \dot{-} x \subseteq A$ and $x \notin A \dot{-} x$ (unless $\vdash x$). More postulates will be discussed in section "Postulates for Contraction Functions". Intuitively, $A \dot{-} x$ is the minimal, most economical or rational, change of A needed to discard x.

A function $\dot{-}$ is the *partial meet contraction function over A determined by the selection function γ for A* if and only if

$$A \dot{-} x = \begin{cases} \bigcap \gamma(A \perp x) & \text{for every } x \in A - Cn(\emptyset) \\ A & \text{for every } x \notin A \text{ and every } x \in Cn(\emptyset) \end{cases}$$

Given a selection function γ for A, the partial meet contraction it determines will be denoted by $\mathscr{C}(\gamma)$. There is a slight deviation from AGM (1985) in order to avoid the application of γ to $\emptyset$ and to preserve the perfect correspondence between U_A and V_A. The function $\dot{-}$ is called a *partial meet contraction over A* if there is a selection function γ for A such that $\dot{-} = \mathscr{C}(\gamma)$.

A selection function γ for A is called *(transitively, quasi-transitively, connectively, acyclicly) relational over A* if there is a (transitive, quasi-transitive, connected, acyclic) preference relation $\leq$ over U_A (over 2^A in AGM 1985) such that for every $x \in A - Cn(\emptyset)$:

$$\gamma(A \perp x) = \{M \in A \perp x : M' \leq M \text{ for all } M' \in A \perp x\}.$$

This is an application of stringent maximization as discussed in section "Rational Choice", and it is called the *marking off identity* in AGM (1985). Recall that we write $\gamma = \mathscr{S}(\leq)$ if γ is determined by $\leq$ in that way.

A contraction function $\dot{-}$ is *(transitively, quasi-transitively, connectively, acyclicly) relational* if there is a (transitive, quasi-transitive, connective, acyclic) preference relation $\leq$ over U_A such that $\dot{-} = \mathscr{C}(\mathscr{S}(\leq))$.

Using the above functions $\mathscr{W}$ and $\mathscr{M}$, we find that the selection γ of sets in $A \perp x$ can equivalently be viewed as a selection γ_W of maximally consistent sets, or "worlds", in $\llbracket \neg x \rrbracket$. If we define

$$\gamma_W(\llbracket \neg x \rrbracket) = \{\mathscr{W}(M) : M \in \gamma(A \perp x)\}$$

then clearly,

$$\gamma(A \perp x) = \{\mathscr{M}(W) : W \in \gamma_W(\llbracket \neg x \rrbracket)\}$$

In the principal case $x \in A - Cn(\emptyset)$, $A \dot{-} x = A \cap \bigcap(\gamma_W(\llbracket \neg x \rrbracket))$. Conditions for the partial meet mechanism may thus be viewed as *model theoretic* or *semantical* conditions. The "possible worlds" view facilitates a visualization of the conditions on preference relations and selection functions and relates the work on partial meet contraction directly to the minimal models approach as adopted in Lewis (1973, 1981), Kraus et al. (1990), Katsuno and Mendelzon (1991), Lindström (1991), and Lehmann and Magidor (1992).

A set W' of maximal consistent sets of propositions is called *elementary* (Δ-*elementary*) if there is a proposition y (a set of propositions B) such that W' is exactly the set of all maximally consistent sets of propositions which contain y (respectively,

which include B), in symbols $W' = [\![y]\!]$ ($W' = [\![B]\!]$). We call γ *elementary* (Δ-*elementary*) if $\gamma_W([\![\neg x]\!])$ is elementary (Δ-elementary) for every $x \in A - Cn(\emptyset)$. If γ is elementary (Δ-elementary) and $\gamma_W([\![\neg x]\!]) = [\![y]\!]$ ($\gamma_W([\![\neg x]\!]) = [\![B]\!]$), then $A \dot{-} x = Cn(\{z \vee y : z \in A\}) = A \cap Cn(y)$ (respectively, $A \dot{-} x = Cn(\{z \vee z' : z \in A$ and $z' \in B\}) = A \cap Cn(B)$).

Remark 1. γ is complete iff it is Δ-elementary.

Proof. From left to right. We show that $[\![(\bigcap \gamma(A \perp x)) \cup \{\neg x\}]\!] = \gamma_W([\![\neg x]\!])$, for every $x \in A - Cn(\emptyset)$. Clearly, $(\bigcap \gamma(A \perp x)) \cup \{\neg x\} \subseteq W$ for every $W \in \gamma_W([\![\neg x]\!])$. To show the converse, suppose for reductio that $(\bigcap \gamma(A \perp x)) \cup \{\neg x\} \subseteq W$ and $W \notin \gamma_W([\![\neg x]\!])$. Hence, by the latter, $\mathscr{M}(W) \notin \gamma(A \perp x)$, so by the completeness of γ, $\bigcap \gamma(A \perp x) \nsubseteq \mathscr{M}(W)$. But also $(\bigcap \gamma(A \perp x)) \cup \{\neg x\} \subseteq W$, and $\bigcap \gamma(A \perp x) \subseteq A$, so $\bigcap \gamma(A \perp x) \subseteq \mathscr{M}(W)$, and we have a contradiction.

From right to left. Suppose for reductio that there are $x \in A - Cn(\emptyset)$ and $M \in U_A$ such that $\bigcap \gamma(A \perp x) \subseteq M$ and $M \notin \gamma(A \perp x)$. Hence $\mathscr{W}(M) \notin \gamma_W([\![\neg x]\!])$. Since γ is Δ-elementary, there is a set B of propositions such that $\gamma_W([\![\neg x]\!]) = [\![B]\!]$. So $B \nsubseteq \mathscr{W}(M)$. Take some $y \in B - \mathscr{W}(M)$. Since $x \vee y \notin \mathscr{W}(M)$, we get, by the definition of $\mathscr{W}$, $x \vee y \notin M$. So by $\bigcap \gamma(A \perp x) \subseteq M$, $x \vee y \notin \bigcap \gamma(A \perp x)$. But since $x \in A$ and $y \in \bigcap \gamma_W([\![\neg x]\!])$, we get $x \vee y \in A \cap \bigcap \gamma_W([\![\neg x]\!]) = \bigcap \gamma(A \perp x)$, and we have again a contradiction.   Q.E.D.

As the domain $A \perp$ of selection functions for A is finitely additive, subtractive and compact, all domain requirements mentioned in Lemma 4(a)–(d) are satisfied. We can further adapt the conditions (I')–(IV') to our needs. We now know that for all $x, y \in A - Cn(\emptyset)$, $A \perp x \cup A \perp y = A \perp (x \wedge y)$, and $\gamma(A \perp x) \cap A \perp y = \emptyset$ iff $y \in \bigcap \gamma(A \perp x)$. With $\mathscr{X} = A \perp$, the general conditions (I')–(IV') can therefore be transformed into the following conditions on selection functions for theories:

(I'') For all $x, y \in A - Cn(\emptyset)$, $\gamma(A \perp (x \wedge y)) \subseteq \gamma(A \perp x) \cup \gamma(A \perp y)$
(II'') For all $x, y \in A - Cn(\emptyset)$, $\gamma(A \perp x) \cap \gamma(A \perp y) \subseteq \gamma(A \perp (x \wedge y))$
(III'') For all $x, y \in A - Cn(\emptyset)$, if $y \in \bigcap \gamma(A \perp (x \wedge y))$, then $\gamma(A \perp x) \subseteq \gamma(A \perp (x \wedge y))$
(IV'') For all $x, y \in A - Cn(\emptyset)$, if $x \notin \bigcap \gamma(A \perp (x \wedge y))$, then $\gamma(A \perp x) \subseteq \gamma(A \perp (x \wedge y))$

From these conditions, it will be a rather short step to some interesting postulates for rational ("economical") theory contraction.

The Finite Case

A set B of propositions will be called *logically finite* (or *finite modulo Cn*, or simply *finite*) if Cn partitions B into finitely many cells. The finite case is much easier to handle than the general one. This is due to the fact that every selection function over a logically finite theory is ω-covering and even full.

Notice that for a theory A to be finite modulo a common logic Cn, it will be necessary that the underlying language has only finitely many atoms. For suppose

there are infinitely many atoms $p_1, p_2, p_3, \ldots$ in our language. Then for every proposition x from A there are infinitely many atoms p_i not occurring in x. Thus the infinitely many $(x \vee p_i)$'s which are all contained in A will be mutually nonequivalent, so A is not finite. Conversely, if only a finite number of nonequivalent propositional operators of bounded arity is definable in Cn, as in classical propositional logic or in modal logics with finitely many modalities, then the finiteness of the number of propositional atoms is also sufficient for the logical finiteness of theories A phrased in the language in question.

Given a logically finite theory A, it is clear that each of the following sets is finite: $A \perp x$, for every x, $A \perp$, and U_A.

A *representative* of a logically finite set of propositions B is a conjunction of representatives of all the equivalence classes under Cn of propositions in B.

Henceforth, we shall use the following notational convention. For sets of propositions $A, B, \ldots, M, N, \ldots, W, \ldots$ which are finite modulo Cn, the lower case letters $a, b, \ldots, m, n, \ldots, w, \ldots$ denote their representatives. For any two sets A and B such that $B \subseteq A$, $\overline{b}_A$ denotes the disjunction of representatives of those equivalence classes under Cn, the elements of which are in $Cn(A) - Cn(B)$. When we are dealing with a fixed theory A, we simply write $\overline{b}$ instead of $\overline{b}_A$. If $B = A$, then $\overline{b}_A$ is defined to be the falsity, $\perp$. We may call $\overline{b}_A$ the *corepresentative* of B (relative to A).

It is easy to verify in the finite case that for $M \in U_A$ and $W = \mathscr{W}(M)$, w is equivalent with $\neg a \wedge m$, and that for $W \in V_A$ and $M = \mathscr{M}(W)$, m is equivalent with $a \vee w$. This helps us to get the following useful

Lemma 6.   *Let A be a finite theory and $M \in U_A$.*

   (i) *If w is the representative of $\mathscr{W}(M)$ and $\overline{w}_L$ is the corepresentative of $\mathscr{W}(M)$ relative to the set of all propositions in L, then the following four propositions are equivalent: $\overline{m}_A$, $m \to a$, $\overline{w}_L$ and $\neg w$.*
  (ii) *$A \perp \overline{m} = \{M\}$.*
 (iii) *For all $M_1, \ldots, M_n \in U_A$, if $m \vdash m_1 \vee \cdots \vee m_n$, then $M = M_i$ for some i.*

Lemma 6(ii), together with Corollary 1(ii), shows that in the finite case every subset **M** of U_A equals $A \perp x$ for some proposition x, viz., for $x = \bigwedge \{\overline{m} : M \in \mathbf{M}\}$. This in turn means that all selection functions for a finite theory A are ω-covering and in fact full. They are not only complete (Δ-elementary) but even elementary (cf. AGM 1985, Observation 4.6).

Postulates for Contraction Functions

We now turn to a set of rationality criteria which has gained some prominence in the literature on belief change. The *basic AGM postulates* are given by $(\dot{-}1) - (\dot{-}6)$, and the two *supplementary* ones are $(\dot{-}7)$ and $(\dot{-}8)$. For their motivation, see Gärdenfors (1988).

$(\dot{-}1)$ $A \dot{-} x$ is a theory
$(\dot{-}2)$ $A \dot{-} x \subseteq A$

($\div$3) If $x \notin A$ then $A \div x = A$
($\div$4) If $x \in A \div x$ then $\vdash x$
($\div$5) $A \subseteq Cn((A \div x) \cup \{x\})$
($\div$6) If $Cn(x) = Cn(y)$ then $A \div x = A \div y$
($\div$7) $A \div x \cap A \div y \subseteq A \div (x \wedge y)$
($\div$8) If $x \notin A \div (x \wedge y)$ then $A \div (x \wedge y) \subseteq A \div x$.

These postulates, and all following postulates, are understood as quantified over all theories A and all propositions x and y. It follows from ($\div$1) and ($\div$5) that $A \div x = A$ for every $x \in Cn(\emptyset)$. We introduce two new conditions.

($\div$8r) $A \div (x \wedge y) \subseteq Cn(A \div x \cup A \div y)$
($\div$8c) If $y \in A \div (x \wedge y)$, then $A \div (x \wedge y) \subseteq A \div x$.

With two very recent exceptions,[5] I have never seen a condition like ($\div$8r) discussed in writings on the logic of theory change. Given ($\div$4), ($\div$8) implies the "covering condition"

$$A \div (x \wedge y) \subseteq A \div x \text{ or } A \div (x \wedge y) \subseteq A \div y$$

(AGM 1985, Observation 3.4) and hence ($\div$8r).[6] Postulate ($\div$8c) was found to be relevant in Rott (1992b), where it has the same name. The "r" in ($\div$8r) stands for "relational", and "c" stands for "cumulative". The first name will be explained by the present paper, the second one is explained in Rott (1992b). For contraction functions satisfying ($\div$4), ($\div$8c) is also a weakening of ($\div$8). However, there is no logical relationship between ($\div$8c) and ($\div$8r), not even in the finite case and in the presence of ($\div$1) – ($\div$7). To see this, consider the propositional language L over the two atoms p and q. In the following two figures, "$\binom{x}{y}$" is short for "$A \div x = Cn(y)$". It is easily verified that the contraction function $\div$ over $A = Cn(p \wedge q)$ defined in Fig. 16.1 satisfies ($\div$1) – ($\div$7) and ($\div$8r), but it does not satisfy ($\div$8c), because $q \in A \div (p \wedge q)$ but $Cn(q) = A \div (p \wedge q) \not\subseteq A \div p = Cn(\neg p \vee q)$. On the other hand, the contraction function $\div$ over the same theory A defined in Fig. 16.2 satisfies ($\div$1) – ($\div$7) and ($\div$8c), but it does not satisfy ($\div$8r), because $Cn(p \vee q) = A \div (p \wedge q) \not\subseteq Cn((A \div p) \cup (A \div q)) = Cn(\{\neg p \vee q, p \vee \neg q\}) = Cn(p \leftrightarrow q)$.
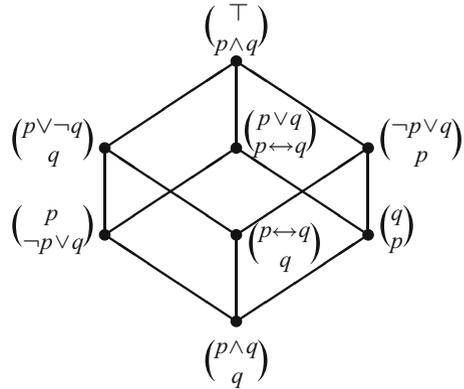
We now relate the abstract postulates for contraction functions to our previous requirements for selection functions in partial meet contraction.

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[5]Both were discovered independently and concern belief revision rather than belief contraction. The first exception is condition (R8) in Katsuno and Mendelzon (1991). The second is the infinitary condition "Gamma" in Lindström (1991) which is labelled (BC7) in its variant for belief revision operations.
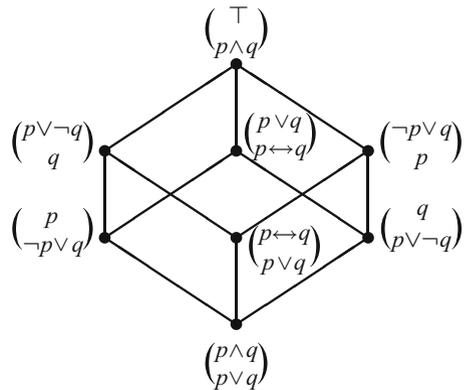
[6]Incidentally, it is proved in AGM (1985, Observation 6.5) that the conjunction of ($\div$7) and ($\div$8) is equivalent to the even stronger "ventilation condition"

$$A \div (x \wedge y) = A \div x \text{ or } A \div (x \wedge y) = A \div y \text{ or } A \div (x \wedge y) = A \div x \cap A \div y.$$

Fig. 16.1 Contraction
function satisfying
$(\dot{-}1)-(\dot{-}7)$ and $(\dot{-}8r)$, but
not $(\dot{-}8c)$

$$\begin{pmatrix}\top\\p\wedge q\end{pmatrix}$$

$$\begin{pmatrix}p\vee\neg q\\q\end{pmatrix}\qquad\begin{pmatrix}p\vee q\\p\leftrightarrow q\end{pmatrix}\qquad\begin{pmatrix}\neg p\vee q\\p\end{pmatrix}$$

$$\begin{pmatrix}p\\\neg p\vee q\end{pmatrix}\qquad\begin{pmatrix}p\leftrightarrow q\\q\end{pmatrix}\qquad\begin{pmatrix}q\\p\end{pmatrix}$$

$$\begin{pmatrix}p\wedge q\\q\end{pmatrix}$$

Fig. 16.2 Contraction
function satisfying
$(\dot{-}1)-(\dot{-}7)$ and $(\dot{-}8c)$, but
not $(\dot{-}8r)$

$$\begin{pmatrix}\top\\p\wedge q\end{pmatrix}$$

$$\begin{pmatrix}p\vee\neg q\\q\end{pmatrix}\qquad\begin{pmatrix}p\vee q\\p\leftrightarrow q\end{pmatrix}\qquad\begin{pmatrix}\neg p\vee q\\p\end{pmatrix}$$

$$\begin{pmatrix}p\\\neg p\vee q\end{pmatrix}\qquad\begin{pmatrix}p\leftrightarrow q\\p\vee q\end{pmatrix}\qquad\begin{pmatrix}q\\p\vee\neg q\end{pmatrix}$$

$$\begin{pmatrix}p\wedge q\\p\vee q\end{pmatrix}$$

Lemma 7. *Let A be a theory. If* $\dot{-} = \mathscr{C}(\gamma)$ *for some selection function*

γ for A and γ satisfies
$\begin{cases}(a) & - \\ (b) & (I'') \\ (c) & (II'')\ and\ \gamma\ is\ complete \\ (d) & (III'') \\ (e) & (IV'')\end{cases}$
then $\dot{-}$ satisfies

$\begin{cases}(a) & (\dot{-}1)-(\dot{-}6) \\ (b) & (\dot{-}1)-(\dot{-}7) \\ (c) & (\dot{-}1)-(\dot{-}6)\ and\ (\dot{-}8r) \\ (d) & (\dot{-}1)-(\dot{-}6)\ and\ (\dot{-}8c) \\ (e) & (\dot{-}1)-(\dot{-}6)\ and\ (\dot{-}8)\end{cases}$.

Proof. Let $\dot{-}$ be a partial meet contraction function over A determined by γ.

(a) It is proved in AGM (1985, Observation 2.5), that $\dot{-}$ satisfies $(\dot{-}1)-(\dot{-}6)$.

It is easy to verify that ($\dot-$7) and ($\dot-$8), and thus also ($\dot-$8r) and ($\dot-$8c), are satisfied whenever one of the limiting cases $x \notin A - Cn(\emptyset)$ or $y \notin A - Cn(\emptyset)$ holds. In the rest of this proof, we always presume the principal case $x, y \in A - Cn(\emptyset)$.

(b) Now let γ satisfy (I''), i.e., let $\gamma(A \bot (x \wedge y)) \subseteq \gamma(A \bot x) \cup \gamma(A \bot y)$. Hence $\bigcap(\gamma(A \bot x)) \cap \bigcap(\gamma(A \bot y)) = \bigcap(\gamma(A \bot x) \cup \gamma(A \bot y)) \subseteq \bigcap \gamma(A \bot (x \wedge y))$, i.e., $A \dot- x \cap A \dot- y \subseteq A \dot- (x \wedge y)$. That is, $\dot-$ satisfies ($\dot-$7).

(c) Now let γ be complete and satisfy (II''). Let $z \in A \dot- (x \wedge y) = \bigcap \gamma(A \bot (x \wedge y))$. So by (II''), $z \in \bigcap(\gamma(A \bot x) \cap \gamma(A \bot y))$. Now suppose for reductio that $z \notin Cn(A \dot- x \cup A \dot- y)$. Then there is an $M \in A \bot z$ such that $(\bigcap \gamma(A \bot x)) \cup (\bigcap \gamma(A \bot y)) \subseteq M$. Since γ is complete, we get $M \in \gamma(A \bot x)$ and $M \in \gamma(A \bot y)$, so $M \in \gamma(A \bot x) \cap \gamma(A \bot y)$. But $z \notin M$, so $z \notin \bigcap(\gamma(A \bot x) \cap \gamma(A \bot y))$ which gives us a contradiction. So $\dot-$ satisfies ($\dot-$8r).

(d) Now let γ satisfy (III''). Since the antecedents of (III'') and ($\dot-$8c) are identical and $\gamma(A \bot x) \subseteq \gamma(A \bot (x \wedge y))$ entails $\bigcap \gamma(A \bot (x \wedge y)) \subseteq \bigcap \gamma(A \bot x)$, it is obvious that $\dot-$ satisfies ($\dot-$8c).

(e) Now let γ satisfy (IV''). By exactly the same argument as in (d), $\dot-$ satisfies ($\dot-$8).   Q.E.D.

Given a contraction function $\dot-$ over A, we can derive from it a selection function $\gamma_{\dot-}$ for A. The idea is that an element of $A \bot x$ is a best element of $A \bot x$ if it includes everything which is included in the contraction of A with respect to x. So we define $\gamma_{\dot-}(A \bot x)$ as $\{M \in A \bot x : A \dot- x \subseteq M\}$, for every $x \in A - Cn(\emptyset)$. Instead of $\gamma_{\dot-}$, we shall also write $\mathscr{S}(\dot-)$.

Lemma 8. *Let A be a theory. If $\dot-$ is a contraction function over A satisfying*

$$
\left\{
\begin{array}{ll}
(a) & (\dot-1) - (\dot-6) \\
(b) & (\dot-1) - (\dot-7) \\
(c) & (\dot-1) - (\dot-6) \text{ and } (\dot-8r) \\
(d) & (\dot-1) - (\dot-6) \text{ and } (\dot-8c) \\
(e) & (\dot-1) - (\dot-6) \text{ and } (\dot-8)
\end{array}
\right\}
$$

then $\dot-$ is a partial meet contraction function determined by the selection function $\mathscr{S}(\dot-)$, i.e., $\dot- = \mathscr{C}(\mathscr{S}(\dot-))$, and $\mathscr{S}(\dot-)$ is complete and satisfies

$$
\left\{
\begin{array}{ll}
(a) & - \\
(b) & (I'') \\
(c) & (II'') \\
(d) & (III'') \\
(e) & (IV'')
\end{array}
\right\}.
$$

Proof. Let $\dot-$ satisfy ($\dot-$1) – ($\dot-$6).

(a) It is proved in (AGM 1985, Observation 2.5) that $\mathscr{S}(\dot-)$ is a selection function for A and that $\dot-$ is determined by $\mathscr{S}(\dot-)$. We add the check for well-definedness.

In the principal case $x, y \in A - Cn(\emptyset)$, $A \bot x = A \bot y$ entails that $Cn(x) = Cn(y)$; for if it were the case that $x \nvdash y$, say, then there would be an element in $A \bot y$ containing x, contrary to $A \bot x = A \bot y$. But from $Cn(x) = Cn(y)$ and $(\dot{-}6)$ it follows by the definition of $\mathscr{S}(\dot{-})$ that $\gamma(A \bot x) = \gamma(A \bot y)$, as desired. By construction, $\mathscr{S}(\dot{-})$ is complete for every contraction function $\dot{-}$. It remains to prove the various additional properties of $\mathscr{S}(\dot{-})$.

(b) Let $\dot{-}$ in addition satisfy $(\dot{-}7)$, and let $M \in \gamma_-(A \bot (x \wedge y))$. We want to show that $M \in \gamma_-(A \bot x) \cup \gamma_-(A \bot y)$. By $M \in \gamma_-(A \bot (x \wedge y))$, $A \dot{-} (x \wedge y) \subseteq M$, so by $(\dot{-}7)$, $A \dot{-} x \cap A \dot{-} y \subseteq M$. Now suppose for reductio that $M \notin \gamma_-(A \bot x) \cup \gamma_-(A \bot y)$, i.e., neither $A \dot{-} x \subseteq M$ nor $A \dot{-} y \subseteq M$. Take $z \in (A \dot{-} x) - M$ and $z' \in (A \dot{-} y) - M$. By $z, z' \in A$, $z, z' \notin M$ and the primeness of all elements of U_A, $z \vee z' \notin M$. But on the other hand, since $z \vee z' \in A \dot{-} x \cap A \dot{-} y \subseteq M$, we get $z \vee z' \in M$, so we have a contradiction. So $\mathscr{S}(\dot{-})$ satisfies (I$''$).

(c) Now let $\dot{-}$ satisfy $(\dot{-}1) - (\dot{-}6)$ and $(\dot{-}8r)$, and let $M \in \gamma_-(A \bot x) \cap \gamma_-(A \bot y)$. We want to show that $M \in \gamma_-(A \bot (x \wedge y))$. But as our hypothesis means that $A \dot{-} x \cup A \dot{-} y \subseteq M$ and M is a theory, we have $Cn(A \dot{-} x \cup A \dot{-} y) \subseteq M$, so by $(\dot{-}8r)$ $A \dot{-} (x \wedge y) \subseteq M$, so $M \in \gamma_-(A \bot (x \wedge y))$. So $\mathscr{S}(\dot{-})$ satisfies (II$''$).

(d) Now let $\dot{-}$ satisfy $(\dot{-}1) - (\dot{-}6)$ and $(\dot{-}8c)$, and let $y \in \bigcap \gamma_-(A \bot (x \wedge y)) = A \dot{-} (x \wedge y)$ and $M \in \gamma_-(A \bot x)$. The latter condition means that $A \dot{-} x \subseteq M$, so by $(\dot{-}8c)$ $A \dot{-} (x \wedge y) \subseteq M$, so $M \in \gamma_-(A \bot (x \wedge y))$, as desired. So $\mathscr{S}(\dot{-})$ satisfies (III$''$).

(e) Now let $\dot{-}$ satisfy $(\dot{-}1) - (\dot{-}6)$ and $(\dot{-}8)$. By exactly the same argument as the last one, $\mathscr{S}(\dot{-})$ satisfies (IV$''$).   Q.E.D.

Representation Theorems for Contraction Functions

We get four representation theorems for partial meet contraction as a result of conjoining previously established lemmas. Our Theorems 1 and 4 are proven in AGM (1985) although our construction for obtaining the latter appears to be quite different from the one used there. Theorems 2 and 3 are new; the "completeness" half of the former is generally valid, the remainder of the new results hold only under various additional conditions. All these conditions are satisfied if A is finite. It should be noted that our proofs of the completeness halves use one and the same method in that they all work with Samuelson revealed preferences.

Theorem 1. *A contraction function $\dot{-}$ over A satisfies $(\dot{-}1) - (\dot{-}6)$ if and only if it is a partial meet contraction function.*

Proof. See Lemmas 7(a) and 8(a) above, according to which in the cases indicated $\mathscr{C}(\gamma)$ satisfies $(\dot{-}1) - (\dot{-}6)$ and $\mathscr{S}(\dot{-})$ is a selection function for A such that $\mathscr{C}(\mathscr{S}(\dot{-})) = \dot{-}$. (Theorem 1 is Observation 2.5 of AGM 1985.)   Q.E.D.

It is clear from Lemma 8 that in the proof of the completeness half of Theorem 1 the determining selection function is chosen complete, but we do not think it is

necessary to state this in the theorem. The same comment applies to the following three representation theorems.

Theorem 2. *Every relational partial meet contraction function ÷ over A satisfies (÷1) – (÷7), and if ÷ is determined by a selection function that is both relational and complete (equivalently, Δ-elementary), then it satisfies (÷8r). Conversely, every contraction function ÷ over A satisfying (÷1) – (÷7) and (÷8r) is a relational partial meet contraction function.*

Proof. For the first part, let ÷ be a partial meet contraction function determined by a preference relation ≤. By Lemma 1(a), $\mathscr{S}(\leq)$ satisfies (I) and (II). Since $A\bot$ is subtractive, it also satisfies (I′) and (II′), by Lemma 2(a), and also (I″) and (II″), by reformulation. So by Lemma 7 (b), $\mathscr{C}(\mathscr{S}(\leq))$ satisfies (÷1) – (÷7), and by Lemma 7(c), it satisfies (÷8r), if $\mathscr{S}(\leq)$ is complete, i.e., Δ-elementary.

For the converse, let ÷ satisfy (÷1) – (÷7) and (÷8r). By Lemma 8(b) and (c), $\mathscr{C}(\mathscr{S}(\dot{-}))$ = ÷ and $\mathscr{S}(\dot{-})$ satisfies (I″) and (II″), and also (I′) and (II′), by reformulation. Since $A\bot$ is subtractive, $\mathscr{S}(\dot{-})$ satisfies (I), by Lemma 4(a). Since $A\bot$ is also finitely additive and compact, $\mathscr{S}(\dot{-})$ is relational with respect to $\mathscr{P}(\mathscr{S}(\dot{-}))$, by Lemma 4(b). That is, $\mathscr{S}(\dot{-}) = \mathscr{S}(\mathscr{P}(\mathscr{S}(\dot{-})))$. Hence ÷ = $\mathscr{C}(\mathscr{S}(\mathscr{P}(\mathscr{S}(\dot{-}))))$, i.e., ÷ is relational with respect to $\mathscr{P}(\mathscr{S}(\dot{-}))$.   Q.E.D.

The completeness of γ is necessary in the "soundness" part of Theorem 2. The possible failure without completeness is due to the fact that for sets W' and W'' of maximally consistent sets of propositions, it does not necessarily hold that $\bigcap (W' \cap W'') \subseteq Cn (\bigcap W' \cup \bigcap W'')$.

Theorem 3. *Every relational partial meet contraction function ÷ over A determined by a smooth and negatively transitive preference relation satisfies (÷1) – (÷7) and (÷8c). In partial converse, every contraction function ÷ over A satisfying (÷1) – (÷7), (÷8r), and (÷8c) is a relational partial meet contraction function, and if $A\bot$ is 123-covering, then the determining preference relation can be chosen negatively transitive.*

Proof. For the first part, let ÷ be a partial meet contraction function determined by a smooth and negatively transitive preference relation ≤. We show in the same way as in the proof of Theorem 2 that $\mathscr{S}(\leq)$ satisfies (I′) and (II′), and we know from Lemmas 3(b) and 4(c) that it satisfies (III) and (III′). So by Lemma 7(b) and (d), $\mathscr{C}(\mathscr{S}(\leq))$ satisfies (÷1) – (÷7) and (÷8c).

For the converse, let ÷ satisfy (÷1) – (÷7), (÷8r), and (÷8c). As in the proof of Theorem 2, we get that ÷ is relational with respect to $\mathscr{P}(\mathscr{S}(\dot{-}))$. Moreover, by Lemma 8(b), $\mathscr{S}(\dot{-})$ satisfies (I′), and by Lemma 8(d), $\mathscr{S}(\dot{-})$ satisfies (III′) and also (III), by Lemma 4(c). Hence if $A\bot$ is 123-covering, then $\mathscr{P}(\mathscr{S}(\dot{-}))$ is negatively transitive, by Lemma 2(b).   Q.E.D.

Theorem 4. *A contraction function ÷ over A satisfies (÷1) – (÷6), (÷7), and (÷8) if and only if it is a transitively relational partial meet contraction function.*

Proof. This is Observation 4.4 of AGM (1985), but we shall sketch our proof which is quite different from the construction offered there.

For the first part of the theorem, let $\dot{-}$ be a partial meet contraction function determined by a transitive preference relation $\leq$. We show in the same way as in the proof of Theorem 2 that $\mathscr{S}(\leq)$ satisfies (I′) and (II′), and we know from Lemmas 3(c) and 4(d) that it satisfies (IV) and (IV′). So by Lemma 7(b), $\mathscr{C}(\mathscr{S}(\leq))$ satisfies $(\dot{-}1) - (\dot{-}7)$, and by Lemma 7(e), it satisfies $(\dot{-}8)$.

For the converse, let $\dot{-}$ satisfy $(\dot{-}1) - (\dot{-}7)$ and $(\dot{-}8)$. As in the proof of Theorem 2, we get that $\dot{-}$ is relational with respect to $\mathscr{P}(\mathscr{S}(\dot{-}))$. Moreover, by Lemma 8(e), $\mathscr{S}(\dot{-})$ satisfies (IV′) and also (IV), by Lemma 4(d). Since $A\perp$ is finitely additive, $\mathscr{P}(\mathscr{S}(\dot{-}))$ is transitive, by Lemma 2(c). Q.E.D.

Corollary 2. *Let A be a logically finite theory. Then $\dot{-}$ is a*

$$\left\{ \begin{array}{c} - \\ relational \\ negatively\ transitively\ relational \\ transitively\ relational \end{array} \right\} partial\ meet\ contraction\ function\ iff\ it\ satisfies$$

$$\left\{ \begin{array}{c} (\dot{-}1) - (\dot{-}6) \\ (\dot{-}1) - (\dot{-}7)\ and\ (\dot{-}8r) \\ (\dot{-}1) - (\dot{-}7),\ (\dot{-}8r)\ and\ (\dot{-}8c) \\ (\dot{-}1) - (\dot{-}7)\ and\ (\dot{-}8) \end{array} \right\}.$$

Proof. By Theorems 1, 2, 3, and 4, because for finite theories A, γ over $A\perp$ is complete, every negatively transitive nonstrict preference relation $\leq$ over A is smooth, and $A\perp$ is 123-covering. □

It is easy to locate the impact of the finiteness assumption. In the case of relational partial meet contraction s (Theorem 2), it is a constraint (completeness) imposed by model theory which has to be met in order to make the "soundness" direction work. In the case of negatively transitively relational partial meet contraction s (Theorem 3), it is a constraint imposed by the theory of rational choice (that γ be 123-covering and smooth) which is satisfied without strains on intuitive plausibility only in the finite case.

It should be noted that although $(\dot{-}8)$ in the context of the other postulates implies $(\dot{-}8r)$ and $(\dot{-}8c)$, transitive relationality does not imply negatively transitive relationality. However, transitivity in companion with connectivity implies negative transitivity. And it is known from AGM (1985, §5) that a connectivity requirement on the underlying preference relation changes very little in the partial meet mechanism.[7]

We conclude this section with a direct representation of the Samuelson preferences over U_A revealed by contraction behavior. Let $\leq = \mathscr{P}(\mathscr{S}(\dot{-}))$. Then in the general case, $M \leq M'$ iff there is an x such that $A \dot{-} x \subseteq M'$ and $M \in A \perp x$, or there

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[7]In order to reconcile this with your intuitions, cf. Footnote 1.

is no x such that $A \dot{-} x \subseteq M$. We recall, for contrast, the construction used in the proof of AGM (1985, Observation 4.4). There $M \leq M'$ is defined to hold between sets M and M' from U_A iff for all x such that $A \dot{-} x \subseteq M$ and $M' \in A \perp x$, it holds that $A \dot{-} x \subseteq M'$, and there is an $x \in A$ such that $A \dot{-} x \subseteq M'$.[8] Evidently, this definition is quite different from the one we used, and it is doubtful whether it is suitable for any case but the strongest one which is treated in Theorem 4. It is interesting, however, that AGM employ our construction in their proof of Observation 4.10 of AGM (1985). In the finite case, it is not difficult to find out that according to our definition, $M \leq M'$ iff $A \dot{-} \overline{m} \wedge \overline{m'} \subseteq M'$, or equivalently(!), iff $\overline{m'} \notin A \dot{-} \overline{m} \wedge \overline{m'}$.

The Logic of Prioritized Base Contractions in the Finite Case

In this section we present the concept of prioritized base contraction as studied in particular by Nebel (1989, 1992). On the face of it, these belief change operations are entirely different from the partial meet contractions discussed so far. As Nebel has shown, however, very close connections can be established between the two approaches. We shall further elaborate this theme and determine the logic of prioritized base contractions in terms of the above-mentioned postulates by applying one of our central representation theorems.

Let B be a finite set of propositions. B is called a *(simple) belief base* for the theory A if $Cn(B) = A$. B is called *prioritized* if there is a weak ordering, i.e., a transitive and connected relation $\preceq$ over B (Nebel's 1989, 1992 "epistemic relevance"). The relation $\sim = \preceq \cap \preceq^{-1}$ partitions B into finitely many equivalence classes $B_1, \ldots, B_n$. We label the classes in such a way that $x_i \preceq x_j$ for $x_i \in B_i$ and $x_j \in B_j$ with $i \leq j$. Generalizing slightly, we say that a *prioritized belief base* for A is a finite sequence $\langle B_1, \ldots, B_n \rangle$, or $\langle B_i \rangle$ for short, of finite sets of propositions such that $Cn(\bigcup B_i) = A$. Simple belief bases are now special cases of prioritized belief bases with $n = 1$. We do not require that the B_i's are disjoint or mutually consistent. Intuitively, the propositions in B_n are the most important (most relevant, most valuable, most certain, etc.) pieces of information, and the propositions in B_1 are the least important. One can think of the complement of $\bigcup B_i$ as the set B_0 of propositions for which there is no independent evidence. Although the finiteness of $B = \bigcup B_i$ does not ensure that the generated theory A is logically finite, our central result in this section, Theorem 7, will be restricted to this case.

Prioritized base contractions are contractions of theories A presented in the form of (generated by, associated with) a fixed prioritized belief base $\langle B_i \rangle$. The basic idea is to minimize the loss of propositions at every level of "priority".

For every $i \in \{1, \ldots, n\}$, let $B_{\geq i}$ be the set of all elements of $B = \bigcup B_i$ with a priority of at least i, i.e., $B_{\geq i} = \bigcup \{B_j : j \geq i\}$. Furthermore, let $C_i = C \cap B_i$ and

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[8]This simplified rephrasing of the AGM construction makes use of the fact that for all $M \in U_A$ and contraction functions $\dot{-}$ satisfying the AGM postulates, $A \dot{-} x \subseteq M$ implies $M \in A \perp x$.

$C_{\geq i} = C \cap B_{\geq i}$ for any set of propositions C. Then, in the context of this section, C is a *preferred x-discarding* subset of B if $C \subseteq B$ and for every i, $C_{\geq i}$ is an inclusion maximal subset of $B_{\geq i}$ subject to the condition that x is not implied.

Two kinds of information are used in prioritized base contraction: the syntactical information encoded in the structure of the propositions in B, and the weighting of these propositions expressed by $\preceq$. If the belief base is simple, i.e., if $n = 1$, then we only exploit syntactical information. Prioritized base contractions and revisions are studied in an infinitistic framework by Nebel (1989, 1992).[9] He shows that prioritized base revisions can be represented by partial meet theory revisions. Then he proves that a certain variant of simple base contraction satisfies $(\div 1) - (\div 7)$ but not $(\div 8)$, and a corresponding result for prioritized base revisions. However, Nebel leaves open the question which logic is *characterized* by prioritized base contractions. Building on Theorem 3 and its corollary, we shall answer this question for the finite case. Our concern will be the slightly more complicated case of prioritized base contractions, and we try to make our proof more transparent by introducing the concept of a base-oriented selection function.

Now consider the following strict preference relations between arbitrary sets of propositions. For every $i \in \{1, \ldots, n\}$ and any two sets C and C', we write $C \ll_i C'$ if and only if $C_i \subset C'_i$ and $C_j \subseteq C'_j$ for every $j > i$. We write $C \ll C''$ if there is an i such that $C \ll_i C'$. The relation $\ll$ is to reflect the fact that intuitively a set C' is better than another set C just in case it contains more important beliefs than C. In particular, $\ll$ satisfies a version of Hansson's (1992) maximizing property, because $C \cap B \subset C' \cap B$ implies $C \ll C'$. It is immediately verified that C is a preferred x-discarding subset of B if and only if it is a maximal element in $B \perp x$ under $\ll$. (Recall that $B \perp x$ is the set $\{N \subseteq B : x \notin Cn(N) \text{ and } x \in Cn(D) \text{ for all } D \text{ with } N \subset D \subseteq B\}$.)

Following Nebel, we let $\underline{\ll}$ be the converse complement of $\ll$, i.e., $C \underline{\ll} C'$ iff not $C' \ll C$. We denote this preference relation over arbitrary sets of propositions by $\mathscr{P}(\langle B_i \rangle)$. Clearly, since $\ll$ is irreflexive, asymmetric, conversely well-founded and transitive, $\mathscr{P}(\langle B_i \rangle)$ is reflexive, connected, negatively well-founded and negatively transitive. But $\mathscr{P}(\langle B_i \rangle)$ is not transitive, not even in the special case of simple belief bases ($n = 1$) where $\underline{\ll}$ coincides with $\not\supset$.

As before, a selection function $\gamma = \mathscr{S}(\underline{\ll})$ can be defined by stringent maximization. But this time γ is a selection function for the base B as well as for the theory A, and its domain may in fact be construed as the class of all nonempty sets of sets propositions in L, that is $2^{2^L} - \{\emptyset\}$. In the following, it is understood that $\gamma(B \perp x) = \{N \in B \perp x : N' \underline{\ll} N \text{ for all } N' \in B \perp x\}$, while $\gamma(A \perp x) = \{M \in A \perp x : M' \underline{\ll} M \text{ for all } M' \in A \perp x\}$. Either way, γ is a selection function.

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[9]Nebel's (1992) treatment of the fully general infinite case is not quite correct. Slips have crept into his claim that his $C \Downarrow \phi$ is nonempty, into his definition (9) of $\ll$, and into the proof of Proposition 8. As Nebel (personal communication) has suggested, they can be remedied by imposing a condition of converse well-foundedness on $\preceq$.

Let $\langle B_i \rangle$ be a prioritized belief base for A, and let $\gamma = \mathscr{S}(\mathscr{P}(\langle B_i \rangle))$ be the selection function over $A\bot$ determined by $\langle B_i \rangle$. A straightforward idea to get prioritized base contractions is the following:

$$A \dot{-} x = \begin{cases} \bigcap \{Cn(N) : N \in \gamma(B\bot x)\} & \text{for every } x \in A - Cn(\emptyset) \\ A & \text{for every } x \notin A \text{ and every } x \in Cn(\emptyset) \end{cases}$$

In the special case of simple belief bases ($n = 1$) we have $\gamma(B\bot x) = B\bot x$, so this definition boils down to the *full* meet $\bigcap \{B\bot x\}$.

Contraction functions obtained in this way satisfy most but not all of the postulates we would want them to satisfy.

Theorem 5. *If $\dot{-}$ is a prioritized base contraction function as determined by the above definition, then it satisfies $(\dot{-}1) - (\dot{-}4)$, $(\dot{-}6)$, $(\dot{-}7)$, and $(\dot{-}8c)$. However, even if the base is simple, $\dot{-}$ will in general fail to satisfy $(\dot{-}5)$ and $(\dot{-}8r)$.*

Proof. It is obvious that $\dot{-}$ satisfies $(\dot{-}1) - (\dot{-}4)$ and $(\dot{-}6)$.

For $(\dot{-}7)$, assume that $N \vdash z$ for every $N \in \gamma(B\bot x) \cup \gamma(B\bot y)$. We need to show that $N' \vdash z$ for every $N' \in \gamma(B\bot (x \wedge y))$. Let $N' \in \gamma(B\bot (x \wedge y)) \subseteq B\bot (x \wedge y)$. First we note that every element of $B\bot (x \wedge y)$ is either in $B\bot x$ or in $B\bot y$. Without loss of generality, assume that $N' \in B\bot x$. We are ready if we can show that N' is in $\gamma(B\bot x)$. Suppose it is not. Then there is an $N_1 \in B\bot x$ such that $N' \ll N_1$. N_1 must not be in $B\bot (x \wedge y)$, since N' is in $\gamma(B\bot (x \wedge y))$. Because $N_1 \in B\bot x - B\bot (x \wedge y)$, we get that there is a proper superset N_2 of N_1 such that $N_2 \in B\bot y$. Since $N_1 \in B\bot x$, $N_2 \vdash x$. Since every proper superset of N_2 satisfies both x and y, N_2 is in $B\bot (x \wedge y)$. By the maximizing property, we have $N_1 \ll N_2$. By the transitivity of $\ll$, we get $N' \ll N_2$. This, however, contradicts $N' \in \gamma(B\bot (x \wedge y))$ and $N_2 \in B\bot (x \wedge y)$. Therefore our supposition must have been false.

For $(\dot{-}8c)$, assume that, first, $N \vdash y$ and, second, $N \vdash z$ for every $N \in \gamma(B\bot (x \wedge y))$. We need to show that $N' \vdash z$ for every $N' \in \gamma(B\bot x)$. Let $N' \in \gamma(B\bot x)$ and suppose for reductio that $N' \nvdash z$. If N' is in $B\bot (x \wedge y)$ then there must be, by the second assumption, an N_1 in $B\bot (x \wedge y)$ such that $N' \ll N_1$. In fact, N_1 can be chosen from $\gamma(B\bot (x \wedge y))$, by the transitivity and converse well-foundedness of $\ll$. By the first assumption, $N_1 \in B\bot x$. But this contradicts $N' \ll N_1$ and $N' \in \gamma(B\bot x)$. Hence N' cannot be in $B\bot (x \wedge y)$. But because $N' \nvdash x \wedge y$, there must be a proper superset N_2 of N' in $B\bot (x \wedge y)$. We know that $N_2 \vdash x$, by $N' \in B\bot x$. By the first assumption then, N_2 cannot be in $\gamma(B\bot (x \wedge y))$. By the transitivity and converse well-foundedness of $\ll$, there must be an N_3 in $\gamma(B\bot (x \wedge y))$ such that $N_2 \ll N_3$. By our first assumption, N_3 is in $B\bot x$. By the maximizing property, we have $N' \ll N_2 \ll N_3$, so by transitivity $N' \ll N_3$. This, however, contradicts $N' \in \gamma(B\bot x)$ and $N_3 \in B\bot x$. Therefore our supposition must have been false.

The failure of $(\dot{-}5)$ is obvious. Consider for instance $B = \{p \wedge q\}$, for which $A = Cn(p \wedge q) \nsubseteq Cn((A \dot{-} p) \cup \{p\}) = Cn(p)$.

The failure of $(\dot{-}8r)$ is not so easy to see. The simplest examples involve three atoms p, q, r and look somewhat like the following. Let B consist of the following four propositions:

$$(p \wedge q) \vee (p \wedge r) \vee (q \wedge r), \quad r \to (p \leftrightarrow q),$$

$$q \wedge (p \leftrightarrow r), \quad p \wedge r.$$

Clearly $A = Cn(B) = Cn(p \wedge q \wedge r)$. Note that $q \wedge (p \leftrightarrow r)$ implies $r \to (p \leftrightarrow q)$ and that $p \wedge r$ implies $(p \wedge q) \vee (p \wedge r) \vee (q \wedge r)$. $(p \wedge q) \vee (p \wedge r) \vee (q \wedge r)$ and $r \to (p \leftrightarrow q)$ taken together imply $p \wedge q$. This gives us

$$B \perp p = \{\{(p \wedge q) \vee (p \wedge r) \vee (q \wedge r)\}, \{r \to (p \leftrightarrow q), q \wedge (p \leftrightarrow r)\}\}$$

and

$$B \perp q = \{\{r \to (p \leftrightarrow q)\}, \{(p \wedge q) \vee (p \wedge r) \vee (q \wedge r), p \wedge r\}\}.$$

Therefore, the above account yields $A \dot{-} p = Cn(q \vee (p \wedge r))$ and $A \dot{-} q = Cn(\neg(\neg p \wedge q \wedge r))$, so $Cn(A \dot{-} p \cup A \dot{-} q) = Cn((p \wedge r) \vee (q \wedge \neg r))$. On the other hand,

$$B \perp (p \wedge q) = \{\{(p \wedge q) \vee (p \wedge r) \vee (q \wedge r), p \wedge r\}, \{r \to (p \leftrightarrow q), q \wedge (p \leftrightarrow r)\}\},$$

yielding $A \dot{-} p \wedge q = Cn((p \wedge r) \vee (\neg p \wedge q \wedge \neg r))$. At last we realize that the union of $A \dot{-} p$ and $A \dot{-} q$ does not imply $(p \wedge r) \vee (\neg p \wedge q \wedge \neg r)$, because the former, but not the latter is compatible with $p \wedge q \wedge \neg r$.   Q.E.D.

In so far as one considers the postulates ($\dot{-}5$) and ($\dot{-}8r$) to be plausible principles of theory change—which in fact we do—, Theorem 5 shows that the above definition is defective. For this reason we introduce a slightly more sophisticated concept, and our official definition reads as follows. Let a again be a representative of A.

The *prioritized base contraction* $\dot{-}$ over a logically finite theory A determined by $\langle B_i \rangle$ is then given by

$$A \dot{-} x =$$

$$\begin{cases} \bigcap \{Cn(N \cup \{x \to a\}): N \in \gamma(B \perp x)\}^{10} & \text{for every } x \in A - Cn(\emptyset) \\ A & \text{for every } x \notin A \text{ and } x \in Cn(\emptyset) \end{cases}$$

The singleton $\{x \to a\}$ is inserted in order to make sure that the recovery postulate ($\dot{-}5$) is satisfied. This trick is used by Nebel (1989).[11] We shall see that this modification also solves our problem with ($\dot{-}8r$). As a first indication

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[10]Or equivalently, $Cn((\bigcap\{Cn(N) : N \in \gamma(B \perp x)\}) \cup \{x \to a\}) = Cn(A \dot{-}'x \cup \{x \to a\})$ where $A \dot{-}'x$ follows the first definition.

[11]Nebel's (1992) later paper deals with revisions where this problem does not arise. In fact, if revisions are construed as being generated by the so-called Levi-identity $A * x = Cn((A \dot{-} \neg x) \cup \{x\})$, then the modification made in our official definition does not have any effect on revisions.

of this, note that the official definition cures our above counterexample to $(\dot{-}8r)$ by strengthening $A \dot{-} x$ to $Cn(y \wedge (x \to z))$ and $A \dot{-} y$ to $Cn(y \to (x \wedge z))$, so that $Cn(A \dot{-} x \cup A \dot{-} y) = Cn(x \wedge y \wedge z)$. Since this has been the full theory A, $(\dot{-}8r)$ is clearly satisfied.

We denote the prioritized base contraction over A determined by the prioritized belief base $\langle B_i \rangle$ by $\mathscr{C}(\langle B_i \rangle)$. A contraction function $\dot{-}$ over A is a *simple* (or *prioritized*) *base contraction function* if there is a simple belief base B (a prioritized belief base $\langle B_i \rangle$) for A such that $\dot{-} = \mathscr{C}(B)$ (respectively, $\dot{-} = \mathscr{C}(\langle B_i \rangle)$). In conformity with the terminology of AGM (1985), simple base contractions could also be named *full meet* base contractions.

Lemma 9. *Let $B = \bigcup B_i$, $A = Cn(B)$, $M \in A \perp x$, and let $\gamma = \mathscr{S}(\mathscr{P}(\langle B_i \rangle))$. Then $M \in \gamma(A \perp x)$ iff $N \subseteq M$ for some $N \in \gamma(B \perp x)$.*

Let us call selection functions which harmonize choices in $A \perp$ with choices in $B \perp$ in the way exhibited in Lemma 9 base-oriented. The following result shows that the selection function γ induced by a prioritized belief base leads to the same result when applied directly to belief bases as when applied to the generated theory. Prioritized base contraction thus reduces to a special kind of partial meet theory contraction. A proof of almost the same fact has already been given by Nebel (1989, Theorem 14, 1992, Theorem 7). Our proof is somewhat more general in that it only turns on the fact that γ is base-oriented. Nothing hinges on the particular definition of $\underset{\approx}{\leqslant}$.

Theorem 6. *Let $\langle B_i \rangle$ be a prioritized belief base for a logically finite theory A, and let $\gamma = \mathscr{S}(\mathscr{P}(\langle B_i \rangle))$. Then $\mathscr{C}(\langle B_i \rangle) = \mathscr{C}(\gamma)$.*

Proof. For the principal case $x \in A - Cn(\emptyset)$, we have to show that

$$\bigcap \{Cn(N \cup \{x \to a\}) : N \in \gamma(B \perp x)\} = \bigcap \gamma(A \perp x).$$

By Lemma 9, we know that γ is base-oriented.

"LHS $\subseteq$ RHS": Let $N \cup \{x \to a\} \vdash y$ for every $N \in \gamma(B \perp x)$, and let $M \in \gamma(A \perp x)$. Since γ is base-oriented, there is an $N \in \gamma(B \perp x)$ such that $N \subseteq M$. So $M \cup \{x \to a\} \vdash y$. But also, by the fullness of the elements in $A \perp x$, $x \to a \in M$. So $M \vdash y$, as desired.

"RHS $\subseteq$ LHS": Assume that there is an $N \in \gamma(B \perp x)$ such that $N \cup \{x \to a\} \not\vdash y$. Hence, by the deduction theorem and supraclassicality, $N \cup \{y \to a\} \not\vdash x$. Thus, there is an $M \in A \perp x$ such that $N \cup \{y \to a\} \subseteq M$. Since $N \subseteq M$ and γ is base-oriented, $M \in \gamma(A \perp x)$. Since $y \to a \in M$, $a \vdash x$ and $M \not\vdash x$, we have $y \notin M$. So $y \notin \bigcap \gamma(A \perp x)$, as desired.   Q.E.D.

We are now going to construct, for an arbitrary given preference relation $\leq$ over a logically finite theory A, an equivalent simple belief base B for A. This base will be denoted $\mathscr{B}(\leq)$. For $M \in U_A$, let $\hat{M}$ be the set $\{M' \in U_A : M' \not\leq M\} \cup \{M\}$ of elements in U_A which are not covered by M, together with M itself. Let $\hat{m}$ be an abbreviation for the disjunction $\bigvee \{m' : M' \in \hat{M}\}$, with each m' being a representative of the

respective M'. Then $B = \mathscr{B}(\leq)$ is defined as the set $\{\hat{m} : M \in U_A\}$. Since A is finite modulo Cn, everything else is. This construction and the proof idea for the direction "(iii)$\Rightarrow$(i)" in Theorem 7 is adapted from Lewis (1981).

Lemma 10. *Let A be a logically finite theory and $\leq$ be a reflexive and negatively transitive preference relation over U_A. Then $\mathscr{P}(\mathscr{B}(\leq)) =\leq$.*

Proof. As a preparatory consideration, let us consider what is contained in a set $M \cap B$, when $M \in U_A$ and $B = \mathscr{B}(\leq)$ for a given $\leq$. By definition, $M = Cn(m)$. Now any proposition from B is of the form $\widehat{m^*}$ for some $M^* \in U_A$, or more explicitly, of the form $\bigvee\{m^{**} : M^{**} \in U_A,$ and $M^{**} \not\leq M^*$ or $M^{**} = M^*\}$. We know that a proposition $\widehat{m^*}$ from B is in M if and only if $m \vdash \widehat{m^*}$, i.e., $m \vdash \bigvee\{m^{**} : M^{**} \in U_A,$ and $M^{**} \not\leq M^*$ or $M^{**} = M^*\}$. By Lemma 6(iii), this obtains just in case $M \not\leq M^*$ or $M = M^*$.

Now let $\leq^* = \mathscr{P}(\mathscr{B}(\leq))$. We want to show that $M \leq^* M'$ iff $M \leq M'$. By definition, $M \leq^* M'$ if and only if either $M \cap B \subseteq M' \cap B$ or $M' \cap B \not\subseteq M \cap B$. Applying our preparatory consideration, this means that either

(i)   For every $M^* \in U_A$, if $M \not\leq M^*$ or $M = M^*$, then $M' \not\leq M^*$ or $M' = M^*$,
   or

(ii) There is an $M^* \in U_A$ such that $M' \not\leq M^*$ or $M' = M^*$, and $M \leq M^*$ and $M \neq M^*$.

We need to show that this is equivalent with $M \leq M'$. That the latter implies (i) or (ii) is clear: If $M = M'$, then (i), and if $M \neq M'$, then (ii) is immediate by putting $M^* = M'$.

For the converse, suppose that $M \not\leq M'$. We verify that this is incompatible with each of (i) and (ii). From (i), we get by putting $M^* = M$ that $M' \not\leq M$ or $M' = M$. In the former case we get from $M \not\leq M'$ and negative transitivity that $M \not\leq M$, contradicting the reflexivity of $\leq$. The latter case doesn't fare better; it contradicts $M \not\leq M'$ as well. From (ii), we get that $M' \not\leq M^*$ or $M' = M^*$ for some M^*. In the former case, $M \not\leq M'$ and negative transitivity would give us $M \not\leq M^*$, contradicting (ii). In the latter case, $M \not\leq M'$ would equally well yield $M \not\leq M^*$, again contradicting (ii).   Q.E.D.

Theorem 7. *Let $\dot{-}$ be a contraction function over a logically finite theory A. Then the following conditions are equivalent:*

(i) $\dot{-}$ *is a simple base contraction function;*
(ii) $\dot{-}$ *is a prioritized base contraction function;*
(iii) $\dot{-}$ *satisfies $(\dot{-}1) - (\dot{-}7)$, $(\dot{-}8r)$, and $(\dot{-}8c)$.*

Proof. That (i) entails (ii) is trivial.

To show that (ii) entails (iii), observe that by Theorem 6, $\mathscr{C}(\langle B_i\rangle) = \mathscr{C}(\mathscr{S}(\mathscr{P}(\langle B_i\rangle)))$ and that $\mathscr{P}(\langle B_i\rangle)$ is reflexive and negatively transitive. So by Corollary 2, $\mathscr{C}(\langle B_i\rangle)$ satisfies $(\dot{-}1) - (\dot{-}7)$, $(\dot{-}8r)$, and $(\dot{-}8c)$.

To show that (iii) entails (i), we conclude from (iii) with the help of Corollary 2 that there is a negatively transitive preference relation $\leq$ such that $\dot{-} = \mathscr{C}(\mathscr{S}(\leq))$.

We know from Lemma 10 that the latter is identical with $\mathscr{C}(\mathscr{S}(\mathscr{P}(\mathscr{B}(\leq))))$. So there is a simple belief base B, namely $B = \mathscr{B}(\leq)$, such that $\dot{-} = \mathscr{C}(\mathscr{S}(\mathscr{P}(B)))$. By Theorem 6 then, $\dot{-} = \mathscr{C}(B)$.   Q.E.D.

Theorem 7 tells us that for every prioritized belief base $\langle B_i \rangle$ for A there is a simple belief base B for A such that $\mathscr{C}(B) = \mathscr{C}(\langle B_i \rangle)$. In the finite case, every prioritization can be encoded syntactically. Does this mean that prioritization is superfluous? In answering this question we first have to emphasize that our generation of B from $\langle B_i \rangle$ took a rather roundabout route: $B = \mathscr{B}(\mathscr{P}(\langle B_i \rangle))$. An interesting problem now is whether a more perspicuous construction of B from $\langle B_i \rangle$ is possible. This question, too, is put in informal terms, and as such it permits no absolutely precise answer. Still we think that the answer must be no. Even for the most straightforward prioritized belief bases $\langle B_i \rangle$ the generated simple base $B = \mathscr{B}(\mathscr{P}(\langle B_i \rangle))$ becomes grossly unintuitive, and there is no prospect of finding different solutions to the problem. Consider for example the base containing p and $p \to q$ with the alternative prioritizations $\langle \{p \to q\}, \{p\} \rangle$ and $\langle \{p\}, \{p \to q\} \rangle$. In the former case, $B = \{p \wedge q, p, p \vee q, q \to p\}$, while in the latter case, $B' = \{q, p \leftrightarrow q\}$ will lead to exactly the same results as the prioritized belief base. But in neither case is there anything like a transparent relation to the original base $\langle B_i \rangle$. It appears that prioritization is useful, notwithstanding its formal dispensability in the finite case.

Appendix 1: Relating Theory Change and Nonmonotonic Logic

In a recent paper, Makinson and Gärdenfors (1991) make considerable progress toward linking the areas of theory change and nonmonotonic reasoning. They close with the following problem which we quote in full:

> We end with an important open question. In their (1985), Alchourrón, Gärdenfors and Makinson established a representation theorem for theory revision operations $*$ satisfying conditions $(*1)$–$(*8)$, in terms of "transitively relational partial meet revisions". The proof went via a representation theorem for a contraction function $\dot{-}$ satisfying certain conditions $(\dot{-}1)$ – $(\dot{-}8)$. On the other hand, Kraus et al. (1990) have established a representation theorem for supraclassical, cumulative and distributive nonmonotonic inference relations $\mid\!\sim$ defined between individual propositions, in terms of classical stoppered preferential model structures. The former proof is relatively short and abstract; the latter seems more complex. Also, the latter has not been generalized to a representation theorem for supraclassical, cumulative and distributive inference operations $C : 2^L \to 2^L$ [...] Does the representation theorem for theory change hold the key for a solution to this problem of extending the Kraus/Lehmann/Magidor representation theorem to the infinite case—despite the failure of consistency preservation for preferential model structures? Or do we have two essentially different representation problems? (Makinson and Gärdenfors 1991, pp. 203–204, notation adapted)

This question does not have a simple answer. Three different points have to be taken into consideration.

First, the approaches of AGM and KLM are not as distinct as Makinson and Gärdenfors seemed to assume. AGM contract and revise by single propositions, and similarly KLM consider the nonmonotonic consequences of simple propositions. Makinson and Gärdenfors's equation $y \in A * x$ iff $x \hspace{1pt}\vdash\hspace{-6pt}\sim y$ (iff $y \in C(x)$) fully reflects this. A truly infinitistic stance towards both theory change and nonmonotonic logic is taken only by Lindström (1991). The question of whether the theory A is logically finite has no bearing on this issue. Here Makinson and Gärdenfors saw a difference which simply does not exist.

Secondly, Makinson and Gärdenfors tacitly passed over the fact that in KLM there is no counterpart to ($*8$) or ($\div 8$). But this difference is indeed crucial, as is clear from a later paper of Lehmann and Magidor (1992). As regards the *preferential* logics dealt with by KLM, no "relatively short and abstract" proof seems to be possible. Thus it appears reasonable to construe Makinson and Gärdenfors's question as referring to the *rational* logics treated by Lehmann and Magidor.

But thirdly, Lehmann and Magidor's rational logics still differ from AGM-style theory revisions in that they are not required to satisfy a condition of consistency preservation which corresponds to postulate ($\div 4$) for contractions. In this respect, Makinson and Gärdenfors show a keen sense of the intricacies of the situation. In unpublished notes, we have applied the techniques developed in this paper to the problem of providing rational logics with canonical models. We have found that it *is* possible to transfer our proof to the area of nonmonotonic logic, but that consistency preservation with respect to the underlying monotonic logic *Cn* is, in fact, indispensable for a perfect matching. The reason is that the compactness property presupposed for *Cn* runs idle if there are any "inaccessible worlds" (and this is just what a violation of consistency preservation amounts to). Since the results of our efforts bear considerable similarity with the venerable presentation in Lewis (1973)—except for the fact that the role of Lewis's extra-linguistic propositions (sets of "real" possible worlds) is played by the propositions of the object language—we had better refrain from expounding them here in more detail.

Appendix 2: Some Proofs and Examples[12]

Proof of Lemma 1. (a) Immediate from the definition of relationality. (If x is greatest in S' then it is so in all subsets of S' in which it is contained. If x is greatest in every S_i then it is so in $\bigcup S_i$.)

(b) Let γ be 12-covering and satisfy (I) and (II). We have to show that for every $S \in \mathscr{X}$, $\gamma(S) = \{x \in S : \text{for all } y \in S, \{x, y\} \in \mathscr{X} \text{ and } x \in \gamma(\{x, y\})\}$. *LHS $\subseteq$ RHS*: Let $x \in \gamma(S)$ and $y \in S$. As γ is 12-covering, $\{x, y\} \in \mathscr{X}$ and (I) gives us $x \in \gamma(\{x, y\})$.

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[12]This appendix was not included in the original 1993 publication of this paper.

RHS $\subseteq$ *LHS*: Let $x \in S$ and assume that for every $y \in S$, $\{x, y\} \in \mathcal{X}$ and $x \in \gamma(\{x, y\})$. Note that $\bigcup\{\{x, y\} : y \in S\} = S \in \mathcal{X}$ and $x \in \bigcup\{\{x, y\} : y \in S\}$. By (II), we get $x \in \gamma(\bigcup\{\{x, y\} : y \in S\}) = \gamma(S)$, as desired.

(c) Let γ be 12-covering and satisfy (I).

    $\mathscr{P}_2(\gamma) \subseteq \mathscr{P}(\gamma)$: This direction is always valid.

    $\mathscr{P}(\gamma) \subseteq \mathscr{P}_2(\gamma)$: Let $x \leq_\gamma y$. Then there is an $S \in \mathcal{X}$ such that $y \in \gamma(S) \subseteq S$ and $x \in S$. Because γ is 12-covering, $\{x, y\} \in \mathcal{X}$. But as $\{x, y\} \subseteq S$, (I) gives us $y \in \gamma(\{x, y\})$, so $x \leq_{\gamma, 2} y$.

(d) Follows from (b) and (c).

(e) Follows from (a), (b), and (d).

(f) Let γ be additive and satisfy (I) and (II). We have to show that for every $S \in \mathcal{X}$, $\gamma(S) = \{x \in S : \text{for all } y \in S \text{ there is a } T \in \mathcal{X} \text{ such that } x \in \gamma(T) \text{ and } y \in T\}$.

    LHS $\subseteq$ *RHS*: Take $T = S$.

    RHS $\subseteq$ *LHS*: Let $x \in S$ and assume that for all $y \in S$ there is a T^y such that $x \in \gamma(T^y)$ and $y \in T^y$. Thus $x \in \bigcap \gamma(T^y)$ and $S \subseteq \bigcup T^y$. From the former and the additivity of γ we get $x \in \gamma(\bigcup T^y)$, by (II), and now the latter and (I) yield $x \in \gamma(S)$, as desired.   Q.E.D.

Proof of Lemma 2. (a) Let γ be 12n-covering and satisfy (I). We show the claim for $\mathscr{P}_2(\gamma)$, which is identical with $\mathscr{P}(\gamma)$, by Lemma 1(c). Suppose for reductio that $x_1 <_\gamma^2 x_2 <_\gamma^2 \cdots <_\gamma^2 x_n <_\gamma^2 x_1$ for some $x_1, \ldots x_n \in X$. That is, $\gamma(\{x_i, x_{i+1}\}) = \{x_{i+1}\}$ with $+$ denoting addition modulo n. Now consider $\gamma(\{x_1, \ldots, x_n\}) \neq \emptyset$. Let $x_k \in \gamma(\{x_1, \ldots, x_n\})$. But $x_k \notin \gamma(\{x_k, x_k + 1\})$. This is in contradiction with (I). The rest of (a) is trivial.

(b) Let γ be 123-covering and satisfy (I) and (III). We show the claim for $\mathscr{P}_2(\gamma)$, which is identical with $\mathscr{P}(\gamma)$, by Lemma 1(c). Assume that $x \nleq_{\gamma, 2} y$ and $y \nleq_{\gamma, 2} z$. By definition of $\leq_{\gamma, 2}$, this means that $y \notin \gamma(\{x, y\})$ and $z \notin \gamma(\{y, z\})$. Now consider $\gamma(\{x, y, z\})$. By (I), $y \notin \gamma(\{x, y, z\})$ and $z \notin \gamma(\{x, y, z\})$, so $\gamma(\{x, y, z\}) = \{x\}$ since $\gamma(\{x, y, z\})$ must be a non-empty subset of $\{x, y, z\}$. By (I), $x \in \gamma(\{x, z\})$, so $\gamma(\{x, y, z\}) \subseteq \gamma(\{x, z\})$. Hence, by (III), $\gamma(\{x, z\}) \subseteq \gamma(\{x, y, z\}) = \{x\}$, so $z \notin \gamma(\{x, z\})$, i.e. $x \nleq_{\gamma, 2} z$, as desired.

(c) Let γ be finitely additive and satisfy (IV), and let $x \leq_\gamma y$ and $y \leq_\gamma z$. This means that there is an $S \in \mathcal{X}$ such that $y \in \gamma(S)$ and $x \in S$ and a $T \in \mathcal{X}$ such that $z \in \gamma(T)$ and $y \in T$. Now consider $S \cup T$. Obviously, $x \in S \cup T$. In order to show that $x \leq_\gamma z$ it suffices to show that $z \in \gamma(S \cup T)$. By finite additivity, $S \cup T \in \mathcal{X}$. By $z \in \gamma(T)$ and (IV), it suffices to show that $\gamma(S \cup T) \cap T \neq \emptyset$. Suppose for reductio that $\gamma(S \cup T) \cap T = \emptyset$. Then, since $\emptyset \neq \gamma(S \cup T) \subseteq S \cup T$, $\gamma(S \cup T) \cap S \neq \emptyset$. So by (IV), $\gamma(S) \subseteq \gamma(S \cup T)$. So $y \in \gamma(S \cup T)$. But since also $y \in T$, $\gamma(S \cup T) \cap T \neq \emptyset$ after all, and we have a contradiction.

    (Notice that an attempted proof of the transitivity of $\leq_{\gamma, 2}$ would also need, apart from the 123-covering condition, (I) in order to come from $z \in \gamma(\{x, y, z\})$ to $z \in \gamma(\{x, z\})$, so we can rest content with (c).)   Q.E.D.

Proof of Lemma 3. (a) Assume that $\mathscr{S}(\leq)$ is no selection function over $\mathcal{X}$. By the definition of $\mathscr{S}(\leq)$, this can only happen if some $S \in \mathcal{X}$ possesses no greatest element under $\leq$. Thus for every $x_i \in S$ there is an $x_{i+1} \in S$ such that $x_{i+1} \nless x_1$.

This, however, contradicts smoothness. That $\mathscr{S}(\leq)$ satisfies (I) and (II) follows from Lemma 1(a).

(b) Let γ be relational with respect to some negatively transitive and negatively well-founded relation $\leq$. (Or alternatively, let γ be subtractive and relational with respect to some negatively transitive and $\mathscr{X}$-smooth relation $\leq$.) Let $S, S' \in \mathscr{X}, S \subseteq S'$ and $\gamma(S') \subseteq \gamma(S)$. We want to show that $\gamma(S) \subseteq \gamma(S')$. Suppose for reductio that this is not the case, i.e., that there is some x which is in $\gamma(S)$ but not in $\gamma(S')$. The latter means that there is some y_1 in S' such that $y_1 \not\leq x$. As $x \in \gamma(S)$, $y_1 \in S' - S$. But because $\gamma(S') \subseteq \gamma(S) \subseteq S$, $y_1 \notin \gamma(S')$. So there is some $y_2 \in S'$ such that $y_2 \not\leq y_1$. By negative transitivity, $y_2 \not\leq x$. So by the same reasoning as before, $y_2 \in S' - S$ and $y_2 \notin \gamma(S')$. So there is some $y_3 \in S'$ such that $y_3 \not\leq y_2$. By negative transitivity again, $y_3 \not\leq x$, and the same reasoning can be continued again and again. What we get is an infinite chain $y_1, y_2, y_3, \ldots$ in $S' - S$ such that $\cdots \not\leq y_3 \not\leq y_2 \not\leq y_1$. But this contradicts the negative well-foundedness of $\leq$ (or the subtractivity of γ, which guarantees that $S' - S \in \mathscr{X}$, and smoothness of $\leq$).

(c) Let γ be relational with respect to some transitive relation $\leq$. Let $S, S' \in \mathscr{X}, S \subseteq S'$ and $x \in \gamma(S') \cap S$. By relationality, the latter conditions says that $y \leq x$ for all $y \in S'$. Let $z \in \gamma(S)$. We have to show that $z \in \gamma(S')$, i.e., by relationality, that $y \leq z$ for all $y \in S'$. But since $x \in S$ and $z \in \gamma(S)$, $x \leq z$, so since $y \leq x$ for all $y \in S'$, the desired conclusion follows from the transitivity of $\leq$.   Q.E.D.

Proof of Lemma 4. (a) Let γ be subtractive.

(I) implies (I′): Let $S, S', S \cup S' \in \mathscr{X}$ and $x \in \gamma(S \cup S')$. As $S, S' \subseteq S \cup S'$ and x is either in S or in S', we get $x \in \gamma(S) \cup \gamma(S')$, by (I), as desired. *(I′) implies (I)*: Let $S, S' \in \mathscr{X}$ and $S \subseteq S'$. By subtractivity, $S' - S \in \mathscr{X}$ as well. Using $S' = S \cup (S' - S)$ and (I′), we get $\gamma(S') \subseteq \gamma(S) \cup \gamma(S' - S)$. Intersecting both sides with S, then, we get $\gamma(S') \cap S \subseteq (\gamma(S) \cap S) \cup (\gamma(S' - S) \cap S) = \gamma(S)$, since $\gamma(S) \subseteq S$

(b) First we note that for finitely additive γ, (II′) is equivalent to

(II$_{fin}$) For all $\{S_i : i \in I, I \text{ finite}\} \subseteq \mathscr{X}$ such that $\bigcup\{S_i : i \in I\} \in \mathscr{X}$, $\bigcap\{\gamma(S_i) : i \in I\} \subseteq \gamma(\bigcup\{S_i : i \in I\})$

Now we simply modify the proof of Lemma 1(f) by taking only finitely many of the T^y's into consideration. This is possible because compactness can be applied to the inclusion $S \subseteq \bigcup T^y$ in that proof.

(c) Let γ satisfy (I).

(III) implies (III′): Let $S, S \cup S' \in \mathscr{X}$ and $\gamma(S \cup S') \cap S' = \emptyset$. From the latter condition and $\gamma(S \cup S') \subseteq S \cup S'$ it follows that $\gamma(S \cup S') \subseteq S$. Hence, by (I), $\gamma(S \cup S') = S \cap \gamma(S \cup S') \subseteq \gamma(S)$. Thus, by (III), $\gamma(S) \subseteq \gamma(S \cup S')$, as desired. *(III′) implies (III)*: Let $S, S' \in \mathscr{X}, S \subseteq S'$ and $\gamma(S') \subseteq \gamma(S)$. Since $S' = S \cup (S' - S) \in \mathscr{X}$ and $\gamma(S') \subseteq \gamma(S) \subseteq S$, it holds that $\gamma(S \cup (S' - S)) \cap (S' - S) = \emptyset$. By (III′), we get $\gamma(S) \subseteq \gamma(S')$, as desired.

(d) Immediate.

(e) That (I&II) implies (I) is immediate if we put $I = \{i\}$ and $S_i = S'$, and that (I&II) implies (II) is immediate if we put $S = \bigcup S_i$ and observe that $\gamma(S_i) \subseteq S$.

In order to see that (I) and (II) taken together imply (I&II), we note that $\bigcup\{S_i : i \in I\} \in \mathcal{X}$, by additivity. So (II) gives us $S \cap \bigcap \gamma(S_i) \subseteq S \cap \gamma(\bigcup S_i)$, and (I) gives us $S \cap \gamma(\bigcup S_i) \subseteq \gamma(S)$, whenever $S \subseteq \bigcup S_i$. So in this case $S \cap \bigcap \gamma(S_i) \subseteq \gamma(S)$, as desired.

(f) Similar to (e).   Q.E.D.

Re section "Representation Theorems for Contraction Functions": Example of a relational partial meet contraction function which does not satisfy ($\dot{-}8r$).

Consider a propositional language L with denumerably many atoms $p_1, p_2, p_3, \ldots$ and q and r. Let $A = Cn(p_1, p_2, p_3, \ldots, q, r)$. Let W_i, W_i^q, W_i^r, for $i = 1, 2, 3, 4, \ldots$, be the maximally consistent set of propositions defined by

$p_i \in W_i, W_i^q, W_i^r$, for every i,
$\neg p_j \in W_i, W_i^q, W_i^r$, for every i and every $j \neq i$,
$q \in W_i^q$ and $\neg q \in W_i, W_i^r$, for every i,
$r \in W_i^r$ and $\neg r \in W_i, W_i^q$, for every i.

Let $\mathbf{W}^{qr} = \bigcup\{\{W_i, W_i^q, W_i^r\} : i = 1, 2, 3, \ldots\}$. Remember that $V_A = \mathbf{W} - [\![A]\!]$. Let the relation $<_W$ over V_A be defined as follows:

$W <_W W'$ for every $W \in \mathbf{W}^{qr}$ and $W' \in V_A - \mathbf{W}^{qr}$
$W_j^q <_W W_{j+1}$, $W_j^q <_W W_{j+1}^q$, and $W_j^q <_W W_{j+1}^r$, for every $j \in \{0, 2, 4, 6, \ldots\}$
$W_k^r <_W W_{k+1}$, $W_k^r <_W W_{k+1}^q$, and $W_k^r <_W W_{k+1}^r$, for every $k \in \{1, 3, 5, \ldots\}$
These are all pairs standing in the relation $<_W$ (see Fig. 16.3).

Then define, for the purposes of this example, $M \leq M'$ iff $M' \not< M$ iff $\mathscr{W}(M) \not<_W \mathscr{W}(M')$. Then clearly, the $\leq$-greatest elements of $A \perp x$ correspond to the $<$-minimal elements of $[\![\neg x]\!]$. Note that $\leq$ is connected, acyclic and conversely well-founded (no infinite ascending chains), but not negatively transitive. Let $\dot{-} = \mathscr{C}(\mathscr{S}(\leq))$ be the relational partial meet contraction determined by $\leq$.

Now observe that the greatest elements of $A \perp (q \wedge r) = A \perp q \cup A \perp r$ are exactly $A \cap W_0$, $A \cap W_0^q$, and $A \cap W_0^r$. Since p_0 is an element of all these sets, we have $p_0 \in A \dot{-} (q \wedge r)$.

The greatest elements of $A \perp q$ are $A \cap W_k$ and $A \cap W_k^r$ for every $k \in \{0, 1, 3, 5, \ldots\}$. Obviously, $q \rightarrow x \in A \dot{-} q$ for every $x \in A$. Furthermore, $\neg p_n \vee \neg p_m \in A \dot{-} q$ for every $n \neq m$. Informally speaking, $A \dot{-} q$ contains the information that exactly one proposition p_k, for $k \in \{0, 1, 3, 5, \ldots\}$, is true. But this cannot be expressed in our first-order propositional language. Note in particular that $A \dot{-} q$ does not contain any finite disjunction of p_i's.

The greatest elements of $A \perp r$ are $A \cap W_j$ and $A \cap W_j^q$ for every $j \in \{0, 2, 4, 6, \ldots\}$. The information contained in $A \dot{-} r$ is analogous with that in $A \dot{-} q$, with the even positive integers playing the role of the odd ones in the previous case.

We find that $A \dot{-} q$ and $A \dot{-} r$ do not even jointly imply p_0. For consider the maximally consistent set W^* containing $\neg p_i$ for every $i \geq 0$, together with $\neg q$ and $\neg r$. Considering what we have just said about $A \dot{-} q$ and $A \dot{-} r$, it becomes clear that W^* contains both of these contracted theories, as well as $\neg p_0$.

Since $p_0 \in A \dot{-} (q \wedge r)$ but $p_0 \notin Cn(A \dot{-} q \cup A \dot{-} r)$, we have a violation of ($\dot{-}8r$).
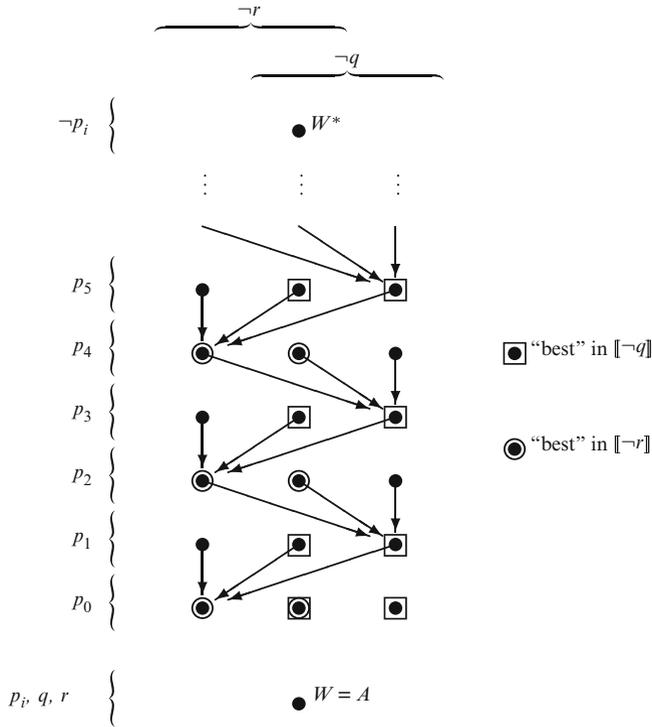
Fig. 16.3 Relational partial meet contraction not satisfying (÷8r)

Proof of Lemma 9. We show that for every $M \in A \perp x$,

(i) M is maximal in $A \perp x$ under $\ll$

      is equivalent to

(ii) $N \subseteq M$ for some N which is maximal in $B \perp x$ under $\ll$.

     (i) implies (ii): (i) means that M is such that for no M' and i it holds that $M \ll_i M'$. Now take $N = M \cap B$. We have to show that N is maximal in $B \perp x$. First, it is clear that $N \nvdash x$. Next, we verify that N is a maximal subset of B which does not imply x. Suppose there is a $y \in B - N$ such that $N \cup \{y\} \nvdash x$, but then, with $y \in B_k$, $N_{\geq k} \cup \{y\} \nvdash x$. So there would be a superset M' of $N_{\geq k} \cup \{y\}$ in $A \perp x$ for which $M \ll_{\geq k} M'$, contradicting (i). Finally, we have to show that there is no $N' \in B \perp x$ which is better than N under $\ll$. But by the same argument as just used, it is impossible that there be some $y \in B_k$ such that $y \in N' - N$ and $N \ll_k N'$.

     (ii) implies (i): Suppose for reductio that (ii) but not (i). By the latter, there is an M' such that $M \ll M'$. But then $M \cap B \ll M' \cap B$. Take, on the other hand, the N from (ii). Since $N \subseteq M \cap B$, we have $N \ll M' \cap B$. But since $M' \cap B \nvdash x$, N cannot be maximal in $B \perp x$, contradicting (ii).   Q.E.D.

Re section "The Logic of Prioritized Base Contractions in the Finite Case": Illustration of the counterexample to ($\dotdiv$8r) in straightforward prioritized base contractions and its solution by the official definition. Let $B = \{(p \wedge q) \vee (p \wedge r) \vee (q \wedge r), r \rightarrow (p \leftrightarrow q), q \wedge (p \leftrightarrow r), p \wedge r\}$, $A = Cn(B)$, $A \dotdiv' x = \bigcap \{Cn(N): N \in \gamma(B \perp x)\}$ and $A \dotdiv x = \bigcap \{Cn(N \cup \{x \rightarrow a\}): N \in \gamma(B \perp x)\}$. See Fig. 16.4

Re section "The Logic of Prioritized Base Contractions in the Finite Case": Example illustrating how prioritized belief bases can be replaced by simple ones.

Consider the propositional language L over the two atoms p and q. Let the belief base contain the propositions p and $p \rightarrow q$. We contrast the prioritizations $\langle \{p \rightarrow q\}, \{p\} \rangle$ and $\langle \{p\}, \{p \rightarrow q\} \rangle$. In the former case, $B = \mathscr{B}(\mathscr{P}(\langle \{p \rightarrow q\}, \{p\} \rangle)) = \{p \wedge q, p, p \vee q, q \rightarrow p\}$, while in the latter case, $B' = \mathscr{B}(\mathscr{P}(\langle \{p\}, \{p \rightarrow q\} \rangle)) = \{p \wedge q, q, p \leftrightarrow q, \top\}$.

In Figs. 16.5 and 16.6, "$\binom{x}{y}{z}$" should be read as "$A \dotdiv x = Cn(y)$ according to prioritized base contraction, while $A \dotdiv x = Cn(z)$ according to the corresponding simple base contraction". In the last two lines, the left argument of '$\wedge$' comes from straightforward base contraction and the right argument is the recovery-guaranteeing appendage used in our official definition of prioritized base contractions.
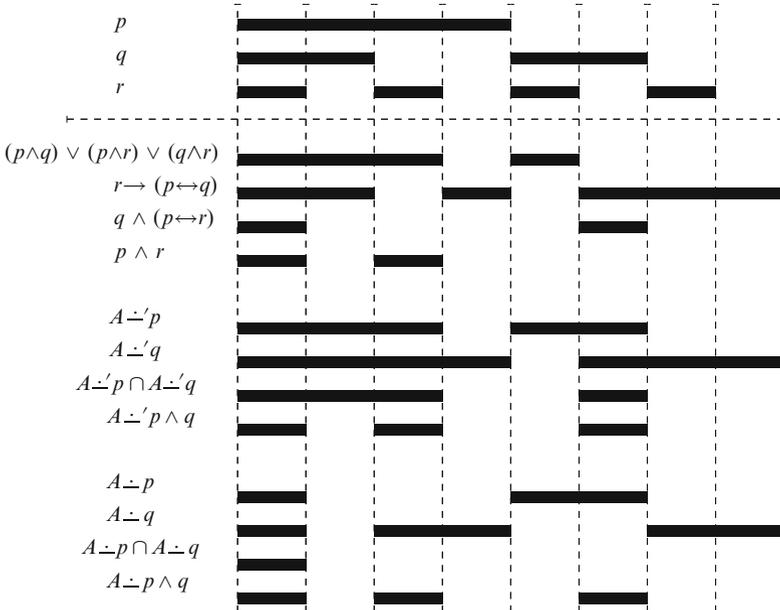

Fig. 16.4 Prioritized base contraction $\dotdiv'$ with simple base violating ($\dotdiv$8r), and strengthened contraction $\dotdiv$ satisfying ($\dotdiv$8r)
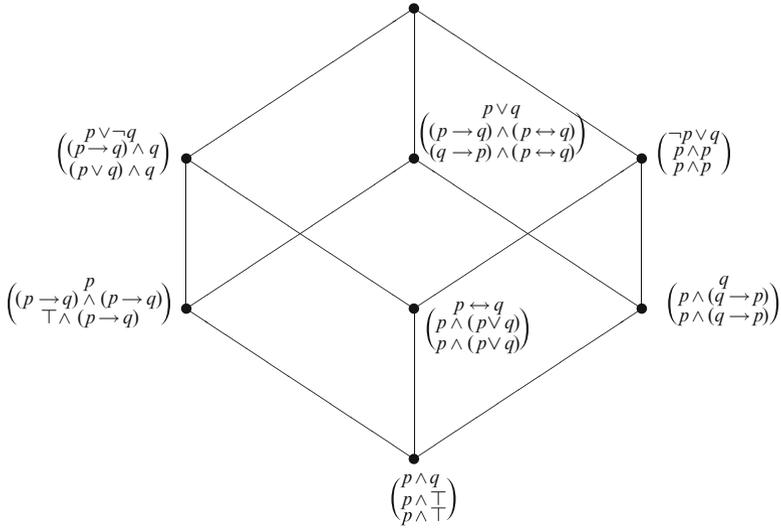
Fig. 16.5 Prioritized base contraction of a two-element base, with p having priority over $p \rightarrow q$
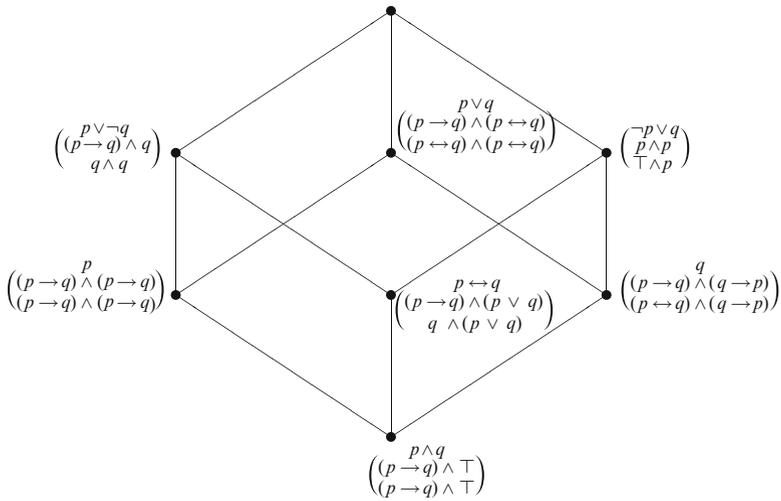

Fig. 16.6 Prioritized base contraction of a two-element base, with $p \rightarrow q$ having priority over p

Remarks. Prioritized and corresponding simple base contractions indeed lead to the same results—as they should, according to the proof of Theorem 7. But note that the recovery-guaranteeing appendage is essential in quite a few cases. As for the contraction function $\dot{-}$, the differences in prioritization are effective only in the case of $A \dot{-} (p \wedge q)$, $A \dot{-} q$ and $A \dot{-} (p \leftrightarrow q)$.

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