# Chapter 19
# Allais's Paradox

**Leonard Savage**

Introspection about certain hypothetical decision situations suggests that the sure-thing principle and, with it, the theory of utility are normatively unsatisfactory. Consider an example based on two decision situations each involving two gambles.[1]

Situation 1. Choose between

Gamble 1. $500,000 with probability 1; and
Gamble 2. $2,500,000 with probability 0.1,
$500,000 with probability 0.89, status quo with probability 0.01.

Situation 2. Choose between

Gamble 3. $500,000 with probability 0.11, status quo with probability 0.89; and
Gamble 4. $2,500,000 with probability 0.1, status quo with probability 0.9.

Many people prefer Gamble 1 to Gamble 2, because, speaking qualitatively, they do not find the chance of winning a *very* large fortune in place of receiving a large fortune outright adequate compensation for even a small risk of being left in the status quo. Many of the same people prefer Gamble 4 to Gamble 3; because, speaking qualitatively, the chance of winning is nearly the same in both gambles, so the one with the much larger prize seems preferable. But the intuitively acceptable pair of preferences, Gamble 1 preferred to Gamble 2 and Gamble 4 to Gamble 3, is not compatible with the utility concept or, equivalently, the sure-thing

Leonard Savage was deceased at the time of publication.

[1]This particular example is due to Allais (1953). Another interesting example was presented somewhat earlier by Georges Morlat (1954).

L. Savage (deceased)
Princeton University, New York, NY, USA



H. Arló-Costa et al. (eds.), *Readings in Formal Epistemology*, Springer Graduate Texts in Philosophy 1, DOI 10.1007/978-3-319-20451-2_19

principle. Indeed that pair of preferences implies the following inequalities for any hypothetical utility function.

$$
\begin{aligned}
&U(\$500{,}000) > 0.1U(\$2{,}500{,}000) + 0.89U(\$500{,}000) + 0.1U(\$0),\\
(3)\quad &\\
&0.1U(\$2{,}500{,}000) + 0.9U(\$0) > 0.11U(\$500{,}000) + 0.89U(\$0);
\end{aligned}
$$

and these are obviously incompatible.

Examples[2] like the one cited do have a strong intuitive appeal; even if you do not personally feel a tendency to prefer Gamble 1 to Gamble 2 and simultaneously Gamble 4 to Gamble 3, I think that a few trials with other prizes and probabilities will provide you with an example appropriate to yourself.

If, after thorough deliberation, anyone maintains a pair of distinct preferences that are in conflict with the sure-thing principle, he must abandon, or modify, the principle; for that kind of discrepancy seems intolerable in a normative theory. Analogous circumstances forced D. Bernoulli to abandon the theory of mathematical expectation for that of utility (Bernoulli 1738). In general, a person who has tentatively accepted a normative theory must conscientiously study situations in which the theory seems to lead him astray; he must decide for each by reflection—deduction will typically be of little relevance—whether to retain his initial impression of the situation or to accept the implications of the theory for it.

To illustrate, let me record my own reactions to the example with which this heading was introduced. When the two situations were first presented, I immediately expressed preference for Gamble 1 as opposed to Gamble 2 and for Gamble 4 as opposed to Gamble 3, and I still feel an intuitive attraction to those preferences. But I have since accepted the following way of looking at the two situations, which amounts to repeated use of the sure-thing principle.

One way in which Gambles 1–4 could be realized is by a lottery with a hundred numbered tickets and with prizes according to the schedule shown in Table 19.1.

**Table 19.1** Prizes in units of \$100,000 in a lottery realizing gambles 1–4

| | | Ticket number | | |
|---|---|---|---|---|
| | | 1 | 2–11 | 12–100 |
| Situation 1 | Gamble 1 | 5 | 5 | 5 |
| | Gamble 2 | 0 | 25 | 5 |
| Situation 2 | Gamble 3 | 5 | 5 | 0 |
| | Gamble 4 | 0 | 25 | 0 |

[2] Allais has announced (but not yet published) an empirical investigation of the responses of prudent, educated people to such examples (Allais 1953).

Now, if one of the tickets numbered from 12 through 100 is drawn, it will not matter, in either situation, which gamble I choose. I therefore focus on the possibility that one of the tickets numbered from 1 through 11 will be drawn, in which case Situations 1 and 2 are exactly parallel. The subsidiary decision depends in both situations on whether I would sell an outright gift of $500,000 for a 10-to-1 chance to win $2,500,000—a conclusion that I think has a claim to universality, or objectivity. Finally, consulting my purely personal taste, I find that I would prefer the gift of $500,000 and, accordingly, that I prefer Gamble 1 to Gamble 2 and (contrary to my initial reaction) Gamble 3 to Gamble 4.

It seems to me that in reversing my preference between Gambles 3 and 4 I have corrected an error. There is, of course, an important sense in which preferences, being entirely subjective, cannot be in error; but in a different, more subtle sense they can be. Let me illustrate by a simple example containing no reference to uncertainty. A man buying a car for $2,134.56 is tempted to order it with a radio installed, which will bring the total price to $2,228.41, feeling that the difference is trifling. But, when he reflects that, if he already had the car, he certainly would not spend $93.85 for a radio for it, he realizes that he has made an error.

One thing that should be mentioned before this chapter is closed is that the law of diminishing marginal utility plays no fundamental role in the von Neumann-Morgenstern theory of utility, viewed either empirically or normatively. Therefore the possibility is left open that utility as a function of wealth may not be concave, at least in some intervals of wealth. Some economic-theoretical consequences of recognition of the possibility of non-concave segments of the utility function have been worked out by Friedman and myself (1948), and by Friedman alone (1953). The work of Friedman and myself on this point is criticized by Markowitz (1952).[3]

## References

Allais, M. (1953). Le comportement de l'homme rationnel devant le risque: Critique des postulats et axioms de l'école Americaine. *Econometrica, 21*, 503–546.

Archibald, G. C. (1959). Utility, risk, and linearity. *Journal of Political Economy, 67,* 437–450.

Bernoulli, D. (1738). Specimen theoriae novae de mensura sortis. *Commentarii academiae scientiarum imperialis Petropolitanae (for 1730 and 1731)*, 5, 175–192.

Centre National de Recherche Scientifique Fondements et applications de la théorie du risque en économetrie, Paris, Centre National de la Recherche Scientifique. (1954) *Report of an international econometric colloquium on risk, in which there was much discussion of utility*, held in Paris, May 12–17, 1952.

Friedman, M. (1953). Choice, chance, and personal distribution of income. *Journal of Political Economy, 61,* 277–290.

Friedman, M., & Savage, L. J. (1948). The utility analysis of choices involving risk. *Journal of Political Economy, 56*, 279–304, Reprinted, with a correction, in Stigler and Boulding (1952).

[3]See also Archibald (1959) and Hakansson (1970).

Hakansson, N. H. (1970). Friedman-Savage utility functions consistent with risk aversion. *Quarterly Journal of Economics, 84,* 472–487.

Markowitz, H. (1952). The utility of wealth. *Journal of Political Economy, 60,* 151–158, Marshall, Alfred.

Stigler, G. J., & Boulding, K. E. (Eds.). (1952). *Readings in price theory*. Chicago: Richard D. Irwin.