Chapter 3
Applications of the Concepts of Analytical Mechanics

3.1 Introduction

This chapter provides a number of important examples of application of the principles of Analytical Mechanics. The examples are chosen with reference to the applications to Quantum Mechanics shown in later chapters. The first sections treat the problems of the square well, linear harmonic oscillator, and central motion. The subsequent sections deal with the two-particle interaction: first, the description of the collision is given, along with the calculation of the energy exchange involved in it, with no reference to the form of the potential energy; this is followed by the treatment of the collision when the potential energy is of the repulsive-Coulomb type. The chapter continues with the treatment of a system of strongly-bound particles: the diagonalization of its Hamiltonian function shows that the motion of the particles is a superposition of harmonic oscillations. Finally, the motion of a particle subjected to a periodic potential energy is analyzed, including the case where a weak perturbation is superimposed to the periodic part. A number of complements are also given, that include the treatment of the collision with a potential energy of the attractive-Coulomb type, and that of the collision of two relativistic particles.

3.2 Particle in a Square Well

As a first example of linear motion consider the case of a potential energy V of the form shown in Fig. 3.1. Such a potential energy is called *square well* and is to be understood as the limiting case of a potential-energy well whose sides have a finite, though very large, slope. It follows that the force $F = -\mathrm{d}V/\mathrm{d}x$ is everywhere equal to zero with the exception of the two points $-x_M$ and $+x_M$, where it tends to $+\infty$ and $-\infty$, respectively. From the discussion of Sect. 2.9 it follows that the case $E < 0$

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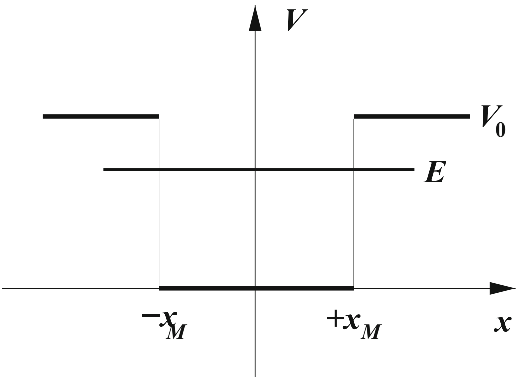


Fig. 3.1 The example of the square well analyzed in Sect. 3.2. Only the case $0 \le E \le V_0$ is shown

is forbidden. The motion of the particle is finite for $0 \le E \le V_0$, while it is infinite for $E > V_0$.

Considering the $0 \le E \le V_0$ case first, the motion is confined within the well and the velocity of the particle is constant in the interval $-x_M < x < +x_M$, where the Hamiltonian function yields $m\,\dot{x}^2/2 = E$. If the particle's motion is oriented to the right, the velocity is $\dot{x} = \sqrt{2\,E/m}$. When the particle reaches the position x_M its velocity reverses instantly to become $\dot{x} = -\sqrt{2\,E/m}$. The motion continues at a constant velocity until the particle reaches the position $-x_M$ where is reverses again, and so on. As the spatial interval corresponding to a full cycle is $4\,x_M$, the oscillation period is $T = \sqrt{8\,m/E}\,x_M$.

To treat the $E > V_0$ case assume that the particle is initially at a position $x < -x_M$ with a motion oriented to the right. The Hamiltonian function outside the well yields $m\,\dot{x}^2/2 + V_0 = E$. The constant velocity is $\dot{x} = \sqrt{2\,(E - V_0)/m}$ until the particle reaches the position $-x_M$. There the velocity increases abruptly to $\dot{x} = \sqrt{2\,E/m}$ and keeps this value until the particle reaches the other edge of the well, $+x_M$. There, the velocity decreases abruptly back to the initial value $\dot{x} = \sqrt{2\,(E - V_0)/m}$, and the particle continues its motion at a constant velocity in the positive direction.

3.3 Linear Harmonic Oscillator

An important example of linear motion is found when the force derives from a potential energy of the form $V = c\,x^2/2$, with $c > 0$. The force acting on the particle turns out to be $F = -\mathrm{d}V/\mathrm{d}x = -c\,x$, namely, it is linear with respect to x, vanishes at $x = 0$, and has a modulus that increases as the particle departs from the origin. Also, due to the positiveness of c, the force is always directed towards the origin. A force of this type is also called *linear elastic force*, and c is called *elastic constant* (Fig. 3.2).

From the discussion of Sect. 2.9 it follows that the case $E < 0$ is forbidden. The motion of the particle is always finite because for any $E \ge 0$ it is confined between the two zeros $x_M = \pm\sqrt{2\,E/c}$ of the equation $V = E$. The Hamiltonian function reads

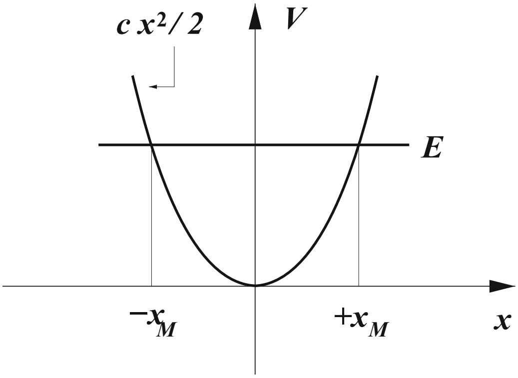


Fig. 3.2 The example of the linear harmonic oscillator analyzed in Sect. 3.3

$$H = \frac{1}{2\,m}\,p^2 + \frac{1}{2}\,cx^2 = E = \text{const}, \tag{3.1}$$

yielding the motion's equation $\dot{p} = m\ddot{x} = -\partial H/\partial x = -c\,x$ whose solution is

$$x(t) = x_M\,\cos(\omega\,t + \alpha_0), \quad \dot{x}(t) = -\omega\,x_M\,\sin(\omega\,t + \alpha_0), \quad \omega = \sqrt{c/m}. \tag{3.2}$$

Due to the form of (3.2), a particle whose motion is derived from the Hamiltonian function (3.1) is called *linear harmonic oscillator*. The maximum elongation $x_M > 0$ and the initial phase α_0 are readily expressed in terms of the initial conditions $x_0 = x(t=0)$, $\dot{x}_0 = \dot{x}(t=0)$. In fact, letting $t = 0$ in (3.2) yields $x_M^2 = x_0^2 + \dot{x}_0^2/\omega^2$ and $\tan\alpha_0 = -\dot{x}_0/(\omega\,x_0)$. The total energy in terms of the initial conditions reads $E = m\,\dot{x}_0^2/2 + c\,x_0^2/2$ and, finally, the oscillation's period is $T = 2\,\pi/\omega$. Note that T depends on the two parameters m, c appearing in the Hamiltonian function, but not on the total energy (in other terms, for a given pair m, c the oscillation period does not depend on the initial conditions). As mentioned in Sect. 2.9, this is an exceptional case.

3.4 Central Motion

Consider the case of a particle of mass m acted upon by a force that derives from a potential energy of the form $V = V(r)$, where $r = \sqrt{x^2 + y^2 + z^2}$ is the modulus of the position vector $\mathbf{r}$ of the particle. The force

$$\mathbf{F} = -\text{grad}\,V = -\frac{\mathrm{d}V}{\mathrm{d}r}\,\frac{\mathbf{r}}{r}, \tag{3.3}$$

depends on r only and is oriented along $\mathbf{r}$. For this reason it is called *central force*. In turn, the point whence the force originates (in this case, the origin of the reference) is called *center of force*. The corresponding Lagrangian function

$$L = \frac{1}{2}\,m\dot{r}^2 - V(r), \qquad \mathbf{p} = m\dot{\mathbf{r}}, \qquad \dot{r} = |\dot{\mathbf{r}}| \tag{3.4}$$

turns out to be invariant under any rotation (Sect. 2.6.4). Remembering (2.28, 2.29), this type of invariance means that the projection of the angular momentum $\mathbf{M}$ onto any direction is conserved. It follows that, for a particle acted upon by a central force, the vector $\mathbf{M}$ itself is conserved. On the other hand it is $\mathbf{M} = \mathbf{r} \wedge m\dot{\mathbf{r}}$, so the constant angular momentum is fixed by the initial conditions of the motion.

As $\mathbf{M}$ is normal to the plane defined by $\mathbf{r}$ and $\dot{\mathbf{r}}$, the trajectory of the particle lies always on such a plane. It is then useful to select the Cartesian reference by aligning, e.g., the z axis with $\mathbf{M}$. In this way, the trajectory belongs to the x, y plane and two coordinates eventually suffice to describe the motion. Turning to the spherical coordinates (B.1) and using (2.40) yields the Hamiltonian function

$$H = \frac{1}{2m}\left(p_r^2 + \frac{M^2}{r^2}\right) + V(r) = \frac{p_r^2}{2m} + V_e(r), \quad V_e = V + \frac{M^2}{2mr^2}, \tag{3.5}$$

with $M^2 = p_\vartheta^2 + p_\varphi^2/\sin^2\vartheta$, $\mathbf{M} = \text{const}$, and $p_r = m\dot{r}$, $p_\vartheta = mr^2\dot{\vartheta}$, $p_\varphi = M_z = mr^2\dot{\varphi}\,\sin\vartheta$. However, the z axis has been aligned with $\mathbf{M}$, which is equivalent to letting $\vartheta = \pi/2$. It turns out $M_z = M$, and $p_r = m\dot{r}$, $p_\vartheta = 0$, $p_\varphi = M = mr^2\dot{\varphi}$, so that

$$H = \frac{1}{2m}\left(p_r^2 + \frac{p_\varphi^2}{r^2}\right) + V(r) = \frac{p_r^2}{2m} + V_e(r), \quad V_e = V + \frac{p_\varphi^2}{2mr^2}. \tag{3.6}$$

As the total energy is conserved it is $H = E$, where E is known from the initial conditions. The intervals of r where the motion can actually occur are those in which $E \geq V_e$. Letting $r_0 = r(t = 0)$, the time evolution of the radial part is found from $p_r^2 = m^2(\mathrm{d}r/\mathrm{d}t)^2 = 2\,m\,(E - V_e)$, namely

$$t(r) = \pm\sqrt{\frac{m}{2}}\int_{r_0}^{r}\frac{\mathrm{d}\xi}{\sqrt{E - V_e(\xi)}}. \tag{3.7}$$

From $p_\varphi = mr^2\dot{\varphi} = \text{const}$ it follows that φ depends monotonically on time, and also that $\mathrm{d}t = (mr^2/p_\varphi)\,\mathrm{d}\varphi$. Combining the latter with (3.7) written in differential form, $\mathrm{d}t = \pm\sqrt{m/2}\,[E - V_e(r)]^{-1/2}\,\mathrm{d}r$, yields the equation for the trajectory,

$$\varphi(r) = \varphi_0 \pm \frac{p_\varphi}{\sqrt{2m}}\int_{r_0}^{r}\frac{\mathrm{d}\xi}{\xi^2\sqrt{E - V_e(\xi)}}, \tag{3.8}$$

with $\varphi_0 = \varphi(t = 0)$. Finally, elimination of r from $t(r)$ and $\varphi(r)$ provides the time evolution of φ. It is convenient to let the initial time $t = 0$ correspond to an extremum of the possible values of r. In this way the sign of t and $\varphi - \varphi_0$ changes at $r = r_0$. By this choice the trajectory is symmetric with respect to the line drawn from the origin to the point of coordinates r_0, φ_0, and the evolution of the particle's motion over each half of the trajectory is symmetric with respect to time.

3.5 Two-Particle Collision

Consider a system made of two particles whose masses are m_1, m_2. The system is isolated, namely, the particles are not subjected to any forces apart those due to the mutual interaction. As a consequence, the Lagrangian function is invariant under coordinate translations, and the Hamiltonian function is invariant under time translations. Thus, as shown in Sect. 2.6, the total momentum and total energy of the system are conserved.

The type of motion that is considered is such that the distance between the particles is initially so large as to make the interaction negligible. The interaction becomes significant when the particles come closer to each other; when they move apart, the interaction becomes negligible again. This type of interaction is called *collision*. The values of the dynamical quantities that hold when the interaction is negligible are indicated as *asymptotic values*. The labels a and b will be used to mark the asymptotic values before and after the interaction, respectively.

It is worth specifying that it is assumed that the collision does not change the internal state of the particles (for this reason it is more appropriately termed *elastic collision*, [67, Sect. 17]). When the distance is sufficiently large, the particles can be considered as isolated: they move at a constant velocity and the total energy of the system is purely kinetic, $E_a = T_a$ and $E_b = T_b$. On the other hand, due to the invariance under time translation the total energy of the system is conserved, $E_b = E_a$. In conclusion it is $T_b = T_a$, namely, in an elastic collision the asymptotic kinetic energy of the system is conserved.

An analysis of the collision based only on the asymptotic values is incomplete because it does not take into account the details of the interaction between the two particles. However it provides a number of useful results, so it is worth pursuing. Letting $\mathbf{r}_1$ and $\mathbf{r}_2$ be the positions of the particles in a reference O, the position of the center of mass and the relative position of the particles are

$$\mathbf{R} = \frac{m_1\,\mathbf{r}_1 + m_2\,\mathbf{r}_2}{m_1 + m_2}, \qquad \mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2. \tag{3.9}$$

The corresponding velocities are $\mathbf{v}_1 = \dot{\mathbf{r}}_1$, $\mathbf{v}_2 = \dot{\mathbf{r}}_2$, and $\mathbf{v} = \dot{\mathbf{r}}$. The relations between the velocities are obtained by differentiating (3.9) with respect to time. Solving for $\mathbf{v}_1$, $\mathbf{v}_2$ yields

$$\mathbf{v}_1 = \dot{\mathbf{R}} + \frac{m_2}{m_1 + m_2}\,\mathbf{v}, \qquad \mathbf{v}_2 = \dot{\mathbf{R}} - \frac{m_1}{m_1 + m_2}\,\mathbf{v}. \tag{3.10}$$

Letting $\dot{R} = |\dot{\mathbf{R}}|$, the system's kinetic energy before the interaction is

$$T_a = \frac{1}{2}\,m_1\,v_{1a}^2 + \frac{1}{2}\,m_2\,v_{2a}^2 = \frac{1}{2}\,(m_1 + m_2)\,\dot{R}_a^2 + \frac{1}{2}\,m\,v_a^2, \tag{3.11}$$

where $m = m_1\,m_2/(m_1 + m_2)$ is called *reduced mass*. The expression of the kinetic energy after the interaction is obtained from (3.11) by replacing a with b.

The total momentum before the collision is $\mathbf{P}_a = m_1\,\mathbf{v}_{1a} + m_2\,\mathbf{v}_{2a} = (m_1 + m_2)\,\dot{\mathbf{R}}_a$. The conservation of $\mathbf{P}$ due to the invariance under coordinate translations yields $\mathbf{P}_b = \mathbf{P}_a$, whence $\dot{\mathbf{R}}_b = \dot{\mathbf{R}}_a$. Using (3.11) in combination with the conservation rules $\dot{\mathbf{R}}_b = \dot{\mathbf{R}}_a$ and $T_b = T_a$ yields $v_b = v_a$, namely, the asymptotic modulus of the relative velocity is conserved.

The analysis is now repeated in a new reference B in which the particles' positions are defined as

$$\mathbf{s}_1 = \mathbf{r}_1 - \mathbf{R} = \frac{m_2}{m_1 + m_2}\,(\mathbf{r}_1 - \mathbf{r}_2), \qquad \mathbf{s}_2 = \mathbf{r}_2 - \mathbf{R} = \frac{m_1}{m_1 + m_2}\,(\mathbf{r}_2 - \mathbf{r}_1). \quad (3.12)$$

By construction, the origin of B coincides with the system's center of mass. The relative position in B is the same as in O, in fact

$$\mathbf{s} = \mathbf{s}_1 - \mathbf{s}_2 = \mathbf{r}_1 - \mathbf{r}_2 = \mathbf{r}. \quad (3.13)$$

From (3.12, 3.13) one finds

$$m_1\,\mathbf{s}_1 = -m_2\,\mathbf{s}_2, \qquad \mathbf{s}_1 = \frac{m_2}{m_1 + m_2}\,\mathbf{s}, \qquad \mathbf{s}_2 = -\frac{m_1}{m_1 + m_2}\,\mathbf{s}. \quad (3.14)$$

The velocities in reference B are $\mathbf{u}_1 = \dot{\mathbf{s}}_1$, $\mathbf{u}_2 = \dot{\mathbf{s}}_2$, and $\mathbf{u} = \dot{\mathbf{s}}$. The relations among the latter are found by differentiating (3.12, 3.13) and read

$$\mathbf{u}_1 = \mathbf{v}_1 - \dot{\mathbf{R}}, \qquad \mathbf{u}_2 = \mathbf{v}_2 - \dot{\mathbf{R}}, \qquad \mathbf{u} = \mathbf{v}, \quad (3.15)$$

$$\mathbf{u}_1 = \frac{m_2}{m_1 + m_2}\,\mathbf{u}, \qquad \mathbf{u}_2 = -\frac{m_1}{m_1 + m_2}\,\mathbf{u}, \quad (3.16)$$

which in turn yield

$$\mathbf{v}_1 = \dot{\mathbf{R}} + \frac{m_2}{m_1 + m_2}\,\mathbf{u}, \qquad \mathbf{v}_2 = \dot{\mathbf{R}} - \frac{m_1}{m_1 + m_2}\,\mathbf{u}. \quad (3.17)$$

Thanks to (3.16) the system's kinetic energy before and after the interaction, in reference B, is

$$K_a = \frac{1}{2}\,m_1 u_{1a}^2 + \frac{1}{2}\,m_2 u_{2a}^2 = \frac{1}{2}\,m u_a^2, \qquad K_b = \frac{1}{2}\,m u_b^2. \quad (3.18)$$

The conservation of the kinetic energy, $K_b = K_a$, yields $u_b = u_a$. Using the third of (3.15) then yields

$$u_b = u_a = v_b = v_a, \quad (3.19)$$

that is, the asymptotic modulus of the relative velocity is conserved and has the same value in the two references. Moreover, (3.16) show that it is also $u_{1b} = u_{1a}$ and $u_{2b} = u_{2a}$, namely, in reference B the asymptotic kinetic energy is conserved for each particle separately.

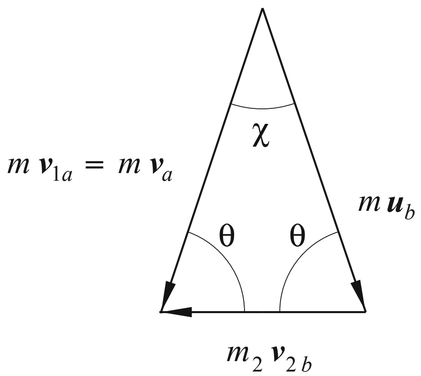


Fig. 3.3 Graphic representation of the vector relation (3.20)

3.6 Energy Exchange in the Two-Particle Collision

To complete the asymptotic analysis of the two-particle collision it is useful to choose for O a reference such that $\mathbf{v}_{2a} = 0$. In this case (3.10) yield $\mathbf{v}_a = \mathbf{v}_{1a}$, whence the total momentum reads $(m_1 + m_2)\,\dot{\mathbf{R}}_a = m_1\mathbf{v}_{1a} = m_1\mathbf{v}_a$. Remembering that $\dot{\mathbf{R}}_b = \dot{\mathbf{R}}_a$ one finds $\dot{\mathbf{R}}_b = m_1\,\mathbf{v}_a/(m_1 + m_2)$. Using the latter relation in the second of (3.17) specified with the b label yields, after multiplying both sides by m_2,

$$m_2\mathbf{v}_{2b} = m\mathbf{v}_a - m\mathbf{u}_b. \tag{3.20}$$

The triangle formed by the vectors $m_2\mathbf{v}_{2b}$, $m\mathbf{v}_a$, and $m\mathbf{u}_b$ is isosceles because $\mathbf{v}_a$ and $\mathbf{u}_b$ have the same modulus (Fig. 3.3). Letting χ, θ be the angle between $m\,\mathbf{v}_a$ and $m\,\mathbf{u}_b$ and, respectively, the common value of the other two angles, a scalar multiplication of (3.20) by $\mathbf{v}_a$ yields $m_2\,v_{2b}\,\cos\theta = mv_a - mu_b\,\cos\chi = mv_a\,(1 - \cos\chi)$. Using $2\theta + \chi = \pi$ and $v_a = v_{1a}$ transforms the latter into

$$m_2 v_{2b} = m v_{1a}\,\frac{1 - \cos\chi}{\cos\left[(\pi - \chi)/2\right]} = 2 m v_{1a}\,\sin(\chi/2). \tag{3.21}$$

This relation allows one to calculate, in reference O where the particle of mass m_2 is initially at rest, the modulus of the final velocity of this particle in terms of the initial velocity of the other particle and of angle χ. As only v_{1a} is prescribed, the two quantities v_{2b} and χ can not be determined separately. The reason for this (already mentioned in Sect. 3.5) is that (3.21) is deduced using the motion's asymptotic conditions without considering the interaction's details. In fact, the calculation is based only on the momentum and total-energy conservation and on the hypothesis that the collision is elastic. From (3.21) one derives the relation between the kinetic energies $T_{1a} = (1/2)\,m_1 v_{1a}^2$ and $T_{2b} = (1/2)\,m_2 v_{2b}^2$,

$$T_{2b}(\chi) = \frac{4\,m_1\,m_2}{(m_1 + m_2)^2}\,T_{1a}\,\sin^2(\chi/2). \tag{3.22}$$

As in reference O the particle of mass m_2 is initially at rest, T_{2b} is the variation of the kinetic energy of this particle due to to the collision, expressed in terms of χ.

The maximum variation is $T_{2b}(\chi = \pm\pi)$. The conservation relation for the kinetic energy $T_{1b} + T_{2b} = T_{1a}$ coupled with (3.22) yields the kinetic energy of the particle of mass m_1 after the collision,

$$T_{1b} = T_{1a} - T_{2b} = \left[1 - \frac{4\, m_1\, m_2}{(m_1 + m_2)^2} \sin^2(\chi/2)\right] T_{1a}. \tag{3.23}$$

Expressing T_{1a} and T_{1a} in (3.23) in terms of the corresponding velocities yields the modulus of the final velocity of the particle of mass m_1 as a function of its initial velocity and of angle χ,

$$v_{1b} = [(m_1^2 + m_2^2 + 2m_1 m_2 \cos\chi)^{1/2}/(m_1 + m_2)]\, v_{1a}. \tag{3.24}$$

Although expressions (3.21–3.24) are compact, the use of angle χ is inconvenient. It is preferable to use the angle, say ψ, between vectors $\mathbf{v}_{1b}$ and $\mathbf{v}_{1a} = \mathbf{v}_a$ that belong to the same reference O (Fig. 3.7).[1] A scalar multiplication by $\mathbf{v}_{1a}$ of the conservation relation for momentum, $m_1\mathbf{v}_{1b} = m_1\mathbf{v}_{1a} - m_2\mathbf{v}_{2b}$, followed by the replacement of the expressions of v_{2b} and v_{1b} extracted from (3.21, 3.24), eventually yields $\cos\psi = (m_1 + m_2\cos\chi)/(m_1^2 + m_2^2 + 2m_1 m_2\cos\chi)^{1/2}$. Squaring both sides of the latter provides the relation between χ and ψ,

$$\tan\psi = \frac{\sin\chi}{m_1/m_2 + \cos\chi}. \tag{3.25}$$

Using (3.25) one expresses (3.21–3.24) in terms of the deflection ψ (in reference O) of the particle of mass m_1. If $m_1 > m_2$, then $\psi < \pi/2$, while it is $\psi = \chi/2$ if $m_1 = m_2$. When $m_1 < m_2$ and $\chi = \arccos(-m_1/m_2)$, then $\psi = \pi/2$; finally, if $m_1 \ll m_2$ it is $\psi \simeq \chi$ and, from (3.21–3.24), it follows $v_{2b} \simeq 0$, $T_{2b} \simeq 0$, $v_{1b} \simeq v_{1a}$, $T_{1b} \simeq T_{1a}$. In other terms, when $m_1 \ll m_2$ the particle of mass m_2 remains at rest; the other particle is deflected, but its kinetic energy is left unchanged.

In reference O, the angle between the final velocity of the particle of mass m_2, initially at rest, and the initial velocity of the other particle has been defined above as $\theta = (\pi - \chi)/2$. Replacing the latter in (3.25) provides the relation between ψ and θ,

$$\tan\psi = \frac{\sin(2\theta)}{m_1/m_2 - \cos(2\theta)}. \tag{3.26}$$

[1] Note that the angle γ between two momenta $\mathbf{p}' = m'\mathbf{v}'$ and $\mathbf{p}'' = m''\mathbf{v}''$ is the same as that between the corresponding velocities because the masses cancel out in the angle's definition $\gamma = \arccos[\mathbf{p}' \cdot \mathbf{p}''/(p' p'')]$. In contrast, a relation like (3.20) involving a triad of vectors holds for the momenta but not (with the exception of the trivial cases of equal masses) for the corresponding velocities.

3.7 Central Motion in the Two-Particle Interaction

Consider an isolated two-particle system where the force acting on each particle derives from a potential energy $V = V(\mathbf{r}_1, \mathbf{r}_2)$. Using the symbols defined in Sect. 3.5 yields the Lagrangian function $L = m_1 v_1^2/2 + m_2 v_2^2/2 - V(\mathbf{r}_1, \mathbf{r}_2)$. Now assume that the potential energy V depends on the position of the two particles only through the modulus of their distance, $r = |\mathbf{r}_1 - \mathbf{r}_2|$. In this case it is convenient to use the coordinates and velocities relative to the center of mass, (3.12–3.17), to find

$$L = \frac{1}{2}(m_1 + m_2)\,\dot{R}^2 + \frac{1}{2}\,m\,\dot{s}^2 - V(s), \qquad \dot{s} = |\dot{\mathbf{s}}| = |\mathbf{u}|. \tag{3.27}$$

As discussed in Sects. 2.6.3 and 3.5 the total momentum is conserved, whence $\dot{R}$ is constant. Another way of proving this property is noting that the components of $\mathbf{R}$ are cyclic (Sect. 1.6). The first term at the right hand side of (3.27), being a constant, does not influence the subsequent calculations. The remaining terms, in turn, are identical to those of (3.4). This shows that, when in a two-particle system the potential energy depends only on the relative distance, adopting suitable coordinates makes the problem identical to that of the central motion. One can then exploit the results of Sect. 3.4. Once the time evolution of $\mathbf{s}$ is found, the description of the motion of the individual particles is recovered from (3.12–3.17), where the constant $\dot{\mathbf{R}}$ is determined by the initial conditions.

The total energy of the two-particle system is conserved and, in the new reference, it reads

$$\frac{1}{2}\,m\,\dot{s}^2 + V(s) = E_B, \qquad E_B = E - \frac{1}{2}(m_1 + m_2)\,\dot{R}^2. \tag{3.28}$$

The total angular momentum is constant as well. Starting from the original reference and using (3.12–3.17) yields

$$\mathbf{M} = \mathbf{r}_1 \wedge m_1 \mathbf{v}_1 + \mathbf{r}_2 \wedge m_2 \mathbf{v}_2 = (m_1 + m_2)\,\mathbf{R} \wedge \dot{\mathbf{R}} + m\,\mathbf{s} \wedge \mathbf{u}. \tag{3.29}$$

The constancy of $\dot{\mathbf{R}}$ yields $\mathbf{R} = \mathbf{R}_0 + \dot{\mathbf{R}}\,t$, with $\mathbf{R}_0$ the initial value of $\mathbf{R}$, whence $(\mathbf{R}_0 + \dot{\mathbf{R}}\,t) \wedge \dot{\mathbf{R}} = \mathbf{R}_0 \wedge \dot{\mathbf{R}}$. Thus, the first term at the right hand side of (3.29) is constant, which makes $\mathbf{M}_B = m\,\mathbf{s} \wedge \mathbf{u}$ a constant as well. The latter vector is parallel to $\mathbf{M}$ because the motion is confined to a fixed plane (Sect. 3.4). Then, aligning the z axis with $\mathbf{M}$, turning to polar coordinates over the x, y plane ($s_x = s\,\cos\varphi$, $s_y = s\,\sin\varphi$), and using (3.8), one finds

$$\varphi(s) = \varphi_0 \pm \frac{M_B}{\sqrt{2m}} \int_{s_0}^{s} \frac{\mathrm{d}\xi}{\xi^2 \sqrt{E_B - V_e(\xi)}}, \tag{3.30}$$

with $V_e(s) = V(s) + M_B^2/(2ms^2)$. It is important to note that the factor M_B in (3.30) is the scalar coefficient of $\mathbf{M}_B = M_B\mathbf{k}$, with $\mathbf{k}$ the unit vector of the z axis. As a consequence, M_B may have sign. As observed in Sect. 2.9, the admissible values of s are those belonging to the interval such that $E_B \geq V_e(s)$. If two or more disjoint

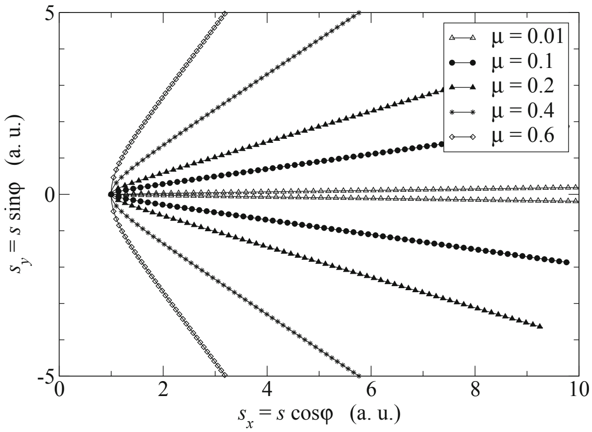


Fig. 3.4 Graphic representation of the trajectory (3.35) for different values of the angular momentum. The curves have been obtained by setting the parameters' values to $s_0 = 1$, $\varphi_0 = 0$, $\lambda = 0.5$, and $\mu = 0.01, \dots, 0.6$ (the units are arbitrary)

intervals exist that have this property, the actual interval of the motion is determined by the initial conditions. The motion is limited or unlimited, depending on the extent of this interval.

The analysis can not be pursued further unless the form of the potential energy V is specified. This is done in Sect. 3.8 with reference to the Coulomb case.

3.8 Coulomb Field

An important example is that of a potential energy of the form $V \propto 1/r$, that occurs for the gravitational and for the electrostatic force. In the latter case the term *Coulomb potential energy* is used for V, that reads

$$V(s) = \frac{\kappa\, Z_1\, Z_2\, q^2}{4\pi\, \varepsilon_0\, s}, \qquad s > 0, \tag{3.31}$$

with $q > 0$ the elementary electric charge, $Z_1\, q$ and $Z_2\, q$ the absolute value of the net charge of the first and second particle, respectively, ε_0 the vacuum permittivity and, finally, $\kappa = 1\,(-1)$ in the repulsive (attractive) case. The form of V fixes the additive constant of the energy so that $V(\infty) = 0$. The repulsive case only is considered here, whence V_e is strictly positive and $E_B \geq V_e > 0$. Defining the lengths

$$\lambda = \frac{Z_1\, Z_2\, q^2}{8\pi\, \varepsilon_0\, E_B} > 0, \qquad \mu = \frac{M_B}{\sqrt{2m\, E_B}} \tag{3.32}$$

yields $V_e/E_B = 2\lambda/s + \mu^2/s^2$. The zeros of $E_B - V_e = E_B\,(s^2 - 2\,\lambda\, s - \mu^2)/s^2$ are

$$s_A = \lambda - \sqrt{\lambda^2 + \mu^2}, \qquad s_B = \lambda + \sqrt{\lambda^2 + \mu^2}, \tag{3.33}$$

where s_A is negative and must be discarded as s is strictly positive. The only acceptable zero is then $s_B \geq 2\lambda > 0$, that corresponds to the position where the radial velocity $\dot{s} = \pm\sqrt{2\,(E_B - V)/m}$ reverses, and must therefore be identified with s_0 (Sect. 3.7). The definitions (3.32, 3.33) are now inserted into the expression (3.30) of the particle's trajectory. To calculate the integral it is convenient to use a new variable w such that $(s_0 - \lambda)/(\xi - \lambda) = (\mu^2 - s_0^2\, w^2)/(\mu^2 + s_0^2\, w^2)$. The range of w corresponding to $s_0 \le \xi \le s$ is

$$0 \le w \le \frac{|\mu|}{s_0}\sqrt{\frac{s - s_0}{s + s_0 - 2\lambda}}, \qquad s \ge s_0 \ge 2\lambda > 0. \tag{3.34}$$

From (3.30) the trajectory in the s, φ reference is thus found to be

$$\varphi(s) = \varphi_0 \pm 2\ \arctan\left(\frac{\mu}{s_0}\sqrt{\frac{s - s_0}{s + s_0 - 2\lambda}}\right). \tag{3.35}$$

Next, the trajectory in the Cartesian reference s_x, s_y is found by replacing (3.35) into $s_x = s\ \cos\varphi$, $s_y = s\ \sin\varphi$ and eliminating s from the pair $s_x(s)$, $s_y(s)$ thus found. A graphic example is given in Fig. 3.4. It is worth observing that in the derivation of (3.35) the factor $|\mu|$ appears twice, in such a way as to compensate for the sign of μ. The result then holds irrespective of the actual sign of $M_B = \sqrt{2m\,E_B}\,\mu$. It still holds for $M_B = 0$, that yields $\varphi(s) = \varphi_0$; such a case corresponds to a straight line crossing the origin of the s, φ reference: along this line the modulus s of the relative position decreases until it reaches s_0, then it increases from this point on.

When $M_B \neq 0$ the angles corresponding to the asymptotic conditions of the motion are found by letting $s \to \infty$, namely, $\varphi_a = \varphi_0 - 2\ \arctan(\mu/s_0)$ and $\varphi_b = \varphi_0 + 2\ \arctan(\mu/s_0)$. The total deflection is then $\varphi_b - \varphi_a$ which, in each curve of Fig. 3.4, is the angle between the two asymptotic directions. Now one combines the definition of angle χ given in Sect. 3.6 with the equality $\mathbf{u} = \mathbf{v}$ taken from the last of (3.15); with the aid of Fig. 3.5 one finds

$$\chi = \pi - (\varphi_b - \varphi_a) = \pi - 4\ \arctan\left(\frac{\mu}{s_0}\right). \tag{3.36}$$

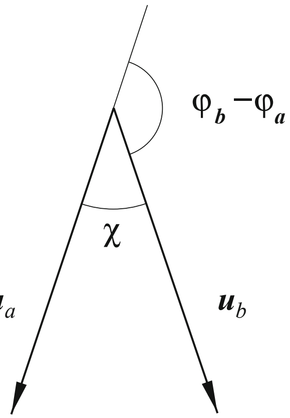
The definitions (3.32, 3.33) show that (3.36) eventually provides the relation $\chi = \chi(E_B, M_B)$. In contrast with the approach of Sect. 3.6, where the asymptotic conditions only were considered, here the analysis has been brought to the end by considering a specific type of interaction.

When μ ranges from $-\infty$ to $+\infty$ at a fixed E_B, the definitions (3.32, 3.33) make the ratio $\mu/s_0 = \mu/s_A$ to range from -1 to $+1$. If $\mu/s_0 = 1\ (-1)$, then $\chi = 0\ (2\pi)$, namely, no deflection between $\mathbf{u}_a$ and $\mathbf{u}_b$ occurs. If $\mu/s_0 = 0$, then $\chi = \pi$, namely, the motion's direction reverses at s_0 as noted above. From $\mathbf{u} = \mathbf{v}$ one finds that χ is also the angle between $\mathbf{v}_a = \mathbf{v}_{1a} - \mathbf{v}_{2a}$ and $\mathbf{v}_b = \mathbf{v}_{1b} - \mathbf{v}_{2b}$.

3.9 System of Particles near an Equilibrium Point

Consider a system of N particles, not necessarily identical to each other, subjected to conservative forces. The mass and instantaneous position of the jth particle are indicated with m_j and $\mathbf{R}_j = (X_{j1}, X_{j2}, X_{j3})$, respectively. It is assumed that there

Fig. 3.5 Graphic representation of (3.36)



are no constraints, so that the number of degrees of freedom of the system is $3N$. The Hamiltonian function reads

$$H_a = T_a + V_a = \sum_{j=1}^{N} \frac{P_j^2}{2\,m_j} + V_a(X_{11}, X_{12}, \dots), \tag{3.37}$$

with $P_j^2 = m_j^2\,(\dot X_{j1}^2 + \dot X_{j2}^2 + \dot X_{j3}^2)$. The force acting on the jth particle along the kth axis is $F_{jk} = -\partial V_a/\partial X_{jk}$. The Hamilton equations (Sect. 1.6) read

$$\dot X_{jk} = \frac{\partial H_a}{\partial P_{jk}} = \frac{P_{jk}}{m_j}, \qquad \dot P_{jk} = -\frac{\partial H_a}{\partial X_{jk}} = -\frac{\partial V_a}{\partial X_{jk}} = F_{jk}. \tag{3.38}$$

They show that the relation $F_{jk} = m_j\,\ddot X_{jk}$, which yields the dynamics of the jth particle along the kth axis, involves the positions of all particles in the system due to the coupling of the latter.

Define the $3N$-dimensional vector $\mathbf{R} = (X_{11}, \dots, X_{N3})$ that describes the instantaneous position of the system in the configuration space, and let $\mathbf{R}_0$ be a position where the potential energy V_a has a minimum, namely, $(\partial V_a/\partial X_{jk})_{\mathbf{R}_0} = 0$ for all j, k. Such a position is called *equilibrium point* of the system. To proceed, assume that the instantaneous displacement $\mathbf{R} - \mathbf{R}_0$ with respect to the equilibrium point is small. In this case one approximates V with a second-order Taylor expansion around $\mathbf{R}_0$. To simplify the notation new symbols are adopted, namely, $s_1 = X_{11}$, $s_2 = X_{12}, \dots, s_{3j+k-3} = X_{jk}, \dots$, and $h_n = s_n - s_{n0}$, with $n = 1, 2, \dots, 3N$ and s_{n0} the equilibrium position. Remembering that the first derivatives of V_a vanish at $\mathbf{R}_0$ one finds

$$V_a \simeq V_{a0} + \frac{1}{2} \sum_{k=1}^{3N} h_k \sum_{n=1}^{3N} c_{kn}\,h_n, \qquad c_{kn} = \left(\frac{\partial^2 V_a}{\partial h_k\,\partial h_n}\right)_{\mathbf{R}_0}. \tag{3.39}$$

In (3.39) it is $V_{a0} = V_a(\mathbf{R}_0)$, and the terms c_{kn} are called *elastic coefficients*. As the approximate form of the potential energy is quadratic in the displacements, each component of the force is a linear combination of the latter,

$$F_r = -\frac{\partial V_a}{\partial s_r} = -\frac{\partial V_a}{\partial h_r} = -\sum_{n=1}^{3N} c_{rn}\, h_n, \qquad r = 3j + k - 3. \tag{3.40}$$

To recast the kinetic energy in terms of the new symbols it is necessary to indicate the masses with μ_n, $n = 1, \ldots, 3N$, where $\mu_{3j-2} = \mu_{3j-1} = \mu_{3j} = m_j$, $j = 1, \ldots, N$. Observing that $\dot{X}_{jk} = \dot{s}_{3j+k-3} = \dot{h}_{3j+k-3}$, one finds a quadratic form in the derivatives of the displacements,

$$T_a = \frac{1}{2}\sum_{j=1}^{N}\sum_{k=1}^{3} \mu_{3j+k-3}\dot{X}_{jk}^2 = \frac{1}{2}\sum_{j=1}^{N}\sum_{k=1}^{3} \mu_{3j+k-3}\dot{h}_{3j+k-3}^2 = \frac{1}{2}\sum_{n=1}^{3N} \mu_n \dot{h}_n^2. \tag{3.41}$$

The relations obtained so far are readily recast in matrix form. First, one defines the *mass matrix* $\mathbf{M}$ as the real, $3N \times 3N$ diagonal matrix whose entries are $[\mathbf{M}]_{kn} = \mu_n\,\delta_{kn} > 0$, with δ_{kn} the Kronecker symbol (A.18). By construction, the mass matrix is symmetric and positive definite; the entries of its inverse are $[\mathbf{M}^{-1}]_{kn} = \delta_{kn}/\mu_n$. Then, one defines the *elastic matrix* $\mathbf{C}$ as the real, $3N \times 3N$ matrix whose entries are $[\mathbf{C}]_{kn} = c_{kn}$. The entries of the elastic matrix are the second derivatives of the potential energy V_a; as the order of the derivation is irrelevant, the matrix is symmetric. Also, the derivatives are calculated in a minimum of V_a; from the first of (3.39) it follows that the quadratic form at the right hand side equals $V_a - V_{a0}$ which, by construction, is positive. It follows that the elastic matrix is positive definite, namely, for any choice of the displacements (excluding the case where all displacements are zero) the quadratic form generated by the matrix is positive. Finally, let $\mathbf{h}$ be the column vector of entries $h_1, h_2, \ldots$, and $\mathbf{h}^T$ its transpose. Combining (3.37, 3.39, 3.41) expresses the Hamiltonian function in terms of the sum of two quadratic forms,

$$H_a - V_{a0} = \frac{1}{2}\,\dot{\mathbf{h}}^T\,\mathbf{M}\,\dot{\mathbf{h}} + \frac{1}{2}\,\mathbf{h}^T\,\mathbf{C}\,\mathbf{h}. \tag{3.42}$$

3.10 Diagonalization of the Hamiltonian Function

Thanks to the properties of the matrices $\mathbf{M}$ and $\mathbf{C}$, the right hand side of (3.42) can be set in diagonal form. To this purpose one considers the eigenvalue equation

$$\mathbf{C}\,\mathbf{g}_\sigma = \lambda_\sigma\,\mathbf{M}\,\mathbf{g}_\sigma, \qquad \sigma = 1, \ldots, 3N, \tag{3.43}$$

where the eigenvalues λ_σ are real because $\mathbf{C}$ and $\mathbf{M}$ are real and symmetric. As all coefficients of (3.43) are real, the eigenvectors $\mathbf{g}_\sigma$ are real as well. Also, due to the positive definiteness of $\mathbf{C}$ and $\mathbf{M}$, the eigenvalues are positive and the eigenvectors are linearly independent. They can also be selected in order to fulfill the property of being orthonormal with respect to $\mathbf{M}$, namely, $\mathbf{g}_\sigma^T\,\mathbf{M}\,\mathbf{g}_\tau = \delta_{\sigma\tau}$.

Each of the $3N$ eigenvectors $\mathbf{g}_\sigma$ has $3N$ entries. Thus, the set of eigenvectors can be arranged to form a $3N \times 3N$ real matrix $\mathbf{G}$, whose σth column is the σth eigenvector. The inverse matrix $\mathbf{G}^{-1}$ exists because, by construction, the columns of $\mathbf{G}$ are linearly independent. The orthonormality relation between the eigenvectors can now be expressed in matrix form as

$$\mathbf{G}^T\,\mathbf{M}\,\mathbf{G} = \mathbf{I}, \tag{3.44}$$

with $\mathbf{I}$ the identity matrix. Equation (3.44) is the basic ingredient for the diagonalization of (3.42). From it one preliminarily derives four more relations,

$$\mathbf{G}^T\,\mathbf{M} = \mathbf{G}^{-1}, \qquad \mathbf{G}\,\mathbf{G}^T\,\mathbf{M} = \mathbf{I}, \qquad \mathbf{M}\,\mathbf{G} = \left(\mathbf{G}^T\right)^{-1}, \qquad \mathbf{M}\,\mathbf{G}\,\mathbf{G}^T = \mathbf{I}. \tag{3.45}$$

The first of (3.45) is obtained by right multiplying (3.44) by $\mathbf{G}^{-1}$ and using $\mathbf{G}\,\mathbf{G}^{-1} = \mathbf{I}$. Left multiplying by $\mathbf{G}$ the first of (3.45) yields the second one. The third of (3.45) is obtained by left multiplying (3.44) by $(\mathbf{G}^T)^{-1}$. Finally, right multiplying by $\mathbf{G}^T$ the third of (3.45) yields the fourth one. To complete the transformation of the equations into a matrix form one defines the eigenvalue matrix $\mathbf{L}$ as the real, $3N \times 3N$ diagonal matrix whose entries are $[\mathbf{L}]_{\sigma\tau} = \lambda_\tau\,\delta_{\sigma\tau} > 0$. The set of $3N$ eigenvalue Eqs. (3.43) then takes one of the two equivalent forms

$$\mathbf{C}\,\mathbf{G} = \mathbf{M}\,\mathbf{G}\,\mathbf{L}, \qquad \mathbf{G}^T\,\mathbf{C}\,\mathbf{G} = \mathbf{L}. \tag{3.46}$$

The first of (3.46) is the analogue of (3.43), while the second form is obtained from the first one by left multiplying by $\mathbf{G}^T$ and using (3.44). The diagonalization of (3.42) is now accomplished by inserting the second and fourth of (3.45) into the potential-energy term of (3.42) to obtain

$$\mathbf{h}^T\,\mathbf{C}\,\mathbf{h} = \mathbf{h}^T\,(\mathbf{M}\,\mathbf{G}\,\mathbf{G}^T)\,\mathbf{C}\,(\mathbf{G}\,\mathbf{G}^T\,\mathbf{M})\,\mathbf{h} = (\mathbf{h}^T\,\mathbf{M}\,\mathbf{G})\,(\mathbf{G}^T\,\mathbf{C}\,\mathbf{G})\,(\mathbf{G}^T\,\mathbf{M}\,\mathbf{h}), \tag{3.47}$$

where the associative law has been used. At the right hand side of (3.47), the term in the central parenthesis is replaced with $\mathbf{L}$ due to the second of (3.46). The term in the last parenthesis is a column vector for which the short-hand notation $\mathbf{b} = \mathbf{G}^T\,\mathbf{M}\,\mathbf{h}$ is introduced. Note that $\mathbf{b}$ depends on time because $\mathbf{h}$ does. The first of (3.45) shows that $\mathbf{h} = \mathbf{G}\,\mathbf{b}$, whence $\mathbf{h}^T = \mathbf{b}^T\,\mathbf{G}^T$. Finally, using (3.44), transforms the term in the first parenthesis at the right hand side of (3.47) into $\mathbf{h}^T\,\mathbf{M}\,\mathbf{G} = \mathbf{b}^T\,\mathbf{G}^T\,\mathbf{M}\,\mathbf{G} = \mathbf{b}^T$. In conclusion, the potential-energy term of (3.42) is recast in terms of $\mathbf{b}$ as $\mathbf{h}^T\,\mathbf{C}\,\mathbf{h} = \mathbf{b}^T\,\mathbf{L}\,\mathbf{b}$, which is the diagonal form sought. By a similar procedure one finds for the kinetic-energy term $\dot{\mathbf{h}}^T\,\mathbf{M}\,\dot{\mathbf{h}} = \dot{\mathbf{b}}^T\,\mathbf{G}^T\,\mathbf{M}\,\mathbf{G}\,\dot{\mathbf{b}} = \dot{\mathbf{b}}^T\,\dot{\mathbf{b}}$.

The terms $\dot{\mathbf{b}}^T\,\dot{\mathbf{b}}$ and $\mathbf{b}^T\,\mathbf{L}\,\mathbf{b}$ have the same units. As a consequence, the units of $\mathbf{L}$ are the inverse of a time squared. Remembering that the entries of $\mathbf{L}$ are positive, one introduces the new symbol $\omega_\sigma^2 = \lambda_\sigma$ for the eigenvalues, $\omega_\sigma > 0$. In conclusion, the diagonal form of (3.42) reads

$$H_a - V_{a0} = \sum_{\sigma=1}^{3N} H_\sigma, \qquad H_\sigma = \frac{1}{2}\,\dot{b}_\sigma^2 + \frac{1}{2}\,\omega_\sigma^2\,b_\sigma^2. \tag{3.48}$$

Apart from the constant V_{a0}, the Hamiltonian function H_a is given by a sum of terms, each associated with a single degree of freedom. A comparison with (3.1) shows that the individual summands H_σ are identical to the Hamiltonian function of a linear harmonic oscillator with $m = 1$. As a consequence, the two canonical variables of the σth degree of freedom are $q_\sigma = b_\sigma$, $p_\sigma = \dot{b}_\sigma$, and the time evolution of b_σ is the same as that in (3.2),

$$b_\sigma(t) = \beta_\sigma \cos(\omega_\sigma t + \varphi_\sigma) = \frac{1}{2}\left[\tilde{\beta}_\sigma \exp(-\mathrm{i}\omega_\sigma t) + \tilde{\beta}_\sigma^* \exp(\mathrm{i}\omega_\sigma t)\right]. \quad (3.49)$$

The constants β_σ, φ_σ depend on the initial conditions $b_\sigma(0)$, $\dot{b}_\sigma(0)$. The complex coefficients are related to the above constants by $\tilde{\beta}_\sigma = \beta_\sigma \exp(-\mathrm{i}\varphi_\sigma)$. In turn, the initial conditions are derived from those of the displacements, $\mathbf{b}(0) = \mathbf{G}^{-1}\,\mathbf{h}(0)$, $\dot{\mathbf{b}}(0) = \mathbf{G}^{-1}\,\dot{\mathbf{h}}(0)$.

The $3N$ functions $b_\sigma(t)$ are called *normal coordinates* or *principal coordinates*. Once the normal coordinates have been found, the displacements of the particles are determined from $\mathbf{h} = \mathbf{G}\,\mathbf{b}$. It follows that such displacements are superpositions of oscillatory functions. Despite the complicacy of the system, the approximation of truncating the potential energy to the second order makes the Hamiltonian function completely separable in the normal coordinates. The problem then becomes a generalization of that of the linear harmonic oscillator (Sect. 3.3), and the frequencies of the oscillators are determined by combining the system parameters, specifically, the particle masses and elastic constants. The Hamiltonian function associated with each degree of freedom is a constant of motion, $H_\sigma = E_\sigma$, whose value is prescribed by the initial conditions. The total energy of the system is also a constant and is given by

$$E = V_{a0} + \sum_{\sigma=1}^{3N} E_\sigma, \qquad E_\sigma = \frac{1}{2}\,\dot{b}_\sigma^2(0) + \frac{1}{2}\,\omega_\sigma^2\, b_\sigma^2(0). \quad (3.50)$$

The oscillation of the normal coordinate of index σ is also called *mode* of the vibrating system.

3.11 Periodic Potential Energy

An interesting application of the action-angle variables introduced in Sect. 2.10 is found in the case of a conservative motion where the potential energy V is periodic. For simplicity a linear motion is considered (Sect. 2.9), whence $V(x + a) = V(x)$, with $a > 0$ the spatial period. Letting E be the total energy and m the mass of the particle, an unlimited motion is assumed, namely, $E > V$; it follows that the momentum $p = \sqrt{2\,m\,[E - V(x)]}$ is a spatially-periodic function of period a whence, according to the definition of Sect. 2.10, the motion is a rotation. For any position g, the time τ necessary for the particle to move from g to $g + a$ is found from (2.47), where the positive sign is provisionally chosen:

$$\tau = \sqrt{\frac{m}{2}} \int_g^{g+a} \frac{\mathrm{d}x}{\sqrt{E - V(x)}} > 0. \tag{3.51}$$

The integral in (3.51) is independent of g due to the periodicity of V. As a consequence, for any g the position of the particle grows by a during the time τ. The action variable is found from (2.49):

$$J(E) = \int_g^{g+a} p\,\mathrm{d}x = \sqrt{2m} \int_g^{g+a} \sqrt{E - V(x)}\,\mathrm{d}x = \text{const}. \tag{3.52}$$

In turn, the derivative of the angle variable is found from (2.51). It reads $\dot{w} = \partial H/\partial J = 1/(\mathrm{d}J/\mathrm{d}E) = \text{const}$, with H the Hamiltonian function. The second form of $\dot{w}$ holds because H does not depend on w, and $H = E$. Using (3.52) and comparing with (3.51) one finds

$$\frac{1}{\dot{w}} = \frac{\mathrm{d}J}{\mathrm{d}E} = \int_g^{g+a} \frac{m}{[2\,m\,(E - V(x))]^{1/2}}\,\mathrm{d}x = \tau. \tag{3.53}$$

As expected, $1/\tau$ is the rotation frequency. In conclusion, the time evolution of the action-angle variables is given by $w = t/\tau + w_0$, $J = \text{const}$. Note that the relation (3.52) between E and J holds when the positive sign is chosen in (2.47); if the above calculations are repeated after choosing the negative sign, one finds that $-J$ is associated to the same E. As a consequence, E is an even function of J.

Another observation is that the action-angle variables can be scaled by letting, e.g., $w\,J = (a\,w)\,(J/a)$. In this way the property that the product of two canonically-conjugate variables is dimensionally an action is still fulfilled. A comparison with (3.52) shows that, thanks to this choice of the scaling factor, $P = J/a$ is the average momentum over a period, while $X = a\,w$ is a length. The Hamilton equations and the time evolution of the new variables are then

$$\dot{X} = \frac{\partial H}{\partial P}, \quad \dot{P} = -\frac{\partial H}{\partial X} = 0, \qquad X = \frac{a}{\tau}\,t + X_0, \quad P = P_0 = \text{const}, \tag{3.54}$$

where a/τ is the average velocity of the particle over the spatial period, and $X_0 = X(0)$, $P_0 = P(0)$. In conclusion, in the new canonical variables no force is acting ($\dot{P} = 0$), and the motion of the new position X is uniform in time. However, the relation between E and $P = J/a$, given by (3.52), is not quadratic as it would be in free space.[2]

In many cases it is of interest to investigate the particle's dynamics when a perturbation δH is superimposed to the periodic potential energy V. It is assumed that δH depends on x only, and that E is the same as in the unperturbed case (the latter assumption is not essential). The Hamiltonian function of the perturbed case is then

[2] Compare with comments made in Sect. 19.6.1.

written as the sum of the unperturbed one and of the perturbation; also in this case an unlimited motion is assumed, specifically, $E > V$ and $E > V + \delta H$. Still using the positive sign for the momentum, one finds

$$H(x, p) = \frac{p^2}{2m} + V(x) + \delta H(x) = E, \qquad p(x, E) = \sqrt{2\,m\,(E - V - \delta H)}. \tag{3.55}$$

As in the unperturbed case one defines the average momentum over a,

$$\tilde{P}(g, E) = \frac{\sqrt{2m}}{a} \int_g^{g+a} \sqrt{E - V - \delta H}\,\mathrm{d}x, \tag{3.56}$$

which depends also on g because δH is not periodic. Differentiating (3.56) with respect to E and comparing with (3.51) shows that

$$\frac{\partial \tilde{P}}{\partial E} = \frac{\tilde{\tau}}{a}, \qquad \tilde{\tau}(g) = \int_g^{g+a} \frac{m}{[2\,m\,(E - V(x) - \delta H)]^{1/2}}\,\mathrm{d}x, \tag{3.57}$$

with $\tilde{\tau}$ the time necessary for the particle to move from g to $g + a$ in the perturbed case. Using $H = E$ in the above yields

$$\frac{\partial H}{\partial \tilde{P}} = \frac{a}{\tilde{\tau}} = \frac{(g + a) - g}{\tilde{\tau}}. \tag{3.58}$$

So far no hypothesis has been made about the perturbation. Now one assumes that δH is weak and varies little over the period a. The first hypothesis implies $|\delta H| \ll E - V$ so that, to first order, $[2\,m\,(E - V - \delta H)]^{1/2} \simeq [2\,m\,(E - V)]^{1/2} - m\,[2\,m\,(E - V)]^{-1/2}\,\delta H$. Using $P = J/a$ and (3.52), the average momentum (3.56) becomes

$$\tilde{P}(g, E) \simeq P(E) - \frac{1}{a} \int_g^{g+a} \frac{m\,\delta H}{[2\,m\,(E - V)]^{1/2}}\,\mathrm{d}x. \tag{3.59}$$

In turn, the hypothesis that the perturbation varies little over the period a implies that in the interval $[g, g + a]$ one can approximate $\delta H(x)$ with $\delta H(g)$, which transforms (3.59), due to (3.53), into

$$\tilde{P}(g, E) \simeq P(E) - \frac{\tau}{a}\,\delta H(g). \tag{3.60}$$

If the procedure leading to (3.60) is repeated in the interval $[g + a, g + 2\,a]$ and the result is subtracted from (3.60), the following is found:

$$\frac{\tilde{P}(g + a, E) - \tilde{P}(g, E)}{\tau} = -\frac{\delta H(g + a) - \delta H(g)}{a}. \tag{3.61}$$

The above shows that the perturbed momentum $\tilde{P}$ varies between g and $g + a$ due to the corresponding variation in δH. Fixing the time origin at the position g and letting $\tau \simeq \tilde{\tau}$ in the denominator transforms (3.61) into

$$\frac{\tilde{P}(\tilde{\tau}, E) - \tilde{P}(0, E)}{\tilde{\tau}} \simeq -\frac{\delta H(g + a) - \delta H(g)}{a}. \tag{3.62}$$

The relations (3.58, 3.62) are worth discussing. If one considers g as a position coordinate and $\tilde{P}$ as the momentum conjugate to it, (3.58, 3.62) become a pair of Hamilton equations where some derivatives are replaced with difference quotients. Specifically, (3.62) shows that the average momentum varies so that its "coarse-grained" variation with respect to time, $\Delta \tilde{P}/\Delta \tilde{\tau}$, is the negative coarse-grained variation of the Hamiltonian function with respect to space, $-\Delta H/\Delta g = -\Delta \delta H/\Delta g$. In turn, (3.58) shows that the coarse-grained variation of position with respect to time, $\Delta g/\Delta \tilde{\tau}$, is the derivative of the Hamiltonian function with respect to the average momentum. In conclusion, (3.58, 3.62) are useful when one is not interested in the details of the particle's motion within each spatial period, but wants to investigate on a larger scale how the perturbation influences the average properties of the motion.

3.12 Energy-Momentum Relation in a Periodic Potential Energy

It has been observed, with reference to the non-perturbed case, that the relation (3.52) between the total energy and the average momentum is not quadratic. In the perturbed case, as shown by (3.56), the momentum depends on both the total energy and the coarse-grained position. To investigate this case it is then necessary to fix g and consider the dependence of $\tilde{P}$ on E only. To proceed one takes a small interval of $\tilde{P}$ around a given value, say, $\tilde{P}_s$, corresponding to a total energy E_s, and approximates the $E(\tilde{P})$ relation with a second-order Taylor expansion around $\tilde{P}_s$,

$$E \simeq E_s + \left(\frac{\mathrm{d}E}{\mathrm{d}\tilde{P}}\right)_s (\tilde{P} - \tilde{P}_s) + \frac{1}{2}\left(\frac{\mathrm{d}^2 E}{\mathrm{d}\tilde{P}^2}\right)_s (\tilde{P} - \tilde{P}_s)^2. \tag{3.63}$$

Although in general the $\tilde{P}(E)$ relation (3.56) can not be inverted analytically, one can calculate the derivatives that appear in (3.63). The latter are worked out in the unperturbed case $\delta H = 0$ for simplicity. Using (3.53), the first derivative is found to be $(\mathrm{d}E/\mathrm{d}\tilde{P})_s \simeq (\mathrm{d}E/\mathrm{d}P)_s = a/\tau_s$, with $\tau_s = \tau(E_s)$. For the second derivative,

$$\frac{\mathrm{d}^2 E}{\mathrm{d}\tilde{P}^2} \simeq \frac{\mathrm{d}^2 E}{\mathrm{d}P^2} = \frac{\mathrm{d}(a/\tau)}{\mathrm{d}P} = -\frac{a}{\tau^2}\frac{\mathrm{d}\tau}{\mathrm{d}E}\frac{\mathrm{d}E}{\mathrm{d}P} = -\frac{a^3}{\tau^3}\frac{\mathrm{d}^2 P}{\mathrm{d}E^2}. \tag{3.64}$$

On the other hand, using (3.53) again, it is

$$\frac{\mathrm{d}^2 P}{\mathrm{d}E^2} = \frac{\mathrm{d}(\tau/a)}{\mathrm{d}E} = -\frac{m^2}{a}\int_g^{g+a} K^3\,\mathrm{d}x, \qquad K = [2m\,(E - V(x))]^{-1/2}. \tag{3.65}$$

Combining (3.64) with (3.65) and defining the dimensionless parameter

$$r_s = \int_{g/a}^{g/a+1} K^3\,\mathrm{d}(x/a) \times \left[\int_{g/a}^{g/a+1} K\,\mathrm{d}(x/a)\right]^{-3}, \tag{3.66}$$

transforms (3.63) into

$$E \simeq E_s + \frac{a}{\tau_s}\,(\tilde{P} - \tilde{P}_s) + \frac{r_s}{2\,m}\,(\tilde{P} - \tilde{P}_s)^2, \tag{3.67}$$

where the coefficients, thanks to the neglect of the perturbation, do not depend on g. The linear term is readily eliminated by shifting the origin of the average momentum; in fact, letting $\tilde{P} - \tilde{P}_s = \tilde{p} - (m/r_s)\,(a/\tau_s)$ yields

$$E - E(0) = \frac{r_s}{2\,m}\,\tilde{p}^2, \qquad E(0) = E_s - \frac{1}{2}\,\frac{m}{r_s}\,\left(\frac{a}{\tau_s}\right)^2. \tag{3.68}$$

In conclusion, in a small interval of $\tilde{p}$ the relation between energy and average momentum of a particle of mass m subjected to a periodic potential has the same form as that of a free particle of mass m/r_s. In other terms, the ratio m/r_s acts as an *effective mass* within the frame of the coarse-grained dynamics.

A bound for r_s is obtained from Hölder's inequality (C.110). Letting $|F| = K$, $G = 1$, $b = 3$, $x_1 = g/a$, $x_2 = g/a + 1$ in (C.110), and using the definition (3.66), yields $r_s \geq 1$, whence $m/r_s \leq m$: the effective mass can never exceed the true mass. The equality between the two masses is found in the limiting case $E - V_M \gg V_M - V - \delta H$, with V_M the maximum of V. In fact, (3.66) yields $r \simeq 1$ and, from (3.56), it is $\tilde{P} \simeq \sqrt{2mE}$. As expected, this limiting case yields the dynamics of a free particle.

3.13 Complements

3.13.1 Comments on the Linear Harmonic Oscillator

The paramount importance of the example of the linear harmonic oscillator, shown in Sect. 3.3, is due to the fact that in several physical systems the position of a particle at any instant happens to depart little from a point where the potential energy V has a minimum. As a consequence, the potential energy can be approximated with a second-order expansion around the minimum, that yields a positive-definite quadratic form for the potential energy and a linear form for the force. The theory depicted in this section is then applicable to many physical systems, as shown by the examples of Sects. 3.9 and 5.6. The approximation of the potential energy with a second-order expansion, like that discussed in Sects. 3.9, 3.10, is called *harmonic approximation*. The terms beyond the second order in the expansion are called *anharmonic*.

3.13.2 Degrees of Freedom and Coordinate Separation

With reference to the analysis of the central motion carried out in Sect. 3.4, it is worth noting that the constancy of **M** reduces the number of degrees of freedom of the problem from three to two. Also, the form (3.6) of the Hamiltonian function is such as to provide a relation containing only r and the corresponding momentum p_r. Thus, the coordinate r is separable according to the definition of Sect. 2.4. This allows one to independently find the time evolution (3.7) of r by solving an ordinary differential equation of the first order. Then one finds (3.8), that is, the trajectory $\varphi(r)$, through another equation of the same type. Finally, combining (3.7) with (3.8) yields the time evolution of the remaining coordinate φ.

It has been noted in Sect. 3.4 that, thanks to the constancy of the angular momentum, the adoption of spherical coordinates allows one to separate the radial coordinate r. This simplifies the problem, whose solution is in fact reduced to the successive solution of the evolution equations for r and φ. The same problem, instead, is not separable in the Cartesian coordinates. In other terms, separability may hold in some coordinate reference, but does not hold in general in an arbitrarily-chosen reference.

Another example of separability is that illustrated in Sects. 3.9, 3.10. In general the Hamiltonian function is not separable in the Cartesian coordinates, whereas it is completely separable in the normal coordinates, no matter how large the number of the degrees of freedom is. Moreover, after the separation has been accomplished, one finds that all the equations related to the single degrees of freedom (the second relation in (3.48)) have the same form. In fact, they differ only in the numerical value of the angular frequency ω_σ. As a consequence, the expression of the solution is the same for all. Also, as the energy H_σ of each degree of freedom is independently conserved, no exchange of energy among the normal coordinates occurs: therefore, the distribution of energy among the normal coordinates that is present at $t = 0$ is maintained forever. This result is baffling because, for instance, it seems to prevent the condition of thermal equilibrium from being established; actually it is due to the fact that the system under investigation is isolated: if it were put in contact with a thermal reservoir, the exchanges of energy occurring with the reservoir would eventually bring the energy distribution of the system to the condition of thermal equilibrium.

Still with reference to the system discussed in Sects. 3.9, 3.10 it is important to underline the formal analogy between the modes of a mechanical, vibrating system and those of the electromagnetic field *in vacuo* described in Sect. 5.6. In both cases the energy of each mode is that of a linear harmonic oscillator of unit mass (Eq. (3.48) and, respectively, (5.40)).

3.13.3 Comments on the Normal Coordinates

It has been shown in Sect. 3.9 that the elastic matrix **C** is positive definite. One may argue that in some cases the matrix is positive semi-definite. Consider, for instance, the case where the potential energy depends on the relative distance of the particles, $V_a = V_a(\mathbf{R}_1 - \mathbf{R}_2, \mathbf{R}_1 - \mathbf{R}_3, \dots)$. For any set of positions $\mathbf{R}_1, \mathbf{R}_2, \dots$, a uniform displacement $\mathbf{R}_\delta$ of all particles, that tranforms each $\mathbf{R}_j$ into $\mathbf{R}_j + \mathbf{R}_\delta$, leaves V_a unchanged. As a consequence, if the positions prior to the displacement correspond

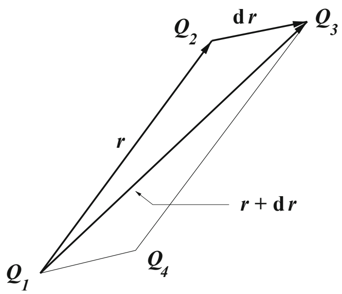


Fig. 3.6 Graphic representation of (3.69)

to the equilibrium point $\mathbf{R}_{01}, \mathbf{R}_{02}, \ldots$, it is $V_a(\mathbf{R}_{01} + \mathbf{R}_\delta, \ldots) = V_a(\mathbf{R}_{01}, \ldots) = V_{a0}$. In such a case all terms beyond the zero-order term in the Taylor expansion of V_a around the equilibrium point vanish, which implies that the elastic matrix $\mathbf{C}$ is positive semi-definite. In the case examined in Sect. 3.9 the eigenvalues are real and positive; here, instead, they are real and non-negative. Remembering (3.48), one finds that the Hamiltonian function of the degree of freedom corresponding to the null eigenvalue reads $H = \dot{b}_\sigma^2/2$, whence $\ddot{b}_\sigma = 0$, $b_\sigma = b_\sigma(0) + a\,t$, with a a constant.

The problem tackled in Sect. 3.10 is that of diagonalizing the right hand side of (3.42). The diagonalization of a quadratic form entails a linear transformation over the original vector ($\mathbf{h}$ in this case) using a matrix formed by eigenvectors. One may observe that, in (3.42), the kinetic energy $\dot{\mathbf{h}}^T\,\mathbf{M}\,\dot{\mathbf{h}}/2$ is already diagonal in the original vector, while the potential energy $\mathbf{h}^T\,\mathbf{C}\,\mathbf{h}/2$ is not. If the diagonalization were carried out using the matrix formed by the eigenvalues of $\mathbf{C}$ alone, the outcome of the process would be that of making the potential energy diagonal while making the kinetic energy non-diagonal (both in the transformed vector). The problem is solved by using the eigenvalue Eq. (3.43), that involves both matrices $\mathbf{M}$ and $\mathbf{C}$ in the diagonalization process. In fact, as shown in Sect. 3.10, in the transformed vector $\mathbf{b}$ the potential energy becomes diagonal, and the kinetic energy remains diagonal.

One may observe that, given the solutions of the eigenvalue Eq. (3.43), the process of diagonalizing (3.42) is straightforward. The real difficulty lies in solving (3.43). When the number of degrees of freedom is large, the solution of (3.43) must be tackled by numerical methods and may become quite cumbersome. In practical applications the elastic matrix $\mathbf{C}$ exhibits some structural properties, like symmetry or periodicity (e.g., Sect. 17.7.1), that are exploited to ease the problem of solving (3.43).

3.13.4 Areal Velocity in the Central-Motion Problem

Consider the central-motion problem discussed in Sect. 3.4. In the elementary time-interval $\mathrm{d}t$ the position vector changes from $\mathbf{r}$ to $\mathbf{r} + \mathrm{d}\mathbf{r}$. The area $\mathrm{d}A$ of the triangle whose sides are $\mathbf{r}$, $\mathbf{r} + \mathrm{d}\mathbf{r}$, and $\mathrm{d}\mathbf{r}$ is half the area of the parallelogram $Q_1 Q_2 Q_3 Q_4$

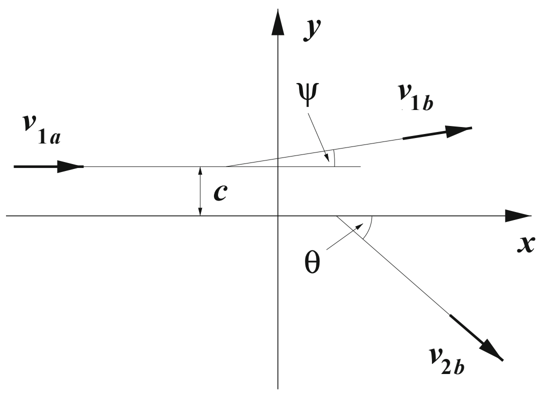


Fig. 3.7 Definition of the angles used in Sects. 3.6 and 3.13.5

whose consecutive sides are, e.g., $\mathbf{r}$ and $\mathrm{d}\mathbf{r}$ (Fig. 3.6). Thus,

$$\mathrm{d}A = \frac{1}{2}\,|\mathbf{r} \wedge \mathrm{d}\mathbf{r}| = \frac{1}{2}\,|\mathbf{r} \wedge \dot{\mathbf{r}}\,\mathrm{d}t| = \frac{|\mathbf{M}|}{2\,m}\,\mathrm{d}t, \qquad \frac{\mathrm{d}A}{\mathrm{d}t} = \frac{|\mathbf{M}|}{2\,m}, \tag{3.69}$$

with $\mathbf{M}$ the angular momentum. The derivative $\mathrm{d}A/\mathrm{d}t$ is called *areal velocity*. The derivation of (3.69) is based purely on definitions, hence it holds in general. For a central motion the angular momentum $\mathbf{M}$ is constant, whence the areal velocity is constant as well: the area swept out by the position vector $\mathbf{r}$ in a given time interval is proportional to the interval itself (*Kepler's second law*). If the particle's trajectory is closed, the time T taken by $\mathbf{r}$ to complete a revolution and the area A enclosed by the orbit are related by

$$A = \int_0^T \frac{\mathrm{d}A}{\mathrm{d}t}\,\mathrm{d}t = \frac{|\mathbf{M}|}{2\,m}\,T, \qquad T = \frac{2\,m\,A}{|\mathbf{M}|}. \tag{3.70}$$

3.13.5 Initial Conditions in the Central-Motion Problem

The theory of the central motion for a two-particle system has been worked out in Sects. 3.7,3.8 without specifying the initial conditions. To complete the analysis it is convenient to use the same prescription as in Sect. 3.6, namely, to select an O reference where the particle of mass m_2 is initially at rest ($\mathbf{v}_{2a} = 0$). Moreover, here reference O is chosen in such a way as to make the initial position of the particle of mass m_2 to coincide with the origin ($\mathbf{r}_{2a} = 0$), and the initial velocity $\mathbf{v}_a = \mathbf{v}_{1a}$ of the particle of mass m_1 to be parallel to the x axis (Fig. 3.7), so that $\mathbf{v}_{1a} = (\mathbf{v}_{1a} \cdot \mathbf{i})\,\mathbf{i}$. From (3.11) one finds $E = E_a = T_a = T_{1a} = m_1 v_{1a}^2/2$ and, from Sect. 3.5, $(m_1 + m_2)\,\dot{\mathbf{R}}_a = m_1\,\mathbf{v}_{1a}$. Using $\mathbf{r}_{1a} = x_{1a}\,\mathbf{i} + y_{1a}\,\mathbf{j}$ and (3.28) then yields

$$E_B = \frac{1}{2}\,m\,v_{1a}^2, \qquad \mathbf{M} = \mathbf{r}_{1a} \wedge m_1\,\mathbf{v}_{1a} = -m_1\,y_{1a}\,(\mathbf{v}_{1a} \cdot \mathbf{i})\,\mathbf{k}, \tag{3.71}$$

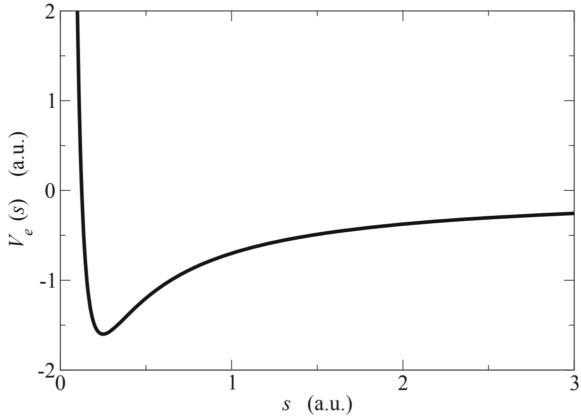


Fig. 3.8 Dependence of V_e on the distance s from the center of force, as given by (3.74) in arbitrary units

with $\mathbf{i}, \mathbf{j}, \mathbf{k}$ the unit vectors of the x, y, z axes and m the reduced mass. On the other hand, (3.29) shows that $\mathbf{M} = (m_1 + m_2)\,\mathbf{R}_a \wedge \dot{\mathbf{R}} + \mathbf{M}_B$ whence, writing $\mathbf{R}_a$, $\dot{\mathbf{R}}$ in terms of $\mathbf{r}_{1a}$, $\mathbf{v}_{1a}$ and equating the two expressions of $\mathbf{M}$ provides

$$M_B = -y_{1a}\, m\, \mathbf{v}_{1a} \cdot \mathbf{i}. \tag{3.72}$$

Replacing (3.72) and the first of (3.71) in (3.32, 3.33) yields

$$\mu = \mp y_{1a}, \qquad s_0 = \lambda + \sqrt{\lambda^2 + c^2}, \qquad c = |y_{1a}|. \tag{3.73}$$

The distance c between the x axis and the direction of $\mathbf{v}_{1a}$ (Fig. 3.7) is called *impact parameter*. The outcome of the calculation demonstrates the usefulness of choosing reference O as described above. In fact, for a given form of the potential energy V, the angle χ defined in Sect. 3.6 becomes a function of two easily-specified quantities: kinetic energy ($E = m_1\, v_{1a}^2/2$ or $E_B = m\, v_{1a}^2/2$) and impact parameter c (compare, e.g., with (3.36)). Once χ is determined, the final kinetic energies T_{1b}, T_{2b} and the angles ψ, θ are recovered from (3.22, 3.23) and (3.25, 3.26), respectively. Another property of reference O is that θ turns out to be the angle between the x axis and the final direction of the particle of mass m_2.

3.13.6 The Coulomb Field in the Attractive Case

To treat the attractive case one lets $\kappa = -1$ in (3.31). The trajectory lies in the x, y plane; in polar coordinates it is still given by (3.30), with

$$V_e(s) = \frac{M_B^2}{2ms^2} - \frac{Z_1\, Z_2\, q^2}{4\pi\, \varepsilon_0\, s}, \qquad s > 0. \tag{3.74}$$

In this case V_e becomes negative for some values of s. As a consequence, E_B may also be negative, provided the condition $E_B \geq V_e$ is fulfilled. Then, it is not possible to use the definitions (3.32) because E_B is positive there. The following will be used instead,

$$\alpha = \frac{Z_1 Z_2 q^2}{8\pi \varepsilon_0} > 0, \qquad \beta = \frac{M_B}{\sqrt{2m}}, \tag{3.75}$$

so that $V_e(s) = (\beta/s)^2 - 2\alpha/s$. Like in Sect. 3.8 it is assumed that M_B differs from zero and has either sign. It is found by inspection that V_e has only one zero at $s = s_c = \beta^2/(2\alpha)$ and only one minimum at $s = 2s_c$, with $\min(V_e) = V_e(2s_c) = -\alpha^2/\beta^2$. Also, it is $\lim_{s\to 0} V_e = \infty$, $\lim_{s\to\infty} V_e = 0$ (Fig. 3.8). The motion is unlimited when $E_B \ge 0$, while it is limited when $\min(V_e) \le E_B < 0$. The case $E_B = \min(V_e)$ yields $s = 2s_c = \text{const}$, namely, the trajectory is a circumference. When $\min(V_e) = -\alpha^2/\beta^2 < E_B < 0$ it is $\alpha^2 > \beta^2 |E_B|$. Then, the difference $E_B - V_e = -(|E_B| s^2 - 2\alpha s + \beta^2)/s^2$ has two real, positive zeros given by

$$s_0 = \frac{\alpha - \sqrt{\alpha^2 - \beta^2 |E_B|}}{|E_B|}, \qquad s_1 = \frac{\alpha + \sqrt{\alpha^2 - \beta^2 |E_B|}}{|E_B|}, \qquad s_0 < s_1. \tag{3.76}$$

Using the zeros one finds $s^2 \sqrt{E_B - V_e} = \sqrt{|E_B|}\, s \sqrt{(s - s_0)(s_1 - s)}$, that is replaced within (3.30) after letting $s \leftarrow \xi$. The upper limit of the integral belongs to the interval $s_0 \le s \le s_1$. To calculate the integral it is convenient to use a new variable w such that $2 s_0 s_1/\xi = (s_1 - s_0) w + s_1 + s_0$. The range of w corresponding to the condition $s_0 \le \xi \le s$ is

$$\frac{2 s_0 s_1 - (s_0 + s_1) s}{(s_1 - s_0) s} \le w \le 1, \qquad w(s_1) = -1. \tag{3.77}$$

From (3.30) the trajectory in the s, φ reference is thus found to be

$$\varphi(s) = \varphi_0 \pm \frac{M_B}{|M_B|} \arccos\left[\frac{2 s_0 s_1 - (s_0 + s_1) s}{(s_1 - s_0) s}\right]. \tag{3.78}$$

As noted in Sect. 3.4, the trajectory is symmetric with respect to φ_0. When (3.78) is inverted, the $\pm M_B/|M_B|$ factor is irrelevant because the cosine is an even function of the argument. Thus,

$$\frac{1}{s} = \frac{s_1 + s_0}{2 s_0 s_1}\left[1 + \frac{s_1 - s_0}{s_1 + s_0} \cos(\varphi - \varphi_0)\right]. \tag{3.79}$$

When $\varphi = \varphi_0$ it is $s = s_0$; when $\varphi = \varphi_0 + \pi$ it is $s = s_1$. The $s(\varphi)$ relation (3.79) is the equation of an ellipse of eccentricity $e = (s_1 - s_0)/(s_1 + s_0)$, where the center of force $s = 0$ is one of the foci. The distance between the foci is $s_1 - s_0$. With the aid of Fig. 3.9 one finds that the semimajor and semiminor axes are obtained, respectively, from $a = (s_1 + s_0)/2$, $b^2 = a^2 - (s_1 - s_0)^2/4$ whence, using (3.76),

$$a = \frac{\alpha}{|E_B|}, \qquad b = \frac{|\beta|}{|E_B|} = \frac{|M_B|}{\sqrt{2m\,|E_B|}}. \tag{3.80}$$

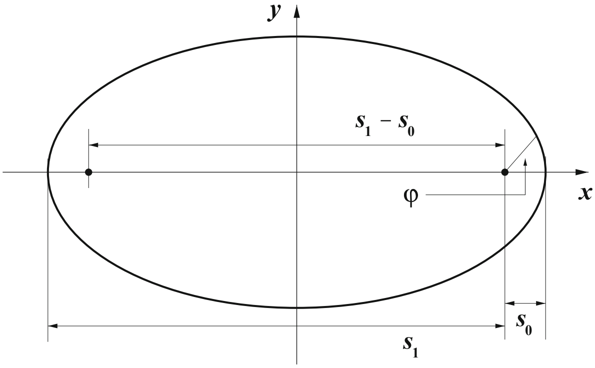


Fig. 3.9 The elliptical trajectory described by (3.79) with $\varphi_0 = 0$

As the particle's trajectory is two-dimensional, the problem has four constants of motion (Sect. 2.11.2); the total energy E_B and the angular momentum M_B are two of such constants. As shown by (3.80), the semimajor axis of the elliptical trajectory depends only on E_B; the area of the ellipse in terms of the constants of motion is $A = \pi\, a\, b = (\pi\, \alpha/\sqrt{2\, m})\, |M_B|\, |E_B|^{-3/2}$. The position vector $\mathbf{s}$ completes a full orbit in a period T given by (3.70); combining the latter with the expression of A yields

$$T = \frac{\pi\, \alpha\, \sqrt{2\, m}}{|E_B|^{3/2}} = \pi\, \sqrt{\frac{2\, m}{\alpha}}\, a^{3/2}, \tag{3.81}$$

namely, the period depends on the total energy, but not on the angular momentum. Thus, the period is still given by (3.81) in the limiting case $M_B \to 0$, which makes the trajectory to shrink into a segment of length a crossing the origin of the s, φ reference, and the position vector to oscillate along this segment (compare with problem 3.2). The second form of (3.81) shows that $T^2 \propto a^3$ (*Kepler's third law*).

3.13.7 Dynamic Relations of Special Relativity

The dynamic relations considered in this book refer almost invariably to situations where the particle velocity is small with respect to that of light. For this reason it is sufficient to use the non-relativistic relations. The only exception where the velocities of the particles involved do not allow for such an approximation is considered in Sects. 3.13.8 and 7.4.3. For this reason a set of relations of the Special-Relativity Theory are given here, that apply to the case of a free particle. The first of them is the relation between velocity $\mathbf{u}$ and momentum $\mathbf{p}$,

$$\mathbf{p} = \frac{m_0\, \mathbf{u}}{\sqrt{1 - u^2/c^2}}, \qquad u = |\mathbf{u}|, \tag{3.82}$$

with c the velocity of light and m_0 a constant mass. The second relation involves energy and velocity and reads

$$E = m\,c^2, \qquad m = \frac{m_0}{\sqrt{1 - u^2/c^2}}\,, \tag{3.83}$$

where E is a kinetic energy because a free particle is considered. In the above, $m = m(u)$ is called *relativistic mass* and $m_0 = m(0)$ is called *rest mass*. The latter is the mass measured in a reference where the particle is at rest, and is the value of the mass that is used in non-relativistic mechanics. From (3.82, 3.83) it follows

$$\mathbf{p} = m\,\mathbf{u}, \qquad m^2\,c^2 - m^2\,u^2 = m_0^2\,c^2, \qquad m^2\,c^2 = E^2/c^2, \tag{3.84}$$

whence the elimination of u provides the relation between E and the modulus of $\mathbf{p}$:

$$E^2/c^2 - p^2 = m_0^2\,c_2, \qquad p = \sqrt{E^2/c^2 - m_0^2\,c^2}. \tag{3.85}$$

In general the $p = p(E)$ relation is non linear. However, for particles with $m_0 = 0$ the expressions (3.85) simplify to the linear relation $p = E/c$. An example of this is found in the theory of the electromagnetic field, where the same momentum-energy relation is derived from the Maxwell equations (Sect. 5.7). It is worth observing that if a particle has $m_0 = 0$ and $p \neq 0$, its velocity is necessarily equal to c; in fact, the first of (3.84) yields $\lim_{m_0 \to 0} \mathbf{p} = 0$ when $u < c$. Another limiting case is found when $u/c \ll 1$. In fact, the second of (3.83) simplifies to

$$m \simeq \frac{m_0}{1 - u^2/(2\,c^2)} \simeq m_0 \left(1 + \frac{u^2}{2\,c^2}\right). \tag{3.86}$$

Inserting the last form into the first of (3.83) yields

$$E \simeq m_0\,c^2 + \frac{1}{2}\,m_0\,u^2. \tag{3.87}$$

The constant $m_0\,c^2$ is called *rest energy*. The limiting case $u/c \ll 1$ then renders for $E - m_0\,c^2$ the non-relativistic expression of the kinetic energy.

3.13.8 Collision of Relativistic Particles

This section illustrates the collision between two relativistic particles that constitute an isolated system. The same approach of Sect. 3.5 is used here, namely, the asymptotic values only are considered. Also, the case where the particles' trajectories belong to the same plane, specifically, the x, y plane, is investigated. The initial conditions are the same as in Sect. 3.6: the asymptotic motion of the first particle before the collision is parallel to the x axis, while the second particle is initially at rest. Finally, it is assumed that the rest mass of the first particle is zero, so that the momentum-energy relation of this particle is $p = E/c$ as shown in Sect. 3.13.7, while the rest mass of the second particle is $m_0 \neq 0$.

The collision is treated by combining the conservation laws of energy and momentum of the two-particle system. Let E_a, E_b be the asymptotic energies of the first particle before and after the collision, respectively. As for the second particle, which is initially at rest, the energy before the collision is its rest energy $m_0 c^2$, while that after the collision is $m\,c^2$ (Sect. 3.13.7). The conservation of energy then reads

$$E_a + m_0\,c^2 = E_b + mc^2, \tag{3.88}$$

while the conservation of momentum reads, respectively for the x and y components,

$$\frac{E_a}{c} = \frac{E_b}{c}\,\cos\psi + mu\,\cos\theta, \qquad 0 = \frac{E_b}{c}\,\sin\psi - mu\,\sin\theta. \tag{3.89}$$

The angles ψ and θ in (3.89) are the same as in Fig. 3.7. Extracting $m\,c^2$ from (3.88) and squaring both sides yields

$$(E_a - E_b)^2 + 2\,m_0\,c^2\,(E_a - E_b) = m^2\,c^4 - m_0^2\,c^4 = m^2\,u^2\,c^2, \tag{3.90}$$

where the last equality derives from the second of (3.84). Then, the momentum-conservation relations (3.89) are used to eliminate θ by squaring and adding up the results, to find

$$m^2\,u^2\,c^2 = E_a^2 + E_b^2 - 2\,E_a\,E_b\,\cos\psi = (E_a - E_b)^2 + 4\,E_a\,E_b\,\sin^2(\psi/2). \tag{3.91}$$

Eliminating $m^2\,u^2\,c^2 - (E_a - E_b)^2$ between (3.90) and (3.91) yields

$$\frac{1}{E_b} - \frac{1}{E_a} = \frac{2}{m_0\,c^2}\,\sin^2\left(\frac{\psi}{2}\right), \tag{3.92}$$

that provides the asymptotic energy after the collision of the first particle, as a function of the asymptotic energy before the collision, the deflection angle of the same particle, and the rest energy of the second particle. Equation (3.92) is used in Sect. 7.4.3 for the explanation of the Compton effect.

The non-relativistic analogue of the above procedure is illustrated in Sect. 3.6. It is interesting to note that the calculation carried out here seems rather less involved than the non-relativistic one. This surprising fact is actually due to the special choice of the first particle, whose rest energy is zero. In this case, in fact, the relation between momentum and energy becomes linear. That of the second particle, which is non linear, is eliminated from the equations. On the contrary, in the non-relativistic case treated in Sect. 3.6 the energy-momentum relations are non linear for both particles, this making the calculation more laborious.

3.13.9 Energy Conservation in Charged-Particles' Interaction

The two-particle interaction considered in Sect. 3.8 involves charged particles. As the particles' velocity during the interaction is not constant, the particles radiate (Sect. 5.11.2) and, consequently, lose energy. This phenomenon is not considered in the analysis carried out in Sect. 3.8, where the total energy of the two-particle system is assumed constant. The assumption is justified on the basis that the radiated power is relatively small. This subject is further discussed in Sect. 5.11.3.

Problems

3.1 Given the Hamiltonian function of the one-dimensional harmonic oscillator of the general form $H = p^2/(2\,m) + (c/s)\,|x|^s$, $m, c, s > 0$, find the oscillator's period.

3.2 Given the Hamiltonian function of the one-dimensional harmonic oscillator of the general form $H = p^2/(2\,m) - (k/s)\,|x|^{-s} = E < 0$, $m, k, s > 0$, find the oscillator's period.

3.3 Consider the collision between two particles in the repulsive Coulomb case. Calculate the relation $T_{1b}(T_{1a}, c)$, with c the impact parameter (hint: follow the discussion of Sect. 3.13.5 and use (3.23, 3.36), (3.32, 3.33), and (3.73)).