Chapter 2
Coordinate Transformations and Invariance Properties

2.1 Introduction

An important generalization of the subject of coordinate transformation is that of canonical transformation, which leads to the concept of generating function and, through it, to the definition of the principal function and characteristic function of Hamilton. The principal function is found to coincide with the solution of the Hamilton–Jacobi equation introduced in the previous chapter, this showing the equivalence of the approach based on the variational principles with that based on the canonical transformations. Connected with the Hamiltonian formalism is also the important concept of separability. Still based on Hamilton's principal function is the concept of phase velocity applied to a mechanical system, that brings about an analogy with the electromagnetic theory. The aspects mentioned above give another indication about the generality that is reached by the formalism of Analytical Mechanics illustrated in this part of the book.

Another fundamental issue is that of the invariance properties of the mechanical systems. It is shown that, basing only on the observation of symmetries possessed by the Lagrangian function or other functions connected to it, one derives the existence of invariance properties of the system. Among these are the constants of motion, namely, the dynamical properties that are constant in time and are therefore known from the motion's initial condition.

Of special relevance among the constants of motion are the total energy, the total momentum, and the total angular momentum of the system. The conservation of the total energy is related to the uniformity of time, that of the total momentum is related to the uniformity of space and, finally, that of the total angular momentum is related to the isotropy of space. Besides the theoretical interest connected to it, the knowledge of a constant of motion is important also for practical purposes: by introducing a known constraint among the canonical coordinates, it is of use in the separation procedure.

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This chapter completes the illustration of the basic principles of Analytical Mechanics started in the previous one. The purpose of the two chapters is to introduce a number of concepts that are not only useful *per se*, but also constitute a basis for the concepts of Quantum Mechanics that are introduced in later chapters. The first subject is that of the canonical transformations, followed by the definition and properties of the Hamilton characteristic function and of the phase velocity. Then, the invariance properties that derive from the symmetries of the Lagrangian function are discussed. The chapter continues with a short description of the Maupertuis principle and of the expression of the angular momentum in spherical coordinates. The last paragraphs deal with the linear motion and the action-angle variables.

2.2 Canonical Transformations

Section 1.4 introduced the generalized coordinates $q_1, \ldots, q_n$, that are defined by the first of (1.26) starting from a set of Cartesian coordinates $x_1, \ldots, x_n$. From this, one defines the generalized velocities $\dot{q}_i$ and, from the second of (1.26), calculates the Lagrangian function in the new variables $L(\mathbf{q}, \dot{\mathbf{q}}, t)$. The conjugate momenta p_i are derived from the latter using the first of (1.30) and, finally, the new Hamiltonian function is determined from the second of (1.32). From this, the Hamilton equations (1.42) in the new coordinates are deduced. The process depicted here is a series of replacement, elimination, and differentiation steps.

Relations like (1.26), that transform a set of coordinates into another one, are called *point transformations*. It has been observed in Sect. 1.4 that the canonical coordinates $q_1, \ldots, q_n, \ p_1, \ldots, p_n$ are mutually independent. It follows that the point transformations are not the most general coordinate transformations, because they act on the $q_1, \ldots, q_n$ only. The most general transformations act simultaneously on the generalized coordinate and momenta, hence they have the form

$$\begin{cases} \tilde{q}_1 = \tilde{q}_1(q_1, \ldots, q_n, p_1, \ldots, p_n, t) \\ \quad \vdots \\ \tilde{q}_n = \tilde{q}_n(q_1, \ldots, q_n, p_1, \ldots, p_n, t) \\ \tilde{p}_1 = \tilde{p}_1(q_1, \ldots, q_n, p_1, \ldots, p_n, t) \\ \quad \vdots \\ \tilde{p}_n = \tilde{p}_n(q_1, \ldots, q_n, p_1, \ldots, p_n, t) \end{cases} \tag{2.1}$$

where q_i, p_i indicate the old canonical coordinates and $\tilde{q}_i, \tilde{p}_i$ indicate the new ones. If H is the Hamiltonian function in the old coordinates, introducing into H the inverse transformations of (2.1) yields a function $\tilde{H}$ that depends on the new coordinates and on time.

For an arbitrary choice of the form of (2.1) it is not possible in general to deduce from $\tilde{H}$ the Hamilton equations in the new coordinates. In fact it is necessary to

limit the choice of the transformation (2.1) to the cases where the resulting $\tilde{H}$ is a Hamiltonian function proper, namely, it is such that

$$\frac{\mathrm{d}\tilde{q}_i}{\mathrm{d}t} = \frac{\partial \tilde{H}}{\partial \tilde{p}_i}, \qquad \frac{\mathrm{d}\tilde{p}_i}{\mathrm{d}t} = -\frac{\partial \tilde{H}}{\partial \tilde{q}_i}, \qquad i = 1, \ldots, n \tag{2.2}$$

are fulfilled. The transformations (2.1) that make (2.2) to hold are called *canonical transformations*. The procedure by which the Hamilton equations in the old coordinates are found has been illustrated in Sect. 1.4 and is based on the derivation of the extremum equation of the action integral (1.45). To obtain (2.2) the same calculation must be repeated based on the action integral defined in the new coordinates. It follows that for two sets of coordinates q_i, p_i and $\tilde{q}_i$, $\tilde{p}_i$ connected by a canonical transformation, the following must hold simultaneously:

$$S = \int_a^b \left(\sum_{i=1}^n p_i \frac{\mathrm{d}q_i}{\mathrm{d}t} - H \right) \mathrm{d}t, \qquad \tilde{S} = \int_a^b \left(\sum_{i=1}^n \tilde{p}_i \frac{\mathrm{d}\tilde{q}_i}{\mathrm{d}t} - \tilde{H} \right) \mathrm{d}t. \tag{2.3}$$

The difference between the two integrals in (2.3) can be set equal to an arbitrary constant because the calculation uses only the variations of S or $\tilde{S}$. As the limits of the two integrals are fixed to the same values a and b, the constant can in turn be written as the integral between a and b of the total time derivative of an arbitrary function K. In this way the relation between the two integrands in (2.3) reads

$$\sum_{i=1}^n p_i \frac{\mathrm{d}q_i}{\mathrm{d}t} - H = \sum_{i=1}^n \tilde{p}_i \frac{\mathrm{d}\tilde{q}_i}{\mathrm{d}t} - \tilde{H} + \frac{\mathrm{d}K}{\mathrm{d}t}. \tag{2.4}$$

It is worth reminding that in the derivation of the Hamilton equations in Sect. 1.6 it is assumed that all variations of generalized coordinates and momenta vanish at the integration limits. Here this applies to both the old and new sets of coordinates. As a consequence, K can be made to depend on all $4n$ coordinates q_i, p_i, $\tilde{q}_i$, $\tilde{p}_i$, and on time t. Due to the $2n$ relations (2.1) that define the canonical transformation, only $2n$ coordinates are independent. Thus, K can be made to depend on $2n$ coordinates chosen among the $4n$ available ones, and on time. The most interesting cases are those where K has one of the following forms [42, Sect. 9-1]:

$$K_1 = K_1(\mathbf{q}, \tilde{\mathbf{q}}, t), \quad K_2 = K_2(\mathbf{q}, \tilde{\mathbf{p}}, t), \quad K_3 = K_3(\mathbf{p}, \tilde{\mathbf{q}}, t), \quad K_4 = K_4(\mathbf{p}, \tilde{\mathbf{p}}, t). \tag{2.5}$$

By way of example, select the first form: replacing K_1 into (2.4), calculating $\mathrm{d}K_1/\mathrm{d}t$, and multiplying both sides by $\mathrm{d}t$ yields

$$\sum_{i=1}^n \left(\frac{\partial K_1}{\partial q_i} - p_i \right) \mathrm{d}q_i + \sum_{i=1}^n \left(\frac{\partial K_1}{\partial \tilde{q}_i} + \tilde{p}_i \right) \mathrm{d}\tilde{q}_i + \left(\frac{\partial K_1}{\partial t} + H - \tilde{H} \right) \mathrm{d}t = 0, \tag{2.6}$$

where the left hand side is a total differential in the independent variables q_i, $\tilde{q}_i$, and t. To fulfill (2.6) the parentheses must vanish independently from each other, whence

$$p_i = \frac{\partial K_1}{\partial q_i}, \qquad \tilde{p}_i = -\frac{\partial K_1}{\partial \tilde{q}_i}, \qquad \tilde{H} = H + \frac{\partial K_1}{\partial t}, \qquad i = 1, \dots, n. \tag{2.7}$$

As K_1 is prescribed, the first two equations in (2.7) provide $2n$ relations involving the $4n$ coordinates q_i, p_i, $\tilde{q}_i$, $\tilde{p}_i$, that constitute the canonical transformation sought. Using the latter one expresses the right hand side of the third of (2.7) in terms of $\tilde{q}_i$, $\tilde{p}_i$, this yielding the new Hamiltonian function $\tilde{H}(\tilde{\mathbf{q}}, \tilde{\mathbf{p}}, t)$. The procedure is the same for the other functions listed in (2.5), that can all be defined starting from K_1. In fact, letting

$$K_2(\mathbf{q}, \tilde{\mathbf{p}}, t) = K_1(\mathbf{q}, \tilde{\mathbf{q}}, t) + \sum_{i=1}^{n} \tilde{p}_i \tilde{q}_i, \tag{2.8}$$

and applying the same procedure used to determine (2.7), yields

$$p_i = \frac{\partial K_2}{\partial q_i}, \qquad \tilde{q}_i = \frac{\partial K_2}{\partial \tilde{p}_i}, \qquad \tilde{H} = H + \frac{\partial K_2}{\partial t}, \qquad i = 1, \dots, n. \tag{2.9}$$

In (2.8) the independent variables are q_i, $\tilde{p}_i$, so that the coordinates $\tilde{q}_i$ are expressed through them. Similarly, when K_3 is used one lets

$$K_3(\mathbf{p}, \tilde{\mathbf{q}}, t) = K_1(\mathbf{q}, \tilde{\mathbf{q}}, t) - \sum_{i=1}^{n} p_i q_i, \tag{2.10}$$

to find

$$q_i = -\frac{\partial K_3}{\partial p_i}, \qquad \tilde{p}_i = -\frac{\partial K_3}{\partial \tilde{q}_i}, \qquad \tilde{H} = H + \frac{\partial K_3}{\partial t}, \qquad i = 1, \dots, n. \tag{2.11}$$

Finally, in the case of K_4 one lets

$$K_4(\mathbf{p}, \tilde{\mathbf{p}}, t) = K_1(\mathbf{q}, \tilde{\mathbf{q}}, t) + \sum_{i=1}^{n} \tilde{p}_i \tilde{q}_i - \sum_{i=1}^{n} p_i q_i, \tag{2.12}$$

whence

$$q_i = -\frac{\partial K_4}{\partial p_i}, \qquad \tilde{q}_i = \frac{\partial K_4}{\partial \tilde{p}_i}, \qquad \tilde{H} = H + \frac{\partial K_4}{\partial t}, \qquad i = 1, \dots, n. \tag{2.13}$$

Regardless of the choice of K, the relation between the old and new Hamiltonian function is always of the form $\tilde{H} = H + \partial K / \partial t$. As the canonical transformation is completely determined when K is prescribed, the latter is called *generating function of the canonical transformation*. Two interesting examples are those produced by the generating functions $F_1 = \sum_{i=1}^{n} q_i \tilde{q}_i$ and $F_2 = \sum_{i=1}^{n} q_i \tilde{p}_i$. Applying (2.7) to

F_1 yields $\tilde{q}_i = p_i$ and $\tilde{p}_i = -q_i$. As a consequence, the effect of the transformation generated by F_1 is that of exchanging the roles of the generalized coordinates and momenta. This result shows that the distinction between coordinates and momenta is not fundamental, namely, these two groups of variables globally constitute a set of $2n$ independent coordinates. Applying (2.9) to F_2 provides the identical transformation $\tilde{q}_i = q_i$, $\tilde{p}_i = p_i$. A generalization of this example is found using $F_2 = \sum_{i=1}^{n} z_i(\mathbf{q}, t)\tilde{p}_i$, where z_i are arbitrary functions. The new coordinates are in this case $\tilde{q}_i = z_i(\mathbf{q}, t)$ which, as indicated at the beginning of this section, are point transformations. This example shows that all point transformations are canonical.

2.3 An Application of the Canonical Transformation

The discussion of Sect. 2.2 has shown that a canonical transformation based on an arbitrary generating function K brings a Hamiltonian function $H(\mathbf{q}, \mathbf{p}, t)$ into a new one $\tilde{H}(\tilde{\mathbf{q}}, \tilde{\mathbf{p}}, t)$. One may then exploit the arbitrariness of K to obtain the form of $\tilde{H}$ that is most convenient for solving the problem in hand. For instance, remembering the definition of cyclic coordinate given in Sect. 1.42, one may seek a transformation such that the new canonical coordinates $\tilde{q}_i$, $\tilde{p}_i$ are all cyclic. In this case, thanks to (2.2), it is $\mathrm{d}\tilde{q}_i/\mathrm{d}t = \partial\tilde{H}/\partial\tilde{p}_i = 0$, $\mathrm{d}\tilde{p}_i/\mathrm{d}t = -\partial\tilde{H}/\partial\tilde{q}_i = 0$, namely, each new canonical coordinate is a constant of motion.

The simplest way to obtain this result is to set the new Hamiltonian function equal to zero. Remembering from Sect. 2.2 that in every canonical transformation the relation between the old and new Hamiltonian function is $\tilde{H} = H + \partial K/\partial t$, one finds in this case the relation $\partial K/\partial t + H = 0$. To proceed it is convenient to choose a generating function of the $K_2 = K_2(\mathbf{q}, \tilde{\mathbf{p}}, t)$ type in which, as noted above, the new momenta $\tilde{p}_i$ are constants of motion. Given that the aim is to obtain the relation $\tilde{H} = 0$, the generating function of this problem is the particular function of the coordinates q_i, $\tilde{p}_i$, and t, that fulfills the equation $\partial K_2/\partial t + H = 0$. In other terms, the generating function becomes the problem's unknown. A comparison with (1.51) shows that the equation to be solved is that of Hamilton–Jacobi, and that K_2 coincides with Hamilton's principal function S.

As mentioned in Sect. 1.7, (1.51) is a first-order, partial differential equation in the unknown S and in the $n+1$ variables $q_1, \ldots, q_n, t$. As one of the $n+1$ integration constants can be set to zero, the actual number of integration constants is n. This seems contradictory, because the Hamilton–Jacobi equation is expected to be equivalent to those of Hamilton or Lagrange for the description of the system's motion. As a consequence, the number of constants involved should be $2n$. The contradiction is easily removed by observing that n more constants appear in S, to be identified with the new momenta $\tilde{p}_1, \ldots, \tilde{p}_n$: remembering the canonical transformations (2.9) to be used in connection with the generating function of the K_2 type one finds

$$p_i = \frac{\partial S}{\partial q_i}, \qquad \tilde{q}_i = \frac{\partial S}{\partial \tilde{p}_i}, \qquad i = 1, \ldots, n\,. \tag{2.14}$$

Calculating the first of (2.14) at the initial time $t = a$ yields a set of n algebraic equations in the n unknowns $\tilde{p}_1, \dots, \tilde{p}_n$. In fact, at $t = a$ the old canonical coordinates q_i, p_i are known because they are the problem's initial conditions. The solution of such algebraic equations yields the first set of motion's constants $\tilde{p}_1, \dots, \tilde{p}_n$. Then, one considers the second of (2.14) at $t = a$, whose right hand sides, at this point of the procedure, are known. As a consequence, the second of (2.14) yield the new generalized coordinates $\tilde{q}_1, \dots, \tilde{q}_n$, that are the second set of motion's constants.

It is worth observing that the procedure depicted above provides also the time evolution of the old canonical coordinates. In fact, after all constants have been calculated, Eqs. (2.14) form $2n$ relations in the $2n+1$ variables $q_1, \dots, q_n, p_1, \dots, p_n, t$. From them one extracts the relations $q_1 = q_1(t), \dots, p_n = p_n(t)$. This shows that the Hamilton–Jacobi picture is equivalent to those based on the Hamilton or Lagrange equations for the solution of the mechanical problem.

2.4 Separation—Hamilton's Characteristic Function

The Hamilton–Jacobi equation (1.51) can be recast in a more symmetric form by letting $q_{n+1} = t$ and incorporating $\partial S/\partial t = \partial S/\partial q_{n+1}$ into the other term:

$$C\left(q_1, \dots, q_{n+1}, \frac{\partial S}{\partial q_1}, \dots, \frac{\partial S}{\partial q_{n+1}}\right) = 0. \tag{2.15}$$

Solving (2.15) becomes simpler if one of the coordinates, say q_i, and the corresponding momentum $p_i = \partial S/\partial q_i$ appear in (2.15) only through a relation $c_i = c_i(q_i, \partial S/\partial q_i)$ that does not contain any other coordinate, nor derivatives with respect to them, nor time. In this case q_i is called *separable coordinate* and the solution of (2.15) can be written as $S = S_i + W_i$, where S_i depends only on q_i and W_i depends on the other coordinates and time [67, Sect. 48]. Replacing this expression of S into (2.15) and extracting c_i yields a relation of the form $C_i = c_i$ with

$$C_i = C_i\left(q_1, \dots, q_{i-1}, q_{i+1}, \dots, q_{n+1}, \frac{\partial W_i}{\partial q_1}, \dots, \frac{\partial W_i}{\partial q_{i-1}}, \frac{\partial W_i}{\partial q_{i+1}}, \dots, \frac{\partial W_i}{\partial q_{n+1}}\right),$$

$$c_i = c_i\left(q_i, \frac{\partial S_i}{\partial q_i}\right). \tag{2.16}$$

The equality $C_i = c_i$ must hold for any value of the coordinates. As this is possible only if the two sides are constant, $C_i = c_i$ separates and yields the pair

$$c_i\left(q_i, \frac{\partial S_i}{\partial q_i}\right) = c_{i0}, \tag{2.17}$$

$$C\left(q_1, \dots, q_{i-1}, q_{i+1}, \dots, q_{n+1}, \frac{\partial W_i}{\partial q_1}, \dots, \frac{\partial W_i}{\partial q_{i-1}}, \frac{\partial W_i}{\partial q_{i+1}}, \dots, \frac{\partial W_i}{\partial q_{n+1}}, c_{i0}\right) = 0,$$

where C does not contain q_i nor the corresponding derivative. The solution of the first of (2.17) provides $S_i(q_i)$. The latter contains two constants, namely, c_{i0} and the

integration constant. As noted earlier, the latter can be set to zero because an additive constant on S is irrelevant. In conclusion, c_{i0} is the only constant that remains after this step. In turn, the solution of the second of (2.17), which is an n-variable differential equation, contains n more constant, one of which is additive and can be disregarded. It follows that the total number of integration constants in the set (2.17) is still n.

If all coordinates are separable one has $S = \sum_{i=1}^{n} S_i(q_i)$ and the problem is solved by n individual integrations (an example is given in Sect. 3.10). In this case one says that the Hamilton–Jacobi equation is *completely separable*. A special case of separable coordinate is that of the cyclic coordinate. If q_i is cyclic, in fact, (2.17) reduces to $\partial S_i/\partial q_i = c_{i0}$, whence $S_i = c_{i0} q_i$ and

$S = c_{i0} q_i + W_i$. If the cyclic coordinate is $q_{n+1} = t$, the above becomes

$$\frac{\partial S_{n+1}}{\partial t} = -E, \qquad S_{n+1} = -Et, \tag{2.18}$$

where the symbol E is used for the constant $c_{n+1,0}$. It is worth noting that the units of E are always those of an energy regardless of the choice of the generalized coordinates q_i. Comparing (2.18) with (1.51) yields $H = E =$ cost, consistently with the hypothesis that H does not depend on t. Using the symbol W instead of W_{n+1} provides the pair

$$H\left(q_1, \ldots, q_n, \frac{\partial W}{\partial q_1}, \ldots, \frac{\partial W}{\partial q_n}\right) = E, \qquad S = W - Et, \tag{2.19}$$

that holds when H is a constant of motion. The first of (2.19) is a differential equation in the generalized coordinates only, called *time-independent Hamilton–Jacobi equation*. The unknown function W is called *Hamilton's characteristic function*.

2.5 Phase Velocity

The dynamics of a mechanical system can be obtained from Hamilton's principal function $S(\mathbf{q}, \tilde{\mathbf{p}}, t)$ as shown in Sect. 2.3. After S has been determined it is possible to build up an interesting geometrical construction, that is shown below. The indication of the constants $\tilde{p}_i$ is omitted for the sake of conciseness.

To begin, fix the time t and let $S(\mathbf{q}, t) = S_0$, where S_0 is some constant. This relation describes a surface belonging to the configuration space $q_1, \ldots, q_n$. Now change the time by $\mathrm{d}t$: the corresponding variation in S is obtained from (1.50) and reads $\mathrm{d}S = \sum_{i=1}^{n} p_i \mathrm{d}q_i - H\mathrm{d}t = \mathbf{p} \cdot \mathrm{d}\mathbf{q} - H\mathrm{d}t$. In this relation each component $p_i = \partial S/\partial q_i$ is calculated in terms of the coordinates $q_1, \ldots, q_n$ at the instant t, hence the vector $\mathbf{p} = \mathrm{grad}_{\mathbf{q}} S$ is a function of $\mathbf{q}$ calculated at that instant. If $\mathbf{q}$ belongs to the surface $S = S_0$, then $\mathbf{p}$ is normal to the surface at $\mathbf{q}$. Now let $S' = S + \mathrm{d}S = S_0$, where S_0 is the same constant as before. The relation $S'(\mathbf{q}, t) = S_0$ provides the new surface into which $S = S_0$ evolves in the interval $\mathrm{d}t$. As both $S = S_0$, $S + \mathrm{d}S = S_0$ hold, it must be $\mathrm{d}S = 0$, namely, $\mathbf{p} \cdot \mathrm{d}\mathbf{q} = H\mathrm{d}t$.

When H has no explicit dependence on t, thanks to (2.19) the relation $\mathbf{p}\cdot\mathrm{d}\mathbf{q} = H\,\mathrm{d}t$ becomes $\mathbf{p}\cdot\mathrm{d}\mathbf{q} = E\,\mathrm{d}t$, with $\mathbf{p} = \mathrm{grad}_{\mathbf{q}} W$. In this case, letting φ be the angle between the vectors $\mathbf{p}$ and $\mathrm{d}\mathbf{q}$ (whose moduli are indicated with p, $\mathrm{d}q$), and excluding the points where $\mathbf{p} = 0$, one obtains

$$\cos\varphi\,\frac{\mathrm{d}q}{\mathrm{d}t} = \frac{E}{p}. \tag{2.20}$$

The product $\cos\varphi\,\mathrm{d}q$ in (2.20) is the projection of $\mathrm{d}\mathbf{q}$ over the direction of $\mathbf{p}$, hence it provides the variation of $\mathbf{q}$ in the direction normal to the surface $S = S_0$. When the Cartesian coordinates are used, the product $\cos\varphi\,\mathrm{d}q$ is a length and the left hand side of (2.20) is a velocity, that provides the displacement of the point $\mathbf{q}$ during the time interval $\mathrm{d}t$ and in the direction normal to the surface $S = S_0$.

As shown above, the vector $\mathbf{p}$ is normal to the $S = S_0$ surface at each point of the latter. Consider for simplicity the case of a single particle of mass m in the conservative case, and use the Cartesian coordinates; from $\mathbf{p} = m\,\dot{\mathbf{q}}$ one finds that at each instant the surface $S = S_0$ is normal to the particle's trajectory. This makes the surface $S = S_0$ the analogue of the constant-phase surface found in the wave theory (e.g., Sect. 5.9). For this reason, $\cos\varphi\,\mathrm{d}q/\mathrm{d}t$ is called *phase velocity*.

Due to its definition, the phase velocity depends on the position $\mathbf{q}$ and is the velocity with which each point of the $S = S_0$ surface moves. It is worth adding that the phase velocity does not coincide with the actual velocity of any of the system's particles. To show this it suffices to consider the single particle's case with $\mathbf{p} = m\,\dot{\mathbf{q}}$: from (2.20) one finds that the modulus of the phase velocity is in fact inversely proportional to that of the particle.

2.6 Invariance Properties

A number of dynamical properties of the system of particles under consideration can be inferred directly from the form of the Lagrangian or Hamiltonian function, without the need of solving the motion's equations. An example is the conservation of the momentum conjugate to a cyclic coordinate (Sect. 1.4). Other properties are discussed below.

2.6.1 Time Reversal

It is found by inspection that the expression (1.22) of the system's kinetic energy in Cartesian coordinates is invariant when t is replaced with $-t$ (time reversal). A rather lengthy calculation based on the chain-differentiation rule shows that this property still holds after a coordinate transformation.

In some cases the whole Lagrangian function is invariant under time reversal. This makes the Lagrange equations (1.28) invariant as well. Assume that (1.28) are solved starting from a given initial condition at $t = a$ to the final instant $t = b$. Then,

replace t with $t' = -t$ and solve the Lagrange equations again, using $q_i(t = b)$ and $-\dot{q}_i(t = b)$ as initial conditions. Letting $q_i' = \mathrm{d}q_i/\mathrm{d}t' = -\dot{q}_i$, (1.28) become

$$\frac{\mathrm{d}}{\mathrm{d}(-t')}\frac{\partial L}{\partial(-\,q_i')} = \frac{\mathrm{d}}{\mathrm{d}t'}\frac{\partial L}{\partial q_i'} = \frac{\partial L}{\partial q_i}, \qquad i = 1, \ldots, n \tag{2.21}$$

where, due to the hypothesis of invariance, the Lagrangian function is the same as that used to describe the motion from $t = a$ to $t = b$. It follows that the trajectories of the second motion are equal to those of the first one. Moreover, the initial velocities of the second motion, calculated at $t' = -b$, are opposite to those of the first motion at the same point. Due to the arbitrariness of b, at each point of a trajectory the velocity described by the time t' is opposite to that described by t. A motion having this property is called *reversible*.

Taking the examples of Sect. 1.3 and remembering the form (1.12, 1.18) of the corresponding Lagrangian functions one finds that, in the first example, the motion is reversible if the potential energy V is invariant under time reversal, namely, $V(-t) = V(t)$, while in the second example the motion is reversible if $\varphi(-t) = \varphi(t)$ and $\mathbf{A}(-t) = -\mathbf{A}(t)$.

2.6.2 Translation of Time

Consider the case where the Hamiltonian function is invariant with respect to translations of the origin of time. The invariance holds also for an infinitesimal translation $\mathrm{d}t$, hence it is $\mathrm{d}H/\mathrm{d}t = 0$. In other terms H is a constant of motion. When this happens, as illustrated in Sect. 1.4, the Lagrangian and Hamiltonian functions have no explicit dependence on time, and vice versa.

2.6.3 Translation of the Coordinates

Another interesting case occurs when the Lagrangian function is invariant with respect to translations of the coordinates' origin. By way of example consider an N-particle system with no constraints, whence $n = 3N$, and use the Cartesian coordinates x_{js}. Here the first index is associated with the particles and the second one with the axes. Then, choose an infinitesimal translation $\mathrm{d}h_1$ in the direction of the first axis and, similarly, infinitesimal translations $\mathrm{d}h_2$ and $\mathrm{d}h_3$ in the other two directions. Thus, each coordinate x_{j1}, $j = 1, \ldots, N$ within the Lagrangian function is replaced by $x_{j1} + \mathrm{d}h_1$, and so on. The translational invariance then yields

$$\mathrm{d}L = \mathrm{d}h_1 \sum_{j=1}^{N} \frac{\partial L}{\partial x_{j1}} + \mathrm{d}h_2 \sum_{j=1}^{N} \frac{\partial L}{\partial x_{j2}} + \mathrm{d}h_3 \sum_{j=1}^{N} \frac{\partial L}{\partial x_{j3}} = 0\,. \tag{2.22}$$

Each sum in (2.22) vanishes independently of the others due to the arbitrariness of the translations. Taking the sum multiplying $\mathrm{d}h_1$ and using (1.28) yields

$$\sum_{j=1}^{N} \frac{\partial L}{\partial x_{j1}} = \sum_{j=1}^{N} \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial L}{\partial \dot{x}_{j1}} = \frac{\mathrm{d}}{\mathrm{d}t} \sum_{j=1}^{N} p_{j1} = \frac{\mathrm{d}P_1}{\mathrm{d}t} = 0, \tag{2.23}$$

where $P_1 = \sum_{j=1}^{N} p_{j1}$ is the first component of the *total momentum*

$$\mathbf{P} = \sum_{j=1}^{N} \mathbf{p}_j \tag{2.24}$$

of the system of particles. The other two components are treated in the same manner. In conclusion, if the Lagrangian function is invariant with respect to translations of the coordinates' origin, then the total momentum of the system is a constant of motion.

The above reasoning applies independently to each axis. As a consequence, if the Lagrangian function is such that the sum $\sum_{j=1}^{N} \partial L/\partial x_{j1}$ vanishes, while the analogous sums associated with the other two axes do not vanish, then P_1 is a constant of motion, while P_2, P_3 are not.

An important example of a Langrangian function, that is invariant with respect to translations of the coordinates' origin, is found when the force $\mathbf{F}_i$ acting on the ith particle derives from a potential energy V that depends only on the relative distances $r_{jk} = |\mathbf{r}_j - \mathbf{r}_k|$ among the particles, $k \neq j$. An example is given in Sect. 3.7.

2.6.4 Rotation of the Coordinates

Consider the case where the Lagrangian function is invariant with respect to rotations of the coordinates around an axis that crosses the origin. Like in Sect. 2.6.3 a system of N particles with no constraints is assumed, and the Cartesian coordinates are used. Let π be the plane that contains the origin and is normal to the rotation axis. It is convenient to use on π a polar reference (Sect. B.2) in which the rotation is defined over π by the angle φ. In turn, let ϑ_j be the angle between the rotation axis and the position vector $\mathbf{r}_j = (x_{j1}, x_{j2}, x_{j3})$ of the jth particle. The meaning of the angles is the same as in Fig. B.1, where the axes x, y define plane π; then, ϑ_j and $\mathbf{r}_j$ are represented by ϑ and $\mathbf{r}$ of the figure, respectively. If an infinitesimal rotation $\mathrm{d}\varphi$ is considered, the position vector $\mathbf{r}_j$ undergoes a variation $\mathrm{d}\mathbf{r}_j$ parallel to π and of magnitude $|\mathrm{d}\mathbf{r}_j| = r_j \sin\vartheta_j \, \mathrm{d}\varphi$. To specify the direction of $\mathrm{d}\mathbf{r}_j$ one takes the unit vector $\mathbf{a}$ of the rotation axis and associates to the rotation the vector $\mathbf{a}\,\mathrm{d}\varphi$ such that

$$\mathrm{d}\mathbf{r}_j = \mathbf{a}\,\mathrm{d}\varphi \wedge \mathbf{r}_j. \tag{2.25}$$

The corresponding variations $\mathrm{d}\dot{\mathbf{r}}_j$ of the velocities are found by differentiating (2.25) with respect to time. The variation of the Lagrangian function is

$$\mathrm{d}L = \sum_{j=1}^{N} \sum_{s=1}^{3} \left(\frac{\partial L}{\partial x_{js}} \mathrm{d}x_{js} + \frac{\partial L}{\partial \dot{x}_{js}} \mathrm{d}\dot{x}_{js} \right), \tag{2.26}$$

where the variations of the components are found from (2.25) and read $\mathrm{d}x_{js} = \mathrm{d}\varphi \left(\mathbf{a} \wedge \mathbf{r}_j\right)_s$, $\mathrm{d}\dot{x}_{js} = \mathrm{d}\varphi \left(\mathbf{a} \wedge \dot{\mathbf{r}}_j\right)_s$. Replacing the latter in (2.26) and using (1.28) yields

$$\mathrm{d}L = \mathrm{d}\varphi \sum_{j=1}^{N} \left(\mathbf{a} \wedge \mathbf{r}_j \cdot \dot{\mathbf{p}}_j + \mathbf{a} \wedge \dot{\mathbf{r}}_j \cdot \mathbf{p}_j\right). \tag{2.27}$$

Due to the rotational invariance, (2.27) vanishes for any $\mathrm{d}\varphi$. Letting the sum to vanish after exchanging the scalar and vector products, and remembering that $\mathbf{a}$ is constant, one finds

$$\mathbf{a} \cdot \sum_{j=1}^{N} \left(\mathbf{r}_j \wedge \dot{\mathbf{p}}_j + \dot{\mathbf{r}}_j \wedge \mathbf{p}_j\right) = \mathbf{a} \cdot \frac{\mathrm{d}}{\mathrm{d}t} \sum_{j=1}^{N} \mathbf{r}_j \wedge \mathbf{p}_j = \mathbf{a} \cdot \frac{\mathrm{d}\mathbf{M}}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t} \left(\mathbf{M} \cdot \mathbf{a}\right) = 0, \tag{2.28}$$

where

$$\mathbf{M} = \sum_{j=1}^{N} \mathbf{r}_j \wedge \mathbf{p}_j \tag{2.29}$$

is the *total angular momentum* of the system of particles. In conclusion, if the Lagrangian function is invariant with respect to rotations of the coordinates around an axis that crosses the origin, then the projection of the system's total angular momentum $\mathbf{M}$ over the rotation axis is a constant of motion.

2.7 Maupertuis Principle

Besides the Hamilton principle described in Sect. 1.3.4, other variational principles exist. Among them is the *Maupertuis*, or *least action*, principle, that applies to a particle subjected to conservative forces. Let $V = V(x_1, x_2, x_3)$ be the potential energy and $E = \text{const}$ the total energy, and let A and B indicate the two points of the (x_1, x_2, x_3) space that limit the trajectory of the particle. The Maupertuis principle states that the natural trajectory between A and B is the one that minimizes the functional

$$G = \int_{AB} \sqrt{E - V}\, \mathrm{d}s, \qquad \mathrm{d}s^2 = \mathrm{d}x_1^2 + \mathrm{d}x_2^2 + \mathrm{d}x_3^2, \tag{2.30}$$

where the integral is carried out along the trajectory. The form of (2.30) explains the term "least action": in fact, the relation $p^2/(2m) = m\,u^2/2 = E - V$ shows that the integrand $\sqrt{E - V}$ is proportional to the particle's momentum p; as a multiplicative constant is irrelevant for calculating the extremum functions, the minimization of G is equivalent to that of the action $\int_{AB} p\, \mathrm{d}s$.

To calculate the extremum functions of (2.30) it is convenient to parametrize the coordinates in the form $x_i = x_i(\xi)$, where the parameter ξ takes the same limiting values, say, $\xi = a$ at A and $\xi = b$ at B, for the natural and all the virtual trajectories. Letting $\dot{x}_i = \mathrm{d}x_i/\mathrm{d}\xi$ one finds $(\mathrm{d}s/\mathrm{d}\xi)^2 = \dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_3^2$ which, remembering (1.7), yields the extremum condition for (2.30):

$$\delta \int_a^b \theta \, \mathrm{d}\xi = 0, \qquad \theta = \sqrt{E - V}\sqrt{\dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_3^2}, \qquad \frac{\mathrm{d}}{\mathrm{d}\xi}\frac{\partial \theta}{\partial \dot{x}_i} = \frac{\partial \theta}{\partial x_i}. \tag{2.31}$$

The following relations are useful to work out the last of (2.31): $\mathrm{d}s/\mathrm{d}t = u = \sqrt{(2/m)(E - V)}$, $\mathrm{d}x_i = \dot{x}_i \, \mathrm{d}\xi$, $\mathrm{d}x_i/\mathrm{d}t = u_i$. One finds

$$\frac{\partial \theta}{\partial \dot{x}_i} = \sqrt{E - V}\,\frac{\dot{x}_i}{\mathrm{d}s/\mathrm{d}\xi} = \sqrt{\frac{m}{2}}\,u\,\frac{\dot{x}_i\,\mathrm{d}\xi}{\mathrm{d}s} = \sqrt{\frac{m}{2}}\,\frac{\mathrm{d}x_i}{\mathrm{d}t} = \sqrt{\frac{m}{2}}\,u_i, \tag{2.32}$$

$$\frac{\partial \theta}{\partial x_i} = \frac{\mathrm{d}s}{\mathrm{d}\xi}\,\frac{-\partial V/\partial x_i}{2\sqrt{E - V}} = \frac{\mathrm{d}s}{\mathrm{d}\xi}\,\frac{F_i}{2\sqrt{m/2}u} = \frac{\mathrm{d}t}{\mathrm{d}\xi}\,\frac{F_i}{\sqrt{2m}}, \tag{2.33}$$

with F_i the ith component of the force. The last of (2.31) then yields

$$\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\sqrt{\frac{m}{2}}\,u_i\right) = \frac{\mathrm{d}t}{\mathrm{d}\xi}\,\frac{F_i}{\sqrt{2m}}, \qquad F_i = m\,\frac{\mathrm{d}u_i}{\mathrm{d}\xi}\,\frac{\mathrm{d}\xi}{\mathrm{d}t} = m\,\frac{\mathrm{d}u_i}{\mathrm{d}t}. \tag{2.34}$$

In conclusion, the equation that provides the extremum condition for functional G is equivalent to Newton's second law $\mathbf{F} = m\mathbf{a}$.

2.8 Spherical Coordinates—Angular Momentum

Consider a single particle of mass m and use the transformation from the Cartesian (x, y, z) to the spherical (r, ϑ, φ) coordinates shown in Sect. B.1. The kinetic energy is given by (B.7), namely,

$$T = \frac{m}{2}\,(\dot{r}^2 + r^2\dot{\vartheta}^2 + r^2\dot{\varphi}^2 \sin^2\vartheta). \tag{2.35}$$

If the force acting onto the particle is derivable from a potential energy $V = V(x, y, z, t)$, the Lagrangian function in the spherical reference is $L = T - V(r, \vartheta, \varphi, t)$, where T is given by (2.35). The momenta conjugate to the spherical coordinates are

$$\begin{cases} p_r = \partial L/\partial \dot{r} = m\dot{r} \\ p_\vartheta = \partial L/\partial \dot{\vartheta} = mr^2\,\dot{\vartheta} \\ p_\varphi = \partial L/\partial \dot{\varphi} = mr^2\,\dot{\varphi}\,\sin^2\vartheta \end{cases} \tag{2.36}$$

Using (2.36), the kinetic energy is recast as

$$T = \frac{1}{2m}\left(p_r^2 + \frac{p_\vartheta^2}{r^2} + \frac{p_\varphi^2}{r^2\,\sin^2\vartheta}\right). \tag{2.37}$$

The components of the momentum $\mathbf{p}$ derived from the Lagrangian function written in the Cartesian coordinates are $m\dot{x}$, $m\dot{y}$, $m\dot{z}$. It follows that the components of the angular momentum $\mathbf{M} = \mathbf{r} \wedge \mathbf{p}$ written in the Cartesian and spherical references are

$$\begin{cases} M_x &= m\,(y\,\dot{z} - z\,\dot{y}) &= -mr^2\,(\dot{\vartheta}\,\sin\varphi + \dot{\varphi}\,\sin\vartheta\,\cos\vartheta\,\cos\varphi) \\ M_y &= m\,(z\,\dot{x} - x\,\dot{z}) &= mr^2\,(\dot{\vartheta}\,\cos\varphi - \dot{\varphi}\,\sin\vartheta\,\cos\vartheta\,\sin\varphi) \\ M_z &= m\,(x\,\dot{y} - y\,\dot{x}) &= mr^2\,\dot{\varphi}\,\sin^2\vartheta \end{cases} \tag{2.38}$$

The square modulus of the angular momentum in spherical coordinates reads

$$M^2 = m^2 r^4\,\left(\dot{\vartheta}^2 + \dot{\varphi}^2\,\sin^2\vartheta\right) = p_\vartheta^2 + \frac{p_\varphi^2}{\sin^2\vartheta}, \tag{2.39}$$

where the last equality is due to (2.36). From (2.37, 2.39) one finds

$$T = \frac{1}{2m}\left(p_r^2 + \frac{M^2}{r^2}\right). \tag{2.40}$$

If M is a constant of motion, (2.40) shows that the kinetic energy depends on r and $\dot{r}$ only. Comparing (2.36) with (2.38) one also notices that

$$p_\varphi = M_z, \tag{2.41}$$

namely, the component along the z axis of the angular momentum turns out to be the momentum conjugate to the φ coordinate. The latter describes the rotations along the same axis. In contrast, the other two components of $\mathbf{M}$ are not conjugate momenta. This result is due to the asymmetry of the relations (B.1) that connect the Cartesian to the spherical coordinates, and does not ascribe any privileged role to the z axis. In fact, by exchanging the Cartesian axes one makes p_φ to coincide with M_x or M_y.

Another example refers to a particle of mass m and charge e subjected to an electromagnetic field. Remembering (1.33) one has $L = (1/2)\,mu^2 - eU + e\mathbf{u}\cdot\mathbf{A}$, where the scalar potential is indicated with U to avoid confusion with the φ coordinate. It follows

$$L = \frac{1}{2}\,m\,\left(\dot{r}^2 + r^2\,\dot{\vartheta}^2 + r^2\,\dot{\varphi}^2\,\sin^2\vartheta\right) - e\,U + e\mathbf{u}\cdot\mathbf{A}, \tag{2.42}$$

where the components of $\mathbf{u} = \dot{\mathbf{r}}$ are given by (B.6), and U, A depend on the coordinates and time. Indicating the components of $\mathbf{A}$ with A_x, A_y, A_z, the momenta read

$$\begin{cases} p_r = \partial L/\partial\dot{r} = m\,\dot{r} + e\,A_x\,\sin\vartheta\,\cos\varphi + e\,A_y\,\sin\vartheta\,\sin\varphi + e\,A_z\,\cos\vartheta\,\cos\varphi \\ p_\vartheta = \partial L/\partial\dot{\vartheta} = m\,r^2\,\dot{\vartheta} + e\,A_x\,r\,\cos\vartheta\,\cos\varphi + e\,A_y\,r\,\cos\vartheta\,\sin\varphi - e\,A_z\,r\,\sin\vartheta \\ p_\varphi = \partial L/\partial\dot{\varphi} = m\,r^2\,\dot{\varphi}\,\sin^2\vartheta - e\,A_x\,r\,\sin\vartheta\,\sin\varphi + e\,A_y\,r\,\sin\vartheta\,\cos\varphi \end{cases} \tag{2.43}$$

Thanks to (B.1, B.6), the third of (2.43) can be written as

$$p_\varphi = m\,(x\,\dot{y} - y\,\dot{x}) + e\,(x\,A_y - y\,A_x) = x\,(m\,\dot{y} + e\,A_y) - y\,(m\,\dot{x} + e\,A_x), \tag{2.44}$$

that coincides with the component of the angular momentum $\mathbf{M} = \mathbf{r} \wedge \mathbf{p} = \mathbf{r} \wedge (m\,\mathbf{u} + e\mathbf{A})$ along the z axis. This result shows that (2.41) holds also when the force acting onto the particle derives from an electromagnetic field.

2.9 Linear Motion

The *linear motion* is the motion of a system having only one degree of freedom. Using the Cartesian coordinate x, and assuming the case where the force acting onto the particle derives from a potential energy $V(x)$, gives the Hamiltonian function (1.32) the form $H = p^2/(2\,m) + V(x)$. As shown in Sect. 1.4, a Hamiltonian function of this type is a constant of motion whence, remembering that here it is $p = m\,\dot{x}$,

$$\frac{1}{2}\,m\,\dot{x}^2 + V(x) = E = \text{const.} \tag{2.45}$$

The constant E is called *total energy*. Its value is given by the initial conditions $x_0 = x(t = a)$, $\dot{x}_0 = \dot{x}(t = a)$. As the kinetic energy $m\,\dot{x}^2/2$ can not be negative, the motion is possible only in the intervals of the x axis such that $V(x) \leq E$. In particular, the velocity $\dot{x}$ vanishes at the points where $V = E$. Instead, the intervals where $V > E$ can not be reached by the particle. Equation (2.45) is separable and provides a relation of the form $t = t(x)$,

$$t = a \pm \sqrt{\frac{m}{2}} \int_{x_0}^{x} \frac{\mathrm{d}\xi}{\sqrt{E - V(\xi)}}. \tag{2.46}$$

By way of example consider a situation like that shown in Fig. 2.1, where it is assumed that to the right of x_C the potential energy V keeps decreasing as $x \to \infty$. If the initial position of the particle is $x_0 = x_C$, there the velocity vanishes and the particle is subjected to a positive force $F = -\mathrm{d}V/\mathrm{d}x > 0$. As a consequence, the particle's motion will always be oriented to the right starting from x_C. Such a motion is called *unlimited*. If the initial position is $x_0 > x_C$ and the initial velocity is negative, the particle moves to the left until it reaches the position x_C, where it bounces back. The subsequent motion is the same as described above.

A different situation arises when the initial position of the particle belongs to an interval limited by two zeros of the function $E - V(x)$ like, e.g., x_A and x_B in Fig. 2.1. The motion is confined between x_A and x_B and, for this reason, is called *limited*. The particle bounces back and forth endlessly under the effect of a force that does not depend on time. As a consequence, the time necessary to complete a cycle $x_A \to x_B \to x_A$ is the same for each cycle. In other terms, the motion is periodic in time. Also, from (2.46) it is found by inspection that the time spent by the particle

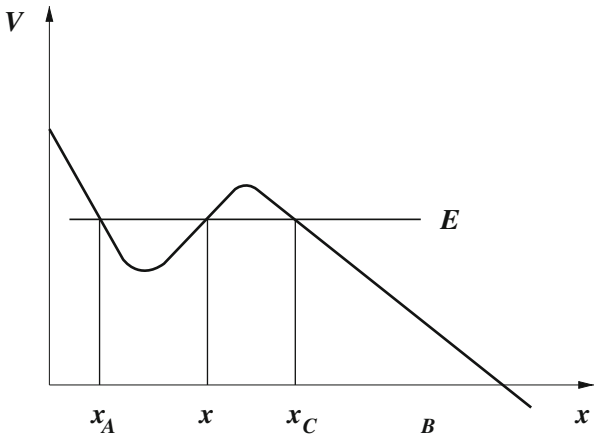

Fig. 2.1 Example of potential energy discussed in Sect. 2.9

in the $x_A \to x_B$ part of the cycle is the same as that spent in the $x_B \to x_A$ part. The period of the oscillation is then found to be

$$T = 2\sqrt{\frac{m}{2}} \int_{x_A}^{x_B} \frac{\mathrm{d}x}{\sqrt{E - V(x)}}. \tag{2.47}$$

Note that the period depends on the total energy E. However there are exceptions, as the example of Sect. 3.3 shows.

2.10 Action-Angle Variables

Consider a linear, conservative motion of constant energy E (Sect. 2.9) and let q, p be two canonical coordinates describing it. The following hold

$$H(q, p) = E, \qquad p = p(q, E). \tag{2.48}$$

The second of (2.48) is derived from the first one by solving for the momentum, and provides the phase trajectory (Sect. 1.9) starting from the initial conditions q_0, p_0 of the motion. As shown below, in several mechanical systems of interest the phase trajectory has some special characteristic that is worth examining.

Consider, first, the situation where the phase trajectory is closed: in this case, after a time T has elapsed from $t = 0$, the canonical coordinates take again the values q_0, p_0. As a consequence, for $t > T$ the motion repeats itself, and so on. It follows that both q and p are periodic functions of time with the same period T. As discussed in Sect. 2.9, this type of periodic motion, in which both q and p are bounded, is typically found when the initial position q_0 lies between two zeros of $E - V$, and is of the oscillatory type. It is also indicated with the astronomical term *libration*.

A second important situation occurs when p is a periodic function of q. In this type of motion q is unbounded. However, when q increases by a period the configuration

of the mechanical system remains practically unchanged. In fact, in this type of motion the canonical coordinate q is always an angle of rotation: the motion is still periodic and is referred to as *rotation*. Note that the same mechanical system may give rise to a libration or a rotation, depending on the motion's initial conditions: a typical example is that of the simple pendulum where q is identified with the angle of deflection [42, Chap. 10.6]. The *action variable* is defined as

$$J(E) = \oint p(q, E)\, \mathrm{d}q, \tag{2.49}$$

where the integral is carried out over a complete period of libration or rotation, depending on the case under investigation. The name given to J stems from the fact that, as mentioned in Sect. 1.5, the product $q\,p$ has the units of an action in all coordinate sets. The action variable is a constant of motion because it depends on E only. Inverting $J(E)$ yields $H = H(J)$. Now one applies a canonical transformation generated by a Hamilton characteristic function of the form $W = W(q, J)$. Remembering the procedure depicted in Sect. 2.4, W is the solution of

$$H\left(q, \frac{\partial W}{\partial q}\right) = E. \tag{2.50}$$

Applying (2.14) one finds the generalized coordinate $w = \partial W/\partial J$, called *angle variable*, conjugate to J. The pair J, w constitutes the set of canonical coordinates called *action-angle variables*. Finally, the Hamilton equations in the new coordinates read

$$\dot{w} = \frac{\partial H}{\partial J} = \text{const}, \qquad \dot{J} = -\frac{\partial H}{\partial w} = 0. \tag{2.51}$$

The time evolution of the action-angle variables is then $w = \dot{w}\, t + w_0$, $J = \text{const}$. From the first of (2.51) it also follows that the units of $\dot{w}$ are those of a frequency. The usefulness of the action-angle variables becomes apparent when one calculates the change Δw over a complete libration or rotation cycle of q:

$$\Delta w = \oint \mathrm{d}w = \oint \frac{\partial w}{\partial q}\, \mathrm{d}q = \oint \frac{\partial^2 W}{\partial q\, \partial J}\, \mathrm{d}q = \frac{\mathrm{d}}{\mathrm{d}J} \oint \frac{\partial W}{\partial q}\, \mathrm{d}q = 1, \tag{2.52}$$

where the last equality derives from combining $p = \partial W/\partial q$ with (2.49). On the other hand, if T is the time necessary for completing a cycle of q, then it is $\Delta w = w(T) - w(0) = \dot{w}\, T$, whence $\dot{w} = 1/T$. Thus, the frequency $\nu = \dot{w}$ is that associated with the periodic motion of q. In conclusion, the action-angle variables provide a straightforward method to determine the frequency of a periodic motion without the need of solving the motion equation. The method is applicable also to conservative systems having more than one degree of freedom, provided there exists at least one set of coordinates in which the Hamilton–Jacobi equation is completely separable [42, Chap. 10.7].

2.11 Complements

2.11.1 Infinitesimal Canonical Transformations

Consider a system with n degrees of freedom whose Hamiltonian function is $H(q_1, \dots, q_n, p_1, \dots, p_n, t)$. Remembering (1.42) one finds that the canonical coordinates at $t + \mathrm{d}t$ are expressed, in terms of the same coordinates at t, by the relations

$$q_i + \mathrm{d}q_i = q_i + \frac{\partial H}{\partial p_i}\,\mathrm{d}t, \qquad p_i + \mathrm{d}p_i = p_i - \frac{\partial H}{\partial q_i}\,\mathrm{d}t. \tag{2.53}$$

Letting $\tilde{q}_i = q_i + \mathrm{d}q_i$, $\tilde{p}_i = p_i + \mathrm{d}p_i$ gives (2.53) the same form as (2.1), namely, that of a coordinate transformation. It is interesting to check whether such a transformation is canonical. For this, one notes that the transformation (2.53) differs by infinitesimal quantities from the identical transformation $\tilde{q}_i = q_i$, $\tilde{p}_i = p_i$; as a consequence one expects the generating function of (2.53), if it exists, to differ by an infinitesimal function from $F_2 = \sum_{i=1}^{n} q_i\, \tilde{p}_i$ which, as shown in Sect. 2.2, generates the identical transformation. One then lets

$$K_2 = \sum_{i=1}^{n} q_i\, \tilde{p}_i + \epsilon\, G(\mathbf{q}, \tilde{\mathbf{p}}, t), \tag{2.54}$$

where ϵ is an infinitesimal quantity. From the first two equations in (2.9) it follows

$$p_i = \frac{\partial K_2}{\partial q_i} = \tilde{p}_i + \epsilon\, \frac{\partial G}{\partial q_i}, \qquad \tilde{q}_i = \frac{\partial K_2}{\partial \tilde{p}_i} = q_i + \epsilon\, \frac{\partial G}{\partial \tilde{p}_i}. \tag{2.55}$$

In the last term of (2.55) one may replace $\tilde{p}_i$ with p_i on account of the fact that the difference between $\epsilon\, \partial G/\partial \tilde{p}_i$ and $\epsilon\, \partial G/\partial p_i$ is infinitesimal of a higher order. Then, letting $\epsilon = \mathrm{d}t$, $G(\mathbf{q}, \mathbf{p}, t) = H(\mathbf{q}, \mathbf{p}, t)$, and $K_2 = \sum_{i=1}^{n} q_i\, \tilde{p}_i + H(\mathbf{q}, \mathbf{p}, t)\,\mathrm{d}t$, makes (2.55) identical to (2.53). Note that this replacement transforms the third of (2.9) into $\tilde{H} = H + (\partial H/\partial t)\,\mathrm{d}t$, as should be (compare with (1.44)).

The above reasoning shows that the $H\,\mathrm{d}t$ term in (2.54) generates a canonical transformation that produces the variations of the canonical coordinates in the time interval $\mathrm{d}t$. Such a transformation is called *infinitesimal canonical transformation*. On the other hand, as the application of more than one canonical transformation is still canonical, the evolution of the coordinates q_i, p_i during a finite interval of time can be thought of as produced by a succession of infinitesimal canonical transformations generated by the Hamiltonian function. In other terms, the Hamiltonian function generates the motion of the system.

2.11.2 Constants of Motion

It has been shown in Sect. 2.11.1 that a succession of infinitesimal canonical transformation generated by the Hamiltonian function determines the time evolution of

the canonical coordinates q_i, p_i. If such a succession starts with the initial conditions q_{i0}, p_{i0}, at some later time t the transformations Eqs. (2.53) take the form

$$q_i = q_i(\mathbf{q}_0, \mathbf{p}_0, t), \qquad p_i = p_i(\mathbf{q}_0, \mathbf{p}_0, t). \tag{2.56}$$

The relations (2.56) are nothing else than the solution of the mechanical problem; in fact, they express the canonical coordinates at time t, given the initial conditions. From another viewpoint, they show that the solution of the problem contains $2\,n$ constants. They are not necessarily constants of motion, in fact, their values at $t > 0$ are in general different from those at $t = 0$. If the system has extra properties (like, e.g., the invariance properties discussed in Sect. 2.6), it also has one or more constants of motion. The latter keep the value that they possessed at $t = 0$, so they are expressible as combinations of the canonical coordinates at $t = 0$; by way of example, the total energy E of a single particle subjected to a conservative force reads $E = (p_{01}^2 + p_{02}^2 + p_{03}^2)/(2\,m) + V(x_{10}, x_{20}, x_{30})$.

For a system having n degrees of freedom the total number of independent combinations of the initial conditions can not exceed the number of the initial conditions themselves. As a consequence, for such a system the maximum number of independent constants of motion is $2n$.

Problems

2.1 Given the Hamiltonian function $H = p^2/(2\,m) + (c/2)\,x^2$, $m, c > 0$ (that describes the *linear harmonic oscillator*, Sect. 3.3), find the oscillation frequency ν using the action-angle variables.