Chapter 5
Applications of the Concepts of Electromagnetism

5.1 Introduction

This chapter provides a number of important applications of the concepts of Electromagnetism. The solution of the wave equation found in Chap. 4 is used to calculate the potentials generated by a point-like charge; this result is exploited later to analyze the decay of atoms in the frame of the classical model, due to the radiated power. Next, the continuity equations for the energy and momentum of the electromagnetic field are found. As an application, the energy and momentum of the electromagnetic field are calculated in terms of modes in a finite domain, showing that the energy of each mode has the same expression as that of a linear harmonic oscillator. The analysis is extended also to an infinite domain. The chapter is concluded by the derivation of the eikonal equation, leading to the approximation of Geometrical Optics, followed by the demonstration that the eikonal equation is generated by a variational principle, namely, the Fermat principle. The complements show the derivation of the fields generated by a point-like charge and the power radiated by it. It is found that the planetary model of the atom is inconsistent with electromagnetism because it contradicts the atom's stability. Finally, a number of analogies are outlined and commented between Mechanics and Geometrical Optics, based on the comparison between the Maupertuis and Fermat principles. The course of reasoning deriving from the comparison hints at the possibility that mechanical laws more general than Newton's law exist.

5.2 Potentials Generated by a Point-Like Charge

The calculation of φ and $\mathbf{A}$ based upon (4.58, 4.59) has the inconvenience that, as $\mathbf{q}$ varies over the space, it is necessary to consider the sources ρ and $\mathbf{J}$ at different time instants. This may be avoided by recasting a in the form

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$$a(\mathbf{q}, t - |\mathbf{r} - \mathbf{q}|/c) = \int_{-\infty}^{+\infty} a(\mathbf{q}, t')\, \delta(t' - t + |\mathbf{r} - \mathbf{q}|/c)\, \mathrm{d}t', \tag{5.1}$$

and interchanging the integration over $\mathbf{q}$ in (4.58, 4.59) with that over t'. This procedure is particularly useful when the source of the field is a single point-like charge. Remembering (4.20), one replaces ρ and $\mathbf{J}$ with

$$\rho_c(\mathbf{q}, t') = e\, \delta\left(\mathbf{q} - \mathbf{s}(t')\right), \qquad \mathbf{J}_c(\mathbf{q}, t') = e\, \delta\left(\mathbf{q} - \mathbf{s}(t')\right)\, \mathbf{u}(t'), \tag{5.2}$$

where e is the value of the point-like charge, $\mathbf{s} = \mathbf{s}(t')$ its trajectory, and $\mathbf{u}(t') = \mathrm{d}\mathbf{s}/\mathrm{d}t'$ its velocity. First, the integration over space fixes $\mathbf{q}$ at $\mathbf{s}' = \mathbf{s}(t')$, this yielding

$$\varphi(\mathbf{r}, t) = \frac{e}{4\,\pi\,\varepsilon_0} \int_{-\infty}^{+\infty} \frac{\delta[\beta(t')]}{|\mathbf{r} - \mathbf{s}'|}\, \mathrm{d}t', \quad \mathbf{A}(\mathbf{r}, t) = \frac{e\,\mu_0}{4\pi} \int_{-\infty}^{+\infty} \frac{\delta[\beta(t')]\, \mathbf{u}(t')}{|\mathbf{r} - \mathbf{s}'|}\, \mathrm{d}t', \tag{5.3}$$

with $\beta(t') = t' - t + |\mathbf{r} - \mathbf{s}'|/c$. Next, the integration over t' fixes the latter to the value that makes the argument of δ to vanish. Such a value is the solution of

$$|\mathbf{r} - \mathbf{s}(t')| = c\,(t - t'), \tag{5.4}$$

where t, $\mathbf{r}$, and the function $\mathbf{s}(t')$ are prescribed. As $|\mathbf{u}| < c$ it can be shown that the solution of (5.4) exists and is unique [68, Sect. 63]. Observing that the argument of δ in (5.3) is a function of t', to complete the calculation one must follow the procedure depicted in Sect. C.5, which involves the derivative

$$\frac{\mathrm{d}\beta}{\mathrm{d}t'} = 1 + \frac{1}{c}\,\frac{\mathrm{d}|\mathbf{r} - \mathbf{s}'|}{\mathrm{d}t'} = 1 + \frac{\mathrm{d}[(\mathbf{r} - \mathbf{s}') \cdot (\mathbf{r} - \mathbf{s}')]}{2c\,|\mathbf{r} - \mathbf{s}'|\,\mathrm{d}t'} = 1 - \frac{\mathbf{r} - \mathbf{s}'}{|\mathbf{r} - \mathbf{s}'|}\,\frac{\mathbf{u}}{c}. \tag{5.5}$$

Then, letting $t' = \tau$ be the solution of (5.4) and $\dot{\beta} = (\mathrm{d}\beta/\mathrm{d}t')_{t'=\tau}$, one applies (C.57) to (5.3). The use of the absolute value is not necessary here, in fact one has $[(\mathbf{r} - \mathbf{s}')/|\mathbf{r} - \mathbf{s}'|] \cdot (\mathbf{u}/c) \le u/c < 1$, whence $|\dot{\beta}| = \dot{\beta}$. In conclusion one finds

$$\varphi(\mathbf{r}, t) = \frac{e/(4\,\pi\,\varepsilon_0)}{|\mathbf{r} - \mathbf{s}(\tau)| - (\mathbf{r} - \mathbf{s}(\tau)) \cdot \mathbf{u}(\tau)/c}, \tag{5.6}$$

$$\mathbf{A}(\mathbf{r}, t) = \frac{e\,\mu_0/(4\,\pi)\,\mathbf{u}(\tau)}{|\mathbf{r} - \mathbf{s}(\tau)| - (\mathbf{r} - \mathbf{s}(\tau)) \cdot \mathbf{u}(\tau)/c}, \tag{5.7}$$

that provide the potentials generated in $\mathbf{r}$ and at time t by a point-like charge that follows the trajectory $\mathbf{s} = \mathbf{s}(\tau)$. The relation between t and τ is given by $t = \tau + |\mathbf{r} - \mathbf{s}(\tau)|/c$, showing that $t - \tau$ is the time necessary for the electromagnetic perturbation produced by the point-like charge at $\mathbf{s}$ to reach the position $\mathbf{r}$. The expressions (5.6, 5.7) are called *Liénard and Wiechert potentials*. In the case $\mathbf{u} = 0$ they become

$$\varphi(\mathbf{r}) = \frac{e}{4\,\pi\,\varepsilon_0\,|\mathbf{r} - \mathbf{s}|}, \qquad \mathbf{A}(\mathbf{r}) = 0, \tag{5.8}$$

$\mathbf{s} = \text{const}$, the first of which is the Coulomb potential. The fields $\mathbf{E}$, $\mathbf{B}$ generated by a point-like charge are obtained from (5.6, 5.7) using (4.26). The calculation is outlined in Sect. 5.11.1.

5.3 Energy Continuity—Poynting Vector

The right hand side of (4.63) is recast in terms of the fields by replacing $\mathbf{J}$ with the left hand side of the second equation in (4.19); using $\mathbf{D} = \varepsilon_0\,\mathbf{E}$,

$$\frac{\partial w}{\partial t} = \mathbf{E}\cdot\left(\mathrm{rot}\mathbf{H} - \varepsilon_0\,\frac{\partial \mathbf{E}}{\partial t}\right) = \mathbf{E}\cdot\mathrm{rot}\mathbf{H} - \frac{\varepsilon_0}{2}\,\frac{\partial E^2}{\partial t}. \tag{5.9}$$

The above expression is given a more symmetric form by exploiting the first equation in (4.19). In fact, a scalar multiplication of the latter by $\mathbf{H}$ along with the relation $\mathbf{B} = \mu_0\,\mathbf{H}$ provides $0 = \mathbf{H}\cdot\mathrm{rot}\mathbf{E} + \mu_0\,\partial(H^2/2)/\partial t$ which, subtracted from (5.9), finally yields

$$\frac{\partial w}{\partial t} = \mathbf{E}\cdot\mathrm{rot}\mathbf{H} - \mathbf{H}\cdot\mathrm{rot}\mathbf{E} - \frac{\partial w_{\mathrm{em}}}{\partial t}, \qquad w_{\mathrm{em}} = \frac{1}{2}\,\left(\varepsilon_0\,E^2 + \mu_0\,H^2\right). \tag{5.10}$$

Then, using the second identity in (A.36) transforms (5.10) into

$$\frac{\partial}{\partial t}\,(w + w_{\mathrm{em}}) + \mathrm{div}\mathbf{S} = 0, \qquad \mathbf{S} = \mathbf{E}\wedge\mathbf{H}. \tag{5.11}$$

As $w + w_{\mathrm{em}}$ is an energy density, $\mathbf{S}$ (called *Poynting vector*) is an energy-flux density ($[\mathbf{S}] = \mathrm{J\ m^{-2}\ s^{-1}}$). To give w_{em} and $\mathbf{S}$ a physical meaning one notes that (5.11) has the form of a continuity equation (compare, e.g., with (23.3) and (4.23)) where two interacting systems are involved, namely, the charges and the electromagnetic field. Integrating (5.11) over a volume V yields

$$\frac{\mathrm{d}}{\mathrm{d}t}(W + W_{\mathrm{em}}) = -\int_\Sigma \mathbf{S}\cdot\mathbf{n}\,\mathrm{d}\Sigma, \qquad W = \int_V w\,\mathrm{d}V, \quad W_{\mathrm{em}} = \int_V w_{\mathrm{em}}\,\mathrm{d}V, \tag{5.12}$$

where Σ is the boundary of V, $\mathbf{n}$ is the unit vector normal to $\mathrm{d}\Sigma$ oriented in the outward direction with respect to V, and W, W_{em} are energies. If V is let expand to occupy all space, the surface integral in (5.12) vanishes because the fields $\mathbf{E}$, $\mathbf{H}$ vanish at infinity; it follows that for an infinite domain the sum $W + W_{\mathrm{em}}$ is conserved in time, so that $\mathrm{d}W_{\mathrm{em}}/\mathrm{d}t = -\mathrm{d}W/\mathrm{d}t$. Observing that W is the kinetic energy of the charges, and that the latter exchange energy with the electromagnetic field, gives W_{em} the meaning of energy of the electromagnetic field; as a consequence, w_{em} is the energy density of the electromagnetic field, and the sum $W + W_{\mathrm{em}}$ is the constant energy of the two interacting systems.

When V is finite, the surface integral in (5.12) may be different from zero, hence the sum $W + W_{\mathrm{em}}$ is not necessarily conserved. This allows one to give the surface integral the meaning of energy per unit time that crosses the boundary Σ, carried by the electromagnetic field. In this reasoning it is implied that, when V is finite, it is chosen in such a way that no charge is on the boundary at time t. Otherwise the kinetic energy of the charges crossing Σ during $\mathrm{d}t$ should also be accounted for.

5.4 Momentum Continuity

The procedure used in Sect. 5.3 to derive the continuity equation for the charge energy can be replicated to obtain the continuity equation for the charge momentum per unit volume, $\mathbf{m}$. Remembering (4.61) one finds the relation $\mathbf{f} = \dot{\mathbf{m}} = \sum_j' \dot{\mathbf{p}}_j/\Delta V$, with $\mathbf{p}_j$ the momentum of the jth charge contained within ΔV. Using (4.19) along with $\mathbf{J} = \rho\,\mathbf{v}$ yields

$$\dot{\mathbf{m}} = \rho\,\mathbf{E} + \mathbf{J} \wedge \mathbf{B} = \mathbf{E}\,\mathrm{div}\mathbf{D} + \left(\mathrm{rot}\mathbf{H} - \frac{\partial \mathbf{D}}{\partial t}\right) \wedge \mathbf{B}. \tag{5.13}$$

Adding $\mathbf{D} \wedge \partial\mathbf{B}/\partial t$ to both sides of (5.13), using $\partial\mathbf{B}/\partial t = -\mathrm{rot}\mathbf{E}$, and rearranging:

$$\dot{\mathbf{m}} + \frac{\partial \mathbf{D}}{\partial t} \wedge \mathbf{B} + \mathbf{D} \wedge \frac{\partial \mathbf{B}}{\partial t} = \mathbf{E}\,\mathrm{div}\mathbf{D} + (\mathrm{rot}\mathbf{H}) \wedge \mathbf{B} + (\mathrm{rot}\mathbf{E}) \wedge \mathbf{D}. \tag{5.14}$$

Poynting vector's definition (5.11) transforms the left hand side of (5.14) into $\dot{\mathbf{m}} + \varepsilon_0\,\mu_0\,\partial(\mathbf{E} \wedge \mathbf{H})/\partial t = \partial(\mathbf{m} + \mathbf{S}/c^2)/\partial t$. In turn, the kth component of $\mathbf{E}\,\mathrm{div}\mathbf{D} + (\mathrm{rot}\mathbf{E}) \wedge \mathbf{D}$ can be recast as

$$\varepsilon_0\,(E_k\,\mathrm{div}\mathbf{E} + \mathbf{E} \cdot \mathrm{grad}E_k) - \frac{\varepsilon_0}{2}\,\frac{\partial E^2}{\partial x_k} = \varepsilon_0\,\mathrm{div}(E_k\,\mathbf{E}) - \frac{\varepsilon_0}{2}\,\frac{\partial E^2}{\partial x_k}. \tag{5.15}$$

Remembering that $\mathrm{div}\mathbf{B} = 0$, the kth component of the term $(\mathrm{rot}\mathbf{H}) \wedge \mathbf{B} = \mathbf{H}\,\mathrm{div}\mathbf{B} + (\mathrm{rot}\mathbf{H}) \wedge \mathbf{B}$ is treated in the same manner. Adding up the contributions of the electric and magnetic parts and using the definition (5.10) of w_{em} yields

$$\frac{\partial}{\partial t}\left(m_k + \frac{1}{c^2}\,S_k\right) + \mathrm{div}\mathbf{T}_k = 0, \qquad \mathbf{T}_k = w_{\mathrm{em}}\,\mathbf{i}_k - \varepsilon_0\,E_k\,\mathbf{E} - \mu_0\,H_k\mathbf{H}, \tag{5.16}$$

with $\mathbf{i}_k$ the unit vector of the kth axis. As $m_k + S_k/c^2$ is a momentum density, $\mathbf{T}_k$ is a momentum-flux density ($[\mathbf{T}_k] = \mathrm{J\,m^{-3}}$). Following the same reasoning as in Sect. 5.3 one integrates (5.16) over a volume V, to find

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_V \left(m_k + \frac{1}{c^2}\,S_k\right)\,\mathrm{d}V = -\int_\Sigma \mathbf{T}_k \cdot \mathbf{n}\,\mathrm{d}\Sigma, \tag{5.17}$$

where Σ and $\mathbf{n}$ are defined as in (5.12). If V is let expand to occupy all space, the surface integral in (5.17) vanishes because the fields $\mathbf{E}$, $\mathbf{H}$ vanish at infinity; it follows that for an infinite domain the sum

$$\int_V \left(\mathbf{m} + \frac{1}{c^2}\,\mathbf{S}\right)\,\mathrm{d}V = \mathbf{p} + \int_V \frac{1}{c^2}\,\mathbf{S}\,\mathrm{d}V, \qquad \mathbf{p} = \int_V \mathbf{m}\,\mathrm{d}V \tag{5.18}$$

is conserved in time. As $\mathbf{p}$ is the momentum of the charges, $\int_V \mathbf{S}/c^2\,\mathrm{d}^3r$ takes the meaning of momentum of the electromagnetic field within V. As a consequence, $\mathbf{S}/c^2$ takes the meaning of momentum per unit volume of the electromagnetic field.

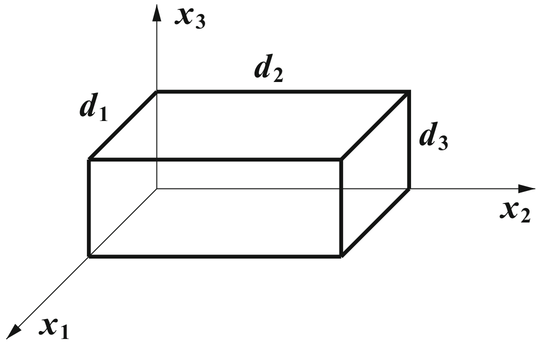

Fig. 5.1 The domain used for the expansion of the vector potential into a Fourier series (Sect. 5.5)

When V is finite, the surface integral in (5.17) may be different from zero, hence the sum (5.18) is not necessarily conserved. This allows one to give the surface integral in (5.17) the meaning of momentum per unit time that crosses the boundary Σ, carried by the electromagnetic field. In this reasoning it is implied that, when V is finite, it is chosen in such a way that no charge is on the boundary at time t. Otherwise the momentum of the charges crossing Σ during $\mathrm{d}t$ should also be accounted for.

5.5 Modes of the Electromagnetic Field

The expressions of the energy and momentum of the electromagnetic field, worked out in Sects. 5.3 and 5.4, take a particularly interesting form when a spatial region free of charges is considered. In fact, if one lets $\rho = 0$, $\mathbf{J} = 0$, equations (4.33) or (4.35) that provide the potentials become homogeneous. To proceed one takes a finite region of volume V; the calculation will be extended in Sect. 5.8 to the case of an infinite domain. As the shape of V is not essential for the considerations illustrated here, it is chosen as that of a box whose sides d_1, d_2, d_3 are aligned with the coordinate axes and start from the origin (Fig. 5.1). The volume of the box is $V = d_1\, d_2\, d_3$.

The calculation is based on (4.33), that are the equations for the potentials deriving from the Coulomb gauge (4.32). Letting $\rho = 0$, $\mathbf{J} = 0$ and dropping the primes yields

$$\nabla^2 \varphi = 0, \qquad \nabla^2 \mathbf{A} - \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} = \frac{1}{c^2}\frac{\partial}{\partial t}\operatorname{grad}\varphi, \tag{5.19}$$

the first of which is a Laplace equation. It is shown in Sect. 5.11.4 that a gauge transformation exists such that $\varphi = 0$ here. The system (5.19) then reduces to the linear, homogeneous wave equation for the vector potential $\mathbf{A} = \mathbf{A}(\mathbf{r}, t)$,

$$\nabla^2 \mathbf{A} - \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} = 0. \tag{5.20}$$

As the vector potential is defined within a finite volume and has a finite module as well, one can expand it into the Fourier series

$$\mathbf{A} = \sum_{\mathbf{k}} \mathbf{a}_{\mathbf{k}} \exp(\mathrm{i}\,\mathbf{k}\cdot\mathbf{r}), \qquad \mathbf{a}_{\mathbf{k}} = \frac{1}{V}\int_V \mathbf{A}\exp(-\mathrm{i}\,\mathbf{k}\cdot\mathbf{r})\,\mathrm{d}V, \tag{5.21}$$

where $\mathbf{a}_{\mathbf{k}} = \mathbf{a}(\mathbf{k}, t)$ is complex and the *wave vector* $\mathbf{k}$ is given by

$$\mathbf{k} = n_1\frac{2\pi}{d_1}\mathbf{i}_1 + n_2\frac{2\pi}{d_2}\mathbf{i}_2 + n_3\frac{2\pi}{d_3}\mathbf{i}_3, \qquad n_i = 0, \pm 1, \pm 2, \ldots \tag{5.22}$$

The symbol $\sum_{\mathbf{k}}$ indicates a triple sum over all integers n_1, n_2, n_3. The definition of $\mathbf{a}_{\mathbf{k}}$ yields

$$\mathbf{a}_{-\mathbf{k}} = \mathbf{a}^*_{\mathbf{k}}, \qquad \mathbf{a}_0 = \frac{1}{V}\int_V \mathbf{A}\,\mathrm{d}V, \tag{5.23}$$

with $\mathbf{a}_0$ real. Applying Coulomb's gauge div$\mathbf{A} = 0$ to the expansion (5.21) provides

$$\mathrm{div}\mathbf{A} = \sum_{\mathbf{k}}\sum_{m=1}^{3} a_{\mathbf{k}m}\,\mathrm{i}\,k_m \exp(\mathrm{i}\,\mathbf{k}\cdot\mathbf{r}) = \mathrm{i}\sum_{k}\mathbf{a}_{\mathbf{k}}\cdot\mathbf{k}\exp(\mathrm{i}\,\mathbf{k}\cdot\mathbf{r}) = 0, \tag{5.24}$$

that is, a linear combination of functions of $\mathbf{r}$. As such functions are linearly independent from each other, (5.24) vanishes only if the coefficients vanish, so it is $\mathbf{a}_{\mathbf{k}}\cdot\mathbf{k} = 0$. Replacing $\mathbf{k}$ with $-\mathbf{k}$ and using (5.23) shows that $\mathbf{a}_{-\mathbf{k}}\cdot\mathbf{k} = \mathbf{a}^*_{\mathbf{k}}\cdot\mathbf{k} = 0$. In conclusion, $\mathbf{a}_{\mathbf{k}}$ has no components in the direction of $\mathbf{k}$, namely, it has only two independent (complex) components that lie on the plane normal to $\mathbf{k}$: letting $\mathbf{e}_1$, $\mathbf{e}_2$ be unit vectors belonging to such a plane and normal to each other, one has

$$\mathbf{a}_{\mathbf{k}} = \mathbf{a}_{\mathbf{k}}\cdot\mathbf{e}_1\,\mathbf{e}_1 + \mathbf{a}_{\mathbf{k}}\cdot\mathbf{e}_2\,\mathbf{e}_2. \tag{5.25}$$

Clearly the reasoning above does not apply to $\mathbf{a}_0$; however, it is shown below that eventually this term does not contribute to the fields. The Fourier series (5.21) is now inserted into the wave equation (5.20), whose two summands become

$$\nabla^2\mathbf{A} = \sum_{\mathbf{k}}\mathbf{a}_{\mathbf{k}}\sum_{m=1}^{3}(\mathrm{i}\,k_m)^2\exp(\mathrm{i}\,\mathbf{k}\cdot\mathbf{r}) = -\sum_{\mathbf{k}}\mathbf{a}_{\mathbf{k}}\,k^2\exp(\mathrm{i}\mathbf{k}\cdot\mathbf{r}), \tag{5.26}$$

$$-\frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2} = -\frac{1}{c^2}\sum_{\mathbf{k}}\ddot{\mathbf{a}}_{\mathbf{k}}\exp(\mathrm{i}\,\mathbf{k}\cdot\mathbf{r}). \tag{5.27}$$

Adding up yields $\sum_{\mathbf{k}}\left(\ddot{\mathbf{a}}_{\mathbf{k}} + c^2 k^2\,\mathbf{a}_{\mathbf{k}}\right)\exp(\mathrm{i}\,\mathbf{k}\cdot\mathbf{r}) = 0$ whence, using the same reasoning as that used for discussing (5.24),

$$\ddot{\mathbf{a}}_{\mathbf{k}} + \omega^2\,\mathbf{a}_{\mathbf{k}} = 0, \qquad \omega(\mathbf{k}) = c\,k \geq 0, \qquad \omega(-\mathbf{k}) = \omega(\mathbf{k}). \tag{5.28}$$

The case $\mathbf{k} = 0$ yields $\ddot{\mathbf{a}}_0 = 0$ whence $\mathbf{a}_0(t) = \mathbf{a}_0(t = 0) + \dot{\mathbf{a}}_0(t = 0)\, t$. The constant $\dot{\mathbf{a}}_0(t = 0)$ must be set to zero to prevent $\mathbf{a}_0$ from diverging. When $\mathbf{k} \neq 0$ the solution of (5.28) is readily found to be $\mathbf{a}_\mathbf{k}(t) = \mathbf{c}_\mathbf{k} \exp(-\mathrm{i}\,\omega t) + \mathbf{c}'_\mathbf{k} \exp(\mathrm{i}\,\omega t)$, where the complex vectors $\mathbf{c}_\mathbf{k}$, $\mathbf{c}'_\mathbf{k}$ depend on $\mathbf{k}$ only and lie on the plane normal to it. Using the first relation in (5.23) yields $\mathbf{c}'_\mathbf{k} = \mathbf{c}^*_{-\mathbf{k}}$ and, finally,

$$\mathbf{a}_\mathbf{k} = \mathbf{s}_\mathbf{k} + \mathbf{s}^*_{-\mathbf{k}}, \qquad \mathbf{s}_\mathbf{k}(t) = \mathbf{c}_\mathbf{k} \exp(-\mathrm{i}\,\omega\, t), \qquad \mathbf{k} \neq 0. \tag{5.29}$$

Thanks to (5.29) one reconstructs the vector potential $\mathbf{A}$ in a form that shows its dependence on space and time explicitly. To this purpose one notes that the sum (5.21) contains all possible combinations of indices n_1, n_2, n_3, so that a summand corresponding to $\mathbf{k}$ is paired with another summand corresponding to $-\mathbf{k}$. One can then rearrange (5.21) as $\mathbf{A} = (1/2) \sum_\mathbf{k} \left[\mathbf{a}_\mathbf{k} \exp(\mathrm{i}\,\mathbf{k}\cdot\mathbf{r}) + \mathbf{a}_{-\mathbf{k}} \exp(-\mathrm{i}\,\mathbf{k}\cdot\mathbf{r})\right]$, where the factor $1/2$ is introduced to eliminate a double counting. Using (5.29), and remembering from (5.28) that $\omega(\mathbf{k})$ is even, renders $\mathbf{A}$ as a sum of real terms,

$$\mathbf{A} = \sum_\mathbf{k} \Re \left\{\mathbf{c}_\mathbf{k} \exp[\mathrm{i}\,(\mathbf{k}\cdot\mathbf{r} - \omega t)] + \mathbf{c}^*_{-\mathbf{k}} \exp[\mathrm{i}\,(\mathbf{k}\cdot\mathbf{r} + \omega t)]\right\}, \tag{5.30}$$

The summands of (5.30) corresponding to $\mathbf{k}$ and $-\mathbf{k}$ describe two plane and monochromatic waves that propagate in the $\mathbf{k}$ and $-\mathbf{k}$ direction, respectively. The two waves together form a *mode* of the electromagnetic field, whose angular frequency is $\omega = c\,k$. The summands corresponding to $\mathbf{k} = 0$ yield the real constant $\mathbf{c}_0 + \mathbf{c}^*_0 = \mathbf{a}_0$. Finally, the $\mathbf{E}$ and $\mathbf{B}$ fields are found by introducing the expansion of $\mathbf{A}$ into (4.26) after letting $\varphi = 0$. For this calculation it is convenient to use the form of the expansion bearing the factor $1/2$ introduced above: from the definition (5.29) of $\mathbf{s}_\mathbf{k}$ and the first identity in (A.35) one finds

$$\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} = \frac{1}{2} \sum_\mathbf{k} \mathrm{i}\,\omega \left[\left(\mathbf{s}_\mathbf{k} - \mathbf{s}^*_{-\mathbf{k}}\right) \exp(\mathrm{i}\,\mathbf{k}\cdot\mathbf{r}) + \left(\mathbf{s}_{-\mathbf{k}} - \mathbf{s}^*_\mathbf{k}\right) \exp(-\mathrm{i}\,\mathbf{k}\cdot\mathbf{r})\right], \tag{5.31}$$

$$\mathbf{B} = \mathrm{rot}\mathbf{A} = \frac{1}{2} \sum_\mathbf{k} \mathrm{i}\,\mathbf{k} \wedge \left[\left(\mathbf{s}_\mathbf{k} + \mathbf{s}^*_{-\mathbf{k}}\right) \exp(\mathrm{i}\,\mathbf{k}\cdot\mathbf{r}) - \left(\mathbf{s}_{-\mathbf{k}} + \mathbf{s}^*_\mathbf{k}\right) \exp(-\mathrm{i}\,\mathbf{k}\cdot\mathbf{r})\right]. \tag{5.32}$$

As anticipated, the constant term $\mathbf{a}_0$ does not contribute to the fields. Also, due to the second relation in (5.29), the vectors $\mathbf{s}_\mathbf{k}$, $\mathbf{s}^*_{-\mathbf{k}}$, $\mathbf{s}_{-\mathbf{k}}$, and $\mathbf{s}^*_\mathbf{k}$ lie over the plane normal to $\mathbf{k}$. Due to (5.31, 5.32) the $\mathbf{E}$ and $\mathbf{B}$ fields lie on the same plane as well, namely, they have no component in the propagation direction. For this reason they are called *transversal*.

5.6 Energy of the Electromagnetic Field in Terms of Modes

The expressions of the $\mathbf{E}$, $\mathbf{B}$ fields within a finite volume V free of charges have been calculated in Sect. 5.5 as superpositions of modes, each of them associated with a wave vector $\mathbf{k}$ and an angular frequency $\omega = ck$. Basing upon such expressions one is able to determine the electromagnetic energy within V in terms of modes. To this purpose one calculates from (5.31, 5.32) the squares $E^2 = \mathbf{E} \cdot \mathbf{E}$ and $B^2 = \mathbf{B} \cdot \mathbf{B}$, inserts the resulting expression into the second relation of (5.10) to obtain the energy per unit volume and, finally, integrates the latter over V (last relation in (5.12)). Letting $\mathbf{I_k}$ be the quantity enclosed within brackets in (5.31), it is $E^2 = -(1/4) \sum_{\mathbf{k}} \sum_{\mathbf{k}'} \omega\, \omega' \,\mathbf{I_k} \cdot \mathbf{I_{k'}}$, where $\omega' = ck'$. The integration over V avails itself of the integrals (C.121), to yield

$$-\frac{1}{4} \sum_{\mathbf{k}'} \omega\, \omega' \int_V \mathbf{I_k} \cdot \mathbf{I_{k'}} \,\mathrm{d}V = V \,\omega^2 \left(\mathbf{s_k} - \mathbf{s}^*_{-\mathbf{k}}\right) \cdot \left(\mathbf{s}^*_{\mathbf{k}} - \mathbf{s}_{-\mathbf{k}}\right), \tag{5.33}$$

so that the part of the electromagnetic energy deriving from $\mathbf{E}$ reads

$$\int_V \frac{\varepsilon_0}{2} E^2 \,\mathrm{d}V = \frac{\varepsilon_0}{2} V \sum_{\mathbf{k}} \omega^2 \left(\mathbf{s_k} - \mathbf{s}^*_{-\mathbf{k}}\right) \cdot \left(\mathbf{s}^*_{\mathbf{k}} - \mathbf{s}_{-\mathbf{k}}\right). \tag{5.34}$$

By the same token one lets $\mathbf{Y_k}$ be the quantity enclosed within brackets in (5.32), whence $B^2 = -(1/4) \sum_{\mathbf{k}} \sum_{\mathbf{k}'} (\mathbf{k} \wedge \mathbf{Y_k}) \cdot (\mathbf{k}' \wedge \mathbf{Y_{k'}})$ and

$$-\frac{1}{4} \sum_{\mathbf{k}'} \int_V (\mathbf{k} \wedge \mathbf{Y_k}) \cdot (\mathbf{k}' \wedge \mathbf{Y_{k'}}) \,\mathrm{d}V = V \left[\mathbf{k} \wedge \left(\mathbf{s_k} + \mathbf{s}^*_{-\mathbf{k}}\right)\right] \cdot \left[\mathbf{k} \wedge \left(\mathbf{s}_{-\mathbf{k}} + \mathbf{s}^*_{\mathbf{k}}\right)\right]. \tag{5.35}$$

The expression at the right hand side of (5.35) simplifies because, due to (5.29), $\mathbf{k}$ is normal to the plane where $\mathbf{s_k}$, $\mathbf{s}^*_{-\mathbf{k}}$, $\mathbf{s}_{-\mathbf{k}}$, and $\mathbf{s}^*_{\mathbf{k}}$ lie, so that $[\mathbf{k} \wedge (\mathbf{s_k} + \mathbf{s}^*_{-\mathbf{k}})] \cdot [\mathbf{k} \wedge (\mathbf{s}_{-\mathbf{k}} + \mathbf{s}^*_{\mathbf{k}})] = k^2 (\mathbf{s_k} + \mathbf{s}^*_{-\mathbf{k}}) \cdot (\mathbf{s}_{-\mathbf{k}} + \mathbf{s}^*_{\mathbf{k}})$. Using the relation $k^2 = \omega^2/c^2 = \varepsilon_0 \,\mu_0 \,\omega^2$ yields the part of the electromagnetic energy deriving from $\mathbf{H} = \mathbf{B}/\mu_0$,

$$\int_V \frac{1}{2\,\mu_0} B^2 \,\mathrm{d}V = \frac{\varepsilon_0}{2} V \sum_{\mathbf{k}} \omega^2 (\mathbf{s_k} + \mathbf{s}^*_{-\mathbf{k}}) \cdot (\mathbf{s}_{-\mathbf{k}} + \mathbf{s}^*_{\mathbf{k}}). \tag{5.36}$$

Adding up (5.34) and (5.36) one finally obtains

$$W_{\mathrm{em}} = \varepsilon_0 \,V \sum_{\mathbf{k}} \omega^2 \left(\mathbf{s_k} \cdot \mathbf{s}^*_{\mathbf{k}} + \mathbf{s}_{-\mathbf{k}} \cdot \mathbf{s}^*_{-\mathbf{k}}\right) = 2\,\varepsilon_0 \,V \sum_{\mathbf{k}} \omega^2 \,\mathbf{s_k} \cdot \mathbf{s}^*_{\mathbf{k}}. \tag{5.37}$$

This result shows that the energy of the electromagnetic field within V is the sum of individual contributions, each associated to a wave vector $\mathbf{k}$ through the complex vector $\mathbf{s_k}$. As the latter lies on the plane normal to $\mathbf{k}$, it is expressed in terms of two scalar components as $\mathbf{s_k} = s_{\mathbf{k}1}\,\mathbf{e}_1 + s_{\mathbf{k}2}\,\mathbf{e}_2$. Such components are related to the polarization of the electromagnetic field over the plane [9, Sect. 1.4.2]. These considerations allow one to count the number of indices that are involved in the representation (5.37) of W_{em}: in fact, the set of $\mathbf{k}$ vectors is described by the triple infinity of indices n_1, n_2, $n_3 \in \mathbf{Z}$ that appear in (5.22), while the two scalar components require another index $\sigma = 1, 2$. The $\mathbf{s_k}$ vectors describe the electromagnetic field through (5.31) and (5.32), hence one may think of each scalar component $s_{\mathbf{k}\sigma}$ as a degree of freedom of the field; the counting outlined above shows that the number of degrees of freedom is $2 \times \mathbf{Z}^3$. In turn, each degree of freedom is made of a real and an imaginary part, $s_{\mathbf{k}\sigma} = R_{\mathbf{k}\sigma} + \mathrm{i}\, I_{\mathbf{k}\sigma}$, this yielding

$$W_{\mathrm{em}} = \sum_{\mathbf{k}\sigma} W_{\mathbf{k}\sigma}, \qquad W_{\mathbf{k}\sigma} = 2\,\varepsilon_0\, V\, \omega^2\, \left(R_{\mathbf{k}\sigma}^2 + I_{\mathbf{k}\sigma}^2\right). \tag{5.38}$$

As $\omega = ck$, the mode with $\mathbf{k} = 0$ does not contribute to the energy. In (5.38) it is $R_{\mathbf{k}\sigma} = |c_{\mathbf{k}\sigma}| \cos[\mathrm{i}\,\omega\,(t_0 - t)]$, $I_{\mathbf{k}\sigma} = |c_{\mathbf{k}\sigma}| \sin[\mathrm{i}\,\omega\,(t_0 - t)]$, where the polar form $|c_{\mathbf{k}\sigma}| \exp(\mathrm{i}\,\omega\, t_0)$ has been used for $c_{\mathbf{k}\sigma}$. One notes that each summand in (5.38) is related to a single degree of freedom and has a form similar to the Hamiltonian function of the linear harmonic oscillator discussed in Sect. 3.3. To further pursue the analogy one defines the new pair

$$q_{\mathbf{k}\sigma}(t) = 2\sqrt{\varepsilon_0\, V}\, R_{\mathbf{k}\sigma}, \qquad p_{\mathbf{k}\sigma}(t) = 2\,\omega\sqrt{\varepsilon_0\, V}\, I_{\mathbf{k}\sigma}, \tag{5.39}$$

whence

$$W_{\mathbf{k}\sigma} = \frac{1}{2}\left(p_{\mathbf{k}\sigma}^2 + \omega^2\, q_{\mathbf{k}\sigma}^2\right), \qquad \frac{\partial W_{\mathbf{k}\sigma}}{\partial p_{\mathbf{k}\sigma}} = p_{\mathbf{k}\sigma}, \qquad \frac{\partial W_{\mathbf{k}\sigma}}{\partial q_{\mathbf{k}\sigma}} = \omega^2\, q_{\mathbf{k}\sigma}. \tag{5.40}$$

On the other hand, the time dependence of $R_{\mathbf{k}\sigma}$, $I_{\mathbf{k}\sigma}$ is such that

$$\dot{q}_{\mathbf{k}\sigma} = p_{\mathbf{k}\sigma} = \frac{\partial W_{\mathbf{k}\sigma}}{\partial p_{\mathbf{k}\sigma}}, \qquad \dot{p}_{\mathbf{k}\sigma} = -\omega^2\, q_{\mathbf{k}\sigma} = -\frac{\partial W_{\mathbf{k}\sigma}}{\partial q_{\mathbf{k}\sigma}}. \tag{5.41}$$

Comparing (5.41) with (1.42) shows that $q_{\mathbf{k}\sigma}$, $p_{\mathbf{k}\sigma}$ are canonically-conjugate variables and $W_{\mathbf{k}\sigma}$ is the Hamiltonian function of the degree of freedom associated to $\mathbf{k}\sigma$. Then, comparing (5.40) with (3.1, 3.2) shows that $W_{\mathbf{k}\sigma}$ is indeed the Hamiltonian function of a linear harmonic oscillator of unit mass.

The energy associated to each degree of freedom is constant in time. In fact, from the second relation in (5.29) one derives $W_{\mathbf{k}\sigma} = 2\,\varepsilon_0\, V\,\omega^2\, \mathbf{c}_{\mathbf{k}\sigma} \cdot \mathbf{c}_{\mathbf{k}\sigma}^*$. The same result can be obtained from the properties of the linear harmonic oscillator (Sect. 3.3). It follows that the total energy W_{em} is conserved. As shown in Sect. 5.11.4 this is due to the periodicity of the Poynting vector (5.11): in fact, the electromagnetic energies the cross per unit time two opposite faces of the boundary of V are the negative of each other.

5.7 Momentum of the Electromagnetic Field in Terms of Modes

It has been shown in Sect. 5.4 that the momentum per unit volume of the electromagnetic field is $\mathbf{S}/c^2 = \mathbf{E} \wedge \mathbf{B}/(\mu_0\, c^2) = \varepsilon_0 \mathbf{E} \wedge \mathbf{B}$. Using the symbols defined in Sect. 5.6 one finds $\varepsilon_0 \mathbf{E} \wedge \mathbf{B} = -(\varepsilon_0/4) \sum_{\mathbf{k}} \sum_{\mathbf{k}'} \omega\, \mathbf{I}_{\mathbf{k}} \wedge (\mathbf{k}' \wedge \mathbf{Y}_{\mathbf{k}'})$ and

$$-\frac{\varepsilon_0}{4} \sum_{\mathbf{k}'} \int_V \omega\, \mathbf{I}_{\mathbf{k}} \wedge (\mathbf{k}' \wedge \mathbf{Y}_{\mathbf{k}'})\, \mathrm{d}V = \frac{\varepsilon_0}{2}\, \omega\, V\, \left(\mathbf{Z}_{\mathbf{k}} + \mathbf{Z}_{-\mathbf{k}}\right), \tag{5.42}$$

with $\mathbf{Z_k} = (\mathbf{s_k} - \mathbf{s}^*_{-\mathbf{k}}) \wedge [\mathbf{k} \wedge (\mathbf{s}_{-\mathbf{k}} + \mathbf{s}^*_{\mathbf{k}})]$. The expression of $\mathbf{Z_k}$ simplifies because, due to (5.29), $\mathbf{k}$ is normal to the plane where $\mathbf{s_k}$, $\mathbf{s}^*_{-\mathbf{k}}$, $\mathbf{s}_{-\mathbf{k}}$, and $\mathbf{s}^*_{\mathbf{k}}$ lie, so that $\mathbf{Z_k} = \mathbf{k}\,(\mathbf{s_k} - \mathbf{s}^*_{-\mathbf{k}}) \cdot (\mathbf{s}_{-\mathbf{k}} + \mathbf{s}^*_{\mathbf{k}})$ and $\mathbf{Z_k} + \mathbf{Z}_{-\mathbf{k}} = 2\,\mathbf{k}\,\mathbf{s_k} \cdot \mathbf{s}^*_{\mathbf{k}} + 2\,(-\mathbf{k})\,\mathbf{s}_{-\mathbf{k}} \cdot \mathbf{s}^*_{-\mathbf{k}}$. In conclusion, observing that $\omega\,\mathbf{k} = (\omega^2/c)\,\mathbf{k}/k$,

$$\int_V \frac{\mathbf{S}}{c^2}\,\mathrm{d}V = 2\,\varepsilon_0\,V \sum_{\mathbf{k}} \omega\,\mathbf{s_k} \cdot \mathbf{s}^*_{\mathbf{k}}\,\mathbf{k} = \sum_{\mathbf{k}\sigma} \frac{1}{c}\,W_{\mathbf{k}\sigma}\,\frac{\mathbf{k}}{k} = \frac{1}{2c} \sum_{\mathbf{k}\sigma} \left(p^2_{\mathbf{k}\sigma} + \omega^2 q^2_{\mathbf{k}\sigma}\right) \frac{\mathbf{k}}{k}, \tag{5.43}$$

where the last two equalities derive from (5.37, 5.38, 5.40). One notes from (5.43) that the momentum of the electromagnetic field is the sum of individual momenta, each related to a single degree of freedom. The modulus of the individual momentum is equal to the energy $W_{\mathbf{k}\sigma}$ pertaining to the same degree of freedom divided by c. The same relation between momentum and energy has been derived in Sect. 3.13.7 with reference to the dynamic relations of Special Relativity. Each summand in (5.43) is constant in time, so the electromagnetic momentum is conserved; as noted in Sects. 5.6, 5.11.4, this is due to the periodicity of the Poynting vector.

5.8 Modes of the Electromagnetic Field in an Infinite Domain

The treatment of Sects. 5.5, 5.6, 5.7 is extended to the case of an infinite domain by means of the Fourier transform (Sect. C.2)[1]

$$\mathbf{A} = \iiint_{-\infty}^{+\infty} \mathbf{b_k}\,\frac{\exp(\mathrm{i}\,\mathbf{k}\cdot\mathbf{r})}{(2\,\pi)^{3/2}}\,\mathrm{d}^3 k, \qquad \mathbf{b_k} = \iiint_{-\infty}^{+\infty} \mathbf{A}\,\frac{\exp(-\,\mathrm{i}\,\mathbf{k}\cdot\mathbf{r})}{(2\,\pi)^{3/2}}\,\mathrm{d}^3 r. \tag{5.44}$$

where $\mathbf{b_k} = \mathbf{b}(\mathbf{k}, t)$ is complex, with $\mathbf{b}_{-\mathbf{k}} = \mathbf{b}^*_{\mathbf{k}}$, and the components of the wave vector $\mathbf{k}$ are continuous. Relations of the same form as (5.29) hold for $\mathbf{b_k}$, yielding

$$\mathbf{b_k} = \tilde{\mathbf{s}}_{\mathbf{k}} + \tilde{\mathbf{s}}^*_{-\mathbf{k}}, \qquad \tilde{\mathbf{s}}_{\mathbf{k}}(t) = \mathbf{d_k}\,\exp(-\,\mathrm{i}\,\omega\,t), \qquad \mathbf{k} \neq 0. \tag{5.45}$$

where the complex vector $\mathbf{d_k}$ depend on $\mathbf{k}$ only and lies on the plane normal to it. Relations similar to (5.30, 5.31, 5.32) hold as well, where $\mathbf{c_k}$ and $\mathbf{s_k}$ are replaced with $\mathbf{d_k}$ and $\tilde{\mathbf{s}}_{\mathbf{k}}$, respectively, and the sum is suitably replaced with an integral over $\mathbf{k}$. To determine the energy of the electromagnetic field one must integrate over the whole space the energy density w_{em}. Using (C.56) and the second relation in (5.45) one finds

$$W_{\mathrm{em}} = \iiint_{-\infty}^{+\infty} w_{\mathrm{em}}\,\mathrm{d}^3 r = 2\,\varepsilon_0 \iiint_{-\infty}^{+\infty} \omega^2\,\mathbf{d_k} \cdot \mathbf{d}^*_{\mathbf{k}}\,\mathrm{d}^3 k. \tag{5.46}$$

[1] For the existence of (5.44) it is implied that the three-dimensional equivalent of condition (C.19) holds.

It is sometimes useful to consider the frequency distribution of the integrand at the right hand side of (5.46). For this one converts the integral into spherical coordinates k, ϑ, γ and uses the relation $k = \omega/c = 2\pi\nu/c$ to obtain $\mathbf{d}_\mathbf{k} = \mathbf{d}(\nu, \vartheta, \gamma)$; then, from (B.3),

$$W_{em} = \int_{-\infty}^{+\infty} U_{em}(\nu)\,d\nu, \quad U_{em} = \frac{2\,\varepsilon_0}{c^2}\,(2\,\pi\,\nu)^4 \int_0^{\pi}\int_0^{2\pi} |\mathbf{d}|^2\,\sin\vartheta\,d\vartheta\,d\gamma, \tag{5.47}$$

where U_{em} (whose units are J s) is called *spectral energy* of the electromagnetic field. By a similar procedure one finds the total momentum, that reads

$$\iiint_{-\infty}^{+\infty} \frac{\mathbf{S}}{c^2}\,d^3r = 2\,\varepsilon_0 \iiint_{-\infty}^{+\infty} \omega\,|\mathbf{d}|^2\,\mathbf{k}\,d^3k. \tag{5.48}$$

5.9 Eikonal Equation

Consider the case of a monochromatic electromagnetic field with angular frequency ω. For the calculation in hand it is convenient to consider the Maxwell equations in complex form; specifically, $\mathrm{rot}\mathbf{H} = \partial\mathbf{D}/\partial t$ yields, in vacuo,

$$\Re\left[(\mathrm{rot}\mathbf{H}_c + i\,\omega\,\varepsilon_0\,\mathbf{E}_c)\,\exp(-i\,\omega\,t)\right] = 0, \tag{5.49}$$

while $\mathrm{rot}\mathbf{E} = -\partial\mathbf{B}/\partial t$ yields

$$\Re\left[(\mathrm{rot}\mathbf{E}_c - i\,\omega\,\mu_0\,\mathbf{H}_c)\,\exp(-i\,\omega\,t)\right] = 0. \tag{5.50}$$

The solution of (5.49, 5.50) has the form $\mathbf{E}_c = \mathbf{E}_{c0}\,\exp(i\,\mathbf{k}\cdot\mathbf{r})$, $\mathbf{H}_c = \mathbf{H}_{c0}\,\exp(i\,\mathbf{k}\cdot\mathbf{r})$, with $\mathbf{E}_{c0}, \mathbf{H}_{c0} = \mathrm{const}$, i.e., a planar wave propagating along the direction of $\mathbf{k}$. In a non-uniform medium it is $\varepsilon = \varepsilon(\mathbf{r})$, $\mu = \mu(\mathbf{r})$, and the form of the solution differs from the planar wave. The latter can tentatively be generalized as

$$\mathbf{E}_c = \mathbf{E}_{c0}(\mathbf{r})\,\exp[i\,k\,S(\mathbf{r})], \qquad \mathbf{H}_c = \mathbf{H}_{c0}(\mathbf{r})\,\exp[i\,k\,S(\mathbf{r})], \tag{5.51}$$

with $k = \omega\sqrt{\varepsilon_0\,\mu_0} = \omega/c$. Function $k\,S$ is called *eikonal* ($[S] = \mathrm{m}$). Replacing (5.51) into (5.49, 5.50) and using the first identity in (A.35) yields

$$\mathrm{grad}S \wedge \mathbf{H}_{c0} + c\,\varepsilon\,\mathbf{E}_{c0} = -\frac{c}{i\,\omega}\,\mathrm{rot}\mathbf{H}_{c0}, \tag{5.52}$$

$$\mathrm{grad}S \wedge \mathbf{E}_{c0} - c\,\mu\,\mathbf{H}_{c0} = -\frac{c}{i\,\omega}\,\mathrm{rot}\mathbf{E}_{c0}. \tag{5.53}$$

Now it is assumed that ω is large enough to make the right hand side of (5.52, 5.53) negligible; in this case $\mathrm{grad}S$, $\mathbf{E}_{c0}$, $\mathbf{H}_{c0}$ become normal to each other. Vector multiplying (5.52) by $\mathrm{grad}S$ and using (5.53) then yields

$$\mathrm{grad}S \wedge (\mathrm{grad}S \wedge \mathbf{H}_{c0}) + \frac{\varepsilon\,\mu}{\varepsilon_0\,\mu_0}\,\mathbf{H}_{c0} = 0. \tag{5.54}$$

Remembering that $c = 1/\sqrt{\varepsilon_0\,\mu_0}$ one defines the *phase velocity*, *refraction index*, and *wavelength* of the medium as

$$u_f(\mathbf{r}) = \frac{1}{\sqrt{\varepsilon\,\mu}}, \qquad n(\mathbf{r}) = \frac{c}{u_f}, \qquad \lambda(\mathbf{r}) = \frac{u_f}{\nu}, \tag{5.55}$$

respectively, so that $\varepsilon\,\mu/(\varepsilon_0\,\mu_0) = n^2$. Using the first identity in (A.33) and remembering that $\mathrm{grad}S \cdot \mathbf{H}_{c0} = 0$ transforms (5.54) into $(|\mathrm{grad}S|^2 - n^2)\,\mathbf{H}_{c0} = 0$. As $\mathbf{H}_{c0} \neq 0$ it follows

$$|\mathrm{grad}S|^2 = n^2, \qquad n = n(\mathbf{r}), \tag{5.56}$$

that is, a partial-differential equation, called *eikonal equation*, in the unknown S. The equation has been derived in the hypothesis that $\omega = 2\,\pi\,\nu$ is large, hence $\lambda = u_f/\nu$ is small; it is shown below that this condition is also described by stating that $\mathbf{E}_{c0}(\mathbf{r})$, $\mathbf{H}_{c0}(\mathbf{r})$, and $S(\mathbf{r})$ vary little over a distance of the order of λ.

The form of (5.51) is such that $S(\mathbf{r}) = \mathrm{const}$ defines the constant-phase surface (the same concept has been encountered in Sect. 2.5 for the case of a system of particles). It follows that the normal direction at each point $\mathbf{r}$ of the surface is that of $\mathrm{grad}S$. Let $\mathbf{t} = \mathrm{d}\mathbf{r}/\mathrm{d}s$ be the unit vector parallel to $\mathrm{grad}S$ in the direction of increasing S. A *ray* is defined as the envelope of the $\mathbf{t}$ vectors, taken starting from a point A in a given direction. The description of rays obtained through the approximation of the eikonal equation is called *Geometrical Optics*.

The eikonal equation (5.56) can be given a different form by observing that from the definition of $\mathbf{t}$ it follows $\mathrm{grad}S = n\,\mathbf{t}$ and $\mathbf{t} \cdot \mathrm{grad}S = \mathrm{d}S/\mathrm{d}s = n$, whence

$$\mathrm{grad}n = \mathrm{grad}\frac{\mathrm{d}S}{\mathrm{d}s} = \frac{\mathrm{d}\,\mathrm{grad}S}{\mathrm{d}s} = \frac{\mathrm{d}\,(n\mathbf{t})}{\mathrm{d}s} = \frac{\mathrm{d}}{\mathrm{d}s}\left(n\,\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}s}\right). \tag{5.57}$$

This form of the eikonal equation is more often used. It shows that the equation is of the second order in the unknown function $\mathbf{r}(s)$, where $\mathbf{r}$ is the point of the ray corresponding to the curvilinear abscissa s along the ray itself. The equation's coefficient and data are given by the refraction index n. As the equation is of the second order, two boundary conditions are necessary to completely define the solution; for instance, the value of $\mathbf{r}(s = 0)$ corresponding to the initial point A, and the direction $\mathbf{t} = \mathrm{d}\mathbf{r}/\mathrm{d}s$ of the ray at the same point. Remembering that $\mathrm{d}\mathbf{t}/\mathrm{d}s = \mathbf{n}/\rho_c$, where ρ_c is the *curvature radius* of the ray at $\mathbf{r}$, and $\mathbf{n}$ the *principal normal unit vector*, the eikonal equation may also be recast as $\mathrm{grad}n = (\mathrm{d}n/\mathrm{d}s)\,\mathbf{t} + (n/\rho_c)\,\mathbf{n}$. Using the curvature radius one can specify in a quantitative manner the approximation upon which the eikonal equation is based; in fact, for the approximation to hold it is necessary that the electromagnetic wave can be considered planar, namely, that its amplitude and direction do not significantly change over a distance of the order of λ. This happens if at each point $\mathbf{r}$ along the ray it is $\rho_c \gg \lambda$.

5.10 Fermat Principle

It is worth investigating whether the eikonal equation (5.57) worked out in Sect. 5.9 is derivable from a variational principle. In fact it is shown below that the *Fermat* (or *least time*) principle holds, stating that, if A and B are two different points belonging to a ray, the natural ray (that is, the actual path followed by the radiation between the given points) is the one that minimizes the time $\int_{AB} \mathrm{d}t$. The principle thus reads

$$\delta \int_{AB} \mathrm{d}t = 0, \tag{5.58}$$

where the integral is carried out along the trajectory. The analysis is similar to that carried out in Sect. 2.7 with reference to the Maupertuis principle.

Using the relations (5.55) and observing that $\mathrm{d}t = \mathrm{d}s/u_f = n\,\mathrm{d}s/c$ transforms (5.58) into $\delta \int_{AB} n\,\mathrm{d}s = 0$. Introducing a parametric description $\mathbf{r} = \mathbf{r}(\xi)$ of the ray, with $\xi = a$ when $\mathbf{r} = A$ and $\xi = b$ when $\mathbf{r} = B$, yields

$$\int_{AB} n\,\mathrm{d}s = \int_a^b g\,\mathrm{d}\xi, \qquad g = n\,\frac{\mathrm{d}s}{\mathrm{d}\xi} = n(x_1, x_2, x_3)\sqrt{\dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_3^2}, \tag{5.59}$$

$$\frac{\partial g}{\partial \dot{x}_i} = n\,\frac{2\dot{x}_i}{2\,\mathrm{d}s/\mathrm{d}\xi} = n\,\frac{\mathrm{d}x_i}{\mathrm{d}s}, \qquad \frac{\partial g}{\partial x_i} = \frac{\partial n}{\partial x_i}\,\frac{\mathrm{d}s}{\mathrm{d}\xi}. \tag{5.60}$$

Remembering (1.7), the Euler equation for the ith coordinate reads

$$\frac{\mathrm{d}}{\mathrm{d}\xi}\left(n\,\frac{\mathrm{d}x_i}{\mathrm{d}s}\right) = \frac{\partial n}{\partial x_i}\,\frac{\mathrm{d}s}{\mathrm{d}\xi}, \qquad i = 1, 2, 3, \tag{5.61}$$

whence

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(n\,\frac{\mathrm{d}x_i}{\mathrm{d}s}\right) = \frac{\partial n}{\partial x_i}. \tag{5.62}$$

As (5.62) is the ith component of the eikonal equation (5.57), such an equation is indeed derivable from the variational principle (5.58). Some comments about the formal analogy between the Maupertuis and Fermat principles are made in Sect. 5.11.6.

5.11 Complements

5.11.1 Fields Generated by a Point-Like Charge

The Liénard and Wiechert expressions (5.6, 5.7) provide the potentials generated in $\mathbf{r}$ at time t by a point-like charge that follows a trajectory $\mathbf{s}$. More specifically, if $\mathbf{s} = \mathbf{s}(\tau)$ is the position occupied by the charge at the instant τ, and $\mathbf{r}$ is the position where the potentials produced by the charge are detected at time $t > \tau$, the relation (5.4) holds, namely, $|\mathbf{r} - \mathbf{s}(\tau)| = c\,(t - \tau)$, that links the spatial coordinates with the time instants. Letting

$$\mathbf{g} = \mathbf{r} - \mathbf{s}(\tau), \qquad g = |\mathbf{g}|, \qquad \mathbf{u} = \frac{\mathrm{d}\mathbf{s}}{\mathrm{d}\tau}, \qquad \dot{\mathbf{u}} = \frac{\mathrm{d}\mathbf{u}}{\mathrm{d}\tau}, \tag{5.63}$$

the fields **E**, **B** are determined by applying (4.26) to (5.6, 5.7), which amounts to calculating the derivatives with respect to t and the components of **r**. This is somewhat complicated because (4.26) introduces a relation of the form $\tau = \tau(\mathbf{r}, t)$, so that $\varphi = \varphi(\mathbf{r}, \tau(\mathbf{r}, t))$ and $\mathbf{A} = \mathbf{A}(\mathbf{r}, \tau(\mathbf{r}, t))$. It is therefore convenient to calculate some intermediate steps first. To this purpose, (5.4) is recast in implicit form as

$$\sigma(x_1, x_2, x_3, t, \tau) = \left[\sum_{i=1}^{3} (x_i - s_i(\tau))^2\right]^{1/2} + c\,(\tau - t) = 0, \tag{5.64}$$

whence $\mathrm{grad}\sigma = \mathbf{g}/g$, $\partial\sigma/\partial t = -c$, $\partial\sigma/\partial\tau = c - \mathbf{u}\cdot\mathbf{g}/g$. The differentiation rule of the implicit functions then yields

$$\frac{\partial\tau}{\partial t} = -\frac{\partial\sigma/\partial t}{\partial\sigma/\partial\tau} = \frac{c}{c - \mathbf{u}\cdot\mathbf{g}/g} \tag{5.65}$$

Basing on (5.64, 5.65) and following the calculation scheme reported, e.g., in [96, Chap. 6] one obtains

$$\mathbf{E} = \frac{e\,(\partial\tau/\partial t)^3}{4\,\pi\,\varepsilon_0\,g^3}\left\{\left(1 - \frac{u^2}{c^2}\right)\left(\mathbf{g} - g\,\frac{\mathbf{u}}{c}\right) + \mathbf{g}\wedge\left[\left(\mathbf{g} - g\,\frac{\mathbf{u}}{c}\right)\wedge\frac{\dot{\mathbf{u}}}{c^2}\right]\right\}, \tag{5.66}$$

$$\mathbf{B} = \frac{\mathbf{g}}{g}\wedge\frac{\mathbf{E}}{c}. \tag{5.67}$$

This result shows that **E** and **B** are the sum of two terms, the first of which decays at infinity like g^{-2}, while the second decays like g^{-1}. The latter term differs from zero only if the charge is accelerated ($\dot{\mathbf{u}} \neq 0$); its contribution is called *radiation field*. Also, **E** and **B** are orthogonal to each other, while **g** is orthogonal to **B** but not to **E**; however, if g is large enough to make the second term in (5.66) dominant, **g** becomes orthogonal also to **E** and (5.67) yields $B = E/c$. In the case $\mathbf{u} = 0$ the relations (5.66, 5.67) simplify to

$$\mathbf{E} = \frac{e}{4\,\pi\,\varepsilon_0\,g^2}\,\frac{\mathbf{g}}{g}, \qquad \mathbf{B} = 0, \tag{5.68}$$

that hold approximately also for $\mathbf{u} = \mathrm{const}$, $u/c \ll 1$.

5.11.2 *Power Radiated by a Point-Like Charge*

The expressions of the **E** and **B** fields worked out in Sect. 5.11.1 are readily exploited to determine the power radiated by a point-like charge. Remembering the results of Sect. 5.3, it suffices to integrate the Poynting vector over a surface Σ surrounding the charge. Introducing (5.67) into the definition (5.11) of Poynting's vector and using the first identity in (A.33) yields

$$\mathbf{S} = \frac{1}{\mu_0\, c}\,\mathbf{E} \wedge \left(\frac{\mathbf{g}}{g} \wedge \mathbf{E}\right) = \frac{\varepsilon_0\, c}{g}\,\left(E^2\,\mathbf{g} - \mathbf{E}\cdot\mathbf{g}\,\mathbf{E}\right). \tag{5.69}$$

The case $\dot{\mathbf{u}} \neq 0$, $u/c \ll 1$ is considered, which is typical of a bound particle. As the surface Σ can be chosen arbitrarily, it is convenient to select it at a large distance from the charge in order to make the second term in (5.66) dominant and **E** practically normal to **g**. This simplifies (5.66) and (5.69) to

$$\mathbf{S} \simeq \varepsilon_0\, c\, E^2\,\frac{\mathbf{g}}{g}, \qquad \mathbf{E} \simeq \frac{e}{4\,\pi\,\varepsilon_0\, g}\left(\frac{\mathbf{g}}{g}\cdot\frac{\dot{\mathbf{u}}}{c^2}\,\frac{\mathbf{g}}{g} - \frac{\dot{\mathbf{u}}}{c^2}\right), \tag{5.70}$$

where the first identity in (A.33) has been used. Letting ϑ be the angle between **g** and $\dot{\mathbf{u}}$, one combines the two expressions in (5.70) to find

$$\mathbf{S} \simeq \varepsilon_0\, c\,\mathbf{E}\cdot\mathbf{E}\,\frac{\mathbf{g}}{g} = \frac{1}{4\,\pi\,\varepsilon_0}\,\frac{e^2\,\dot{u}^2}{4\,\pi\,c^3}\,\frac{\sin^2\vartheta}{g^2}\,\frac{\mathbf{g}}{g}. \tag{5.71}$$

To proceed one chooses for Σ a spherical surface centered at $\mathbf{s}(\tau)$ and shifts the origin to its center. This yields $\mathbf{g} = \mathbf{r}$ at time τ, whence the unit vector normal to Σ becomes $\mathbf{n} = \mathbf{g}/g$. The radiation emitted by the charge reaches Σ at a later time $t = \tau + g/c$; however, thanks to the hypothesis $u/c \ll 1$, during the interval $t - \tau$ the charge moves little with respect to center of the sphere. For this reason, the surface integral can be calculated by keeping the charge fixed in the center of the spherical surface, so that the integral $\int_\Sigma (\sin^2\vartheta/g^2)\,\mathrm{d}\Sigma$ must be evaluated with $g = \text{const}$. Such an integral is easily found to equal $8\,\pi/3$: first, one turns to spherical coordinates and expresses the volume element as $J\,\mathrm{d}\vartheta\,\mathrm{d}\varphi\,\mathrm{d}g = \mathrm{d}\Sigma\,\mathrm{d}g$; then, one finds from (B.3) the ratio $\mathrm{d}\Sigma/g^2 = \sin\vartheta\,\mathrm{d}\theta\,\mathrm{d}\varphi$ and replaces it in the integral. In conclusion, combining the above result with (5.12, 5.71),

$$-\frac{\mathrm{d}(W + W_{\text{em}})}{\mathrm{d}t} = \frac{1}{4\,\pi\,\varepsilon_0}\,\frac{e^2\,\dot{u}^2}{4\,\pi\,c^3}\int_\Sigma \frac{\sin^2\vartheta}{g^2}\,\mathbf{n}\cdot\mathbf{n}\,\mathrm{d}\Sigma = \frac{2\,e^2/3}{4\,\pi\,\varepsilon_0\,c^3}\,\dot{u}^2. \tag{5.72}$$

The expression at the right hand side of (5.72), called *Larmor formula*, gives an approximate expression of the power emitted by a point-like charge, that is applicable when $u/c \ll 1$. As shown by the left hand side, part of the emitted power $(-\mathrm{d}W/\mathrm{d}t)$ is due to the variation in the charge's mechanical energy, while the other part $(-\mathrm{d}W_{\text{em}}/\mathrm{d}t)$ is due to the variation in the electromagnetic energy within the volume enclosed by Σ.

5.11.3 Decay of Atoms According to the Classical Model

The power radiated by a point-like charge has been determined in Sect. 5.11.2 under the approximations $\dot{\mathbf{u}} \neq 0$, $u/c \ll 1$, typical of a bound particle. The radiated power (5.72) is proportional to the square of the particle's acceleration: this result is of a paramount importance, because it shows that the so-called *planetary model* of the atom is not stable. Considering for instance the simple case of hydrogen, the model describes the atom as a planetary system whose nucleus is fixed in the reference's origin while the electron orbits around it. If no power were emitted the motion's description would be that given, e.g., in Sect. 3.13.6, where the Hamiltonian function is a constant of motion. In other terms, the total energy would be conserved. In fact, in the planetary motion the electron's acceleration, hence the emitted power, differ from zero; the emission produces an energy loss which was not considered in the analysis of Sect. 3.13.6. Some comments about this problem were anticipated in Sect. 3.13.9.

To proceed it is useful to carry out a quantitative estimate of the emitted power. The outcome of it is that in the case of a bound electron the emission is relatively weak, so that one can consider it as a perturbation with respect to the conservative case analyzed in Sect. 3.13.6. The estimate starts from the experimental observation of the emission of electromagnetic radiation by excited atoms; here the datum that matters is the minimum angular frequency ω_0 of the emitted radiation, which is found to be in the range $[10^{15}, 10^{16}]$ rad s^{-1}. The simplest model for describing the unperturbed electron's motion is that of the linear harmonic oscillator [4, Vol. II, Sect. 4]

$$\mathbf{s}(\tau) = \mathbf{s}_0 \cos(\omega_0 \tau), \tag{5.73}$$

with $s_0 = |\mathbf{s}_0|$ the maximum elongation with respect to the origin, where the nucleus is placed. Equation (5.73) may be thought of as describing the projection over the direction of $\mathbf{s}_0$ of the instantaneous position of an electron that follows a circular orbit. The product $e\,\mathbf{s}$ is called *electric dipole moment* of the oscillator. Other experimental results, relative to the measure of the atom's size, show that s_0 is of the order of 10^{-10} m so that, calculating $\mathbf{u} = \mathrm{d}\mathbf{s}/\mathrm{d}\tau = -\mathbf{s}_0\,\omega_0 \sin(\omega_0 \tau)$ from (5.73) and letting $\omega_0 = 5 \times 10^{15}$, one finds $u/c \leq s_0\,\omega_0/c \simeq 2 \times 10^{-3}$. This shows that the approximations of Sect. 5.11.2 are applicable.

It is worth noting that the type of motion (5.73) is energy conserving, hence it must be understood as describing the unperturbed dynamics of the electron. Remembering the discussion of Sect. 3.3 one finds for the total, unperturbed energy the expression $E_u = m\,\omega_0^2\,\mathbf{s}_0^2/2$, with $m = 9.11 \times 10^{-31}$ kg the electron mass. To tackle the perturbative calculation it is now necessary to estimate the energy E_r lost by the electron during an oscillation period $2\,\pi/\omega_0$ and compare it with E_u. From $\dot{\mathbf{u}} = \mathrm{d}\mathbf{u}/\mathrm{d}\tau = \omega_0^2\,\mathbf{s}$ one obtains the maximum square modulus of the electron's acceleration, $\dot{u}_M^2 = \omega_0^4\,s_0^2$; inserting the latter into (5.72) and using $e = -q = -1.602 \times 10^{-19}$ C for the electron charge provides the upper bounds

$$E_r \le \frac{2\pi}{\omega_0}\,\frac{2\,e^2/3}{4\pi\,\varepsilon_0\,c^3}\,\omega_0^4\,s_0^2 = \frac{e^2\,s_0^2\,\omega_0^3}{3\,\varepsilon_0\,c^3}, \qquad \frac{E_r}{E_u} \le \frac{2\,e^2\omega_0}{\varepsilon_0\,m\,c^3} \simeq 4\times 10^{-7}. \tag{5.74}$$

This result shows that the energy lost during an oscillation period is indeed small, so that the electron's motion is only slightly perturbed with respect to the periodic case. The equation of motion of the perturbed case can now tentatively be written as

$$m\,\ddot{\mathbf{s}} + m\,\omega_0^2\,\mathbf{s} = \mathbf{F}_r, \tag{5.75}$$

where $\mathbf{F}_r$ is a yet unknown force that accounts for the emitted power. A scalar multiplication of (5.75) by $\mathbf{u}$ yields $m\,\mathbf{u}\cdot\dot{\mathbf{u}} + m\,\omega_0^2\,\mathbf{s}\cdot\mathbf{u} = \mathrm{d}W/\mathrm{d}\tau = \mathbf{u}\cdot\mathbf{F}_r$, with $W = (m/2)\,(u^2 + \omega_0^2\,s^2)$. One notes that W has the same expression as the total energy E_u of the unperturbed case; however, W is not conserved due to the presence of $\mathbf{F}_r \ne 0$ at the right hand side. In fact, $-\mathrm{d}W/\mathrm{d}\tau = -\mathbf{u}\cdot\mathbf{F}_r > 0$ is the power emitted by the electron, and its time average over $2\,\pi/\omega_0$,

$$-\langle\mathbf{u}\cdot\mathbf{F}_r\rangle = -\frac{\omega_0}{2\,\pi}\int_0^{2\,\pi/\omega_0} \mathbf{u}\cdot\mathbf{F}_r\,\mathrm{d}\tau > 0, \tag{5.76}$$

is the variation in the oscillator's energy during a period; a part of it crosses the surface Σ, while the other part is the variation in the electromagnetic energy within Σ (Sects. 5.3 and 5.11.2). The part that crosses Σ is the time average of (5.72);[2] for the sake of simplicity it is assumed that it is dominant with respect to the other one. The factor $\dot{u}^2$ that appears in (5.72) is worked out by taking the time average of the identity $\mathrm{d}(\mathbf{u}\cdot\dot{\mathbf{u}})/\mathrm{d}\tau = \dot{u}^2 + \mathbf{u}\cdot\ddot{\mathbf{u}}$ and observing that $\langle\mathrm{d}(\mathbf{u}\cdot\dot{\mathbf{u}})/\mathrm{d}\tau\rangle$ is negligibly small, whence $\langle\dot{u}^2\rangle = -\langle\mathbf{u}\cdot\ddot{\mathbf{u}}\rangle > 0$. In conclusion, defining a time τ_0 such that $e^2/(6\,\pi\,\varepsilon_0\,c^3) = m\,\tau_0$ and equating (5.76) to the time average of (5.72) yields $\langle\mathbf{u}\cdot m\,\tau_0\,\ddot{\mathbf{u}}\rangle = \langle\mathbf{u}\cdot\mathbf{F}_r\rangle$. It is found $\tau_0 \simeq 6\times 10^{-24}$ s.

As a crude approximation one finally converts the equality of the averages just found into an equality of the arguments, whence $\mathbf{F}_r \simeq m\,\tau_0\,\ddot{\mathbf{u}}$. Replacing the latter into (5.75) yields $\ddot{\mathbf{s}} + \omega_0^2\,\mathbf{s} = \tau_0\,\ddot{\mathbf{u}}$, that is, a linear, homogeneous equation of the third order in $\mathbf{s}$ with constant coefficients. The equation is solved by letting $\mathbf{s} = \mathbf{s}(\tau = 0)\,\exp(\alpha\,\tau)\,\cos(\omega\,\tau)$, with α, ω undetermined. Using the tentative solution provides the system of characteristic algebraic equations

$$\tau_0\,\omega^2 = 3\,\tau_0\,\alpha^2 - 2\,\alpha, \qquad \alpha^2 + \omega_0^2 = \tau_0\,\alpha^3 + (1 - 3\,\tau_0\,\alpha)\,\omega^2, \tag{5.77}$$

whence the elimination of ω^2 yields $8\,\alpha^2 - 2\,\alpha/\tau_0 - \omega_0^2 = 8\,\tau_0\alpha^3$. Thanks to the smallness of τ_0 the latter equation may be solved by successive approximations starting from the zeroth-order solution $\alpha^{(0)} \simeq -\tau_0\,\omega_0^2/2$ (this solution is found by

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[2] Remembering the discussion of Sect. 5.11.2, the use of (5.72) implies that the particle's position departs little from the center of the spherical surface. Thus the radius of Σ must be much larger than the size of the atom.

solving $8\,\alpha^2 - 2\,\alpha/\tau_0 - \omega_0^2 = 0$ and using the binomial approximation; the other possible value of $\alpha^{(0)}$ is positive and must be discarded to prevent **s** from diverging). Replacing $\alpha^{(0)}$ into the first equation in (5.77) yields $\omega^2 = \omega_0^2\,(1 + 3\,\tau_0^2\,\omega_0^2/4) \simeq \omega_0^2$. In conclusion, the zeroth-order solution of the differential equation for **s** reads

$$\mathbf{s}(\tau) \simeq \mathbf{s}(\tau = 0)\,\cos(\omega_0\,\tau)\,\exp(-\tau_0\,\omega_0^2\,\tau/2). \tag{5.78}$$

Basing upon (5.78) one can identify the decay time of the atom with the time necessary for the modulus of **s** to reduce by a factor $1/e$ with respect to the initial value. The decay time is thus found to be $2/(\tau_0\,\omega_0^2) \simeq 13 \times 10^{-9}$ s. As the ratio of the decay time to the period $2\,\pi/\omega_0$ is about 10^7, the perturbative approach is indeed justified.

As anticipated at the beginning of this section, the planetary model of the atom is not stable. The approximate solution (5.78) of the electron's dynamics shows that according to this model the electron would collapse into the nucleus in a very short time due to the radiation emitted by the electron. This behavior is not observed experimentally: in fact, the experiments show a different pattern in the energy-emission or absorption behavior of the atoms. The latter are able to absorb energy from an external radiation and subsequently release it: an absorption event brings the atom to a higher-energy state called *excited state*; the absorbed energy is then released by radiation in one or more steps (*emissions*) until, eventually, the atom reaches the lowest energy state (*ground state*). However, when the atom is in the ground state and no external perturbations is present, the atom is stable and no emission occurs. In conclusion, the experimental evidence shows that the planetary model is not applicable to the description of atoms.[3]

5.11.4 Comments about the Field's Expansion into Modes

The homogeneous wave equation (5.20) used in Sects. 5.5, 5.8 as a starting point for the derivation of the field's expansion into modes is based on the hypothesis that a gauge transformation exists such that $\varphi = 0$. In turn, (5.20) derives from (5.19), that implies the Coulomb gauge $\mathrm{div}\mathbf{A} = 0$. To show that these conditions are mutually compatible one chooses f in (4.30) such that $\varphi' = 0$, whence $\mathbf{E}' = -\partial\mathbf{A}'/\partial t$ due to the second relation in (4.26). In a charge-free space it is $\mathrm{div}\mathbf{D}' = \varepsilon_0\,\mathrm{div}\mathbf{E}' = 0$; it follows $\partial\mathrm{div}\mathbf{A}'/\partial t = 0$, namely, $\mathrm{div}\mathbf{A}'$ does not depend on time. The second equation in (4.19) with $\mathbf{J} = 0$ yields $(1/c^2)\,\partial\mathbf{E}'/\partial t = \mathrm{rot}\mathbf{B}'$, so that $-(1/c^2)\,\partial^2\mathbf{A}'/\partial t^2 = \mathrm{rotrot}\mathbf{A}'$. Now let $\mathbf{A}' = \mathbf{A}'' + \mathrm{grad}g$, where g is an arbitrary function of the coordinates only; the second identity in (A.35) and the first identity in (A.36) then yield $-(1/c^2)\,\partial^2\mathbf{A}''/\partial t^2 = \mathrm{graddiv}\mathbf{A}'' - \nabla^2\mathbf{A}''$, with $\mathrm{div}\mathbf{A}'' = \mathrm{div}\mathbf{A}' -$

[3] In this respect one might argue that the inconsistency between calculation and experiment is due to some flaw in the electromagnetic equations. However, other sets of experiments show that it is not the case.

$\nabla^2 g$. Choosing g such that div$\mathbf{A}'' = 0$ and dropping the double apex finally yields (5.20) [68, Sect. 46].

The vector potential $\mathbf{A}$ has been expressed in (5.21) as a Fourier series and in (5.44) as a Fourier integral. Such expressions produce a separation of the spatial coordinates from the time coordinate: the former appear only in the terms $\exp(\mathrm{i}\,\mathbf{k}\cdot\mathbf{r})$, while the latter appears only in the terms $\mathbf{a}_\mathbf{k}$ and, respectively, $\mathbf{b}_\mathbf{k}$.

The Fourier series (5.21) applies to the case of a finite domain of the form shown in Fig. 5.1 and prescribes the spatial periodicity of $\mathbf{A}$ at all times. By way of example, let $0 \le x_2 \le d_2$, $0 \le x_3 \le d_3$ and consider the point $\mathbf{r}_A = x_2\,\mathbf{i}_2 + x_3\,\mathbf{i}_3$; then, consider a second point $\mathbf{r}_B = d_1\,\mathbf{i}_1 + x_2\,\mathbf{i}_2 + x_3\,\mathbf{i}_3$. By construction, $\mathbf{r}_A$ and $\mathbf{r}_B$ belong to two opposite faces of the domain of Fig. 5.1 and are aligned with each other in the x_1 direction. From (5.22) one obtains for any $\mathbf{k}$

$$\exp(\mathrm{i}\,\mathbf{k}\cdot\mathbf{r}_B) = \exp(\mathrm{i}\,2\pi\,n_1)\,\exp(\mathrm{i}\,\mathbf{k}\cdot\mathbf{r}_A) = \exp(\mathrm{i}\,\mathbf{k}\cdot\mathbf{r}_A) \qquad (5.79)$$

which, combined with (5.21), yields $\mathbf{A}(\mathbf{r}_B, t) = \mathbf{A}(\mathbf{r}_A, t)$. Clearly an equality of this form is found for any pair of opposite boundary points that are aligned along the coordinate direction normal to the faces where the points lie. On the other hand, such an equality is a homogeneous relation among the boundary values of the solution of the differential equation (5.20), namely, it is a homogeneous boundary condition of the Dirichlet type.

The reasoning based on (5.79) is applicable also to the expressions (5.31, 5.32) to yield $\mathbf{E}(\mathbf{r}_B, t) = \mathbf{E}(\mathbf{r}_A, t)$ and $\mathbf{B}(\mathbf{r}_B, t) = \mathbf{B}(\mathbf{r}_A, t)$, namely, the fields have the same periodicity as $\mathbf{A}$. The Poynting vector $\mathbf{S} = \mathbf{E} \wedge \mathbf{H}$ has this property as well, whence $\mathbf{S}(\mathbf{r}_B, t)\cdot\mathbf{n}_B = -\mathbf{S}(\mathbf{r}_A, t)\cdot\mathbf{n}_A$; in fact, the unit vector $\mathbf{n}$ is oriented in the outward direction with respect to the domain (Sect. 5.3), so that when two opposite faces of V are considered it is $\mathbf{n}_B = -\mathbf{n}_A$. Using (5.12) with $W = 0$ shows that $\mathrm{d}W_{\mathrm{em}}/\mathrm{d}t = 0$ namely, as noted in Sect. 5.6, the electromagnetic energy within V is conserved. The same reasoning applies to the conservation of the electromagnetic momentum found in Sect. 5.7. As for the initial condition on $\mathbf{A}$, from (5.21) and (5.29) one derives $\mathbf{A}(\mathbf{r}, t = 0) = \sum_\mathbf{k} (\mathbf{c}_\mathbf{k} + \mathbf{c}^*_{-\mathbf{k}})\exp(\mathrm{i}\,\mathbf{k}\cdot\mathbf{r})$. It follows that the initial condition is provided by the vectors $\mathbf{c}_\mathbf{k}$.

5.11.5 *Finiteness of the Total Energy*

The differential equation (5.20) is linear and homogeneous with respect to the unknown $\mathbf{A}$; when the Fourier series (5.21) is replaced in it, the resulting equation (5.28) is linear and homogeneous with respect to $\mathbf{a}_\mathbf{k}$, hence (due to (5.29)) with respect to $\mathbf{s}_\mathbf{k}$ and $\mathbf{c}_\mathbf{k}$ as well. It follows that the fields (5.31, 5.32) are linear and homogeneous functions of these quantities. The same applies in the case of an infinite domain (Sect. 5.8), in which the fields $\mathbf{E}$, $\mathbf{B}$ are linear and homogeneous functions of $\tilde{\mathbf{s}}_\mathbf{k}$ and $\mathbf{d}_\mathbf{k}$.

In turn, the energy density w_{em} of the electromagnetic field, given by the second relation in (5.11), is a quadratic and homogeneous function of the fields; this explains why the expressions (5.37) and (5.46) are quadratic and homogeneous functions of $\mathbf{s_k}$, $\mathbf{c_k}$ or, respectively, $\tilde{\mathbf{s}}_\mathbf{k}$, $\mathbf{d_k}$.

When (5.37) is considered, the energy associated to the individual degree of freedom is $W_{\mathbf{k}\sigma} = 2\,\varepsilon_0\,V\,\omega^2\,|\mathbf{c}_{\mathbf{k}\sigma}|^2$; as the sum $\sum_{\mathbf{k}\sigma} W_{\mathbf{k}\sigma}$ spans all wave vectors, the factor $\omega^2 = c^2\,k^2$ diverges. On the other hand, the energy of the electromagnetic field within a finite region of space is finite; this means that the term $|\mathbf{c}_{\mathbf{k}\sigma}|^2$ becomes vanishingly small as $|\mathbf{k}| \to \infty$, in such a way as to keep the sum $\sum_{\mathbf{k}\sigma} W_{\mathbf{k}\sigma}$ finite. The same reasoning applies to the term $|\mathbf{d}_{\mathbf{k}\sigma}|^2$ in (5.46); in this case the finiteness of the total energy W_{em} is due to the fact that the vanishing of the fields at infinity makes the Fourier transform in (5.44) to converge.

5.11.6 *Analogies between Mechanics and Geometrical Optics*

A number of analogies exist between the Maupertuis principle, discussed in Sect. 2.7, and the Fermat principle discussed in Sect. 5.10. The principles read, respectively,

$$\delta \int_{AB} \sqrt{E - V}\,\mathrm{d}s = 0, \qquad \delta \int_{AB} n\,\mathrm{d}s = 0, \tag{5.80}$$

and the analogies are:

1. A constant parameter is present, namely, the total energy E on one side, the frequency ν on the other side (in fact, the Fermat principle generates the eikonal equation which, in turn, applies to a monochromatic electromagnetic field, Sect. 5.9).
2. Given the constant parameter, the integrand is uniquely defined by a property of the medium where the physical phenomenon occurs: the potential energy $V(\mathbf{r})$ and the refraction index $n(\mathbf{r})$, respectively.
3. The outcome of the calculation is a curve of the three-dimensional space: the particle's trajectory and the optical ray, respectively. In both cases the initial conditions are the starting position and direction (in the mechanical case the initial velocity is obtained by combining the initial direction with the momentum extracted from $E - V$).

In summary, by a suitable choice of the units, the same concept is applicable to both mechanical and optical problems. In particular it is used for realizing devices able to obtain a trajectory or a ray of the desired form: the control of the ray's shape is achieved by prescribing the refraction index n by means of, e.g., a lens or a system of lenses; similarly, the trajectory of a particle of charge e is controlled by a set of electrodes (*electrostatic lenses*) that prescribe the electric potential $\varphi = V/e$. The design of *electron guns* and of the equipments for *electron-beam lithography* and *ion-beam lithography* is based on this analogy.

It must be emphasized that the Maupertuis principle is derived without approximations: as shown in Sect. 2.7, the principle is equivalent to Newton's law applied to a particle of constant energy. The Fermat principle, instead, is equivalent to the eikonal equation; the latter, in turn, is derived from the Maxwell equations in the hypothesis that at each position along the ray the curvature radius of the ray is much larger than the wavelength. In other terms, the mechanical principle is exact whereas the optical principle entails an approximation. If the exact formulation of electromagnetism given by the Maxwell equation were used, the analogy discussed here would be lost.

The rather surprising asymmetry outlined above could be fixed by speculating that Newton's law is in fact an approximation deriving from more general laws, possibly similar to the Maxwell equations. In this case one could identify in such laws a parameter analogue of the wavelength, and deduce Newton's law as the limiting case in which the parameter is small. It will be shown later that mechanical laws more general than Newton's laws indeed exist: they form the object of Quantum Mechanics.[4]

The analogy between Mechanics and Geometrical Optics discussed here is one of the possible courses of reasoning useful for introducing the quantum-mechanical concepts; however, in this reasoning the analogy should not be pushed too far. In fact, one must observe that the Maupertuis principle given by the first expression in (5.80) provides the non-relativistic form of Newton's law, whereas the Maxwell equations, of which the Fermat principle is an approximation, are intrinsically relativistic. As a consequence, the analogy discussed in this section is useful for generalizing the geometrical properties of the motion, but not the dynamical properties.

Problems

5.1 Solve the eikonal equation (5.57) in a medium whose refraction index depends on one coordinate only, say, $n = n(x_1)$.

5.2 Use the solution of Problem 5.1 to deduce the Descartes law of refraction.

[4] Once Quantum Mechanics is introduced, Newtonian Mechanics is distinguished from it by the designation *Classical Mechanics*.