3

Fixed Income Securities

3.1 Introduction

Corporations finance their operations by selling stock and bonds. Owning a share of stock means partial ownership of the company. Stockholders share in both the profits and losses of the company. Owning a bond is different. When you buy a bond you are loaning money to the corporation, though bonds, unlike loans, are tradeable. The corporation is obligated to pay back the principal and to pay interest as stipulated by the bond. The bond owner receives a fixed stream of income, unless the corporation defaults on the bond. For this reason, bonds are called "fixed income" securities.

It might appear that bonds are risk-free, almost stodgy, but this is not the case. Many bonds are long-term, e.g., 5, 10, 20, or even 30 years. Even if the corporation stays solvent or if you buy a U.S. Treasury bond, where default is for all intents and purposes impossible, your income from the bond is guaranteed only if you keep the bond to maturity. If you sell the bond before maturity, your return will depend on changes in the price of the bond. Bond prices move in opposite direction to interest rates, so a decrease in interest rates will cause a bond "rally," where bond prices increase. Long-term bonds are more sensitive to interest-rate changes than short-term bonds. The interest rate on your bond is fixed, but in the market interest rates fluctuate. Therefore, the market value of your bond fluctuates too. For example, if you buy a bond paying 5 % and the rate of interest increases to 6 %, then your bond is inferior to new bonds offering 6 %. Consequently, the price of your bond will decrease. If you sell the bond, you could lose money.

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The interest rate of a bond depends on its maturity. For example, on March 28, 2001, the interest rate of Treasury bills[1] was 4.23 % for 3-month bills. The yields on Treasury notes and bonds were 4.41 %, 5.01 %, and 5.46 % for 2-, 10-, and 30-year maturities, respectively. The *term structure* of interest rates describes how rates change with maturity.

3.2 Zero-Coupon Bonds

Zero-coupon bonds, also called *pure discount bonds* and sometimes known as "zeros," pay no principal or interest until maturity. A "zero" has a *par value* or *face value*, which is the payment made to the bondholder at maturity. The zero sells for less than the par value, which is the reason it is a discount bond.

For example, consider a 20-year zero with a par value of $1,000 and 6 % interest compounded annually. The market price is the present value of $1,000 with an annual interest rate of 6 % with annual discounting. That is, the market price is

$$\frac{\$1,000}{(1.06)^{20}} = \$311.80.$$

If the annual interest rate is 6 % but compounded every 6 months, then the price is

$$\frac{\$1,000}{(1.03)^{40}} = \$306.56,$$

and if the annual rate is 6 % compounded continuously, then the price is

$$\frac{\$1,000}{\exp\{(0.06)(20)\}} = \$301.19.$$

3.2.1 Price and Returns Fluctuate with the Interest Rate

For concreteness, assume semiannual compounding. Suppose you bought the zero for $306.56 and then 6 months later the interest rate increased to 7 %. The market price would now be

$$\frac{\$1,000}{(1.035)^{39}} = \$261.41,$$

so the value of your investment would drop by ($306.56 − $261.41) = $45.15. You will still get your $1,000 if you keep the bond for 20 years, but if you sold it now, you would lose $45.15. This is a return of

[1] Treasury bills have maturities of 1 year or less, Treasury notes have maturities from 1 to 10 years, and Treasury bonds have maturities from 10 to 30 years.

$$\frac{-45.15}{306.56} = -14.73\,\%$$

for a half-year, or −29.46,% per year. And the interest rate only changed from 6 % to 7 %![2] Notice that the interest rate went up and the bond price went down. This is a general phenomenon. Bond prices always move in the opposite direction of interest rates.

If the interest rate dropped to 5 % after 6 months, then your bond would be worth

$$\frac{\$1,000}{(1.025)^{39}} = \$381.74.$$

This would be an annual rate of return of

$$2\left(\frac{381.74 - 306.56}{306.56}\right) = 49.05\,\%.$$

If the interest rate remained unchanged at 6 %, then the price of the bond would be

$$\frac{\$1,000}{(1.03)^{39}} = \$315.75.$$

The annual rate of return would be

$$2\left(\frac{315.75 - 306.56}{306.56}\right) = 6\,\%.$$

Thus, if the interest rate does not change, you can earn a 6 % annual rate of return, the same return rate as the interest rate, by selling the bond before maturity. If the interest rate does change, however, the 6 % annual rate of return is guaranteed only if you keep the bond until maturity.

General Formula

The price of a zero-coupon bond is given by

$$\text{PRICE} = \text{PAR}(1 + r)^{-T}$$

if T is the time to maturity in years and the annual rate of interest is r with annual compounding. If we assume semiannual compounding, then the price is

$$\text{PRICE} = \text{PAR}(1 + r/2)^{-2T}. \tag{3.1}$$

[2] Fortunately for investors, a rate change as large as going from 6 % to 7 % is rare on a 20-year bond.

3.3 Coupon Bonds

Coupon bonds make regular interest payments. Coupon bonds generally sell at or near the par value when issued. At maturity, one receives a principal payment equal to the par value of the bond and the final interest payment.

As an example, consider a 20-year coupon bond with a par value of $1,000 and 6 % annual coupon rate with semiannual coupon payments, so effectively the 6 % is compounded semiannually. Each coupon payment will be $30. Thus, the bondholder receives 40 payments of $30, one every 6 months plus a principal payment of $1,000 after 20 years. One can check that the present value of all payments, with discounting at the 6 % annual rate (3 % semiannual), equals $1,000:

$$\sum_{t=1}^{40} \frac{30}{(1.03)^t} + \frac{1000}{(1.03)^{40}} = 1000.$$

After 6 months, if the interest rate is unchanged, then the bond (including the first coupon payment, which is now due) is worth

$$\sum_{t=0}^{39} \frac{30}{(1.03)^t} + \frac{1000}{(1.03)^{39}} = (1.03) \left(\sum_{t=1}^{40} \frac{30}{(1.03)^t} + \frac{1000}{(1.03)^{40}} \right) = 1030,$$

which is a semiannually compounded 6 % annual return as expected. If the interest rate increases to 7 %, then after 6 months the bond (plus the interest due) is only worth

$$\sum_{t=0}^{39} \frac{30}{(1.035)^t} + \frac{1000}{(1.035)^{39}} = (1.035) \left(\sum_{t=1}^{40} \frac{30}{(1.035)^t} + \frac{1000}{(1.035)^{40}} \right) = 924.49.$$

This is an annual return of

$$2 \left(\frac{924.49 - 1000}{1000} \right) = -15.1\,\%.$$

If the interest rate drops to 5 % after 6 months, then the investment is worth

$$\sum_{t=0}^{39} \frac{30}{(1.025)^t} + \frac{1000}{(1.025)^{39}} = (1.025) \left(\sum_{t=1}^{40} \frac{30}{(1.025)^t} + \frac{1000}{(1.025)^{40}} \right) = 1{,}153.70, \tag{3.2}$$

and the annual return is

$$2 \left(\frac{1153.7 - 1000}{1000} \right) = 30.72\,\%.$$

3.3.1 A General Formula

Let's derive some useful formulas. If a bond with a par value of PAR matures in T years and makes semiannual coupon payments of C and the yield (rate of interest) is r per half-year, then the value of the bond when it is issued is

$$\sum_{t=1}^{2T} \frac{C}{(1+r)^t} + \frac{\text{PAR}}{(1+r)^{2T}} = \frac{C}{r}\left\{1-(1+r)^{-2T}\right\} + \frac{\text{PAR}}{(1+r)^{2T}}$$
$$= \frac{C}{r} + \left\{\text{PAR} - \frac{C}{r}\right\}(1+r)^{-2T}. \tag{3.3}$$

Derivation of (3.3)

The summation formula for a finite geometric series is

$$\sum_{i=0}^{T} r^i = \frac{1-r^{T+1}}{1-r}, \tag{3.4}$$

provided that $r \neq 1$. Therefore,

$$\sum_{t=1}^{2T} \frac{C}{(1+r)^t} = \frac{C}{1+r}\sum_{t=0}^{2T-1}\left(\frac{1}{1+r}\right)^t = \frac{C\{1-(1+r)^{-2T}\}}{(1+r)\{1-(1+r)^{-1}\}}$$
$$= \frac{C}{r}\{1-(1+r)^{-2T}\}. \tag{3.5}$$

The remainder of the derivation is straightforward algebra.

3.4 Yield to Maturity

Suppose a bond with $T = 30$ and $C = 40$ is selling for \$1,200, \$200 above par value. If the bond were selling at par value, then the interest rate would be 0.04/half-year (= 0.08/year). The 4 %/half-year rate is called the *coupon rate*.

But the bond is *not* selling at par value. If you purchase the bond at \$1,200, you will make *less* than 8 % per year interest. There are two reasons that the rate of interest is less than 8 %. First, the coupon payments are \$40 or 40/1200 = 3.333 %/half-year (or 6.67 %/year) for the \$1,200 investment; 6.67 %/year is called the *current yield*. Second, at maturity you only get back \$1,000, not the entire \$1,200 investment. The current yield of 6.67 %/year, though less than the coupon rate of 8 %/year, overestimates the return since it does not account for this loss of capital.

The *yield to maturity*, often shortened to simply *yield,* is the average rate of return, including the loss (or gain) of capital because the bond was purchased above (or below) par. For this bond, the yield to maturity is the value of r that solves

$$1200 = \frac{40}{r} + \left\{1000 - \frac{40}{r}\right\}(1+r)^{-60}. \tag{3.6}$$

The right-hand side of (3.6) is (3.3) with $C = 40$, $T = 30$, and $\text{PAR} = 1000$. It is easy to solve equation (3.6) numerically. The `R` program in Sect. 3.10.1 does the following:

- computes the bond price for each r value on a grid;
- graphs bond price versus r (this is not necessary, but it is fun to see the graph); and
- interpolates to find the value of r such that the bond value equals \$1,200.

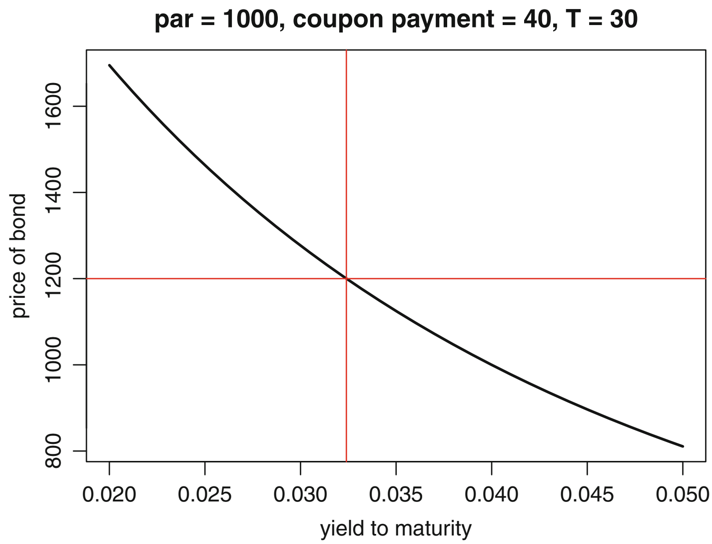

Fig. 3.1. *Bond price versus yield to maturity. The horizontal red line is at the bond price of \$1,200. The price/yield curve intersects this line at 0.0324 as indicated by the vertical red line. Therefore, 0.0324 is the bond's yield.*

One finds that the yield to maturity is 0.0324, that is, 3.24 %/half-year. Figure 3.1 shows the graph of bond price versus the yield (r) and shows that $r = 0.0324$ maps to a bond price of \$1,200.

The yield to maturity of 0.0324 is less than the current yield of 0.0333, which is less than the coupon rate of $40/1000 = 0.04$. (All three rates are rates per half-year.) Whenever, as in this example, the bond is selling above par value, then the coupon rate is greater than the current yield because the bond sells above par value, and the current yield is greater than the yield to maturity because the yield to maturity accounts for the loss of capital when at the maturity date you get back only the par value, not the entire investment. In summary,

$$\text{price} > \text{par} \Rightarrow \text{coupon rate} > \text{current yield} > \text{yield to maturity}.$$

Everything is reversed if the bond is selling below par value. For example, if the price of the bond were only \$900, then the yield to maturity would be 0.0448 (as before, this value can be determined by interpolation), the current yield would be $40/900 = 0.0444$, and the coupon rate would still be $40/1000 = 0.04$. In general,

$$\text{price} < \text{par} \Rightarrow \text{coupon rate} < \text{current yield} < \text{yield to maturity}.$$

3.4.1 General Method for Yield to Maturity

The yield to maturity (on a semiannual basis) of a coupon bond is the value of r that solves

$$\text{PRICE} = \frac{C}{r} + \left\{ \text{PAR} - \frac{C}{r} \right\} (1+r)^{-2T}. \tag{3.7}$$

Here PRICE is the market price of the bond, PAR is the par value, C is the semiannual coupon payment, and T is the time to maturity in years and assumed to be a multiple of 1/2.

For a zero-coupon bond, $C = 0$ and (3.7) becomes

$$\text{PRICE} = \text{PAR}(1+r)^{-2T}. \tag{3.8}$$

3.4.2 Spot Rates

The yield to maturity of a zero-coupon bond of maturity n years is called the n-year *spot rate* and is denoted by y_n. One uses the n-year spot rate to discount a payment n years from now, so a payment of \$1 to be made n years from now has a net present value (NPV) of $\$1/(1+y_n)^n$ if y_n is the spot rate per annum or $\$1/(1+y_n)^{2n}$ if y_n is a semiannual rate.

A coupon bond is a bundle of zero-coupon bonds, one for each coupon payment and a final one for the principal payment. The component zeros have different maturity dates and therefore different spot rates. The yield to maturity of the coupon bond is, thus, a complex “average” of the spot rates of the zeros in this bundle.

Example 3.1. Finding the price and yield to maturity of a coupon bond using spot rates

Consider the simple example of 1-year coupon bond with semiannual coupon payments of \$40 and a par value of \$1,000. Suppose that the one-half-year spot rate is 2.5 %/half-year and the 1-year spot rate is 3 %/half-year. Think of the coupon bond as being composed of two zero-coupon bonds, one with $T = 1/2$ and a par value of \$40 and the second with $T = 1$ and a par value of \$1,040. The price of the bond is the sum of the prices of these two zeros. Applying (3.8) twice to obtain the prices of these zeros and summing, we obtain the price of the zero-coupon bond:

$$\frac{40}{1.025} + \frac{1040}{(1.03)^2} = 1019.32.$$

The yield to maturity on the coupon bond is the value of y that solves

$$\frac{40}{1+y} + \frac{1040}{(1+y)^2} = 1019.32.$$

The solution is $y = 0.0299$/half-year. Thus, the annual yield to maturity is twice 0.0299, or 5.98 %/year. □

General Formula

In this section we will find a formula that generalizes Example 3.1. Suppose that a coupon bond pays semiannual coupon payments of C, has a par value of PAR, and has T years until maturity. Let $y_1, y_2, \ldots, y_{2T}$ be the half-year spot rates for zero-coupon bonds of maturities $1/2, 1, 3/2, \ldots, T$ years. Then the yield to maturity (on a half-year basis) of the coupon bond is the value of y that solves

$$\frac{C}{1+y_1} + \frac{C}{(1+y_2)^2} + \cdots + \frac{C}{(1+y_{2T-1})^{2T-1}} + \frac{\text{PAR}+C}{(1+y_n)^{2T}} \\ = \frac{C}{1+y} + \frac{C}{(1+y)^2} + \cdots + \frac{C}{(1+y)^{2T-1}} + \frac{\text{PAR}+C}{(1+y)^{2T}}. \tag{3.9}$$

The left-hand side of Eq. (3.9) is the price of the coupon bond, and the yield to maturity is the value of y that makes the right-hand side of (3.9) equal to the price.

Methods for solving (3.9) are explored in the `R` lab in Sect. 3.10.

3.5 Term Structure

3.5.1 Introduction: Interest Rates Depend Upon Maturity

On January 26, 2001, the 1-year T-bill rate was 4.83 % and the 30-year Treasury bond rate was 6.11 %. This is typical. Short- and long-term rates usually differ. Often short-term rates are lower than long-term rates. This makes sense since long-term bonds are riskier, because long-term bond prices fluctuate more with interest-rate changes. However, during periods of very high short-term rates, the short-term rates may be higher than the long-term rates. The reason is that the market believes that rates will return to historic levels and no one will commit to the high interest rate for, say, 20 or 30 years. Figure 3.2 shows weekly values of the 90-day, 10-year, and 30-year Treasury rates from 1970 to 1993, inclusive. Notice that the 90-day rate is more volatile than the longer-term rates and is usually less than them. However, in the early 1980s, when interest rates were very high, the short-term rates were higher than the long-term rates. These data were taken from the Federal Reserve Bank of Chicago's website.

The *term structure* of interest rates is a description of how, *at a given time*, yield to maturity depends on maturity.

3.5.2 Describing the Term Structure

Term structure for all maturities up to n years can be described by any one of the following:

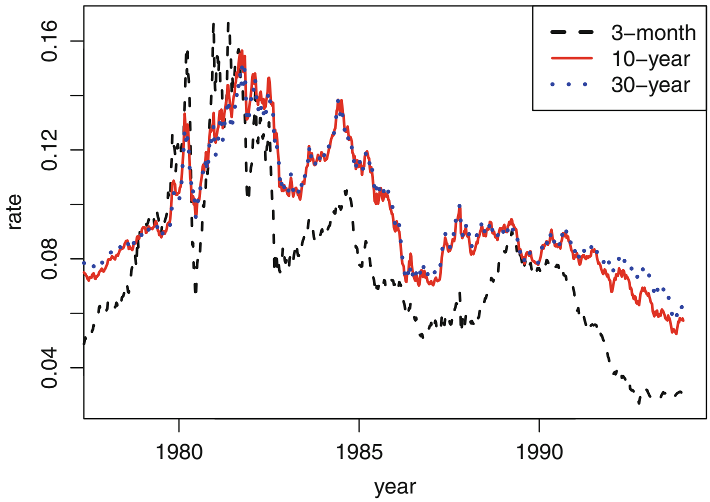

Fig. 3.2. *Treasury rates of three maturities. Weekly time series. The data were taken from the website of the Federal Reserve Bank of Chicago.*

- prices of zero-coupon bonds of maturities 1-year, 2-years, ..., n-years are denoted here by $P(1), P(2), \dots, P(n)$;
- spot rates (yields of maturity of zero-coupon bonds) of maturities 1-year, 2-years, ..., n-years are denoted by $y_1, \dots, y_n$;

- forward rates $r_1, \ldots, r_n$, where r_i is the forward rate that can be locked in now for borrowing in the ith future year ($i = 1$ for next year, and so on).

As discussed in this section, each of the sets $\{P(1), \ldots, P(n)\}$, $\{y_1, \ldots, y_n\}$, and $\{r_1, \ldots, r_n\}$ can be computed from either of the other sets. For example, equation (3.11) ahead gives $\{P(1), \ldots, P(n)\}$ in terms of $\{r_1, \ldots, r_n\}$, and equations (3.12) and (3.13) ahead give $\{y_1, \ldots, y_n\}$ in terms of $\{P(1), \ldots, P(n)\}$ or $\{r_1, \ldots, r_n\}$, respectively.

Term structure can be described by breaking down the time interval between the present time and the maturity time of a bond into short time segments with a constant interest rate within each segment, but with interest rates varying between segments. For example, a 3-year loan can be considered as three consecutive 1-year loans, or six consecutive half-year loans, and so forth.

Example 3.2. Finding prices from forward rates

As an illustration, suppose that loans have the forward interest rates listed in Table 3.1. Using the forward rates in the table, we see that a par \$1,000 1-year zero would sell for

$$\frac{1000}{1 + r_1} = \frac{1000}{1.06} = \$943.40 = P(1).$$

A par \$1,000 2-year zero would sell for

$$\frac{1000}{(1 + r_1)(1 + r_2)} = \frac{1000}{(1.06)(1.07)} = \$881.68 = P(2),$$

since the rate r_1 is paid the first year and r_2 the following year. Similarly, a par \$1,000 3-year zero would sell for

$$\frac{1000}{(1 + r_1)(1 + r_2)(1 + r_3)} = \frac{1000}{(1.06)(1.07)(1.08)} = 816.37 = P(3).$$

Table 3.1. *Forward interest rates used in Examples 3.2 and 3.3*

Year (i)	**Interest rate** (r_i)(%)
1	6
2	7
3	8

□

The general formula for the present value of \$1 paid n periods from now is

$$\frac{1}{(1+r_1)(1+r_2)\cdots(1+r_n)}. \tag{3.10}$$

Here r_i is the *forward interest rate* during the ith period. If the periods are years, then the price of an n-year par \$1,000 zero-coupon bond $P(n)$ is \$1,000 times the discount factor in (3.10); that is,

$$P(n) = \frac{1000}{(1+r_1)\cdots(1+r_n)}. \tag{3.11}$$

Example 3.3. Back to Example 3.2: Finding yields to maturity from prices and from the forward rates

In this example, we first find the yields to maturity from the prices derived in Example 3.2 using the interest rates from Table 3.1. For a 1-year zero, the yield to maturity y_1 solves

$$\frac{1000}{(1+y_1)} = 943.40,$$

which implies that $y_1 = 0.06$. For a 2-year zero, the yield to maturity y_2 solves

$$\frac{1000}{(1+y_2)^2} = 881.68,$$

so that

$$y_2 = \sqrt{\frac{1000}{881.68}} - 1 = 0.0650.$$

For a 3-year zero, the yield to maturity y_3 solves

$$\frac{1000}{(1+y_3)^3} = 816.37,$$

and equals 0.070.

The yields can also be found from the forward rates. First, trivially, $y_1 = r_1 = 0.06$. Next, y_2 is given by

$$y_2 = \sqrt{(1+r_1)(1+r_2)} - 1 = \sqrt{(1.06)(1.07)} - 1 = 0.0650.$$

Also,

$$\begin{aligned} y_3 &= \{(1+r_1)(1+r_2)(1+r_3)\}^{1/3} - 1 \\ &= \{(1.06)(1.07)(1.08)\}^{1/3} - 1 = 0.0700, \end{aligned}$$

or, more precisely, 0.06997. Thus, $(1+y_3)$ is the geometric average of 1.06, 1.07, and 1.08 and very nearly equal to their arithmetic average, which is 1.07. □

Recall that $P(n)$ is the price of a par \$1,000 n-year zero-coupon bond. The general formulas for the yield to maturity y_n of an n-year zero are

$$y_n = \left\{ \frac{1000}{P(n)} \right\}^{1/n} - 1, \tag{3.12}$$

to calculate the yield from the price, and

$$y_n = \{(1 + r_1) \cdots (1 + r_n)\}^{1/n} - 1 \tag{3.13}$$

to obtain the yield from the forward rate.

Equations (3.12) and (3.13) give the yields to maturity in terms of the bond prices and forward rates, respectively. Also, inverting (3.12) gives the formula

$$P(n) = \frac{1000}{(1 + y_n)^n} \tag{3.14}$$

for $P(n)$ as a function of the yield to maturity.

As mentioned before, interest rates for future years are called *forward rates*. A forward contract is an agreement to buy or sell an asset at some fixed future date at a fixed price. Since $r_2, r_3, \ldots$ are rates that can be locked in now for future borrowing, they are forward rates.

The general formulas for determining forward rates from yields to maturity are

$$r_1 = y_1, \tag{3.15}$$

and

$$r_n = \frac{(1 + y_n)^n}{(1 + y_{n-1})^{n-1}} - 1, \quad n = 2, 3, \ldots. \tag{3.16}$$

Now suppose that we only observed bond prices. Then we can calculate yields to maturity and forward rates using (3.12) and then (3.16).

Table 3.2. *Bond prices used in Example 3.4*

Maturity	Price
1 Year	\$920
2 Years	\$830
3 Years	\$760

Example 3.4. Finding yields and forward rates from prices

Suppose that one-, two-, and three-year par \$1,000 zeros are priced as given in Table 3.2. Using (3.12), the yields to maturity are

$$y_1 = \frac{1000}{920} - 1 = 0.087,$$

$$y_2 = \left\{ \frac{1000}{830} \right\}^{1/2} - 1 = 0.0976,$$

$$y_3 = \left\{ \frac{1000}{760} \right\}^{1/3} - 1 = 0.096.$$

Then, using (3.15) and (3.16),

$$r_1 = y_1 = 0.087,$$

$$r_2 = \frac{(1+y_2)^2}{(1+y_1)} - 1 = \frac{(1.0976)^2}{1.0876} - 1 = 0.108, \text{ and}$$

$$r_3 = \frac{(1+y_3)^3}{(1+y_2)^2} - 1 = \frac{(1.096)^3}{(1.0976)^2} - 1 = 0.092.$$

□

The formula for finding r_n from the prices of zero-coupon bonds is

$$r_n = \frac{P(n-1)}{P(n)} - 1, \tag{3.17}$$

which can be derived from

$$P(n) = \frac{1000}{(1+r_1)(1+r_2)\cdots(1+r_n)},$$

and

$$P(n-1) = \frac{1000}{(1+r_1)(1+r_2)\cdots(1+r_{n-1})}.$$

To calculate r_1 using (3.17), we need $P(0)$, the price of a 0-year bond, but $P(0)$ is simply the par value.[3]

Example 3.5. Forward rates from prices

Thus, using (3.17) and the prices in Table 3.2, the forward rates are

$$r_1 = \frac{1000}{920} - 1 = 0.087,$$

$$r_2 = \frac{920}{830} - 1 = 0.108,$$

and

$$r_3 = \frac{830}{760} - 1 = 0.092.$$

□

[3] Trivially, a bond that must be paid back immediately is worth exactly its par value.

3.6 Continuous Compounding

Now assume continuous compounding with forward rates $r_1, \ldots, r_n$. Using continuously compounded rates simplifies the relationships among the forward rates, the yields to maturity, and the prices of zero-coupon bonds.

If $P(n)$ is the price of a \$1,000 par value n-year zero-coupon bond, then

$$P(n) = \frac{1000}{\exp(r_1 + r_2 + \cdots + r_n)}. \tag{3.18}$$

Therefore,

$$\frac{P(n-1)}{P(n)} = \frac{\exp(r_1 + \cdots + r_n)}{\exp(r_1 + \cdots + r_{n-1})} = \exp(r_n), \tag{3.19}$$

and

$$\log\left\{\frac{P(n-1)}{P(n)}\right\} = r_n. \tag{3.20}$$

The yield to maturity of an n-year zero-coupon bond solves the equation

$$P(n) = \frac{1000}{\exp(n y_n)},$$

and is easily seen to be

$$y_n = (r_1 + \cdots + r_n)/n. \tag{3.21}$$

Therefore, $\{r_1, \ldots, r_n\}$ is easily found from $\{y_1, \ldots, y_n\}$ by the relationship

$$r_1 = y_n,$$

and

$$r_n = n y_n - (n-1) y_{n-1} \text{ for } n > 1.$$

Example 3.6. Continuously compounded forward rates and yields from prices

Using the prices in Table 3.2, we have $P(1) = 920$, $P(2) = 830$, and $P(3) = 760$. Therefore, using (3.20),

$$r_1 = \log\left\{\frac{1000}{920}\right\} = 0.083,$$

$$r_2 = \log\left\{\frac{920}{830}\right\} = 0.103,$$

and

$$r_3 = \log\left\{\frac{830}{760}\right\} = 0.088.$$

Also, $y_1 = r_1 = 0.083$, $y_2 = (r_1 + r_2)/2 = 0.093$, and $y_3 = (r_1 + r_2 + r_3)/3 = 0.091$. □

3.7 Continuous Forward Rates

So far, we have assumed that forward interest rates vary from year to year but are constant within each year. This assumption is, of course, unrealistic and was made only to simplify the introduction of forward rates. Forward rates should be modeled as a function varying continuously in time.

To specify the term structure in a realistic way, we assume that there is a function $r(t)$ called the *forward-rate function* such that the current price of a zero-coupon bond of maturity T and with par value equal to 1 is given by

$$D(T) = \exp\left\{ -\int_0^T r(t)dt \right\}. \tag{3.22}$$

$D(T)$ is called the discount function and the price of any zero-coupon bond is given by discounting its par value by multiplication with the discount function; that is,

$$P(T) = \text{PAR} \times D(T), \tag{3.23}$$

where $P(T)$ is the price of a zero-coupon bond of maturity T with par value equal to PAR. Also,

$$\log P(T) = \log(\text{PAR}) - \int_0^T r(t)dt,$$

so that

$$-\frac{d}{dT}\log P(T) = r(T) \text{ for all } T. \tag{3.24}$$

Formula (3.22) is a generalization of formula (3.18). To appreciate this, suppose that $r(t)$ is the piecewise constant function

$$r(t) = r_k \text{ for } k-1 < t \leq k.$$

With this piecewise constant r, for any integer T, we have

$$\int_0^T r(t)dr = r_1 + r_2 + \cdots + r_T,$$

so that

$$\exp\left\{ -\int_0^T r(t)dt \right\} = \exp\{-(r_1 + \cdots + r_T)\}$$

and therefore (3.18) agrees with (3.22) in this special situation. However, (3.22) is a more general formula since it applies to noninteger T and to arbitrary $r(t)$, not only to piecewise constant functions.

The yield to maturity of a zero-coupon bond with maturity date T is defined to be

$$y_T = \frac{1}{T}\int_0^T r(t)\,dt. \tag{3.25}$$

Thinking of the right-hand side of (3.25) as the average of $r(t)$ over the interval $0 \le t \le T$, we see that (3.25) is the analog of (3.21). From (3.22) and (3.25) it follows that the discount function can be obtained from the yield to maturity by the formula

$$D(T) = \exp\{-Ty_T\}, \tag{3.26}$$

so that the price of a zero-coupon bond maturing at time T is the same as it would be if there were a constant forward interest rate equal to y_T. It follows from (3.26) that

$$y_T = -\log\{D(T)\}/T. \tag{3.27}$$

Example 3.7. Finding continuous yield and discount functions from forward rates

Suppose the forward rate is the linear function $r(t) = 0.03 + 0.0005\,t$. Find $r(15)$, y_{15}, and $D(15)$.

Answer: $r(15) = 0.03 + (0.0005)(15) = 0.0375$,

$$\begin{aligned} y_{15} &= (15)^{-1}\int_0^{15}(0.03 + 0.0005\,t)dt \\ &= (15)^{-1}(0.03\,t + 0.0005\,t^2/2)\Big|_0^{15} = 0.03375, \end{aligned}$$

and $D(15) = \exp(-15y_{15}) = \exp\{-(15)(0.03375)\} = \exp(-0.5055) = 0.603$. □

The linear forward rate in Example 3.7 was chosen for simplicity and is not realistic. The Nelson-Siegel and Svensson parametric families of curves introduced in Sect. 11.3 are used in practice to model forward rates and yield curves. The European Community Bank uses the Svensson family. Nonparametric estimation of a forward rate by local polynomial and spline estimation is discussed in Examples 21.1 and 21.3, respectively. The Federal Reserve, the Bank of England, and the Bank of Canada use splines. The European Central Bank uses the Svensson family.

The discount function $D(T)$ and forward-rate function $r(t)$ in formula (3.22) depend on the current time, which is taken to be zero in that formula. However, we could be interested in how the discount function and forward rate function change over time. In that case we define the discount function $D(s,T)$ to be the price at time s of a zero-coupon bond, with a par value of \$1, maturing at time T. Also, if the forward-rate curve at time s is $r(s,t)$, $t \ge s$, then

$$D(s,T) = \exp\left\{ - \int_s^T r(s,t)dt \right\}. \tag{3.28}$$

The yield at time s of a bond maturing at time $T > s$ is

$$y(s,T) = (T-s)^{-1} \int_s^T r(s,u)du.$$

Since $r(t)$ and $D(t)$ in (3.22) are $r(0,t)$ and $D(0,t)$ in our new notation, (3.22) is the special case of (3.28) with $s = 0$. Similarly, y_T is equal to $y(0,T)$ in the new notation. However, for the remainder of this chapter we assume that $s = 0$ and return to the simpler notation of $r(t)$ and $D(t)$.

3.8 Sensitivity of Price to Yield

As we have seen, bonds are risky because bond prices are sensitive to interest rates. This problem is called *interest-rate risk*. This section describes a traditional method of quantifying interest-rate risk.

Using Eq. (3.26), we can approximate how the price of a zero-coupon bond changes if there is a small change in yield. Suppose that y_T changes to $y_T + \delta$, where the change in yield δ is small. Then the change in $D(T)$ is approximately δ times

$$\frac{d}{dy_T} \exp\{-Ty_T\} \approx -T \exp\{-Ty_T\} = -TD(T). \tag{3.29}$$

Therefore, by Eq. (3.23), for a zero-coupon bond of maturity T,

$$\frac{\text{change bond price}}{\text{bond price}} \approx -T \times \text{change in yield}. \tag{3.30}$$

In this equation "≈" means that the ratio of the right- to left-hand sides converges to 1 as $\delta \to 0$.

Equation (3.30) is worth examining. The minus sign on the right-hand side shows us something we already knew, that bond prices move in the opposite direction to interest rates. Also, the relative change in the bond price, which is the left-hand side of the equation, is proportional to T, which quantifies the principle that longer-term bonds have higher interest-rate risks than short-term bonds.

3.8.1 Duration of a Coupon Bond

Remember that a coupon bond can be considered a bundle of zero-coupon bonds of various maturities. The *duration* of a coupon bond, which we will denote by DUR, is the weighted average of these maturities with weights in proportion to the net present value of the cash flows (coupon payments and par value at maturity).

Now assume that all yields change by a constant amount δ, that is, y_T changes to $y_T + \delta$ for all T. This restrictive assumption is needed to define duration. Because of this assumption, Eq. (3.30) applies to each of these cash flows and averaging them with these weights gives us that for a coupon bond,

$$\frac{\text{change bond price}}{\text{bond price}} \approx -\text{DUR} \times \delta. \tag{3.31}$$

The details of the derivation of (3.31) are left as an exercise (Exercise 15). *Duration analysis* uses (3.31) to approximate the effect of a change in yield on bond prices.

We can rewrite (3.31) as

$$\text{DUR} \approx \frac{-1}{\text{price}} \times \frac{\text{change in price}}{\text{change in yield}} \tag{3.32}$$

and use (3.32) as a *definition* of duration. Notice that "bond price" has been replaced by "price." The reason for this is that (3.32) can define the durations of not only bonds but also of derivative securities whose prices depend on yield, for example, call options on bonds. When this definition is extended to derivatives, duration has nothing to do with maturities of the underlying securities. Instead, duration is solely a measure of sensitivity of price to yield. Tuckman (2002) gives an example of a 10-year coupon bond with a duration of 7.79 years and a call option on this bond with a duration of 120.82 years. These durations show that the call is much riskier than the bond since it is 15.5 (= 129.82/7.79) times more sensitive to changes in yield.

Unfortunately, the underlying assumption behind (3.31) that all yields change by the same amount is not realistic, so duration analysis is falling into disfavor and value-at-risk is replacing duration analysis as a method for evaluating interest-rate risk.[4] Value-at-risk and other risk measures are covered in Chap. 19.

3.9 Bibliographic Notes

Tuckman (2002) is an excellent comprehensive treatment of fixed income securities; it is written at an elementary mathematical level and is highly recommended for readers wishing to learn more about this topic. Bodie, Kane, and Marcus (1999), Sharpe, Alexander, and Bailey (1999), and Campbell, Lo, and MacKinlay (1997) provide good introductions to fixed income securities, with the last-named being at a more advanced level. James and Webber (2000) is an advanced book on interest rate modeling. Jarrow (2002) covers

[4] See Dowd (1998).

many advanced topics that are not included in this book, including modeling the evolution of term structure, bond trading strategies, options and futures on bonds, and interest-rate derivatives.

3.10 R Lab

3.10.1 Computing Yield to Maturity

The following R function computes the price of a bond given its coupon payment, maturity, yield to maturity, and par value.

```
bondvalue = function(c, T, r, par)
{
#       Computes bv = bond values (current prices) corresponding
#       to all values of yield to maturity in the
#       input vector r
#
#       INPUT
#        c = coupon payment (semiannual)
#        T = time to maturity (in years)
#        r = vector of yields to maturity (semiannual rates)
#        par = par value
#
bv = c / r + (par - c / r) * (1 + r)^(-2 * T)
bv
}
```

The R code that follows computes the price of a bond for 300 semiannual interest rates between 0.02 and 0.05 for a 30-year par $1,000 bond with coupon payments of $40. Then interpolation is used to find the yield to maturity if the current price is $1,200.

```
price = 1200    #   current price of the bond
C = 40          #   coupon payment
T= 30           #   time to maturity
par = 1000      #   par value of the bond

r = seq(0.02, 0.05, length = 300)
value = bondvalue(C, T, r, par)
yield2M = spline(value, r, xout = price) # spline interpolation
```

The final bit of R code below plots price as a function of yield to maturity and graphically interpolates to show the yield to maturity when the price is $1,200.

```
plot(r, value, xlab = 'yield to maturity', ylab = 'price of bond',
   type = "l", main = "par = 1000, coupon payment = 40,
```

```
    T = 30", lwd = 2)
abline(h = 1200)
abline(v = yield2M)
```

Problem 1 *Use the plot to estimate graphically the yield to maturity. Does this estimate agree with that from spline interpolation?*

As an alternative to interpolation, the yield to maturity can be found using a nonlinear root finder (equation solver) such as `uniroot()`, which is illustrated here:

```
uniroot(function(r) r^2 - .5, c(0.7, 0.8))
```

Problem 2 *What does the code*

```
uniroot(function(r) r^2 - 0.5, c(0.7, 0.8))
```

do?

Problem 3 *Use* uniroot() *to find the yield to maturity of the 30-year par \$1,000 bond with coupon payments of \$40 that is selling at \$1,200.*

Problem 4 *Find the yield to maturity of a par \$10,000 bond selling at \$9,800 with semiannual coupon payments equal to \$280 and maturing in 8 years.*

Problem 5 *Use* uniroot() *to find the yield to maturity of the 20-year par \$1,000 bond with semiannual coupon payments of \$35 that is selling at \$1,050.*

Problem 6 *The yield to maturity is 0.035 on a par \$1,000 bond selling at \$950.10 and maturing in 5 years. What is the coupon payment?*

3.10.2 Graphing Yield Curves

`R`'s `fEcofin` package had many interesting financial data sets but is no longer available. The data sets `mk.maturity.csv` and `mk.zero2.csv` used in this example were taken from this package and are now available on this book's webpage. The data set `mk.zero2` has yield curves of U.S. zero coupon bonds recorded monthly at 55 maturities. These maturities are in the data set `mk.maturity`. The following code plots the yield curves on four consecutive months.

```
mk.maturity = read.csv("mk.maturity.csv", header = T)
mk.zero2 = read.csv("mk.zero2.csv", header = T)
plot(mk.maturity[,1], mk.zero2[5,2:56], type = "l",
  xlab = "maturity", ylab = "yield")
lines(mk.maturity[,1], mk.zero2[6,2:56], lty = 2, type = "l")
lines(mk.maturity[,1], mk.zero2[7,2:56], lty = 3, type = "l")
lines(mk.maturity[,1], mk.zero2[8,2:56], lty = 4, type = "l")
legend("bottomright", c("1985-12-01", "1986-01-01",
   "1986-02-01", "1986-03-01"), lty = 1:4)
```

Run the code above and then, to zoom in on the short end of the curves, rerun the code with maturities restricted to 0 to 3 years; to do that, use `xlim` in the plot function.

Problem 7 *Describe how the yield curve changes between December 1, 1985 and March 1, 1986. Describe the behavior of both the short and long ends of the yield curves.*

Problem 8 *Plot the yield curves from December 1, 1986 to March 1, 1987 and describe how the yield curve changes during this period.*

The next set of code estimates the forward rate for 1 month. Line 1 estimates the integrated forward rate, called `intForward`, which is $Ty_T = \int_0^T r(t)dt$ where $r(t)$ is the forward rate. Line 3 interpolates the estimated integrated forward rate onto a grid of 200 points from 0 to 20. This grid is created on line 2.

If a function f is evaluated on a grid, $t_1, \ldots, t_L$, then $\{f(t_\ell) - f(t_{\ell-1})\}/(t_\ell - t_{\ell-1})$ approximates $f'((t_\ell + t_{\ell-1})/2)$ for $\ell = 2, \ldots, L$. Line 4 numerically differentiates the integrated forward rate to approximate the forward rate on the grid calculated at Line 5.

```
1 intForward = mk.maturity[, 1] * mk.zero2[6, 2:56]
2 xout = seq(0, 20, length = 200)
3 z1 = spline(mk.maturity[ ,1], intForward, xout = xout)
4 forward = diff(z1$y) / diff(z1$x)
5 T_grid = (xout[-1] + xout[-200]) / 2
6 plot(T_grid, forward, type = "l", lwd = 2, ylim = c(0.06, 0.11))
```

Problem 9 *Plot the forward rates on the same dates used before, 1985-12-01, 1986-01-01, 1986-02-01, and 1986-03-01. Describe how the forward rates changed from month to month.*

The approximate forward rates found by numerically differentiating a interpolating spline are "wiggly." The wiggles can be removed, or at least reduced, by using a penalized spline instead of an interpolating spline. See Chap. 21.

3.11 Exercises

1. Suppose that the forward rate is $r(t) = 0.028 + 0.00042t$.
   (a) What is the yield to maturity of a bond maturing in 20 years?
   (b) What is the price of a par \$1,000 zero-coupon bond maturing in 15 years?
2. Suppose that the forward rate is $r(t) = 0.04 + 0.0002t - 0.00003t^2$.
   (a) What is the yield to maturity of a bond maturing in 8 years?
   (b) What is the price of a par \$1,000 zero-coupon bond maturing in 5 years?
   (c) Plot the forward rate and the yield curve. Describe the two curves. Which are convex and which are concave? How do they differ?
   (d) Suppose you buy a 10-year zero-coupon bond and sell it after 1 year. What will be the return if the forward rate does not change during that year?
3. A coupon bond has a coupon rate of 3 % and a current yield of 2.8 %.
   (a) Is the bond selling above or below par? Why or why not?
   (b) Is the yield to maturity above or below 2.8 %? Why or why not?
4. Suppose that the forward rate is $r(t) = 0.032 + 0.001t + 0.0002t^2$.
   (a) What is the 5-year continuously compounded spot rate?
   (b) What is the price of a zero-coupon bond that matures in 5 years?
5. The 1/2-, 1-, 1.5-, and 2-year semiannually compounded spot rates are 0.025, 0.028, 0.032, and 0.033, respectively. A par \$1,000 coupon bond matures in 2 years and has semiannual coupon payments of \$35. What is the price of this bond?
6. Verify the following equality:

$$\sum_{t=1}^{2T} \frac{C}{(1+r)^t} + \frac{\text{PAR}}{(1+r)^{2T}} = \frac{C}{r} + \left\{ \text{PAR} - \frac{C}{r} \right\} (1+r)^{-2T}.$$

7. One year ago a par \$1,000 20-year coupon bond with semiannual coupon payments was issued. The annual interest rate (that is, the coupon rate) at that time was 8.5 %. Now, a year later, the annual interest rate is 7.6 %.
   (a) What are the coupon payments?
   (b) What is the bond worth now? Assume that the second coupon payment was just received, so the bondholder receives an additional 38 coupon payments, the next one in 6 months.
   (c) What would the bond be worth if instead the second payment were just about to be received?
8. A par \$1,000 zero-coupon bond that matures in 5 years sells for \$828. Assume that there is a constant continuously compounded forward rate r.
   (a) What is r?
   (b) Suppose that 1 year later the forward rate r is still constant but has changed to be 0.042. Now what is the price of the bond?

(c) If you bought the bond for the original price of \$828 and sold it 1 year later for the price computed in part (b), then what is the net return?

9. A coupon bond with a par value of \$1,000 and a 10-year maturity pays semiannual coupons of \$21.
   (a) Suppose the yield for this bond is 4 % per year compounded semiannually. What is the price of the bond?
   (b) Is the bond selling above or below par value? Why?
10. Suppose that a coupon bond with a par value of \$1,000 and a maturity of 7 years is selling for \$1,040. The semiannual coupon payments are \$23.
    (a) Find the yield to maturity of this bond.
    (b) What is the current yield on this bond?
    (c) Is the yield to maturity less or greater than the current yield? Why?
11. Suppose that the continuous forward rate is $r(t) = 0.033 + 0.0012t$. What is the current value of a par \$100 zero-coupon bond with a maturity of 15 years?
12. Suppose the continuous forward rate is $r(t) = 0.04 + 0.001t$ when a 8-year zero coupon bond is purchased. Six months later the forward rate is $r(t) = 0.03 + 0.0013t$ and bond is sold. What is the return?
13. Suppose that the continuous forward rate is $r(t) = 0.03 + 0.001t - 0.00021(t - 10)_+$. What is the yield to maturity on a 20-year zero-coupon bond? Here x_+ is the *positive part function* defined by

$$x_+ = \begin{cases} x, & x > 0, \\ 0, & x \leq 0. \end{cases}$$

14. An investor is considering the purchase of zero-coupon bonds with maturities of one, three, or 5 years. Currently the spot rates for 1-, 2-, 3-, 4-, and 5-year zero-coupon bonds are, respectively, 0.031, 0.035, 0.04, 0.042, and 0.043 per year with semiannual compounding. A financial analyst has advised this investor that interest rates will increase during the next year and the analyst expects all spot rates to increase by the amount 0.005, so that the 1-year spot rate will become 0.036, and so forth. The investor plans to sell the bond at the end of 1 year and wants the greatest return for the year. This problem does the bond math to see which maturity, 1, 3, or 5 years, will give the best return under two scenarios: interest rates are unchanged and interest rates increase as forecast by the analyst.
    (a) What are the current prices of 1-, 3-, and 5-year zero-coupon bonds with par values of \$1,000?
    (b) What will be the prices of these bonds 1 year from now if spot rates remain unchanged?
    (c) What will be the prices of these bonds 1 year from now if spot rates each increase by 0.005?
    (d) If the analyst is correct that spot rates will increase by 0.005 in 1 year, which maturity, 1, 3, or 5 years, will give the investor the greatest return when the bond is sold after 1 year? Justify your answer.

(e) If instead the analyst is incorrect and spot rates remain unchanged, then which maturity, 1, 3, or 5 years, earns the highest return when the bond is sold after 1 year? Justify your answer.

(f) The analyst also said that if the spot rates remain unchanged, then the bond with the highest spot rate will earn the greatest 1-year return. Is this correct? Why?

(*Hint:* Be aware that a bond will not have the same maturity in 1 year as it has now, so the spot rate that applies to that bond will change.)

15. Suppose that a bond pays a cash flow C_i at time T_i for $i = 1, \ldots, N$. Then the net present value (NPV) of cash flow C_i is

$$\mathrm{NPV}_i = C_i \exp(-T_i\, y_{T_i}).$$

Define the weights

$$\omega_i = \frac{\mathrm{NPV}_i}{\sum_{j=1}^{N} \mathrm{NPV}_j}$$

and define the duration of the bond to be

$$\mathrm{DUR} = \sum_{i=1}^{N} \omega_i T_i,$$

which is the weighted average of the times of the cash flows. Show that

$$\frac{d}{d\delta} \sum_{i=1}^{N} C_i \exp\{-T_i(y_{T_i} + \delta)\}\bigg|_{\delta=0} = -\mathrm{DUR} \sum_{i=1}^{N} C_i \exp\{-T_i\, y_{T_i}\}$$

and use this result to verify Eq. (3.31).

16. Assume that the yield curve is $Y_T = 0.04 + 0.001\,T$.

(a) What is the price of a par-\$1,000 zero-coupon bond with a maturity of 10 years?

(b) Suppose you buy this bond. If 1 year later the yield curve is $Y_T = 0.042 + 0.001\,T$, then what will be the net return on the bond?

17. A coupon bond has a coupon rate of 3 % and a current yield of 2.8 %.

(a) Is the bond selling above or below par? Why or why not?

(b) Is the yield to maturity above or below 2.8 %? Why or why not?

18. Suppose that the forward rate is $r(t) = 0.03 + 0.001t + 0.0002t^2$

(a) What is the 5-year spot rate?

(b) What is the price of a zero-coupon bond that matures in 5 years?

19. The 1/2-, 1-, 1.5-, and 2-year spot rates are 0.025, 0.029, 0.031, and 0.035, respectively. A par \$1,000 coupon bond matures in 2 years and has semi-annual coupon payments of \$35. What is the price of this bond?

20. Par \$1,000 zero-coupon bonds of maturities of 0.5-, 1-, 1.5-, and 2-years are selling at \$980.39, \$957.41, \$923.18, and \$888.489, respectively.

(a) Find the 0.5-, 1-, 1.5-, and 2-year semiannual spot rates.

(b) A par \$1,000 coupon bond has a maturity of 2 years. The semiannual coupon payment is \$21. What is the price of this bond?

21. A par \$1,000 bond matures in 4 years and pays semiannual coupon payments of \$25. The price of the bond is \$1,015. What is the semiannual yield to maturity of this bond?
22. A coupon bond matures in 4 years. Its par is \$1,000 and it makes eight coupon payments of \$21, one every one-half year. The continuously compounded forward rate is

$$r(t) = 0.022 + 0.005\,t - 0.004\,t^2 + 0.0003\,t^3.$$

(a) Find the price of the bond.
(b) Find the duration of this bond.

References

Bodie, Z., Kane, A., and Marcus, A. (1999) *Investments,* 4th ed., Irwin/McGraw-Hill, Boston.

Campbell, J. Y., Lo, A. W., and MacKinlay, A. C. (1997) *Econometrics of Financial Markets*, Princeton University Press, Princeton, NJ.

Dowd, K. (1998) *Beyond Value at Risk*, Wiley, Chichester.

James, J., and Webber, N. (2000) *Interest Rate Modeling*, Wiley, Chichester.

Jarrow, R. (2002) *Modeling Fixed-Income Securities and Interest Rate Options,* 2nd ed., Stanford University Press, Stanford, CA.

Sharpe, W., Alexander, G., and Bailey, J. (1999) *Investments,* 6th ed., Prentice-Hall, Englewood Cliffs, NJ.

Tuckman, B. (2002) *Fixed Income Securities,* 2nd ed., Wiley, Hoboken, NJ.