

Chapter 4

Electromagnetism

4.1 Introduction

This chapter outlines the basic principles of the electromagnetic theory *in vacuo*. First, the extension of the Lagrangian formalism to functions that depend on more than one variable is tackled: this yields useful tools for the analysis of continuous media. Next, the Maxwell equations are introduced along with the derivation of the electric and magnetic potentials, and the concept of gauge transformation is illustrated. The second part of the chapter is devoted to the Helmholtz and wave equations, both in a finite and infinite domain. The chapter finally introduces the Lorentz force, that connects the electromagnetic field with the particles' dynamics. The complements discuss some invariance properties of the Euler equations, derive the wave equations for the electric and magnetic field, and clarify some issues related to the boundary conditions in the application of the Green method to the boundary-value problem.

4.2 Extension of the Lagrangian Formalism

In Sect. 1.2 the derivation of the extremum functions has been carried out with reference to a functional $G[w]$ of the form (1.1). Such a functional contains one unknown function w that, in turn, depends on one independent variable ξ . The result has been extended to the case where the functional depends on several unknown functions w_1, w_2, \dots , each dependent on one variable only (compare with (1.6)). The extension to more than one independent variable is shown here.

To proceed it suffices to consider a single unknown function w that depends on two independent variables ξ, σ and is differentiable at least twice with respect to each. The first and second derivatives of w are indicated with $w_\xi, w_\sigma, w_{\xi\xi}, w_{\sigma\sigma}$, and $w_{\xi\sigma}$. Letting Ω be the domain over which w is defined, and g the generating function, the functional reads

$$G[w] = \int_{\Omega} g(w, w_\xi, w_\sigma, \xi, \sigma) d\Omega. \tag{4.1}$$

Then, let $\delta w = \alpha \eta$, with $\eta(\xi, \sigma)$ an arbitrary function defined in Ω and differentiable in its interior, and α a real parameter. Like in the case of one independent variable the choice is restricted to those functions η that vanish at the boundary of Ω , so that w and $w + \delta w$ coincide along the boundary for any value of α . If w is an extremum function of G , the extremum condition is found by replacing w with $w + \alpha \eta$ and letting $(dG/d\alpha)_0 = 0$, where suffix 0 indicates that the derivative is calculated at $\alpha = 0$ (compare with Sect. 1.2). Exchanging the integral with the derivative in (4.1) yields

$$\left(\frac{dG}{d\alpha}\right)_0 = \int_{\Omega} \left(\frac{\partial g}{\partial w} \eta + \frac{\partial g}{\partial w_{\xi}} \eta_{\xi} + \frac{\partial g}{\partial w_{\sigma}} \eta_{\sigma} \right) d\Omega = 0. \quad (4.2)$$

The second and third term of the integrand in (4.2) are recast in compact form by defining vector $\mathbf{u} = (\partial g/\partial w_{\xi}, \partial g/\partial w_{\sigma})$ and using the second identity in (A.16), so that the sum of the two terms reads $\mathbf{u} \cdot \text{grad} \eta = \text{div}(\eta \mathbf{u}) - \eta \text{div} \mathbf{u}$. Integrating over Ω and using the divergence theorem (A.23) yields

$$\int_{\Omega} \mathbf{u} \cdot \text{grad} \eta \, d\Omega = \int_{\Sigma} \eta \mathbf{u} \cdot \mathbf{n} \, d\Sigma - \int_{\Omega} \eta \text{div} \mathbf{u} \, d\Omega, \quad (4.3)$$

where Σ is the boundary of Ω and \mathbf{n} the unit vector normal to $d\Sigma$, oriented in the outward direction with respect to Σ . The first term at the right hand side of (4.3) is equal to zero because η vanishes over Σ .

It is important to clarify the symbols that will be used to denote the derivatives. In fact, to calculate $\text{div} \mathbf{u}$ one needs, first, to differentiate $\partial g/\partial w_{\xi}$ with respect to ξ considering also the implicit ξ -dependence within w , w_{ξ} , and w_{σ} ; then, one differentiates in a similar manner $\partial g/\partial w_{\sigma}$ with respect to σ . The two derivatives are summed up to form $\text{div} \mathbf{u}$. For this type of differentiation the symbols $d/d\xi$ and $d/d\sigma$ are used, even if the functions in hand depend on two independent variables instead of one. The symbols $\partial/\partial\xi$ and $\partial/\partial\sigma$ are instead reserved to the derivatives with respect to the explicit dependence on ξ or σ only. With this provision, inserting (4.3) into (4.2) yields the extremum condition

$$\int_{\Omega} \left(\frac{\partial g}{\partial w} - \frac{d}{d\xi} \frac{\partial g}{\partial w_{\xi}} - \frac{d}{d\sigma} \frac{\partial g}{\partial w_{\sigma}} \right) \eta \, d\Omega = 0. \quad (4.4)$$

As (4.4) holds for any η , the term in parentheses must vanish. In conclusion, the extremum condition is

$$\frac{d}{d\xi} \frac{\partial g}{\partial w_{\xi}} + \frac{d}{d\sigma} \frac{\partial g}{\partial w_{\sigma}} = \frac{\partial g}{\partial w}, \quad (4.5)$$

namely, a second-order partial-differential equation in the unknown function w , that must be supplemented with suitable boundary conditions. The equation is linear with respect to the second derivatives of w because g does not depend on such derivatives.

The result is readily extended to the case where g depends on several functions w_1, w_2, \dots, w_l and the corresponding derivatives. Defining the vectors $\mathbf{w}(\xi, \sigma) =$

(w_1, \dots, w_l) , $\mathbf{w}_\xi = (\partial w_1/\partial \xi, \dots, \partial w_l/\partial \xi)$, $\mathbf{w}_\sigma = (\partial w_1/\partial \sigma, \dots, \partial w_l/\partial \sigma)$, the set of the l extremum functions w_i of functional

$$G[\mathbf{w}] = \int_{\Omega} g(\mathbf{w}, \mathbf{w}_\xi, \mathbf{w}_\sigma, \xi, \sigma) \, d\Omega \quad (4.6)$$

is found by solving the equations

$$\frac{d}{d\xi} \frac{\partial g}{\partial(\partial w_i/\partial \xi)} + \frac{d}{d\sigma} \frac{\partial g}{\partial(\partial w_i/\partial \sigma)} = \frac{\partial g}{\partial w_i}, \quad i = 1, \dots, l, \quad (4.7)$$

supplemented with the suitable boundary conditions. It follows that (4.7) are the Euler equations of G . Finally, the case where the independent variables are more than two is a direct extension of (4.7). For instance, for m variables ξ_1, \dots, ξ_m one finds

$$\sum_{j=1}^m \frac{d}{d\xi_j} \frac{\partial g}{\partial(\partial w_i/\partial \xi_j)} = \frac{\partial g}{\partial w_i}, \quad i = 1, \dots, l. \quad (4.8)$$

If g is replaced with $g' = g + \text{div} \mathbf{h}$, where \mathbf{h} is an arbitrary vector of length m whose entries depend on \mathbf{w} and ξ_1, \dots, ξ_m , but not on the derivatives of \mathbf{w} , then (4.8) is still fulfilled. The replacement, in fact, adds the same term to both sides. For instance, the term added to the left hand side is

$$\sum_{j=1}^m \frac{d}{d\xi_j} \frac{\partial}{\partial(\partial w_i/\partial \xi_j)} \sum_{r=1}^m \left(\frac{\partial h_r}{\partial \xi_r} + \sum_{s=1}^l \frac{\partial h_r}{\partial w_s} \frac{\partial w_s}{\partial \xi_r} \right), \quad i = 1, \dots, l, \quad (4.9)$$

where the sum over r is the explicit expression of $\text{div} \mathbf{h}$. Remembering that \mathbf{h} does not depend on the derivatives of w_i one recasts (4.9) as

$$\sum_{j=1}^m \frac{d}{d\xi_j} \sum_{r=1}^m \sum_{s=1}^l \frac{\partial h_r}{\partial w_s} \frac{\partial(\partial w_s/\partial \xi_r)}{\partial(\partial w_i/\partial \xi_j)} = \sum_{j=1}^m \frac{\partial}{\partial \xi_j} \frac{\partial h_j}{\partial w_i}, \quad i = 1, \dots, l, \quad (4.10)$$

where the equality is due to the relation $\partial(\partial w_s/\partial \xi_r)/\partial(\partial w_i/\partial \xi_j) = \delta_{is} \delta_{jr}$, with $\delta_{is(jr)}$ the Kronecker symbol (A.18). Inverting the order of the derivatives at the right hand side of (4.10) yields $\partial \text{div} \mathbf{h}/\partial w_i$, that coincides with the term added to the right hand side of (4.8). Finally, (4.8) is recast in compact form by defining a vector \mathbf{u}_i and a scalar s_i as

$$\mathbf{u}_i = \left[\frac{\partial g}{\partial(\partial w_i/\partial \xi_1)}, \dots, \frac{\partial g}{\partial(\partial w_i/\partial \xi_m)} \right], \quad s_i = \frac{\partial g}{\partial w_i} \quad (4.11)$$

to find

$$\text{div}_\xi \mathbf{u}_i = s_i, \quad i = 1, \dots, l. \quad (4.12)$$

If w_i depends on one variable only, say ξ , (4.8, 4.12) reduce to (1.7). Using the language of the Lagrangian theory, the comparison between the one-dimensional and multi-dimensional case shows that in both cases the functions w_i play the role of generalized coordinates; in turn, the scalar parameter ξ of (1.7) becomes the vector (ξ_1, \dots, ξ_m) of (4.8) and, finally, each generalized velocity \dot{w}_i becomes the set $\partial w_i/\partial \xi_1, \dots, \partial w_i/\partial \xi_m$.

4.3 Lagrangian Function for the Wave Equation

It has been shown in Sect. 1.3 that the relations $\ddot{w}_i = \ddot{w}_i(\mathbf{w}, \dot{\mathbf{w}}, \xi)$, $i = 1, \dots, n$, describing the motion of a system of particles with n degrees of freedom, are the Euler equations of a suitable functional. Then, the analysis of Sect. 4.2 has shown that, when the unknown functions w_1, \dots, w_l depend on more than one variable, the Euler equations are the second-order partial-differential equations (4.8). The form (4.8) is typical of the problems involving continuous media (e.g., elasticity field, electromagnetic field). Following the same reasoning as in Sect. 1.3 it is possible to construct the Lagrangian function whence the partial-differential equation derives. This is done here with reference to the important case of the *wave equation*¹

$$\nabla^2 w - \frac{1}{u^2} \frac{\partial^2 w}{\partial t^2} = s, \quad (4.13)$$

where $u = \text{const}$ is a velocity and, for the sake of simplicity, s is assumed to depend on x and t , but not on w or its derivatives. It is worth noting that, when a differential equation other than Newton's law is considered, the corresponding Lagrangian function is not necessarily an energy. For this reason it will provisionally be indicated with L_e instead of L . To proceed one considers the one-dimensional form of (4.13), $\partial^2 w / \partial x^2 - (1/u^2) \partial^2 w / \partial t^2 = s$ and replaces ξ, σ, g with x, t, L_e , respectively. Then, one makes the one-dimensional form identical to (4.5) by letting

$$\frac{\partial^2 L_e}{\partial w_x^2} = 1, \quad \frac{\partial^2 L_e}{\partial w_t^2} = -\frac{1}{u^2}, \quad \frac{\partial L_e}{\partial w} = s, \quad (4.14)$$

with $w_x = \partial w / \partial x$ and $w_t = \partial w / \partial t$. The second derivatives of $L_e(w, w_x, w_t, x, t)$ with respect to the combinations of the arguments not appearing in the first two equations of (4.14) are set to zero. The third of (4.14) provides $L_e = s w + c$, with c independent of w . Replacing $L_e = s w + c$ into the first two equations in (4.14), and integrating the first one with respect to w_x , yields $\partial c / \partial w_x = w_x + a_{01}$, with a_{01} independent of w_x . Similarly, from the second equation in (4.14), $\partial c / \partial w_t = -w_t / u^2 + a_{02}$, with a_{02} independent of w_t . Also, remembering that c is independent of w , one finds that a_{01} and a_{02} do not depend on w either. Considering that all the second derivatives of L_e not appearing in (4.14) are equal to zero shows that a_{01} depends on t at most, while a_{02} depends on x at most. Integrating $\partial c / \partial w_x = w_x + a_{01}$ and $\partial c / \partial w_t = -w_t / u^2 + a_{02}$ one finds

$$c = \frac{1}{2} w_x^2 + a_{01}(t) w_x + a_{11}, \quad c = -\frac{1}{2u^2} w_t^2 + a_{02}(x) w_t + a_{12}, \quad (4.15)$$

where a_{11} does not depend on w or w_x , while a_{12} does not depend on w or w_t . Also, a_{11} can not depend on both t and w_t due to $\partial^2 L_e / (\partial t \partial w_t) = 0$; similarly, a_{12} can not depend on both x and w_x due to $\partial^2 L_e / (\partial x \partial w_x) = 0$. On the other hand, as both

¹ Also called *D'Alembert equation* in the homogeneous case.

(4.15) hold, a_{11} must coincide (apart from an additive constant) with the first two terms at the right hand side of the second equation in (4.15), and a_{12} must coincide with the first two terms at the right hand side of the first equation. In conclusion,

$$c = \frac{1}{2} w_x^2 - \frac{1}{2u^2} w_t^2 + a_{01}(t) w_x + a_{02}(x) w_t, \quad (4.16)$$

with $a_{01}(t)$, $a_{02}(x)$ arbitrary functions. The last two terms in (4.16) are equal to $d(a_{01} w)/dx + d(a_{02} w)/dt$, namely, they form the divergence of a vector. As shown in Sect. 4.2 such a vector is arbitrary, so it can be eliminated by letting $a_{01} = 0$, $a_{02} = 0$. The relation $L_e = sw + c$ then yields

$$L_e = \frac{1}{2} w_x^2 - \frac{1}{2u^2} w_t^2 + sw. \quad (4.17)$$

The generalization to the three-dimensional case (4.13) is immediate,

$$L_e = \frac{1}{2} |\text{grad}w|^2 - \frac{1}{2u^2} \left(\frac{\partial w}{\partial t} \right)^2 + sw. \quad (4.18)$$

with $|\text{grad}w|^2 = w_x^2 + w_y^2 + w_z^2$.

4.4 Maxwell Equations

The *Maxwell equations*, that describe the electromagnetic field, lend themselves to an interesting application of the results of Sect. 4.3. The first group of Maxwell equations reads

$$\text{div}\mathbf{D} = \rho, \quad \text{rot}\mathbf{H} - \frac{\partial\mathbf{D}}{\partial t} = \mathbf{J}, \quad (4.19)$$

where \mathbf{D} is the electric displacement and \mathbf{H} the magnetic field.² The sources of the electromagnetic field are the charge density ρ and the current density \mathbf{J} . When point-like charges are considered, they read

$$\rho_c = \sum_j e_j \delta(\mathbf{r} - \mathbf{s}_j(t)), \quad \mathbf{J}_c = \sum_j e_j \delta(\mathbf{r} - \mathbf{s}_j(t)) \mathbf{u}_j(t), \quad (4.20)$$

where index c is used to distinguish the case of point-like charges from that of a continuous charge distribution. In (4.20), e_j is the value of the j th charge, \mathbf{s}_j and

² The units in (4.19, 4.23, 4.24) are: $[\mathbf{D}] = \text{C m}^{-2}$, $[\rho] = \text{C m}^{-3}$, $[\mathbf{H}] = \text{A m}^{-1}$, $[\mathbf{J}] = \text{C s}^{-1} \text{m}^{-2} = \text{A m}^{-2}$, $[\mathbf{B}] = \text{V s m}^{-2} = \text{Wb m}^{-2} = \text{T}$, $[\mathbf{E}] = \text{V m}^{-1}$, where ‘‘C’’, ‘‘A’’, ‘‘V’’, ‘‘Wb’’, and ‘‘T’’ stand for Coulomb, Ampere, Volt, Weber, and Tesla, respectively. The coefficients in (4.19, 4.23, 4.24) differ from those of [4] because of the different units adopted there. In turn, the units in (4.25) are $[\epsilon_0] = \text{C V}^{-1} \text{m}^{-1} = \text{F m}^{-1}$, $[\mu_0] = \text{s}^2 \text{F}^{-1} \text{m}^{-1} = \text{H m}^{-1}$, where ‘‘F’’ and ‘‘H’’ stand for Farad and Henry, respectively, and those in (4.26) are $[\varphi] = \text{V}$, $[\mathbf{A}] = \text{V s m}^{-1} = \text{Wb m}^{-1}$.

\mathbf{u}_j its position and velocity at time t , respectively, and \mathbf{r} the independent positional variable. If the spatial scale of the problem is such that one can replace the point-like charges with a continuous distribution, one applies the same procedure as in Sect. 23.2. The number of charges belonging to a cell of volume ΔV centered at \mathbf{r} is $\int_{\Delta V} \rho_c d^3s' = \sum'_j e_j$, where the prime indicates that the sum is limited to the charges that belong to Δ at time t . Then one defines $\rho(\mathbf{r}, t) = \sum'_j e_j / \Delta V$. The continuous distribution of the current density is obtained in a similar manner,

$$\mathbf{J} = \frac{1}{\Delta V} \int_{\Delta V} \mathbf{J}_c d^3s' = \frac{1}{\Delta V} \sum'_j e_j \mathbf{u}_j = \rho \mathbf{v}, \quad \mathbf{v} = \frac{\sum'_j e_j \mathbf{u}_j}{\sum'_j e_j}, \quad (4.21)$$

with $\mathbf{v}(\mathbf{r}, t)$ the average velocity of the charges. If all charges are equal, $e_1 = e_2 = \dots = e$, then $\rho = e N$, with $N(\mathbf{r}, t)$ the concentration, and $\mathbf{J} = e N \mathbf{v} = e \mathbf{F}$, with $\mathbf{F}(\mathbf{r}, t)$ the flux density (compare with the definitions of Sect. 23.2). If the charges are different from each other it is convenient to distribute the sum \sum'_j over the groups made of equal charges. In this case the charge density and current density read

$$\rho = \rho_1 + \rho_2 + \dots, \quad \mathbf{J} = \rho_1 \mathbf{v}_1 + \rho_2 \mathbf{v}_2 + \dots, \quad (4.22)$$

where ρ_1, \mathbf{v}_1 are the charge density and average velocity of the charges of the first group, and so on. Taking the divergence of the second equation in (4.19) and using the third identity in (A.35) yields the *continuity equation*

$$\frac{\partial \rho}{\partial t} + \operatorname{div} \mathbf{J} = 0. \quad (4.23)$$

Apart from the different units of the functions involved, the form of (4.23) is the same as that of (23.3). The meaning of (4.23) is that of conservation of the electric charge. The second group of Maxwell equations is

$$\operatorname{div} \mathbf{B} = 0, \quad \operatorname{rot} \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0, \quad (4.24)$$

where \mathbf{B} and \mathbf{E} are the magnetic induction and the electric field, respectively. Here the Maxwell equations are considered *in vacuo*, so that the following hold

$$\mathbf{D} = \varepsilon_0 \mathbf{E}, \quad \mathbf{B} = \mu_0 \mathbf{H}, \quad \frac{1}{\sqrt{\varepsilon_0 \mu_0}} = c, \quad (4.25)$$

with $\varepsilon_0 \simeq 8.854 \times 10^{-12} \text{ F m}^{-1}$ and $\mu_0 \simeq 1.256 \times 10^{-6} \text{ H m}^{-1}$ the vacuum permittivity and permeability, respectively, and $c \simeq 2.998 \times 10^8 \text{ m s}^{-1}$ the speed of light *in vacuo*.

4.5 Potentials and Gauge Transformations

Thanks to (4.25), the electromagnetic field *in vacuo* is determined by two suitably-chosen vectors—typically, \mathbf{E} and \mathbf{B} —out of the four ones appearing in (4.25). This amounts to using six scalar functions of position and time. However, the number of scalar functions is reduced by observing that, while (4.19) provide relations between the electromagnetic field and its sources, (4.24) provide relations among the field vectors themselves; as a consequence, (4.24) reduce the number of independent vectors. In fact, using the properties illustrated in Sect. A.9, one finds that from $\operatorname{div}\mathbf{B} = 0$ one derives $\mathbf{B} = \operatorname{rot}\mathbf{A}$, where \mathbf{A} is called *vector potential* or *magnetic potential*. In turn, the vector potential transforms the second of (4.24) into $\operatorname{rot}(\mathbf{E} + \partial\mathbf{A}/\partial t) = 0$; using again the results of Sect. A.9 shows that the term in parentheses is the gradient of a scalar function, that is customarily indicated with $-\varphi$. Such a function³ is called *scalar potential* or *electric potential*. In summary,

$$\mathbf{B} = \operatorname{rot}\mathbf{A}, \quad \mathbf{E} = -\operatorname{grad}\varphi - \frac{\partial\mathbf{A}}{\partial t}, \quad (4.26)$$

showing that for determining the electromagnetic field *in vacuo* it suffices to know four scalar functions, namely, φ and the three components of \mathbf{A} . To proceed, one replaces (4.26) into (4.19) and uses the third relation in (4.25), to find

$$\nabla^2\varphi + \frac{\partial}{\partial t}\operatorname{div}\mathbf{A} = -\frac{\rho}{\varepsilon_0}, \quad -\operatorname{rot}\operatorname{rot}\mathbf{A} - \frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2} = -\mu_0\mathbf{J} + \frac{1}{c^2}\operatorname{grad}\frac{\partial\varphi}{\partial t}. \quad (4.27)$$

Thanks to the first identity in (A.36) the second equation in (4.27) becomes

$$\nabla^2\mathbf{A} - \frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2} = -\mu_0\mathbf{J} + \operatorname{grad}\theta, \quad \theta = \operatorname{div}\mathbf{A} + \frac{1}{c^2}\frac{\partial\varphi}{\partial t} \quad (4.28)$$

while, using the definition (4.28) of θ , one transforms the first equation in (4.27) into

$$\nabla^2\varphi - \frac{1}{c^2}\frac{\partial^2\varphi}{\partial t^2} = -\frac{\rho}{\varepsilon_0} - \frac{\partial\theta}{\partial t}. \quad (4.29)$$

In conclusion, (4.29) and the first equation in (4.28) are a set of four scalar differential equations whose unknowns are φ and the components of \mathbf{A} . Such equations are coupled because θ contains all unknowns; however, they become decoupled after suitable transformations, shown below.

To proceed, one observes that only the derivatives of the potentials, not the potential themselves, appear in (4.26); as a consequence, while the fields \mathbf{E} , \mathbf{B} are uniquely defined by the potentials, the opposite is not true. For instance, replacing \mathbf{A}

³ The minus sign in the definition of φ is used for consistency with the definition of the gravitational potential, where the force is opposite to the direction along which the potential grows.

with $\mathbf{A}' = \mathbf{A} + \text{grad} f$, where $f(\mathbf{r}, t)$ is any differentiable scalar function, and using the second identity in (A.35), yields $\mathbf{B}' = \text{rot} \mathbf{A}' = \mathbf{B}$, namely, \mathbf{B} is invariant with respect to such a replacement. If, at the same time, one replaces φ with a yet undetermined function φ' , (4.26) yields $\mathbf{E}' = -\text{grad}(\varphi' + \partial f/\partial t) - \partial \mathbf{A}'/\partial t$. It follows that by choosing $\varphi' = \varphi - \partial f/\partial t$ one obtains $\mathbf{E}' = \mathbf{E}$. The transformation $(\varphi, \mathbf{A}) \rightarrow (\varphi', \mathbf{A}')$ defined by

$$\varphi' = \varphi - \frac{\partial f}{\partial t}, \quad \mathbf{A}' = \mathbf{A} + \text{grad} f. \quad (4.30)$$

is called *gauge transformation*. As shown above, \mathbf{E} and \mathbf{B} are invariant with respect to such a transformation. One also finds that (4.29) and the first equation in (4.28) are invariant with respect to the transformation: all terms involving f cancel each other, so that the equations in the primed unknowns are identical to the original ones. However, the solutions φ', \mathbf{A}' are different from φ, \mathbf{A} because, due to (4.30), their initial and boundary conditions are not necessarily the same. The difference between the primed and unprimed solutions is unimportant because the fields, as shown above, are invariant under the transformation. Using (4.30) in the second equation of (4.28) shows that θ transforms as

$$\theta' = \text{div} \mathbf{A}' + \frac{1}{c^2} \frac{\partial \varphi'}{\partial t} = \theta + \nabla^2 f - \frac{1}{c^2} \frac{\partial^2 f}{\partial t^2}. \quad (4.31)$$

The arbitrariness of f may be exploited to give θ' a convenient form. For instance one may choose f such that $\theta' = (1/c^2) \partial \varphi'/\partial t$, which is equivalent to letting

$$\text{div} \mathbf{A}' = 0, \quad (4.32)$$

called *Coulomb gauge*. The latter yields

$$\nabla^2 \varphi' = -\frac{\rho}{\varepsilon_0}, \quad \nabla^2 \mathbf{A}' - \frac{1}{c^2} \frac{\partial^2 \mathbf{A}'}{\partial t^2} = -\mu_0 \mathbf{J} + \frac{1}{c^2} \frac{\partial}{\partial t} \text{grad} \varphi', \quad (4.33)$$

the first of which (the *Poisson equation*) is decoupled from the second one. After solving the Poisson equation, the last term at the right hand side of the second equation is not an unknown any more, this showing that the equations resulting from the Coulomb gauge are indeed decoupled. Another possibility is choosing f such that $\theta' = 0$, which is equivalent to letting

$$\text{div} \mathbf{A}' = -\frac{1}{c^2} \frac{\partial \varphi'}{\partial t}, \quad (4.34)$$

called *Lorentz gauge*. This transformation yields

$$\nabla^2 \varphi' - \frac{1}{c^2} \frac{\partial^2 \varphi'}{\partial t^2} = -\frac{\rho}{\varepsilon_0}, \quad \nabla^2 \mathbf{A}' - \frac{1}{c^2} \frac{\partial^2 \mathbf{A}'}{\partial t^2} = -\mu_0 \mathbf{J}. \quad (4.35)$$

that are decoupled and have the form of the wave Eq. (4.13). Another interesting application of the gauge transformation is shown in Sect. 5.11.4.

4.6 Lagrangian Density for the Maxwell Equations

To apply the Lagrangian formalism to the Maxwell equations it is useful to use the expressions (4.26) of the fields in terms of the potentials. It follows that the functions playing the role of generalized coordinates and generalized velocities are φ , A_i , and, respectively, $\partial\varphi/\partial x_k$, $\partial A_i/\partial x_k$, $\partial A_i/\partial t$, with $i, k = 1, 2, 3$, $k \neq i$. The Lagrangian density, whose units are J m^{-3} , then reads

$$L_e = \frac{\varepsilon_0}{2} E^2 - \frac{1}{2\mu_0} B^2 - \rho\varphi + \mathbf{J} \cdot \mathbf{A}, \quad (4.36)$$

with

$$E^2 = \left(\frac{\partial\varphi}{\partial x_1} + \frac{\partial A_1}{\partial t} \right)^2 + \left(\frac{\partial\varphi}{\partial x_2} + \frac{\partial A_2}{\partial t} \right)^2 + \left(\frac{\partial\varphi}{\partial x_3} + \frac{\partial A_3}{\partial t} \right)^2 \quad (4.37)$$

and

$$B^2 = \left(\frac{\partial A_3}{\partial x_2} - \frac{\partial A_2}{\partial x_3} \right)^2 + \left(\frac{\partial A_1}{\partial x_3} - \frac{\partial A_3}{\partial x_1} \right)^2 + \left(\frac{\partial A_2}{\partial x_1} - \frac{\partial A_1}{\partial x_2} \right)^2, \quad (4.38)$$

To show that (4.36) is in fact the Lagrangian function of the Maxwell equations one starts with the generalized coordinate φ , to find $\partial L_e/\partial\varphi = -\rho$. Then, considering the k th component,

$$\frac{\partial L_e/\varepsilon_0}{\partial(\partial\varphi/\partial x_k)} = \frac{\partial E^2/2}{\partial(\partial\varphi/\partial x_k)} = \frac{\partial E^2/2}{\partial(\partial A_k/\partial t)} = \frac{\partial\varphi}{\partial x_k} + \frac{\partial A_k}{\partial t} = -E_k = -\frac{D_k}{\varepsilon_0}. \quad (4.39)$$

Using (4.8) after replacing g with L_e , ξ_j with x_j , and w_i with φ yields $\text{div}\mathbf{D} = \rho$, namely, the first equation in (4.19). Turning now to another generalized coordinate, say, A_1 , one finds $\partial L_e/\partial A_1 = J_1$. As L_e depends on the spatial derivatives of A_1 only through B^2 , (4.38) and the first of (4.26) yield

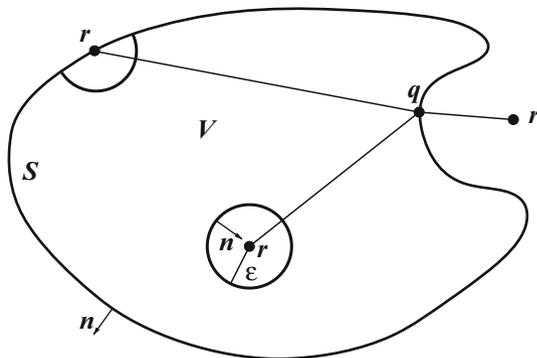
$$\frac{\partial B^2/2}{\partial(\partial A_1/\partial x_3)} = \frac{\partial A_1}{\partial x_3} - \frac{\partial A_3}{\partial x_1} = B_2, \quad \frac{\partial B^2/2}{\partial(\partial A_1/\partial x_2)} = \frac{\partial A_1}{\partial x_2} - \frac{\partial A_2}{\partial x_1} = -B_3. \quad (4.40)$$

In contrast, L_e depends on the time derivative of A_1 only through E^2 , as shown by (4.39). To use (4.8) one replaces g with L_e and w_i with A_1 , then takes the derivative with respect to x_3 in the first relation in (4.40), the derivative with respect to x_2 in the second relation, and the derivative with respect to t of the last term in (4.39). In summary this yields

$$\frac{1}{\mu_0} \left(\frac{\partial B_3}{\partial x_2} - \frac{\partial B_2}{\partial x_3} \right) - \frac{\partial D_1}{\partial t} = J_1, \quad (4.41)$$

namely, the first component of the second equation in (4.19).

Fig. 4.1 The domain V used for the solution of the Helmholtz equation (4.43). The three possible positions of point \mathbf{r} are shown: external to V , internal to V , or on the boundary S



4.7 Helmholtz Equation

Consider the wave equations (4.35) and assume that the charge density ρ and current density \mathbf{J} are given as functions of position and time. In the following, the apex in φ and \mathbf{A} will be dropped for the sake of conciseness. The four scalar equations (4.35) are linear with respect to the unknowns and have the same structure; also, their coefficients and unknowns are all real. The solution of (4.35) will be tackled in this section and in the following ones, basing upon the Fourier transform whose general properties are depicted in Sect. C.2. This solution procedure involves the use of complex functions. The starting assumption is that the condition for the existence of the Fourier transform with respect to time holds (such a condition is found by replacing x with t in (C.19)). Then one obtains

$$\nabla^2 \mathcal{F}_t \varphi + \frac{\omega^2}{c^2} \mathcal{F}_t \varphi = -\frac{1}{\epsilon_0} \mathcal{F}_t \rho, \quad \nabla^2 \mathcal{F}_t \mathbf{A} + \frac{\omega^2}{c^2} \mathcal{F}_t \mathbf{A} = -\mu_0 \mathcal{F}_t \mathbf{J}. \quad (4.42)$$

Indicating with f the transform of φ or A_i , and with b the transform of $-\rho/\epsilon_0$ or $-\mu_0 J_i$, $i = 1, 2, 3$, and letting $k^2 = \omega^2/c^2$, each scalar equation in (4.42) has the form of the *Helmholtz equation*

$$\nabla^2 f + k^2 f = b. \quad (4.43)$$

The solution of (4.43) is sought within a finite domain V (Fig. 4.1), for a given set of boundary conditions defined over the boundary S of V , and for a given right hand side b defined within V and over S . Let $\mathbf{r} = (x, y, z)$ be a point external to V , $\mathbf{q} = (\xi, \eta, \zeta)$ a point internal to V , and

$$\mathbf{g} = \mathbf{r} - \mathbf{q}, \quad g = [(x - \xi)^2 + (y - \eta)^2 + (z - \zeta)^2]^{1/2} \quad (4.44)$$

where, by construction, it is $g > 0$. In the following calculation, \mathbf{r} is kept fixed while \mathbf{q} varies. As a consequence, the derivatives of g are calculated with respect to ξ , η , and ζ . It is easily shown that in a three-dimensional space the auxiliary function

$$G(g) = \frac{1}{g} \exp(-i k g), \quad k \text{ real}, \quad (4.45)$$

fulfills the homogeneous Helmholtz equation $\nabla^2 G + k^2 G = 0$. Using the procedure that leads to the second Green theorem (Sect. A.5, Eq. (A.25)) yields the integral relation

$$\int_S \left(G \frac{\partial f}{\partial n} - f \frac{\partial G}{\partial n} \right) dS = \int_V G b dV, \quad (4.46)$$

where the unit vector \mathbf{n} over S is oriented in the outward direction and, by construction, point \mathbf{r} is external to V .

4.8 Helmholtz Equation in a Finite Domain

The relation (4.46) would not be applicable if \mathbf{r} were internal to V , because G diverges for $g \rightarrow 0$ and, as a consequence, is not differentiable in $\mathbf{q} = \mathbf{r}$. On the other hand, in many applications \mathbf{r} happens to be internal to V . In such cases one must exclude from the integral a suitable portion of volume V ; this is achieved by considering a spherical domain of radius ε centered on \mathbf{r} and internal to V (Fig. 4.1). Letting $V_\varepsilon, S_\varepsilon$ be, respectively, the volume and surface of such a sphere, and considering the new volume $V' = V - V_\varepsilon$, having $S' = S \cup S_\varepsilon$ as boundary, makes (4.46) applicable to V' , to yield

$$\int_S \left(G \frac{\partial f}{\partial n} - f \frac{\partial G}{\partial n} \right) dS + \int_{S_\varepsilon} (\dots) dS_\varepsilon = \int_V G b dV - \int_{V_\varepsilon} G b dV_\varepsilon, \quad (4.47)$$

where the dots indicate that the integrand is the same as in the first integral at the left hand side. Over S_ε it is $G = (1/\varepsilon) \exp(-ik\varepsilon)$, with the unit vector \mathbf{n} pointing from the surface towards the center of the sphere, namely, opposite to the direction along which ε increases. It follows $\partial G/\partial n = -\partial G/\partial \varepsilon = (ik + 1/\varepsilon)G$. Letting $[f]$ and $[\partial f/\partial n]$ be the average values of f and, respectively, $\partial f/\partial n$ over S_ε , and observing that G and $\partial G/\partial \varepsilon$ are constant there, yields

$$\int_{S_\varepsilon} \left(G \frac{\partial f}{\partial n} - f \frac{\partial G}{\partial n} \right) dS_\varepsilon = 4\pi \exp(-ik\varepsilon) \left(\varepsilon \left[\frac{\partial f}{\partial n} \right] - (1 + ik\varepsilon)[f] \right). \quad (4.48)$$

As for the integral $I = \int_{V_\varepsilon} G b dV_\varepsilon$ it is useful to adopt the spherical coordinates (B.1) after shifting the origin to the center of the sphere. In the new reference it is $\mathbf{r} = 0$, so that the radial coordinate coincides with g . It follows

$$I = \int_0^\varepsilon \int_0^\pi \int_0^{2\pi} g \sin \vartheta \exp(-ikg) b(g, \vartheta, \varphi) dg d\vartheta d\varphi. \quad (4.49)$$

Taking the absolute value of I and observing that g and $\sin \vartheta$ are positive yields $|I| \leq 2\pi \varepsilon^2 \sup_{V_\varepsilon} |b|$. To proceed, one assumes that f and b are sufficiently smooth as to fulfill the conditions

$$\lim_{\varepsilon \rightarrow 0} \varepsilon [f] = 0, \quad \lim_{\varepsilon \rightarrow 0} \varepsilon \left[\frac{\partial f}{\partial n} \right] = 0, \quad \lim_{\varepsilon \rightarrow 0} \varepsilon^2 \sup_{V_\varepsilon} |b| = 0. \quad (4.50)$$

Thanks to (4.50) one restores the original volume V by taking the limit $\varepsilon \rightarrow 0$. Observing that $\lim_{\varepsilon \rightarrow 0} [f] = f(\mathbf{r})$, one finds

$$4\pi f(\mathbf{r}) = \int_S \left(G \frac{\partial f}{\partial n} - f \frac{\partial G}{\partial n} \right) dS - \int_V G b dV, \quad (4.51)$$

that renders $f(\mathbf{r})$ as a function of b , of the boundary values of f and $\partial f/\partial n$, and of the auxiliary function G . It is easily found that, if \mathbf{r} were on the boundary S instead of being internal to V , the left hand side of (4.51) would be replaced by $2\pi f(\mathbf{p})$. Similarly, if \mathbf{r} were external to V , the left hand side would be zero. In conclusion one generalizes (4.51) to

$$\omega_r f(\mathbf{r}) = \int_S \left(G \frac{\partial f}{\partial n} - f \frac{\partial G}{\partial n} \right) dS - \int_V G b dV, \quad (4.52)$$

where ω_r is the solid angle under which the surface S is seen from \mathbf{r} considering the orientation of the unit vector \mathbf{n} . Namely, $\omega_r = 0$, $\omega_r = 2\pi$, or $\omega_r = 4\pi$ when \mathbf{r} is external to V , on the boundary of V , or internal to V , respectively.

Letting $k = 0$ in (4.45), namely, taking $G = 1/g$, makes the results of this section applicable to the Poisson equation $\nabla^2 f = b$. It must be noted that (4.52) should be considered as an integral relation for f , not as the solution of the differential equation whence it derives. In fact, for actually calculating (4.52) it is necessary to prescribe both f and $\partial f/\partial n$ over the boundary. This is an overspecification of the problem: in fact, the theory of boundary-value problems shows that the solution of an equation of the form (4.43) is found by specifying over the boundary either the unknown function only (*Dirichlet boundary condition*), or its normal derivative only (*Neumann boundary condition*). To find a solution starting from (4.52) it is necessary to carry out more steps, by which either f or $\partial f/\partial n$ is eliminated from the integral at the right hand side [53, Sect. 1.8–1.10]. In contrast, when the solution is sought in a domain whose boundary extends to infinity, and the contribution of the boundary conditions vanish as shown in Sect. 4.9, the limiting case of (4.52) provides a solution proper. More comments about this issue are made in Sect. 4.12.3.

4.9 Solution of the Helmholtz Equation in an Infinite Domain

The procedure shown in Sect. 4.8 is readily extended to the case $V \rightarrow \infty$. Here one may replace S with a spherical surface of radius $R \rightarrow \infty$, centered on \mathbf{r} ; this makes the calculation of the integral over S similar to that over the sphere of radius ε outlined in Sect. 4.8, the only difference being that the unit vector \mathbf{n} now points in the direction where R increases. Shifting the origin to \mathbf{r} and observing that $\omega_r = 4\pi$ yields

$$\int_S \left(G \frac{\partial f}{\partial n} - f \frac{\partial G}{\partial n} \right) dS = 4\pi \exp(-ikR) \left(R \left[\frac{\partial f}{\partial n} \right] + (1 + ikR)[f] \right), \quad (4.53)$$

where the averages are calculated over S . To proceed one assumes that the following relations hold,

$$\lim_{R \rightarrow \infty} [f] = 0, \quad \lim_{R \rightarrow \infty} R \left(\left[\frac{\partial f}{\partial n} \right] + ik[f] \right) = 0, \quad (4.54)$$

that are called *Sommerfeld asymptotic conditions*. Due to (4.54) the surface integral (4.53) vanishes. Shifting the origin back from \mathbf{r} to the initial position, the solution of the Helmholtz equation (4.43) over an infinite domain finally reads

$$f(\mathbf{r}) = -\frac{1}{4\pi} \int_{\infty} b(\mathbf{q}) \frac{\exp(-ik|\mathbf{r} - \mathbf{q}|)}{|\mathbf{r} - \mathbf{q}|} d^3q, \quad (4.55)$$

where \int_{∞} indicates the integral over the whole three-dimensional \mathbf{q} space. The $k = 0$ case yields the solution of the Poisson equation $\nabla^2 f = b$ in an infinite domain,

$$f(\mathbf{r}) = -\frac{1}{4\pi} \int_{\infty} b(\mathbf{q}) \frac{1}{|\mathbf{r} - \mathbf{q}|} d^3q. \quad (4.56)$$

4.10 Solution of the Wave Equation in an Infinite Domain

The solutions of the Helmholtz equation found in Sects. 4.8, 4.9 allow one to calculate that of the wave equation. In fact, it is worth reminding that the Helmholtz Eq. (4.43) was deduced in Sec. 4.7 by Fourier transforming the wave equation (4.35) and *i*) letting f indicate the transform of the scalar potential φ or of any component A_i of the vector potential, *ii*) letting b indicate the transform of $-\rho/\varepsilon_0$ or $-\mu_0 J_i$, $i = 1, 2, 3$. As a consequence, f and b depend on the angular frequency ω besides the spatial coordinates. From the definition $k^2 = \omega^2/c^2$ one may also assume that both k and ω have the same sign, so that $k = \omega/c$. Considering for simplicity the case $V \rightarrow \infty$, applying (C.17) to antitransform (4.56), and interchanging the order of integrals yields

$$\mathcal{F}^{-1} f = -\frac{1}{4\pi} \int_{\infty} \frac{1}{g} \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} b(\mathbf{q}, \omega) \exp[i\omega(t - g/c)] d\omega \right] d^3q, \quad (4.57)$$

with $g = |\mathbf{r} - \mathbf{q}|$. Now, denote with a the antitransform of b , $a(\mathbf{q}, t) = \mathcal{F}^{-1} b$. It follows that the function between brackets in (4.57) coincides with $a(\mathbf{q}, t - g/c)$. As remarked above, when f represents φ , then a stands for $-\rho/\varepsilon_0$; similarly, when f represents a component of \mathbf{A} , then a stands for the corresponding component of $-\mu_0 \mathbf{J}$. In conclusion,

$$\varphi(\mathbf{r}, t) = \frac{1}{4\pi \varepsilon_0} \int_{\infty} \frac{\rho(\mathbf{q}, t - |\mathbf{r} - \mathbf{q}|/c)}{|\mathbf{r} - \mathbf{q}|} d^3q, \quad (4.58)$$

$$\mathbf{A}(\mathbf{r}, t) = \frac{\mu_0}{4\pi} \int_{\infty} \frac{\mathbf{J}(\mathbf{q}, t - |\mathbf{r} - \mathbf{q}|/c)}{|\mathbf{r} - \mathbf{q}|} d^3q, \quad (4.59)$$

that express the potentials in terms of the field sources ρ and \mathbf{J} , when the asymptotic behavior of the potentials fulfills the Sommerfeld conditions (4.54). The functions rendered by the antitransforms are real, as should be. Note that $|\mathbf{r} - \mathbf{q}|/c > 0$ is the time necessary for a signal propagating with velocity c to cross the distance $|\mathbf{r} - \mathbf{q}|$. As $t - |\mathbf{r} - \mathbf{q}|/c < t$, the above expressions of φ and \mathbf{A} are called *retarded potentials*⁴.

4.11 Lorentz Force

It has been assumed so far that the sources of the electromagnetic field, namely, charge density and current density, are prescribed functions of position and time. This is not necessarily so, because the charges are in turn acted upon by the electromagnetic field, so that their dynamics is influenced by it. Consider a test charge of value e immersed in an electromagnetic field described by the vectors \mathbf{E} , \mathbf{B} generated by other charges. The force acting upon the test charge is the *Lorentz force* [4, Vol. I, Sect. 44]

$$\mathbf{F} = e(\mathbf{E} + \mathbf{u} \wedge \mathbf{B}), \quad (4.60)$$

where \mathbf{u} is the velocity of the test charge and \mathbf{E} , \mathbf{B} are independent of \mathbf{u} . The expression of the Lorentz force does not derive from assumptions separate from Maxwell's equations; in fact, it follows from Maxwell's equations and Special Relativity [109, 39]. The extension of (4.60) to the case of a number of point-like charges follows the same line as in Sect. 4.4: considering the charges belonging to a cell of volume ΔV centered at \mathbf{r} , one writes (4.60) for the j th charge and takes the sum over j , to find

$$\mathbf{f} = \frac{\sum_j \mathbf{F}_j}{\Delta V} = \rho(\mathbf{E} + \mathbf{v} \wedge \mathbf{B}), \quad (4.61)$$

where ρ , \mathbf{v} are defined in (4.21) and \mathbf{f} is the force density ($[\mathbf{f}] = \text{N m}^{-3}$). The fields in (4.61) are calculated in \mathbf{r} and t .

Consider a small time interval δt during which the charge contained within ΔV is displaced by $\delta \mathbf{r} = \mathbf{v} \delta t$. The work per unit volume exchanged between the charge and the electromagnetic field due to such a displacement is

$$\delta w = \mathbf{f} \cdot \delta \mathbf{r} = \rho(\mathbf{E} + \mathbf{v} \wedge \mathbf{B}) \cdot \mathbf{v} \delta t = \mathbf{E} \cdot \mathbf{J} \delta t, \quad [w] = \text{J m}^{-3}, \quad (4.62)$$

⁴ Expressions of φ and \mathbf{A} obtained from (4.58, 4.59) after replacing $t - |\mathbf{r} - \mathbf{q}|/c$ with $t + |\mathbf{r} - \mathbf{q}|/c$ are also solutions of the wave equations (4.35). This is due to the fact that the Helmholtz equation (4.43) can also be solved by using G^* instead of G , which in turn reflects the time reversibility of the wave equation. However, the form with $t - |\mathbf{r} - \mathbf{q}|/c$ better represents the idea that an electromagnetic perturbation, that is present in \mathbf{r} at the time t , is produced by a source acting in \mathbf{q} at a time prior to t .

where (A.32) and (4.21) have been used. When the scalar product is positive, the charge acquires kinetic energy from the field, and vice versa. Letting $\delta t \rightarrow 0$ yields

$$\frac{\partial w}{\partial t} = \mathbf{E} \cdot \mathbf{J}, \quad (4.63)$$

where the symbol of partial derivative is used because (4.63) is calculated with \mathbf{r} fixed.

4.12 Complements

4.12.1 Invariance of the Euler Equations

It has been shown in Sect. 4.2 that the Euler equations (4.8) are still fulfilled if the generating function g is replaced with $g' = g + \text{div} \mathbf{h}$, where \mathbf{h} is an arbitrary vector of length m whose entries depend on \mathbf{w} and ξ_1, \dots, ξ_m , but not on the derivatives of \mathbf{w} . This property is a generalization of that illustrated in Sect. 1.2 with reference to a system of particles, where it was shown that the solutions $w_i(\xi)$ are invariant under addition to g of the total derivative of an arbitrary function that depends on \mathbf{w} and ξ only.

4.12.2 Wave Equations for the \mathbf{E} and \mathbf{B} Fields

The Maxwell equations can be rearranged in the form of wave equations for the electric and magnetic fields. To this purpose, one takes the rotational of both sides of the second equation in (4.24). Using the first identity in (A.36) and the relation $\mathbf{D} = \varepsilon_0 \mathbf{E}$ provides $-\partial \text{rot} \mathbf{B} / \partial t = \text{rot} \text{rot} \mathbf{E} = \text{grad} \text{div}(\mathbf{D} / \varepsilon_0) - \nabla^2 \mathbf{E}$. Replacing $\text{div} \mathbf{D}$ and $\text{rot} \mathbf{H} = \text{rot} \mathbf{B} / \mu_0$ from (4.19) and using $\varepsilon_0 \mu_0 = 1/c^2$ then yields

$$\nabla^2 \mathbf{E} - \frac{1}{c^2} \frac{\partial^2 \mathbf{E}}{\partial t^2} = \frac{1}{\varepsilon_0} \text{grad} \rho + \mu_0 \frac{\partial \mathbf{J}}{\partial t}. \quad (4.64)$$

Similarly, one takes the rotational of both sides of the second equation in (4.19). Using the relation $\mathbf{B} = \mu_0 \mathbf{H}$ provides $\varepsilon_0 \partial \text{rot} \mathbf{E} / \partial t + \text{rot} \mathbf{J} = \text{rot} \text{rot} \mathbf{H} = \text{grad} \text{div}(\mathbf{B} / \mu_0) - \nabla^2 \mathbf{H}$. Replacing $\text{div} \mathbf{B}$ and $\text{rot} \mathbf{E}$ from (4.24) yields

$$\nabla^2 \mathbf{H} - \frac{1}{c^2} \frac{\partial^2 \mathbf{H}}{\partial t^2} = -\text{rot} \mathbf{J}. \quad (4.65)$$

4.12.3 Comments on the Boundary-Value Problem

Considering relation (4.52) derived in Sect. 4.8, one notes that the right hand side is made of the difference between two terms; the first one depends on the boundary values of f , $\partial f/\partial n$, but not on b , while the second one depends only on the values of b within V and over the boundary. In these considerations it does not matter whether point \mathbf{r} is external to V , on the boundary of V , or internal to it. In latter case the two terms at the right hand side of (4.52) balance each other.

If b is replaced with a different function \tilde{b} , and thereby the value of the second integral changes, it is possible to modify the boundary values in such a way as to balance the variation of the second integral with that of the first one; as a consequence, $f(\mathbf{r})$ is left unchanged. A possible choice for the modified b is $\tilde{b} = 0$; by this choice one eliminates the data of the differential equation and suitably modifies the boundary values, leaving the solution unaffected. An observer placed at \mathbf{r} would be unable to detect that the data have disappeared. The same process can also be carried out in reverse, namely, by eliminating the boundary values and suitably changing the data.

An example is given in Prob. 4.4 with reference to a one-dimensional Poisson equation where the original charge density differs from zero in a finite interval $[a, b]$. The charge density is removed and the boundary values at a are modified so that the electric potential φ is unaffected for $x \geq b$. Obviously φ changes for $a < x < b$ because both the charge density and boundary conditions are different, and also for $x \leq a$ because the boundary conditions are different.

Problems

- 4.1 Solve the one-dimensional Poisson equation $d^2\varphi/dx^2 = -\rho(x)/\varepsilon_0$, with ρ given, using the integration by parts to avoid a double integral. The solution is prescribed at $x = a$ while the first derivative is prescribed at $x = c$.
- 4.2 Let $c = a$ in the solution of Prob. 4.1 and assume that the charge density ρ differs from zero only in a finite interval $a \leq x \leq b$. Find the expression of φ for $x > b$ when both the solution and the first derivative are prescribed at $x = a$.
- 4.3 In Prob. 4.2 replace the charge density ρ with a different one, say, $\tilde{\rho}$. Discuss the conditions that leave the solution unchanged.
- 4.4 In Prob. 4.2 remove the charge density ρ and modify the boundary conditions at a so that the solution for $x > b$ is left unchanged.
- 4.5 Using the results of Probs. 4.2 and 4.3, and assuming that both M_0 and M_1 are different from zero, replace the ratio ρ/ε_0 with $\mu \delta(x-h)$ and find the parameters μ, h that leave M_0, M_1 unchanged. Noting that h does not necessarily belong to the interval $[a, b]$, discuss the outcome for different positions of h with respect to a .