# Chapter 19
# Advection

*Transport processes are very important in physics and engineering sciences. Transport of a conserved quantity like energy or concentration of a certain substance (e.g. salt) in a moving fluid is due to the effects of diffusion* (Chap. 21) *and advection (which denotes transport by the bulk motion). The combination of these two transport mechanisms is usually called convection.*

*In this chapter we investigate the advection equation in one spatial dimension*

$$\frac{\partial}{\partial t} f(x,t) = -c \frac{\partial}{\partial x} f(x,t). \tag{19.1}$$

*Numerical solutions are obtained with simple and more elaborate methods using finite differences, finite volumes and finite elements. Accuracy and stability of different methods are compared. The linear advection equation is an ideal test case but the methods are also useful for general nonlinear advection equations including the famous system of Navier–Stokes equations.*

## 19.1 The Advection Equation

Consider a fluid moving with velocity $\mathbf{u}(\mathbf{r})$ and let $f(\mathbf{r},t)$ denote the concentration of the substance. Its time dependence obeys the conservation law

$$\frac{\partial}{\partial t} f = \operatorname{div}\left(D \operatorname{grad} f - \mathbf{u} f\right) + S(\mathbf{r},t) = -\operatorname{div}\left(\mathbf{J}_{diff} + \mathbf{J}_{adv}\right) + S(\mathbf{r},t) \tag{19.2}$$

or in integral form

$$\frac{\partial}{\partial t} \int_V dV\, f(\mathbf{r},t) + \oint_{\partial V} \mathbf{J}(\mathbf{r},t) d\mathbf{A} = \int_V dV\, S(\mathbf{r},t). \tag{19.3}$$

Without diffusion and sources or sinks, the flux of the substance is given by



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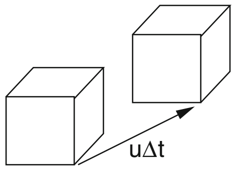



**Fig. 19.1** Advection in an incompressible fluid

$$\mathbf{J}(\mathbf{r}, t) = \mathbf{u}(\mathbf{r}, t) f(\mathbf{r}, t) \tag{19.4}$$

and the continuity equation for the substance concentration reads

$$\frac{\partial}{\partial t} f + \text{div}\,(f\,\mathbf{u}) = 0. \tag{19.5}$$

Introducing the substantial derivative we obtain

$$0 = \frac{\partial}{\partial t} f + \text{div}\,(f\,\mathbf{u}) = \frac{\partial}{\partial t} f + (\mathbf{u}\text{ grad})\, f + f\ \text{div}\mathbf{u} \tag{19.6}$$

$$= \frac{\mathrm{d}f}{\mathrm{d}t} + f\ \text{div}\mathbf{u}. \tag{19.7}$$

For the common case of an incompressible fluid div $\mathbf{u} = 0$ and the advection equation simplifies to

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\partial}{\partial t} f(\mathbf{r}, t) + (\mathbf{u}(\mathbf{r}, t)\text{ grad})\, f(\mathbf{r}, t) = 0 \tag{19.8}$$

which has a very simple interpretation. Consider a small element of the fluid (Fig. 19.1), which during a time interval $\Delta t$ moves from the position $\mathbf{r}$ to $\mathbf{r} + \Delta\mathbf{r} = \mathbf{r} + \mathbf{u}\Delta t$. The amount of substance does not change and we find

$$f(\mathbf{r}, t) = f(\mathbf{r} + \mathbf{u}\Delta t, t + \Delta t) = f(\mathbf{r}, t) + \frac{\partial f}{\partial t}\Delta t + \mathbf{u}\Delta t\text{ grad } f + \cdots \tag{19.9}$$

which in the limit of small $\Delta t$ becomes (19.8).

## 19.2 Advection in One Dimension

In one dimension div $\mathbf{u} = \frac{\partial u_x}{\partial x}$=0 implies constant velocity $u_x = c$. The differential equation

$$\frac{\partial f(x, t)}{\partial t} + c\frac{\partial f(x, t)}{\partial x} = 0 \tag{19.10}$$

can be solved exactly with d' Alembert's method. After substitution

$$x' = x - ct \quad t' = t \tag{19.11}$$

$$f(x,t) = f(x' + ct', t') = \phi(x', t')$$

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial x'} \quad \frac{\partial}{\partial t} = \frac{\partial}{\partial t'} - c\frac{\partial}{\partial x'} \tag{19.12}$$

it becomes

$$0 = \left(\frac{\partial}{\partial t'} - c\frac{\partial}{\partial x'} + c\frac{\partial}{\partial x'}\right)\phi = \frac{\partial}{\partial t'}\phi \tag{19.13}$$

hence $\phi$ does not depend on time and the solution has the

$$f(x,t) = \phi(x') = \phi(x - ct) \tag{19.14}$$

where the constant envelope is determined by the initial values

$$\phi(x') = f(x, t = 0). \tag{19.15}$$

After spatial Fourier transformation

$$\hat{f}(k,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}kx} f(x,t)dx \tag{19.16}$$

the advection equation becomes an ordinary differential equation

$$\frac{\mathrm{d}\hat{f}(t,k)}{\mathrm{d}t} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}kx}\mathrm{i}ck\, f(x,t)dx = \mathrm{i}ck\,\hat{f}(t,k) \tag{19.17}$$

quite similar to the example of a simple rotation (p. 13). Therefore we have to expect similar problems with the simple Euler integration methods (p. 293).

For a Fourier component of $f$ in space and time (i.e. a plane wave moving in x-direction)

$$g_{\omega k} = \mathrm{e}^{\mathrm{i}(\omega t - kx)} \tag{19.18}$$

we find a linear dispersion relation, i.e. all Fourier components move with the same velocity

$$\omega = ck. \tag{19.19}$$

### 19.2.1 Spatial Discretization with Finite Differences

The simplest discretization (p. 259) is obtained by introducing a regular grid

$$x_m = m\Delta x \quad m = 1, 2 \ldots M \tag{19.20}$$

$$f_m(t) = f(x_m, t) \tag{19.21}$$

and approximating the gradient by a finite difference quotient.

In the following we use periodic boundary conditions by setting $f_0 \equiv f_M$, $f_{M+1} \equiv f_1$ which are simplest to discuss and allow us to simulate longer times on a finite domain.

#### 19.2.1.1 First Order Forward and Backward Differences (Upwind Scheme)

First we use a first order backward difference in space

$$\frac{\mathrm{d}f_m(t)}{\mathrm{d}t} = c\frac{f_m(t) - f_{m-1}(t)}{\Delta x}. \tag{19.22}$$

From a Taylor series expansion

$$f(x - \Delta x) = f(x) - \frac{\partial f}{\partial x}\Delta x + \frac{(\Delta x)^2}{2}\frac{\partial^2 f}{\partial x^2} \cdots = \exp\left\{-\Delta x \frac{\partial}{\partial x}\right\} f(x) \tag{19.23}$$

we see that the leading error of the finite difference approximation

$$\frac{\partial f}{\partial t} - c\frac{f(x) - f(x - \Delta x)}{\Delta x} = \frac{\partial f}{\partial t} - c\frac{\partial f}{\partial x} + c\frac{\Delta x}{2}\frac{\partial^2 f}{\partial x^2} + \cdots \tag{19.24}$$

looks like a diffusion term for positive velocity $c$ and is therefore called “numerical diffusion”. Negative velocities, instead lead to an unphysical sharpening of the function f.

For $c < 0$ we have to reverse the space direction and use a forward difference

$$\frac{\mathrm{d}f_m(t)}{\mathrm{d}t} = c\frac{f_{m+1}(t) - f_m(t)}{\Delta x} \tag{19.25}$$

for which the sign of the second derivative changes

$$\frac{\partial f}{\partial t} - c\frac{f(x+\Delta x) - f(x)}{\Delta x} = \frac{\partial f}{\partial t} - c\frac{\partial f}{\partial x} - c\frac{\Delta x}{2}\frac{\partial^2 f}{\partial x^2} + \cdots . \quad (19.26)$$

Using the backward difference we obtain a system of ordinary differential equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_{M-1} \\ f_M \end{pmatrix} = -\frac{c}{\Delta x}\begin{pmatrix} 1 & & & & -1 \\ -1 & 1 & & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 1 & \\ & & & -1 & 1 \end{pmatrix}\begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_{M-1} \\ f_M \end{pmatrix} \quad (19.27)$$

or shorter

$$\frac{\mathrm{d}\mathbf{f}}{\mathrm{d}t} = -\frac{c}{\Delta x}M\mathbf{f} \quad (19.28)$$

with the formal solution

$$\mathbf{f}(t) = \exp\left\{-\frac{c}{\Delta x}M\,t\right\}\mathbf{f}(t=0). \quad (19.29)$$

The eigenpairs of $M$ are easily found (see p. 221). Inserting the Ansatz

$$\mathbf{f_k} = \begin{pmatrix} \mathrm{e}^{-\mathrm{i}k\Delta x} \\ \vdots \\ \mathrm{e}^{-M\mathrm{i}k\Delta x} \end{pmatrix} \quad (19.30)$$

corresponding to a Fourier component (19.18) into the eigenvalue equation we obtain

$$M\mathbf{f_k} = \begin{pmatrix} \mathrm{e}^{-\mathrm{i}k\Delta x} - \mathrm{e}^{-M\mathrm{i}k\Delta x} \\ \mathrm{e}^{-2\mathrm{i}k\Delta x} - \mathrm{e}^{-1\mathrm{i}k\Delta x} \\ \vdots \\ \mathrm{e}^{-(M-1)\mathrm{i}k\Delta x} - \mathrm{e}^{-(M-2)\mathrm{i}k\Delta x} \\ \mathrm{e}^{-M\mathrm{i}k\Delta x} - \mathrm{e}^{-(M-1)\mathrm{i}k\Delta x} \end{pmatrix}. \quad (19.31)$$

Solutions are found for values of $k$ given by

$$\mathrm{e}^{-M\mathrm{i}k\Delta x} = 1,\ k = 0, \frac{2\pi}{M\Delta x}, \ldots (M-1)\frac{2\pi}{M\Delta x} \quad (19.32)$$

or, reducing k-values to the first Brillouin zone (p. 132)

$$k = -\left(\frac{M}{2} - 1\right)\frac{2\pi}{M\Delta x} - \frac{2\pi}{M\Delta x}, 0, \frac{2\pi}{M\Delta x}, \ldots \frac{M}{2}\frac{2\pi}{M\Delta x} \quad M \text{ even} \quad (19.33)$$

$$k = -\frac{M}{2}\frac{2\pi}{M\Delta x} - \frac{2\pi}{M\Delta x}, 0, \frac{2\pi}{M\Delta x}, \ldots \frac{M}{2}\frac{2\pi}{M\Delta x} \quad M \text{ odd} \tag{19.34}$$

for which

$$M\mathbf{f_k} = \lambda_k \mathbf{f}_k = (1 - e^{ik\Delta x})\,\mathbf{f}_k. \tag{19.35}$$

The eigenvalues of $-\frac{c}{\Delta x}M$ are complex valued

$$\sigma_k = -\frac{c}{\Delta x}(1 - e^{ik\Delta x}) = \frac{c}{\Delta x}(\cos k\Delta x - 1) + i\frac{c}{\Delta x}\sin k\Delta x \tag{19.36}$$

and so is the dispersion

$$\omega_k = -i\sigma_k = \frac{c}{\Delta x}\sin k\Delta x - i\frac{c}{\Delta x}(\cos k\Delta x - 1). \tag{19.37}$$

If we take instead the forward difference we find similarly

$$\sigma_k = -\frac{c}{\Delta x}(e^{-ik\Delta x} - 1) = -\frac{c}{\Delta x}(\cos k\Delta x - 1) + i\frac{c}{\Delta x}\sin k\Delta x \tag{19.38}$$

$$\omega_k = -i\lambda_k = \frac{c}{\Delta x}\sin k\Delta x + i\frac{c}{\Delta x}(\cos k\Delta x - 1). \tag{19.39}$$

#### 19.2.1.2 Second Order Symmetric Difference

A symmetric difference quotient has higher error order and no diffusion term

$$\frac{f(x+\Delta x) - f(x-\Delta x)}{2\Delta x} = \frac{\sinh\left(\Delta x \frac{\partial}{\partial x}\right)}{\Delta x} f = \frac{\partial f}{\partial x} + \frac{(\Delta x)^2}{6}\frac{\partial^3 f}{\partial x^3} + \cdots \tag{19.40}$$

It provides the system of ordinary differential equations.

$$\frac{d}{dt}\begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_{M-1} \\ f_M \end{pmatrix} = -\frac{c}{\Delta x}\begin{pmatrix} 0 & 1/2 & & & -1/2 \\ -1/2 & 0 & 1/2 & & \\ & \ddots & \ddots & \ddots & \\ & & -1/2 & 0 & 1/2 \\ 1/2 & & & -1/2 & 0 \end{pmatrix}\begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_{M-1} \\ f_M \end{pmatrix}. \tag{19.41}$$

The eigenpairs of $M$ are easily found (see p. 221) from

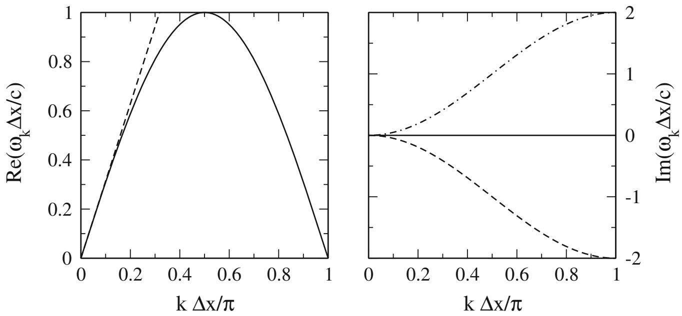



**Fig. 19.2** (Dispersion of finite difference quotients) The dispersion of first order (19.37, 19.39) and second order (19.45) difference quotients is shown. ***Left*** the real part of $\omega_k$ (*full curve*) which is the same in all three cases is compared to the linear dispersion of the exact solution (*broken line*). ***Right*** imaginary part of $\omega_k$ (*dashed curve* = forward difference, *dash-dotted curve* = backward difference, *full line* = second order symmetric difference)

$$M\mathbf{f_k} = \frac{1}{2}\begin{pmatrix} e^{-2ik\Delta x} - e^{-Mik\Delta x} \\ e^{-3ik\Delta x} - e^{-ik\Delta x} \\ \vdots \\ e^{-Mik\Delta x} - e^{-(M-2)ik\Delta x} \\ e^{-ik\Delta x} - e^{-(M-1)ik\Delta x} \end{pmatrix} \tag{19.42}$$

$$= \frac{1}{2}(e^{-ik\Delta x} - e^{ik\Delta x})\mathbf{f}_k = -i\sin k\Delta x\,\mathbf{f}_k \tag{19.43}$$

hence the eigenvalues of $-\frac{c}{\Delta x}M$ are purely imaginary and there is no damping (Fig. 19.2)

$$\sigma_k = i\frac{c}{\Delta x}\sin k\Delta x \tag{19.44}$$

$$\omega_k = -i\sigma_k = \frac{c}{\Delta x}\sin k\Delta x. \tag{19.45}$$

## *19.2.2 Explicit Methods*

Time integration with an explicit forward Euler step proceeds according to (p. 293)

$$\mathbf{f}(t+\Delta t) = \mathbf{f}(t) + \frac{\partial f}{\partial t}\Delta t = \mathbf{f}(t) - c\frac{\partial f}{\partial x}\Delta t \tag{19.46}$$

and can be formulated in matrix notation as

$$\mathbf{f}(t+\Delta t) = A\,\mathbf{f}(t) = (1-\alpha M)\mathbf{f}(t) \tag{19.47}$$

where the matrix M depends on the discretization method.

#### 19.2.2.1 Forward in Time, Backward in Space

Combination with the backward difference quotient gives the FTBS (forward in time backward in space) method

$$f(x,t+\Delta t) = f(x,t) - \alpha\,(f(x,t) - f(x-\Delta x,t)) \tag{19.48}$$

with the so called Courant number [243][1]

$$\alpha = c\frac{\Delta t}{\Delta x}. \tag{19.49}$$

The eigenvalues of $1-\alpha M$ are

$$\begin{aligned}\sigma_k &= 1-\alpha(1-e^{ik\Delta x})\\ &= 1-\alpha(1-\cos k\Delta x) + i\alpha\sin k\Delta x\end{aligned} \tag{19.50}$$

with absolute square (Fig. 19.3)

$$|\sigma_k|^2 = 1 + 2(\alpha^2-\alpha)(1-\cos k\Delta x). \tag{19.51}$$

Stability requires that $|\sigma_k| \le 1$, i.e.

$$2(\alpha^2-\alpha)(1-\cos k\Delta x) \le 0 \tag{19.52}$$

and, since $(1-\cos k\Delta x) \ge 0$

$$(\alpha-1)\alpha \le 0 \tag{19.53}$$

with the solution[2]

$$0 \le \alpha \le 1. \tag{19.54}$$

[1] Also known as CFL (after the names of Courant, Friedrichs, Lewy).

[2] The so called Courant– Friedrichs– Lewy condition (or CFL condition).

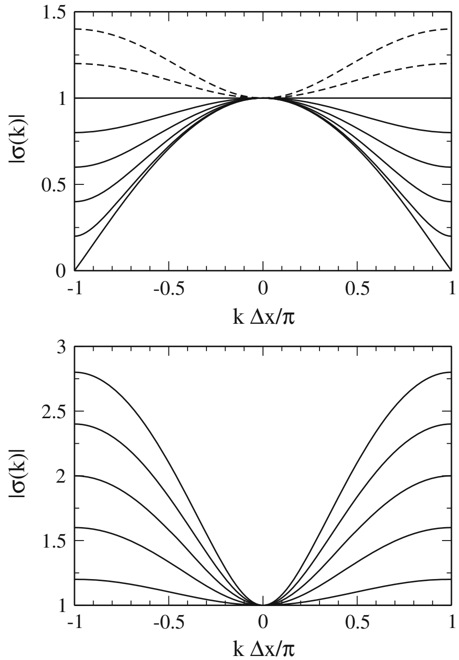



**Fig. 19.3** (Stability of the FTBS method) ***Top*** the magnitude of the eigenvalue $|\sigma_k|$ is shown as a function of k for positive values of the Courant number (from *Bottom* to *Top*) $\alpha = 0.5, 0.6, 0.7, 0.8, 0.9, 1.0, 1.1$. The method is stable for $\alpha \leq 1$ (*full curves*) and unstable for $\alpha > 1$ (*dashed curves*). ***Bottom*** the magnitude of the eigenvalue $|\sigma_k|$ is shown as a function of k for negative values of the Courant number (from *Bottom* to *Top*) $\alpha = -0.1, -0.3, -0.5, -0.7, -0.9$. The method is unstable for all $\alpha < 0$

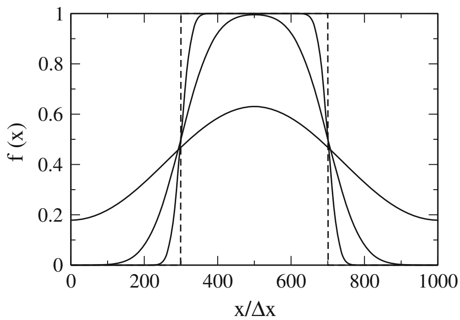



**Fig. 19.4** (Performance of the FTBS method) An initially rectangular pulse (*dashed curve*) is propagated with the FTBS method ($\Delta x = 0.01$, $\Delta t = 0.005$, $\alpha = 0.5$). Due to numerical diffusion the shape is rapidly smoothened. Results are shown after 1,10 and 100 round trips (2000 time steps each)

The FTBS method works, but shows strong damping due to numerical diffusion (Fig. 19.4). Its dispersion relation is

$$\omega_k \Delta t = -\mathrm{i}\ln(\sigma_k) = -\mathrm{i}\ln\left([1 - \alpha(1 - \cos k\Delta x) + \mathrm{i}\alpha \sin k\Delta x]\right). \tag{19.55}$$

#### 19.2.2.2 Forward in Time, Forward in Space

For a forward difference we obtain similarly

$$f(x, t + \Delta t) = f(x, t) - \alpha\,(f(x + \Delta x, t) - f(x, t)) \tag{19.56}$$

$$\sigma_k = 1 - \alpha(e^{-ik\Delta x} - 1) \tag{19.57}$$

$$|\sigma_k|^2 = 1 + 2(\alpha^2 + \alpha)(1 - \cos k\Delta x) \tag{19.58}$$

which is the same result as for the backward difference with $\alpha$ replaced by $-\alpha$.

#### 19.2.2.3 Forward in Time, Centered in Space

For a symmetric difference quotient, the eigenvalues of M are purely imaginary, all eigenvalues of $(1 + \alpha M)$

$$\sigma_k = 1 + i\alpha \sin k\Delta x \tag{19.59}$$

$$|\sigma_k|^2 = 1 + \alpha^2 \sin^2 k\Delta x \tag{19.60}$$

have absolute values $|\sigma_k| > 1$ and this method (FTCS, forward in time centered in space) is unstable (Fig. 19.5).

#### 19.2.2.4 Lax-Friedrichs-Scheme

Stability can be achieved by a modification which is known as Lax-Friedrichs-scheme. The value of $f(x, t)$ is averaged over neighboring grid points

$$\begin{aligned} f(x, t + \Delta t) &= \frac{f(x + \Delta x) + f(x - \Delta x)}{2} - \frac{\alpha}{2}(f(x + \Delta x) - f(x - \Delta x)) \\ &= \left[\frac{1-\alpha}{2}\exp\left(\Delta x \frac{\partial}{\partial x}\right) + \frac{1+\alpha}{2}\exp\left(-\Delta x \frac{\partial}{\partial x}\right)\right] f(x, t) \\ &= 1 - \alpha \frac{\partial f}{\partial x} + \frac{(\Delta x)^2}{2}\frac{\partial^2 f}{\partial x^2} + \cdots . \end{aligned} \tag{19.61}$$

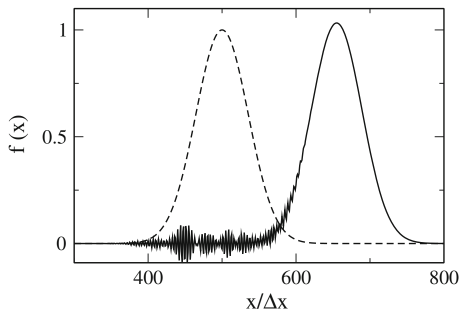



**Fig. 19.5** (Instability of the FTCS method) An initially Gaussian pulse (*dashed curve*) is propagated with the FTCS method ($\Delta x = 0.01$, $\Delta t = 0.005$, $\alpha = 0.5$). Numerical instabilities already show up after 310 time steps and blow up rapidly afterwards

The error order is $O(\Delta x^2)$ as for the FTCS method but the leading error has now diffusive character. We obtain the system of equations

$$\mathbf{f}(t+\Delta t) = \frac{1}{2}\begin{pmatrix} & 1-\alpha & & & 1+\alpha \\ 1+\alpha & & 1-\alpha & & \\ & \ddots & \ddots & \ddots & \\ & & 1+\alpha & & 1-\alpha \\ 1-\alpha & & & 1+\alpha & \end{pmatrix} \mathbf{f}(t). \tag{19.62}$$

The eigenvalues follow from

$$(1-\alpha)\mathrm{e}^{-\mathrm{i}(n+1)k\Delta x} + (1+\alpha)\mathrm{e}^{-\mathrm{i}(n-1)k\Delta x} = \mathrm{e}^{-\mathrm{i}nk\Delta x}\left[(1-\alpha)\mathrm{e}^{-\mathrm{i})k\Delta x} + (1+\alpha)\mathrm{e}^{\mathrm{i}k\Delta x}\right] \tag{19.63}$$

and are given by

$$\sigma_k = \frac{1}{2}\left[(1-\alpha)\mathrm{e}^{-\mathrm{i}k\Delta x} + (1+\alpha)\mathrm{e}^{\mathrm{i}k\Delta x}\right]$$
$$= \cos k\Delta x + \mathrm{i}\alpha \sin k\Delta x. \tag{19.64}$$

The absolute square is

$$|\sigma_k|^2 = \frac{1}{4}\left[(1-\alpha)\mathrm{e}^{-\mathrm{i})k\Delta x} + (1+\alpha)\mathrm{e}^{\mathrm{i}k\Delta x}\right]\left[(1-\alpha)\mathrm{e}^{\mathrm{i})k\Delta x} + (1+\alpha)\mathrm{e}^{-\mathrm{i}k\Delta x}\right]$$
$$= \frac{1}{4}\left[(1-\alpha)^2 + (1+\alpha)^2 + (1-\alpha^2)(\mathrm{e}^{-2\mathrm{i}k\Delta x} + \mathrm{e}^{2\mathrm{i}k\Delta x})\right]$$
$$= \frac{1}{2}\left[1+\alpha^2 + (1-\alpha^2)\cos 2k\Delta x\right]$$
$$= 1 - (1-\alpha^2)(\sin k\Delta x)^2 \tag{19.65}$$

and the method is stable for

$$(1-\alpha^2)(\sin k\Delta x)^2 \geq 0 \tag{19.66}$$

which is the case if the Courant condition holds

$$|\alpha| \leq 1. \tag{19.67}$$

#### 19.2.2.5 Lax-Wendroff Method

The Lax-Friedrichs method can be further improved to reduce numerical diffusion and obtain a method which is higher order in time. It is often used for hyperbolic partial differential equations. From the time derivative of the advection equation

$$\frac{\partial}{\partial t}\left(\frac{\partial f}{\partial t}\right) = -c\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial t}\right) = c^2\frac{\partial^2 f}{\partial x^2} \tag{19.68}$$

we obtain the Taylor expansion

$$f(t+\Delta t) = f(t) - \Delta t\, c\frac{\partial f}{\partial x} + \frac{(\Delta t)^2}{2}c^2\frac{\partial^2 f}{\partial x^2} + \cdots \tag{19.69}$$

which we discretize to obtain the Lax-Wendroff scheme

$$\begin{aligned} f(x,t+\Delta t) &= f(x,t) - \Delta t\, c\frac{f(x+\Delta x,t)-f(x-\Delta x,t)}{2\Delta x} \\ &+ \frac{(\Delta t)^2}{2}c^2\frac{f(x+\Delta x,t)+f(x-\Delta x,t)-2f(x,t)}{(\Delta x)^2}. \end{aligned} \tag{19.70}$$

This algorithm can also be formulated as a predictor-corrector method. First we calculate intermediate values at $t+\Delta t/2$, $x \pm \Delta x/2$ with the Lax method

$$\begin{aligned} f(x+\frac{\Delta x}{2}, t+\frac{\Delta t}{2}) &= \frac{f(x+\Delta x,t)+f(x,t)}{2} - c\Delta t\frac{f(x+\Delta x,t)-f(x,t)}{2\Delta x} \\ f(x-\frac{\Delta x}{2}, t+\frac{\Delta t}{2}) &= \frac{f(x,t)+f(x-\Delta x,t)}{2} - c\Delta t\frac{f(x,t)-f(x-\Delta x,t)}{2\Delta x} \end{aligned} \tag{19.71}$$

which are then combined in a corrector step

$$f(x,t+\Delta t) = f(x,t) - c\Delta t\frac{f(x+\frac{\Delta x}{2},t+\frac{\Delta t}{2}) - f(x-\frac{\Delta t}{2},t+\frac{\Delta t}{2})}{\Delta x}. \tag{19.72}$$

Insertion of the predictor step (19.71) shows the equivalence with (19.70).

$$\begin{aligned} f(x,t) - \frac{c\Delta t}{\Delta x}\Bigg[&\frac{f(x+\Delta x,t)+f(x,t)}{2} - c\Delta t\frac{f(x+\Delta x,t)-f(x,t)}{2\Delta x} \\ &-\frac{f(x,t)+f(x-\Delta x,t)}{2} + c\Delta t\frac{f(x,t)-f(x-\Delta x,t)}{2\Delta x}\Bigg] \end{aligned}$$

$$= f(x,t) - c\frac{\Delta t}{2\Delta x}\left[\frac{f(x+\Delta x,t) - f(x-\Delta x,t)}{2}\right] + \frac{c^2(\Delta t)^2}{2(\Delta x)^2}\left[f(x+\Delta x,t) - 2f(x,t) + f(x-\Delta x,t)\right]. \tag{19.73}$$

In matrix notation the Lax-Wendroff method reads

$$\mathbf{f}(t+\Delta t) = \begin{pmatrix} \ddots & & \\ \frac{\alpha+\alpha^2}{2} & 1-\alpha^2 & \frac{\alpha^2-\alpha}{2} \\ & & \ddots \end{pmatrix} \mathbf{f}(t) \tag{19.74}$$

with eigenvalues

$$\sigma_k = 1 - \alpha^2 + \frac{\alpha^2+\alpha}{2}e^{ik\Delta x} + \frac{\alpha^2-\alpha}{2}e^{-ik\Delta x} = 1 - \alpha^2 + \alpha^2\cos k\Delta x + i\alpha\sin k\Delta x \tag{19.75}$$

and

$$|\sigma_k|^2 = (1 + \alpha^2(\cos(k\Delta x) - 1))^2 + \alpha^2\sin^2 k\Delta x$$

$$= 1 - \alpha^2(1-\alpha^2)(1-\cos k\Delta x)^2 \tag{19.76}$$

which is $\leq 1$ for

$$\alpha^2(1-\alpha^2)(1-\cos k\Delta x)^2 \geq 0 \tag{19.77}$$

which reduces to the CFL condition

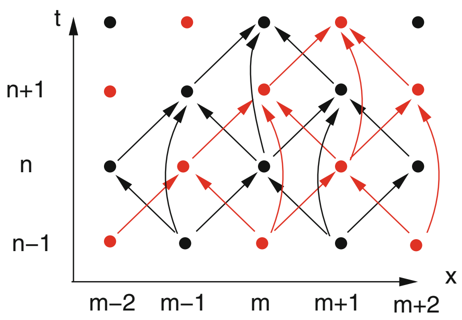
$$|\alpha| \leq 1. \tag{19.78}$$

#### 19.2.2.6 Leapfrog Method

The Leapfrog method uses symmetric differences for both derivatives

$$\frac{f(x,t+\Delta t) - f(x,t-\Delta t)}{2\Delta t} = -c\frac{f(x+\Delta x,t) - f(x-\Delta x,t)}{2\Delta x} \tag{19.79}$$

**Fig. 19.6** Leapfrog method for advection




to obtain a second order two step method

$$f(x, t + \Delta t) = f(x, t - \Delta t) - \alpha [f(x + \Delta x, t) - f(x - \Delta x, t)] \tag{19.80}$$

on a grid which is equally spaced in space and time.

The calculated data form two independent subgrids (Fig. 19.6). For long integration times this can lead to problems if the results on the subgrids become different due to numerical errors. Introduction of a diffusive coupling term can help to avoid such difficulties.

To analyze stability, we write the two step method as a one step method, treating the values at even and odd time steps as independent variables

$$g_m^n = f(m\Delta x, 2n\Delta t) \quad h_m^n = f(m\Delta x, (2n+1)\Delta t) \tag{19.81}$$

for which the Leapfrog scheme becomes

$$h_m^n = h_m^{n-1} - \alpha(g_{m+1}^n - g_{m-1}^n) \tag{19.82}$$

$$g_m^{n+1} = g_m^n - \alpha(h_{m+1}^n - h_{m-1}^n). \tag{19.83}$$

Combining this two equations we obtain the one step iteration

$$g_m^n = f_m^{2n} \quad h_m^n = f_m^{2n+1}$$

$$\begin{pmatrix} h^n \\ g^{n+1} \end{pmatrix} = \begin{pmatrix} 1 & \\ -\alpha M & 1 \end{pmatrix} \begin{pmatrix} 1 & -\alpha M \\ & 1 \end{pmatrix} \begin{pmatrix} h^{n-1} \\ g^n \end{pmatrix} = \begin{pmatrix} 1 & -\alpha M \\ -\alpha M & 1 + \alpha^2 M^2 \end{pmatrix} \begin{pmatrix} h^{n-1} \\ g^n \end{pmatrix}. \tag{19.84}$$

The eigenvalues of the matrix $M$ are $\lambda_k = -2\mathrm{i} \sin k\Delta x$, hence the eigenvalues of the Leapfrog scheme

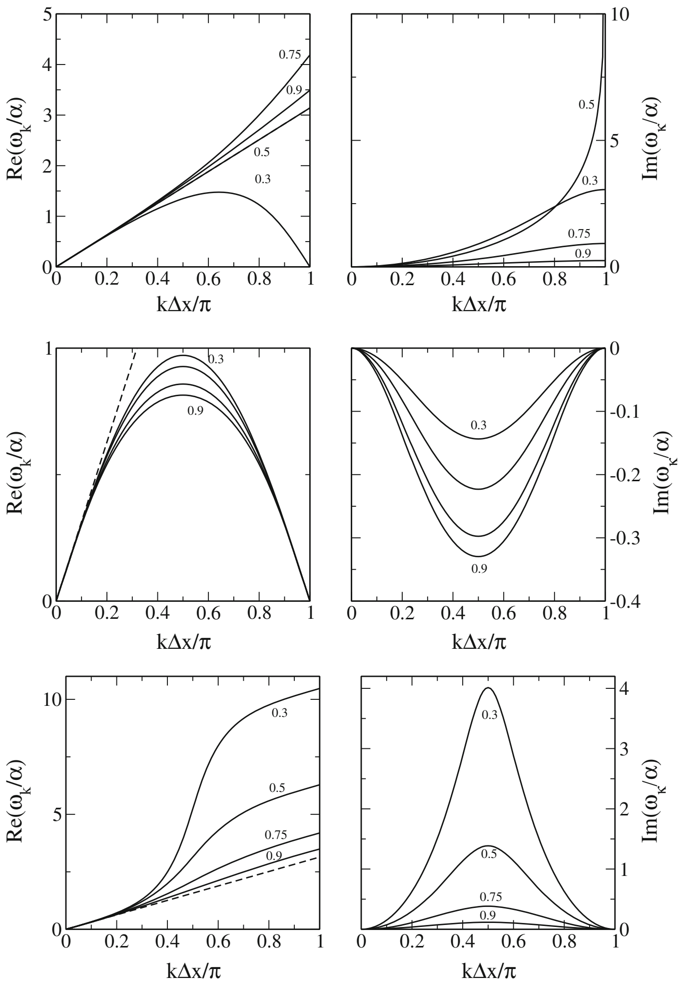



**Fig. 19.7** (Dispersion of first order explicit methods) Real (***Left***) and imaginary (***Right***) part of $\omega_k$ are shown for the first order explicit FTBS (***Top***), FTCS (***Middle***) and Lax-Friedrich (***Bottom***) methods for values of $\alpha = 0.3, 0.5, 0.75, 0.9$

$$\sigma_k = 1 + \frac{\alpha^2\lambda^2}{2} \pm \sqrt{\alpha^2\lambda^2 + \frac{\alpha^4\lambda^4}{4}}$$
$$= 1 - 2\alpha^2 \sin^2 k\Delta x \pm 2\sqrt{\alpha^2 \sin^2 k\Delta x(\alpha^2 \sin^2 k\Delta x - 1)}. \tag{19.85}$$

For $|\alpha| \leq 1$ the squareroot is purely imaginary and

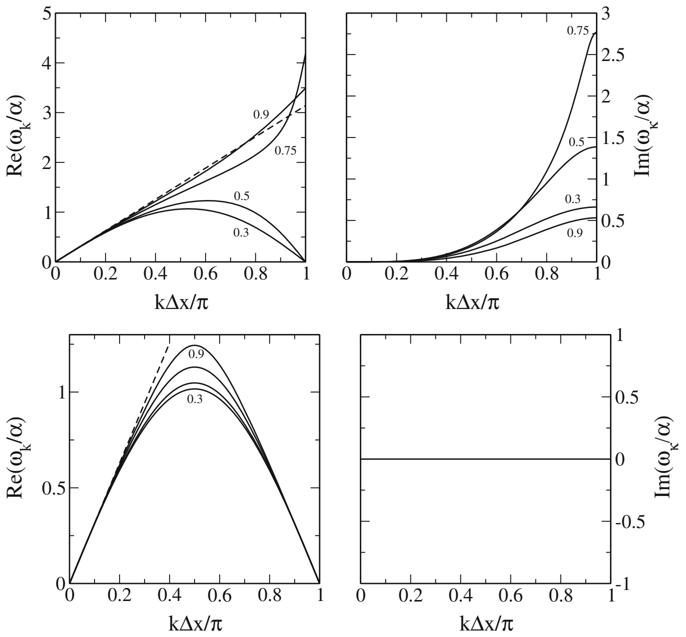



**Fig. 19.8** (Dispersion of second order explicit methods) Real (***Left***) and imaginary (***Right***) part of $\omega_k$ are shown for the second order explicit Lax-Wendroff (***Top***) and leapfrog (***Bottom***) methods for values of $\alpha = 0.3, 0.5, 0.75, 0.9$

$$|\sigma_k|^2 = 1$$

i.e. the method is unconditionally stable and diffusionless. The dispersion

$$2\omega\Delta t = -\mathrm{i}\ln\sigma_k = \arctan\left(\pm\frac{2\sqrt{\alpha^2\sin^2 k\Delta x - \alpha^4\sin^4 k\Delta x}}{1-2\alpha^2\sin^2 k\Delta x}\right) \tag{19.86}$$

has two branches. Expanding for small $k\Delta x$ we find

$$\omega \approx \pm ck + \cdots . \tag{19.87}$$

Only the plus sign corresponds to a physical mode. The negative sign corresponds to the so called computational mode which can lead to artificial rapid oscillations. These can be removed by special filter algorithms [244, 245].

Figures. 19.7, 19.8 and 19.9 show a comparison of different explicit methods.

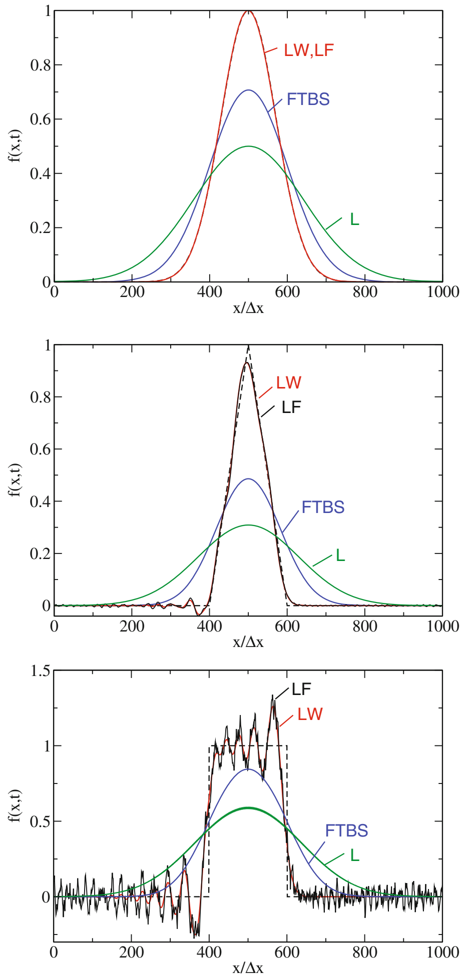



**Fig. 19.9** (Comparison of explicit methods) The results from the FTBS, Lax-Friedrichs(L, *green*), Lax-Wendroff (LW, *black*) and leapfrog (LF, *red*) methods after 10 roundtrips are shown. Initial values (*black dashed curves*) are Gaussian (***Top***), triangular (***Middle***) and rectangular (***Bottom***). $\Delta x = 0.01$, $\Delta t = 0.005$, $\alpha = 0.5$

## *19.2.3 Implicit Methods*

Time integration by implicit methods improves the stability but can be time consuming since inversion of a matrix is involved. A simple Euler backward step (13.4) takes the derivative at $t + \Delta t$

$$\mathbf{f}(t + \Delta t) = \mathbf{f}(t) - \alpha M \mathbf{f}(t + \Delta t) \tag{19.88}$$

which can be formally written as

$$\mathbf{f}(t + \Delta t) = (1 + \alpha M)^{-1} \mathbf{f}(t). \tag{19.89}$$

The Crank–Nicolson method (13.5) takes the average of implicit and explicit Euler step

$$\mathbf{f}(t+\Delta t) = \mathbf{f}(t) - \frac{\alpha}{2} M \left[\mathbf{f}(t+\Delta t) + \mathbf{f}(t)\right] \tag{19.90}$$

$$\mathbf{f}(t+\Delta t) = \left(1 + \frac{\alpha}{2} M\right)^{-1} \left(1 - \frac{\alpha}{2} M\right) \mathbf{f}(t). \tag{19.91}$$

Both methods require to solve a linear system of equations.

#### 19.2.3.1 First Order Implicit Method

Combining the back steps in time and space we obtain the BTBS (backward in time, backward in space) method

$$\mathbf{f}(t+\Delta t) = \left(1 + \alpha \begin{pmatrix} 1 & & & & -1 \\ -1 & 1 & & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 1 & \\ & & & -1 & 1 \end{pmatrix}\right)^{-1} \mathbf{f}(t). \tag{19.92}$$

The tridiagonal structure of the matrix $1 + \alpha M$ simplifies the solution of the system. The eigenvalues of $(1 + \alpha M)^{-1}$ are

$$\sigma_k = \frac{1}{1 + \alpha(1 - \mathrm{e}^{\mathrm{i}k\Delta x})} \tag{19.93}$$

$$|\sigma_k|^2 = \frac{1}{(1+\alpha)^2 + \alpha^2 - 2\alpha(1+\alpha)\cos(k\Delta x)} \le 1 \tag{19.94}$$

and the method is unconditionally stable.

#### 19.2.3.2 Second Order Crank–Nicolson Method

The Crank–Nicolson method with the symmetric difference quotient gives a second order method

$$\left(1 + \frac{\alpha}{2} M\right) \mathbf{f}(t+\Delta t) = \left(1 - \frac{\alpha}{2} M\right) \mathbf{f}(t). \tag{19.95}$$

The eigenvalues of $(1 + \frac{\alpha}{2}M)^{-1}(1 - \frac{\alpha}{2}M)$ are

$$\sigma_k = \frac{1 + \frac{\alpha}{2}\mathrm{i}\sin k\Delta x}{1 - \frac{\alpha}{2}\mathrm{i}\sin k\Delta x} \tag{19.96}$$

$$= \frac{1 - \frac{\alpha^2}{4}\sin^2 k\Delta x + \mathrm{i}\alpha\sin k\Delta x}{1 + \frac{\alpha^2}{4}\sin^2 k\Delta x} \tag{19.97}$$

with

$$|\sigma_k|^2 = 1. \tag{19.98}$$

There is no damping but strong dispersion at larger values of $\alpha$, slowing down partial waves with higher $k-$values (Fig. 19.11)

This method is unconditionally stable (Fig. 19.10). It may, however, show oscillations if the time step is chosen too large (Fig. 19.11). It can be turned into an explicit method by an iterative approach (iterated Crank–Nicolson method, see p. 475), which avoids solution of a linear system but is only stable for $\alpha \leq 1$.

### *19.2.4 Finite Volume Methods*

Finite volume methods [246] are very popular for equations in the form of a conservation law

$$\frac{\partial f(\mathbf{x}, t)}{\partial t} = -\mathrm{div}\mathbf{J}(f(\mathbf{x}, t)). \tag{19.99}$$

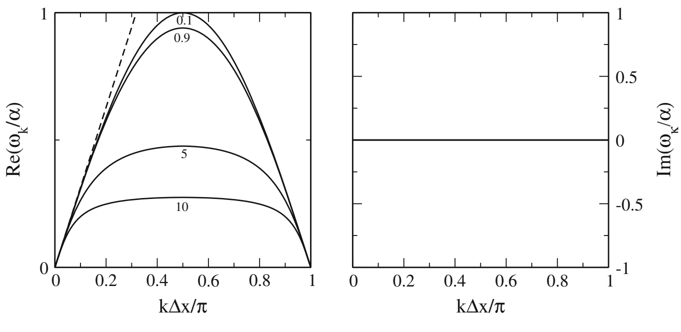



**Fig. 19.10** (Dispersion of the Crank–Nicolson method) Real (***Left***) and imaginary part (***Right***) part of $\omega_k$ are shown for $\alpha = 0.1, 9, 5, 10$. This implicit method is stable for $\alpha > 1$ but dispersion becomes noticeable at higher values

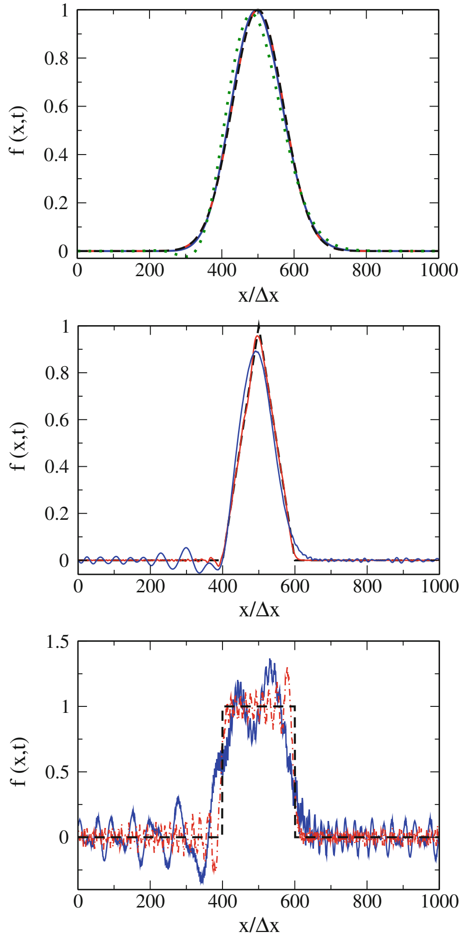



**Fig. 19.11** (Performance of the Crank–Nicolson method) Results of the implicit Crank–Nicolson method after 10 roundtrips are shown. Initial values (*black dashed curves*) are Gaussian (***Top***), triangular (***Middle***) and rectangular (***Bottom***). $\Delta x = 0.01$, $\Delta t = 0.01$ ($\alpha = 1$, *red dash-dotted curve) $\Delta t = 0.1(\alpha = 10$, *blue full curve)* $\Delta t = 0.2(\alpha = 20$, *green dotted curve*, only shown for Gaussian initial values)

In one dimension the control volumes are intervals, in the simplest case centered at equidistant grid points $x_n$

$$V_n = [x_n - \frac{\Delta x}{2}, x_n + \frac{\Delta x}{2}]. \tag{19.100}$$

Integration over one interval gives an equation for the cell average

$$\frac{\partial \overline{f}_n(t)}{\partial t} = \frac{1}{\Delta x}\frac{\partial}{\partial t}\int_{x_n-\Delta x/2}^{x_n+\Delta x/2} f(x,t) = -\frac{1}{\Delta x}\int_{x_n-\Delta x/2}^{x_n+\Delta x/2}\frac{\partial}{\partial x}J(x,t)$$

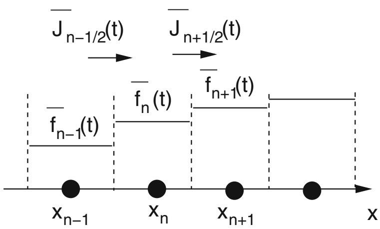



**Fig. 19.12** (Finite volume method) The change of the cell average $\overline{f}_n$ is balanced by the fluxes through the cell interfaces $\overline{J}_{n\pm 1/2}$

$$= -\frac{1}{\Delta x}\left[J(x_n + \frac{\Delta x}{2}, t) - J(x_n - \frac{\Delta x}{2}, t)\right]. \tag{19.101}$$

Formally this can be integrated

$$\overline{f}_n(t+\Delta t) - \overline{f}(t) = -\frac{1}{\Delta x}\left[\int_t^{t+\Delta t} J\left(x_n + \frac{\Delta x}{2}, t'\right)dt - \int_t^{t+\Delta t} J\left(x_n - \frac{\Delta x}{2}, t'\right)dt\right] \tag{19.102}$$

and with the temporally averaged fluxes through the control volume boundaries

$$\overline{J}_{n\pm 1/2}(t) = \frac{1}{\Delta t}\int_t^{t+\Delta t} J\left(x_n \pm \frac{\Delta x}{2}, t'\right)dt \tag{19.103}$$

it takes the simple form (Fig. 19.12)

$$\overline{f}_n(t+\Delta t) = \overline{f}_n(t) - \frac{\Delta t}{\Delta x}\left[\overline{J}_{n+1/2}(t) - \overline{J}_{n-1/2}(t)\right]. \tag{19.104}$$

A numerically scheme for a conservation law is called conservative if it can be written in this form with some approximation $\overline{J}_{n\pm 1/2}(t)$ of the fluxes at the cell interfaces. Conservative schemes are known to converge to a weak solution of the conservation law under certain conditions (stability and consistency).

To obtain a practical scheme, we have to approximate the fluxes in terms of the cell averages. Godunov's famous method [247] uses a piecewise constant approximation of $f(x,t)$

$$f(x,t) \approx \overline{f}_n(t) \quad \text{for } x_{n-1/2} \le x \le x_{n+1/2}. \tag{19.105}$$

To construct the fluxes, we have to solve the Riemann problem[3]

$$\frac{\partial f}{\partial t} = -\frac{\partial J}{\partial x} \tag{19.106}$$

[3] A conservation law with discontinuous initial values.

with discontinuous initial values

$$f(x,t) = \begin{cases} \overline{f}_n(t) & \text{if } x \le x_{n+1/2} \\ \overline{f}_{n+1}(t) & \text{if } x \ge x_{n+1/2} \end{cases} \tag{19.107}$$

in the time interval

$$t \le t' \le t + \Delta t. \tag{19.108}$$

For the linear advection equation with $J(x,t) = cf(x,t)$ the solution is easily found as the discontinuity just moves with constant velocity. For $c\Delta t \le \Delta x$, d'Alembert's method gives

$$f(x_n + \frac{\Delta x}{2}, t') = \overline{f}_n(t) \tag{19.109}$$

and the averaged flux is

$$\overline{J}_{n+1/2}(t) = \frac{c}{\Delta t}\int_t^{t+\Delta t} f(x_n + \frac{\Delta x}{2}, t')dt = c\overline{f}_n(t). \tag{19.110}$$

Finally we end up with the FTBS upwind scheme (Sect. 19.2.2)

$$\overline{f}_n(t+\Delta t) = \overline{f}_n(t) - \frac{c\Delta t}{\Delta x}\left[\overline{f}_n(t) - \overline{f}_{n-1}(t)\right]. \tag{19.111}$$

For general conservation laws, approximate methods have to be used to solve the Riemann problem (so called Riemann solvers [248]).

Higher order methods can be obtained by using higher order piecewise interpolation functions. If we interpolate linearly

$$f(x,t) \approx \overline{f}_n(t) + (x - x_n)\sigma_n(t) \quad \text{for } x_{n-1/2} \le x \le x_{n+1/2}$$

the solution to the Riemann problem is

$$f(x_n + \frac{\Delta x}{2}, t') = f(x_n + \frac{\Delta x}{2} - c(t'-t), t) = \overline{f}_n(t) + \left[\frac{\Delta x}{2} - c(t'-t)\right]\sigma_n(t)$$

$$= \overline{f}_n(t) + \left[\frac{\Delta x}{2} - c(t'-t)\right]\frac{\overline{f}_{n+1} - \overline{f}_{n-1}}{2\Delta x}.$$

The time averaged fluxes are

$$\overline{J}_{n+1/2} = c\overline{f}_n(t) + c\left[\frac{\Delta x}{2} - c\frac{\Delta t}{2}\right]\sigma_n(t)$$

and we end up with

$$\overline{f}_n(t+\Delta t) = \overline{f}_n(t) - \frac{c\Delta t}{\Delta x}\left[\overline{f}_n(t) - \overline{f}_{n-1}(t) + \left(\frac{\Delta x}{2} - c\frac{\Delta t}{2}\right)(\sigma_n - \sigma_{n-1})\right]. \tag{19.112}$$

If we take the slopes from the forward differences

$$\sigma_n = \frac{\overline{f}_{n+1} - \overline{f}_n}{\Delta x} \tag{19.113}$$

we end up with

$$\begin{aligned}\overline{f}_n(t+\Delta t) &= \overline{f}_n(t) - \frac{c\Delta t}{\Delta x}\left[\overline{f}_n(t) - \overline{f}_{n-1}(t)\right] - \frac{c\Delta t}{2\Delta x}(\Delta x - c\Delta t)\frac{\overline{f}_{n+1} - 2\overline{f}_n + \overline{f}_{n-1}}{\Delta x} \\ &= \overline{f}_n(t) - \frac{c\Delta t}{\Delta x}\left[\overline{f}_n(t) - \overline{f}_{n-1}(t) + \frac{\overline{f}_{n+1} - 2\overline{f}_n + \overline{f}_{n-1}}{2}\right] + \frac{(c\Delta t)^2}{2\Delta x}\frac{\overline{f}_{n+1} - 2\overline{f}_n + \overline{f}_{n-1}}{\Delta x} \\ &= \overline{f}_n(t) - \frac{c\Delta t}{\Delta x}\left[\frac{\overline{f}_{n+1} - \overline{f}_{n-1}}{2}\right] + \frac{(c\Delta t)^2}{2\Delta x}\frac{\overline{f}_{n+1} - 2\overline{f}_n + \overline{f}_{n-1}}{\Delta x}\end{aligned} \tag{19.114}$$

i.e. we end up with the Lax-Wendroff scheme. Different approximations for the slopes are possible (backward difference, symmetric differences) leading to the schemes of Fromm and Beam-Warming.

### *19.2.5 Taylor–Galerkin Methods*

The error order of finite difference methods can be improved by using a finite element discretization [249, 250]. We start from the Taylor series expansion in the time step

$$f(t+\Delta t) = f(t) + \Delta t\frac{\partial f}{\partial t} + \frac{(\Delta t)^2}{2}\frac{\partial^2 f}{\partial t^2} + \frac{(\Delta t)^3}{6}\frac{\partial^3 f}{\partial t^3} + \cdots \tag{19.115}$$

which is also the basis of the Lax-Wendroff method (19.70) and make use of the advection equation to substitute time derivatives by spatial derivatives

$$f(t+\Delta t) = f(t) - \Delta t\, c\frac{\partial f}{\partial x} + \frac{(\Delta t)^2}{2}c^2\frac{\partial^2 f}{\partial x^2} + \frac{(\Delta t)^3}{6}c^2\frac{\partial^3 f}{\partial t \partial x^2} + \cdots \tag{19.116}$$

where we use a mixed expression for the third derivative to allow the usage of linear finite elements. We approximate the third derivative as

$$\frac{\partial^3 f}{\partial t \partial x^2} = \frac{\partial^2}{\partial x^2}\frac{f(x,t+\Delta t) - f(x,t)}{\Delta t} + \cdots \tag{19.117}$$

and obtain an implicit expression which is a third order accurate extension of the Lax-Wendroff scheme

$$\left[1 - \frac{(\Delta t)^2}{6}c^2\frac{\partial^2}{\partial x^2}\right](f(x,t+\Delta t) - f(x,t)) = -\Delta t\, c\frac{\partial f}{\partial x} + \frac{(\Delta t)^2}{2}c^2\frac{\partial^2 f}{\partial x^2}. \tag{19.118}$$

Application of piecewise linear elements on a regular grid (p. 282) produces the following Lax-Wendroff Taylor-Galerkin scheme

$$\left[1 + \frac{1}{6}\left(1 - \alpha^2\right) D_2\right](\mathbf{f}(t+\Delta t) - \mathbf{f}(t)) = \left[-\alpha M_1 + \frac{\alpha^2}{2}M_2\right]\mathbf{f}(t). \tag{19.119}$$

The Taylor-Galerkin method can be also combined with other schemes like leapfrog or Crank–Nicolson [250]. It can be generalized to advection-diffusion problems and it can be turned into an explicit scheme [251] by series expansion of the inverse in

$$\mathbf{f}(t+\Delta t) = \mathbf{f}(t) + \left[1 + \frac{1}{6}\left(1 - \alpha^2\right) M_2\right]^{-1}\left[-\alpha M_1 + \frac{\alpha^2}{2}M_2\right]\mathbf{f}(t). \tag{19.120}$$

The eigenvalues are

$$\sigma_k = 1 + \frac{\alpha \mathrm{i} \sin k\Delta x - 2\alpha^2 \sin^2\frac{k\Delta x}{2}}{1 - \frac{2}{3}(1-\alpha^2)\sin^2\frac{k\Delta x}{2}}. \tag{19.121}$$

The method is stable for $|\alpha| \leq 1$. Due to its higher error order it shows less dispersion and damping than the Lax-Wendroff method (Fig. 19.13) and provides superior results (Fig. 19.14).

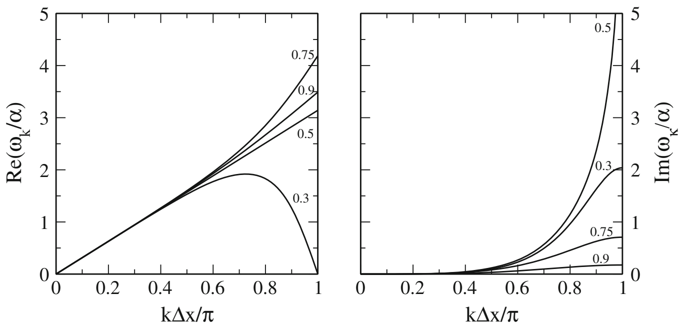



**Fig. 19.13** (Dispersion of the Taylor-Galerkin Lax-Wendroff method) Real (***Left***) and imaginary part (***Right***) part of $\omega_k$ are shown for $\alpha = 0.3, 0.5, 0.75, 0.9$

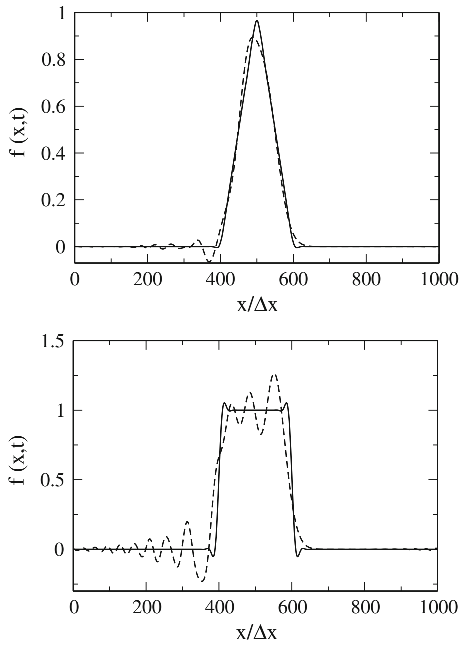



**Fig. 19.14** (Performance of the Taylor-Galerkin Lax-Wendroff method) Results of the Lax-Wendroff (*dashed curves*) and Taylor-Galerkin Lax-Wendroff (*full curves*) methods are compared after 25 roundtrips (2000 steps each). $\Delta x = 0.01$, $\Delta t = 0.005$, $\alpha = 0.5$

## 19.3 Advection in More Dimensions

While in one dimension for an incompressible fluid $c = \text{const}$, this is not necessarily the case in more dimensions.

### 19.3.1 Lax–Wendroff Type Methods

In more dimensions we substitute

$$\frac{\partial f}{\partial t} = -\mathbf{u}\nabla f \tag{19.122}$$

$$\frac{\partial^2 f}{\partial t^2} = \frac{\partial}{\partial t}\left(\frac{\partial f}{\partial t}\right) = -\mathbf{u}\nabla\left(\frac{\partial f}{\partial t}\right) = (\mathbf{u}\nabla)(\mathbf{u}\nabla)f \tag{19.123}$$

in the series expansion

$$f(t+\Delta t) - f(t) = \Delta t\frac{\partial f}{\partial t} + \frac{\Delta t^2}{2}\frac{\partial^2 f}{\partial t^2} + \cdots \tag{19.124}$$

to obtain a generalization of the Taylor expansion (19.69)

$$f(t+\Delta t) - f(t) = -\Delta t\,\mathbf{u}\nabla f + \frac{(\Delta t)^2}{2}(\mathbf{u}\nabla)(\mathbf{u}\nabla)f + \cdots \tag{19.125}$$

which then has to be discretized in space by the usual methods of finite differences or finite elements [250]. Other one-dimensional schemes like leapfrog also can be generalized to more dimensions.

### 19.3.2 Finite Volume Methods

In multidimensions we introduce a, not necessarily regular, mesh of control volumes $V_i$. The surface of $V_i$ is divided into interfaces $A_{i,\alpha}$ to the neighboring cells. Application of the integral form of the continuity equation (19.3) gives

$$\frac{\partial}{\partial t}\int_{V_i} dV\, f(\mathbf{r},t) = -\oint_{\partial V_i} \mathbf{J}(\mathbf{r},t)d\mathbf{A} \tag{19.126}$$

and after time integration

$$\overline{f}_i(t+\Delta t) - \overline{f}(t) = -\Delta t\sum_{\alpha}\overline{J}_{i,\alpha}(t) \tag{19.127}$$

with the cell averages

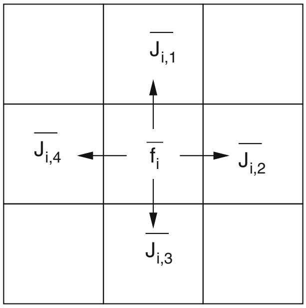



**Fig. 19.15** Averaged fluxes in 2 dimensions

$$\overline{f}_i(t) = \frac{1}{V_i}\frac{\partial}{\partial t}\int_{V_i} dV\, f(\mathbf{r}, t) \tag{19.128}$$

and the flux averages

$$\overline{J}_{i,\alpha}(t) = \frac{1}{\Delta t}\frac{1}{V_i}\sum_{\alpha}\int_t^{t+\Delta t} dt' \quad \oint_{A_{i,\alpha}} \mathbf{J}(\mathbf{r}, t')d\mathbf{A}. \tag{19.129}$$

For a regular mesh with cubic control volumes the sum is over all cell faces

$$\begin{aligned}\overline{f}_{ijk}(t + \Delta t) = \overline{f}_{ijk}(t) - \Delta t \left[\overline{J}_{i+1/2,j,k}(t) + \overline{J}_{i,j+1/2,k}(t) + \overline{J}_{i,j,k+1/2,}(t)\right.\\ \left. -\overline{J}_{i-1/2,j,k}(t) - -\overline{J}_{i,j-1/2,k}(t) - \overline{J}_{i,j,k-1/2}(t)\right].\end{aligned} \tag{19.130}$$

The function values have to be reconstructed from the cell averages, e.g. piecewise constant

$$f(\mathbf{r}, t) = \overline{f}_i(t) \quad \text{for } \mathbf{r} \in V_i \tag{19.131}$$

and the fluxes through the cell surface approximated in a suitable way, e.g. constant over a surface element (Fig. 19.15)

$$\mathbf{J}(\mathbf{r}, t) = \mathbf{J}_{i,\alpha}(t) \quad \text{for } \mathbf{r} \in A_{i,\alpha}. \tag{19.132}$$

Then the Riemann problem has to be solved approximately to obtain the fluxes for times $t \ldots t + \Delta t$. This method is also known as *reconstruct evolve average* (REA) method. An overview of average flux methods is presented in [252].

### 19.3.3 Dimensional Splitting

Splitting methods are very useful to divide a complicated problem into simpler steps. The time evolution of the advection equation can be written as a sum of three contributions[4]

$$\frac{\partial f}{\partial t} = -\mathrm{div}(\mathbf{u} f) = -\frac{\partial(u_x f)}{\partial x} - \frac{\partial(u_y f)}{\partial y} - \frac{\partial(u_z f)}{\partial z} \tag{19.133}$$

or, for an incompressible fluid

$$\frac{\partial f}{\partial t} = -\mathbf{u}\ \mathrm{grad}\ f = -u_x \frac{\partial f}{\partial x} - u_y \frac{\partial f}{\partial y} - u_z \frac{\partial f}{\partial z} \tag{19.134}$$

which has the form

$$\frac{\partial f}{\partial t} = Af = (A_x + A_y + A_z) f. \tag{19.135}$$

The time evolution can be approximated by

$$f(t + \Delta t) = \mathrm{e}^{\Delta t A} f(t) \approx \mathrm{e}^{\Delta t\, A_x} \mathrm{e}^{\Delta t\, A_y} \mathrm{e}^{\Delta t\, A_z} f(t) \tag{19.136}$$

i.e. by a sequence of one-dimensional time evolutions. Accuracy can be improved by applying a symmetrical Strang-splitting

$$f(t + \Delta t) \approx \mathrm{e}^{\Delta t/2\, A_x} \mathrm{e}^{\Delta t/2\, A_y} \mathrm{e}^{\Delta t\, A_z} \mathrm{e}^{\Delta t/2\, A_y} \mathrm{e}^{\Delta t/2\, A_x} f(t). \tag{19.137}$$

## Problems

**Problem 19.1 Advection in one Dimension**

In this computer experiment we simulate 1-dimensional advection with periodic boundary conditions. Different initial values (rectangular, triangular or Gaussian pulses of different widths) and methods (**F**orward in **T**ime **B**ackward in **S**pace, Lax-Friedrichs, leapfrog, Lax-Wendroff, implicit Crank–Nicolson, Taylor-Galerkin Lax-Wendroff) can be compared. See also Figs. 19.4, 19.11, 19.14 and 19.9.

[4]This is also the case if a diffusion term $D\left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}\right)$ is included.