Chapter 20
Waves

Waves are oscillations that move in space and time and are able to transport energy from one point to another. Quantum mechanical wavefunctions are discussed in Chap. 23. In this chapter we simulate classical waves which are, for instance, important in acoustics and electrodynamics. We use the method of finite differences to discretize the wave equation in one spatial dimension

$$\frac{\partial^2}{\partial t^2} f(t, x) = c^2 \frac{\partial^2}{\partial x^2} f(t, x). \tag{20.1}$$

Numerical solutions are obtained by an eigenvector expansion using trigonometric functions or by time integration. Accuracy and stability of different methods are compared. The wave function is second order in time and can be integrated directly with a two step method. Alternatively, it can be converted into a first order system of equations of double dimension. Here, the velocity appears explicitly and velocity dependent damping can be taken into account. Finally, the second order wave equation can be replaced by two coupled first order equations for two variables (like velocity and density in case of acoustic waves), which can be solved by quite general methods. We compare the leapfrog, Lax–Wendroff and Crank–Nicolson methods. Only the Crank–Nicolson method is stable for Courant numbers $\alpha > 1$. It is an implicit method and can be solved iteratively. In a series of computer experiments we simulate waves on a string. We study reflection at an open or fixed boundary and at the interface between two different media. We compare dispersion and damping for different methods.

20.1 Classical Waves

In classical physics there are two main types of waves:

Electromagnetic waves do not require a medium. They are oscillations of the electromagnetic field and propagate also in vacuum. As an example consider a plane wave which propagates in x-direction and is linearly polarized (Fig. 20.1). The electric and magnetic field have the form

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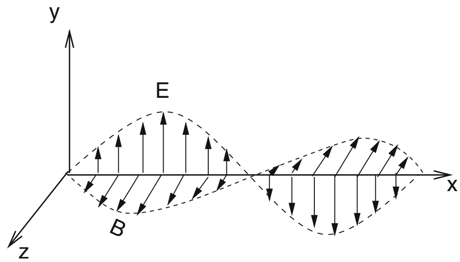

Fig. 20.1 Electromagnetic wave

$$\mathbf{E} = \begin{pmatrix} 0 \\ E_y(x,t) \\ 0 \end{pmatrix} \quad \mathbf{B} = \begin{pmatrix} 0 \\ 0 \\ B_z(x,t) \end{pmatrix}. \tag{20.2}$$

Maxwell's equations read in the absence of charges and currents

$$\mathrm{div}\mathbf{E} = \mathrm{div}\ \mathbf{B} = 0, \quad \mathrm{rot}\ \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \quad \mathrm{rot}\mathbf{B} = \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}. \tag{20.3}$$

The fields (20.2) have zero divergence and satisfy the first two equations. Application of the third and fourth equation gives

$$\frac{\partial E_y}{\partial x} = -\frac{\partial B_z}{\partial t} \quad -\frac{\partial B_z}{\partial x} = \mu_0 \varepsilon_0 \frac{\partial E_y}{\partial t} \tag{20.4}$$

which can be combined to a one-dimensional wave-equation

$$\frac{\partial^2 E_y}{\partial t^2} = c^2 \frac{\partial^2 E_y}{\partial x^2} \tag{20.5}$$

with velocity $c = (\mu_0 \varepsilon_0)^{-1/2}$.

Mechanical waves propagate through an elastic medium like air, water or an elastic solid. The material is subject to external forces deforming it and elastic forces which try to restore the deformation. As a result the atoms or molecules move around their equilibrium positions. As an example consider one-dimensional acoustic waves in an organ pipe (Fig. 20.2):

A mass element

$$dm = \varrho dV = \varrho A dx \tag{20.6}$$

at position x experiences an external force due to the air pressure which, according to Newton's law changes the velocity v of the element as described by Euler's equation[1]

[1] we consider only small deviations from the equilibrium values $\varrho_0, p_0, v_0 = 0$.

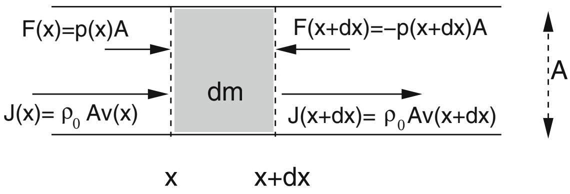

Fig. 20.2 (Acoustic waves in one dimension) A mass element $dm = \varrho A dx$ at position x experiences a total force $F = F(x) + F(x + dx) = -A \frac{\partial p}{\partial x} dx$. Due to the conservation of mass the change of the density $\frac{\partial \varrho}{\partial t}$ is given by the net flux $J = J(x) - J(x + dx) = -\varrho_0 A \frac{\partial v}{\partial x} dx$

$$\varrho_0 \frac{\partial}{\partial t} v = -\frac{\partial p}{\partial x}. \tag{20.7}$$

The pressure is a function of the density

$$\frac{p}{p_0} = \left(\frac{\varrho}{\varrho_0}\right)^n \quad \left(\frac{\mathrm{d}p}{\mathrm{d}\varrho}\right)_0 = n\frac{p_0}{\varrho_0} = c^2 \tag{20.8}$$

where $n = 1$ for an isothermal ideal gas and $n \approx 1.4$ for air under adiabatic conditions (no heat exchange), therefore

$$\varrho_0 \frac{\partial}{\partial t} v = -c^2 \frac{\partial \varrho}{\partial x}. \tag{20.9}$$

From the conservation of mass the continuity equation (12.10) follows

$$\frac{\partial}{\partial t} \varrho = -\varrho_0 \frac{\partial}{\partial x} v. \tag{20.10}$$

Combining the time derivative of (20.10) and the spatial derivative of (20.9) we obtain again the one-dimensional wave equation

$$\frac{\partial^2}{\partial t^2} \varrho = c^2 \frac{\partial^2}{\partial x^2} \varrho. \tag{20.11}$$

The wave-equation can be factorized as

$$\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)\varrho = \left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right)\varrho = 0 \tag{20.12}$$

which shows that solutions of the advection equation

$$\left(\frac{\partial}{\partial t} \pm c\frac{\partial}{\partial x}\right)\varrho = 0 \tag{20.13}$$

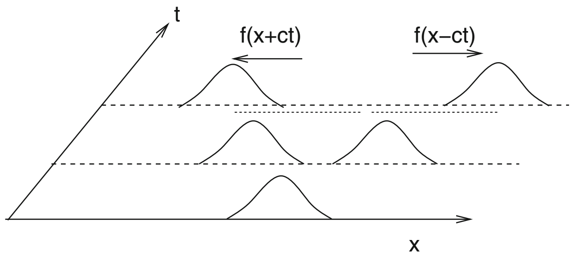

Fig. 20.3 d'Alembert solution to the wave equation

are also solutions of the wave equation, which have the form

$$\varrho = f(x \pm ct). \tag{20.14}$$

In fact a general solution of the wave equation is given according to d'Alembert as the sum of two waves running to the left and right side with velocity c and a constant envelope (Fig. 20.3)

$$\varrho = f_1(x + ct) + f_2(x - ct). \tag{20.15}$$

A special solution of this kind is the plane wave solution

$$f(x, t) = e^{i\omega t \pm ikx}$$

with the dispersion relation

$$\omega = ck. \tag{20.16}$$

20.2 Spatial Discretization in One Dimension

We use the simplest finite difference expression for the spatial derivative (Sects. 3.4 and 12.2)

$$\frac{\partial^2}{\partial x^2} f(x, t) = \frac{f(t, x + \Delta x) + f(t, x - \Delta x) - 2f(t, x)}{\Delta x^2} + O(\Delta x^2) \tag{20.17}$$

and a regular grid

$$x_m = m\Delta x \quad m = 1, 2 \ldots M \tag{20.18}$$

$$f_m = f(x_m). \tag{20.19}$$

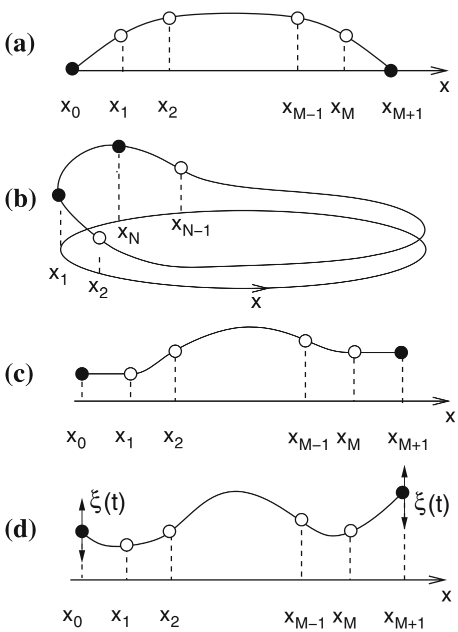

Fig. 20.4 (Boundary Conditions for 1-dimensional waves) Additional boundary points x_0, x_{M+1} are used to realize the boundary conditions. (**a**) **Fixed boundaries** $f(x_0) = 0$ $\frac{\partial^2}{\partial x^2} f(x_1) = \frac{1}{\Delta x^2}(f(x_2) - 2f(x_1))$ or $f(x_{M+1}) = 0, \frac{\partial^2}{\partial x^2} f(x_M) = \frac{1}{\Delta x^2}(f(x_{M-1}) - 2f(x_M))$. (**b**) **Periodic boundary conditions** $x_0 \equiv x_M$, $\frac{\partial^2}{\partial x^2} f(x_1) = \frac{1}{\Delta x^2}(f(x_2) + f(x_M) - 2f(x_1))$ $x_{M+1} \equiv x_1, \frac{\partial^2}{\partial x^2} f(x_M) = \frac{1}{\Delta x^2}(f(x_{M-1}) + f(x_1) - 2f(x_{NM}))$. (**c**) **Open boundaries** $\frac{\partial}{\partial x} f(x_1) = \frac{f(x_2)-f(x_0)}{2\Delta x} = 0, \frac{\partial^2}{\partial x^2} f(x_1) = \frac{1}{\Delta x^2}(2f(x_2) - 2f(x_1))$ or $\frac{\partial}{\partial x} f(x_M) = \frac{f(x_{M+1})-f(x_{M-1})}{2\Delta x} = 0 \frac{\partial^2}{\partial x^2} f(x_M) = \frac{1}{\Delta x^2}(2f((x_{M-1}) - 2f(x_M))$. (**d**) **Moving boundaries** $f(x_0, t) = \xi_0(t) \frac{\partial^2}{\partial x^2} f(x_1) = \frac{1}{\Delta x^2}(f(x_2) - 2f(x_1) + \xi_0(t))$ or $f(x_{M+1}, t) = \xi_{M+1}(t)$, $\frac{\partial^2}{\partial x^2} f(x_M) = \frac{1}{\Delta x^2}(f(x_{M-1}) - 2f(x_M) + \xi_{N+1}(t))$

This turns the wave equation into the system of ordinary differential equations (Sect. 12.2.3)

$$\frac{d^2}{dt^2} f_m = c^2 \frac{f_{m+1} + f_{m-1} - 2f_m}{\Delta x^2} \tag{20.20}$$

where f_0 and f_{M+1} have to be specified by suitable boundary conditions (Fig. 20.4). In matrix notation we have

$$\mathbf{f}(t) = \begin{pmatrix} f_1(t) \\ \vdots \\ f_M(t) \end{pmatrix} \tag{20.21}$$

$$\frac{d^2}{dt^2}\mathbf{f}(t) = A\mathbf{f}(t) + \mathbf{S}(t) \tag{20.22}$$

where for

$$\text{fixed boundaries} \quad A = \begin{pmatrix} -2 & 1 & & & & \\ 1 & -2 & 1 & & & \\ & 1 & -2 & 1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & 1 & -2 & 1 \\ & & & & 1 & -2 \end{pmatrix} \frac{c^2}{\Delta x^2} \quad \mathbf{S}(t) = 0 \tag{20.23}$$

$$\text{periodic boundaries}^2 \; A = \begin{pmatrix} -2 & 1 & & & & 1 \\ 1 & -2 & 1 & & & \\ & 1 & -2 & 1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & 1 & -2 & 1 \\ 1 & & & & 1 & -2 \end{pmatrix} \frac{c^2}{\Delta x^2} \quad \mathbf{S}(t) = 0 \tag{20.24}$$

$$\text{open boundaries} \quad A = \begin{pmatrix} -2 & 2 & & & & \\ 1 & -2 & 1 & & & \\ & 1 & -2 & 1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & 1 & -2 & 1 \\ & & & & 2 & -2 \end{pmatrix} \frac{c^2}{\Delta x^2} \quad \mathbf{S}(t) = 0 \tag{20.25}$$

$$\text{moving boundaries} \quad A = \begin{pmatrix} -2 & 1 & & & & \\ 1 & -2 & 1 & & & \\ & 1 & -2 & 1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & 1 & -2 & 1 \\ & & & & 1 & -2 \end{pmatrix} \frac{c^2}{\Delta x^2} \quad \mathbf{S}(t) = \begin{pmatrix} \xi_0(t) \\ 0 \\ \vdots \\ \vdots \\ 0 \\ \xi_{M+1}(t) \end{pmatrix}. \tag{20.26}$$

A combination of different boundary conditions for both sides is possible.

Equation (20.20) corresponds to a series of mass points which are connected by harmonic springs (Fig. 20.5), a model, which is used in solid state physics to describe longitudinal acoustic waves [253].

[2]This corresponds to the boundary condition $f_0 = f_2$, $\frac{\partial}{\partial x} f(x_1) = 0$. Alternatively we could use $f_0 = f_1$, $\frac{\partial}{\partial x} f(x_{1/2}) = 0$ which replaces the 2 s in the first and last row by 1 s.

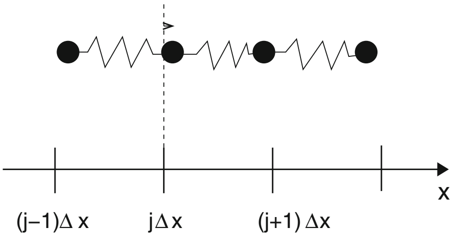

Fig. 20.5 (Atomistic model for longitudinal waves) A set of mass points m is connected by springs with stiffness K. The elongation of mass point number j from its equilibrium position $x_j = j\Delta x$ is ξ_j. The equations of motion $m\ddot{\xi}_j = -K(\xi_j - \xi_{j-1}) - K(\xi_j - \xi_{j+1})$ coincide with (20.20) with a velocity of $c = \Delta x\sqrt{\frac{k\Delta x}{m}}$

20.3 Solution by an Eigenvector Expansion

For fixed boundaries (20.20) reads in matrix form

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\mathbf{f}(t) = A\mathbf{f}(t) \tag{20.27}$$

with the vector of function values:

$$\mathbf{f}(t) = \begin{pmatrix} f_1(t) \\ \vdots \\ f_M(t) \end{pmatrix} \tag{20.28}$$

and the matrix

$$A = \begin{pmatrix} -2 & 1 & & & & \\ 1 & -2 & 1 & & & \\ & 1 & -2 & 1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & 1 & -2 & 1 \\ & & & & 1 & -2 \end{pmatrix} \frac{c^2}{\Delta x^2} \tag{20.29}$$

which can be diagonalized exactly (Sect. 10.3). The two boundary points $f(0) = 0$ and $f((M+1)\Delta x) = 0$ can be added without any changes. The eigenvalues are

$$\lambda = 2\frac{c^2}{\Delta x^2}(\cos(k\Delta x) - 1) = -\frac{4c^2}{\Delta x^2}\sin^2\left(\frac{k\Delta x}{2}\right) = (i\omega_k)^2 \quad k\Delta x = \frac{\pi l}{(M+1)}, l = 1\ldots M \tag{20.30}$$

with the frequencies

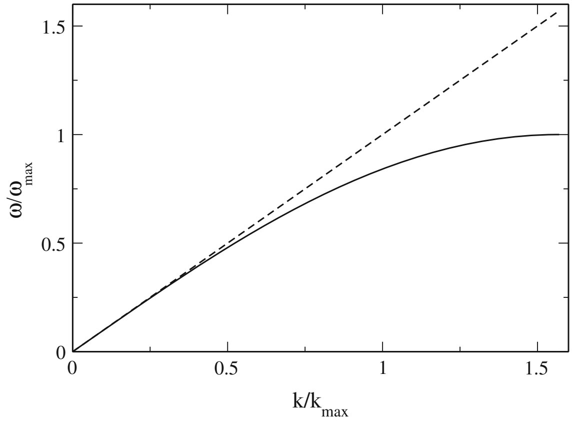

Fig. 20.6 (Dispersion of the discrete wave equation) The dispersion of the discrete wave equation approximates the linear dispersion of the continuous wave equation only at small values of k. At $k_{max} = \pi/\Delta x$ it saturates at $\omega_{max} = 2c/\Delta x = (2/\pi)\, ck_{max}$

$$\omega_k = \frac{2c}{\Delta x} \sin\left(\frac{k\Delta x}{2}\right). \tag{20.31}$$

This result deviates from the dispersion relation of the continuous wave equation (20.11) $\omega_k = ck$ and approximates it only for $k\Delta x \ll 1$ (Fig. 20.6).

The general solution has the form (Sect. 12.2.4)

$$f_n(t) = \sum_{l=1}^{M} \left(C_{l+} e^{i\omega_l t} + C_{l-} e^{-i\omega_l t}\right) \sin\left(m \frac{\pi l}{(M+1)}\right). \tag{20.32}$$

The initial amplitudes and velocities are

$$f_n(t=0) = \sum_{l=1}^{M} (C_{l+} + C_{l-}) \sin\left(m \frac{\pi l}{(M+1)}\right) = F_m$$

$$\frac{\mathrm{d}}{\mathrm{d}t} f_m(t=0, x_m) = \sum_{l=1}^{M} i\omega_l \; (C_{l+} - C_{l-}) \sin\left(m \frac{\pi l}{(M+1)}\right) = G_m \tag{20.33}$$

with F_m and G_m given. Different eigenfunctions of a tridiagonal matrix are mutually orthogonal

$$\sum_{m=1}^{M} \sin\left(m \frac{\pi l}{M+1}\right) \sin\left(m \frac{\pi l'}{M+1}\right) = \frac{M}{2} \delta_{l,l'} \tag{20.34}$$

and the coefficients $C_{l\pm}$ follow from a discrete Fourier transformation:

$$\begin{aligned}\widetilde{F}_l &= \frac{1}{M}\sum_{m=1}^{M}\sin\left(m\frac{\pi l}{N+1}\right)F_m\\ &= \frac{1}{M}\sum_{m=1}^{M}\sum_{l'=1}^{M}(C_{l'+}+C_{l'-})\sin\left(m\frac{\pi l'}{M+1}\right)\sin\left(m\frac{\pi l}{M+1}\right) = \frac{1}{2}(C_{l+}+C_{l-})\end{aligned} \tag{20.35}$$

$$\begin{aligned}\widetilde{G_l} &= \frac{1}{M}\sum_{m=1}^{M}\sin\left(m\frac{\pi l}{N+1}\right)G_n\\ &= \frac{1}{M}\sum_{m=1}^{M}\sum_{l'=1}^{NM} i\omega_l\,(C_{l+}-C_{l-})\sin\left(m\frac{\pi l'}{M+1}\right)\sin\left(m\frac{\pi l}{M+1}\right) = \frac{1}{2}i\omega_l(C_{l+}-C_{l-})\end{aligned} \tag{20.36}$$

$$\begin{aligned}C_{l+} &= \widetilde{F}_l + \frac{1}{i\omega_l}\widetilde{G_l}\\ C_{l-} &= \widetilde{F}_l - \frac{1}{i\omega_l}\widetilde{G_l}.\end{aligned} \tag{20.37}$$

Finally the explicit solution of the wave equation is

$$f_m(t) = \sum_{l=1}^{M} 2(\widetilde{F}_l\cos(\omega_l\, t) + \frac{\widetilde{G_l}}{\omega_l}\sin(\omega_l\, t))\sin\left(m\frac{\pi l}{M+1}\right). \tag{20.38}$$

Periodic or open boundaries can be treated similarly as the matrices can be diagonalized exactly (Sect. 10.3). For moving boundaries the expansion coefficients are time dependent (Sect. 12.2.4).

20.4 Discretization of Space and Time

Using the finite difference expression also for the second time derivative the fully discretized wave equation is

$$\begin{aligned}&\frac{f(t+\Delta t,x)+f(t-\Delta t,x)-2f(t,x)}{\Delta t^2}\\ &= c^2\frac{f(t,x+\Delta x)+f(t,x-\Delta x)-2f(t,x)}{\Delta x^2} + O(\Delta x^2,\Delta t^2).\end{aligned} \tag{20.39}$$

For a plane wave

$$f = \mathrm{e}^{\mathrm{i}(\omega t - kx)} \tag{20.40}$$

we find

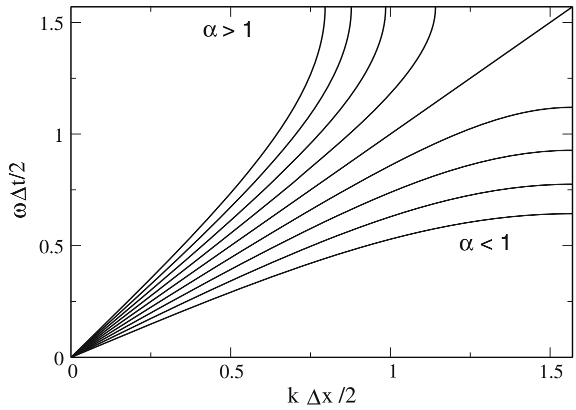

Fig. 20.7 (Dispersion of the discrete wave equation) Only for $\alpha = 1$ or for small values of $k\Delta x$ and $\omega\Delta t$ is the dispersion approximately linear. For $\alpha < 1$ only frequencies $\omega < \omega_{\max} = 2\arcsin(\alpha)/\Delta t$ are allowed whereas for $\alpha > 1$ the range of k-values is bounded by $k_{\max} = 2\arcsin(1/\alpha)/\Delta x$

$$e^{i\omega\Delta t} + e^{-i\omega\Delta t} - 2 = c^2 \frac{\Delta t^2}{\Delta x^2} \left(e^{ik\Delta x} + e^{-ik\Delta x} - 2\right) \tag{20.41}$$

which can be written as

$$\sin \frac{\omega\Delta t}{2} = \alpha \sin \frac{k\Delta x}{2} \tag{20.42}$$

with the so-called Courant-number [243]

$$\alpha = c\frac{\Delta t}{\Delta x}. \tag{20.43}$$

From (20.42) we see that the dispersion relation is linear only for $\alpha = 1$. For $\alpha \neq 1$ not all values of ω and k allowed (Fig. 20.7).

20.5 Numerical Integration with a Two-Step Method

We solve the discrete wave equation (20.39) with fixed or open boundaries for

$$\begin{aligned} f(t+\Delta t, x) = 2f(t,x)(1-\alpha^2) + \alpha^2(f(t, x+\Delta x) + f(t, x-\Delta x)) \\ - f(t-\Delta t, x) + O(\Delta t^2, \Delta x^2) \end{aligned} \tag{20.44}$$

on the regular grids

$$x_m = m\Delta x \quad m = 1, 2 \ldots M \tag{20.45}$$

$$t_n = n\Delta t \quad n = 1, 2 \ldots N \tag{20.46}$$

$$\mathbf{f}_n = \begin{pmatrix} f_1^n \\ \vdots \\ f_M^n \end{pmatrix} = \begin{pmatrix} f(t_n, x_1) \\ \vdots \\ f(t_n, x_M) \end{pmatrix} \tag{20.47}$$

by applying the iteration

$$f_m^{n+1} = 2(1-\alpha^2) f_m^n + \alpha^2 f_{m+1}^n + \alpha^2 f_{m-1}^n - f_m^{n-1}. \tag{20.48}$$

This is a two-step method which can be rewritten as a one-step method of double dimension

$$\begin{pmatrix} \mathbf{f}_{n+1} \\ \mathbf{f}_n \end{pmatrix} = T \begin{pmatrix} \mathbf{f}_n \\ \mathbf{f}_{n-1} \end{pmatrix} = \begin{pmatrix} 2+\alpha^2 M & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{f}_n \\ \mathbf{f}_{n-1} \end{pmatrix} \tag{20.49}$$

with the tridiagonal matrix

$$M = \begin{pmatrix} -2 & a_1 & & & \\ 1 & -2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & -2 & 1 \\ & & & a_N & -2 \end{pmatrix} \tag{20.50}$$

where $a_1$and a_N have the values 1 for a fixed or 2 for an open end.
The matrix M has eigenvalues (Sect. 10.3)

$$\lambda = 2\cos(k\Delta x) - 2 = -4\sin^2\left(\frac{k\Delta x}{2}\right). \tag{20.51}$$

To simulate excitation of waves by a moving boundary we add one grid point with given elongation $\xi_0(t)$ and change the first equation into

$$f(t_{n+1}, x_1) = 2(1-\alpha^2) f(t_n, x_1) + \alpha^2 f(t_n, x_2) + \alpha^2 \xi_0(t_n) - f(t_{n-1}, x_1). \tag{20.52}$$

Repeated iteration gives the series of function values

$$\begin{pmatrix} \mathbf{f}_1 \\ \mathbf{f}_0 \end{pmatrix}, \begin{pmatrix} \mathbf{f}_2 \\ \mathbf{f}_1 \end{pmatrix} = T \begin{pmatrix} \mathbf{f}_1 \\ \mathbf{f}_0 \end{pmatrix}, \begin{pmatrix} \mathbf{f}_3 \\ \mathbf{f}_2 \end{pmatrix} = T^2 \begin{pmatrix} \mathbf{f}_1 \\ \mathbf{f}_0 \end{pmatrix}, \cdots \tag{20.53}$$

A necessary condition for stability is that all eigenvalues of T have absolute values smaller than one. Otherwise small perturbations would be amplified. The eigenvalue equation for T is

$$\begin{pmatrix} 2+\alpha^2 M - \sigma & -1 \\ 1 & -\sigma \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{20.54}$$

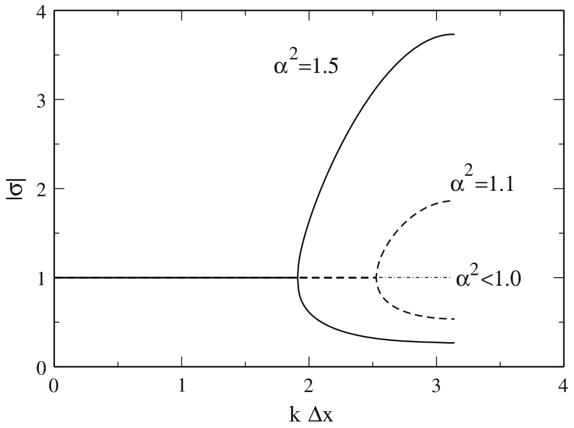

Fig. 20.8 (Stability regions of the two-step method) Instabilities appear for $|\alpha| > 1$. One of the two eigenvalues σ becomes unstable ($|\sigma| > 1$) for waves with large k-values

We substitute the solution of the second equation

$$u = \sigma v \tag{20.55}$$

into the first equation and use the eigenvectors of M (Sect. 10.3) to obtain the eigenvalue equation

$$(2 + \alpha^2\lambda - \sigma)\sigma v - v = 0. \tag{20.56}$$

Hence σ is one of the two roots of

$$\sigma^2 - \sigma(\alpha^2\lambda + 2) + 1 = 0 \tag{20.57}$$

which are given by (Fig. 20.8)

$$\sigma = 1 + \frac{\alpha^2\lambda}{2} \pm \sqrt{\left(\frac{\alpha^2\lambda}{2} + 1\right)^2 - 1}. \tag{20.58}$$

From

$$\lambda = -4\sin^2\left(\frac{k\Delta x}{2}\right)$$

we find

$$-4 < \lambda < 0 \tag{20.59}$$

$$1 - 2\alpha^2 < \frac{\alpha^2\lambda}{2} + 1 < 1 \tag{20.60}$$

and the square root in (20.58) is imaginary if

$$-1 < \frac{\alpha^2\lambda}{2} + 1 < 1 \tag{20.61}$$

which is the case for

$$\sin^2\left(\frac{k\Delta x}{2}\right)\alpha^2 < 1. \tag{20.62}$$

This holds for all k only if

$$|\alpha| < 1. \tag{20.63}$$

But then

$$|\sigma|^2 = \left(1 + \frac{\alpha^2\lambda}{2}\right)^2 + \left(1 - \left(\frac{\alpha^2\lambda}{2} + 1\right)^2\right) = 1 \tag{20.64}$$

and the algorithm is (conditionally) stable. If on the other hand $|\alpha| > 1$ then for some k-values the square root is real. Here we have

$$1 + \frac{\alpha^2\lambda}{2} < -1 \tag{20.65}$$

and finally

$$1 + \frac{\alpha^2\lambda}{2} - \sqrt{\left(1 + \frac{\alpha^2\lambda}{2}\right)^2 - 1} < -1 \tag{20.66}$$

which shows that instabilities are possible in this case.

20.6 Reduction to a First Order Differential Equation

A general method to reduce the order of an ordinary differential equation (or a system of such) introduces the time derivatives as additional variables (Chap. 13). The spatially discretized one-dimensional wave equation (20.22) can be transformed into a system of double dimension

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{f}(t) = \mathbf{v}(t) \tag{20.67}$$

$$\frac{d}{dt}\mathbf{v}(t) = \frac{c^2}{\Delta x^2} M\mathbf{f}(t) + \mathbf{S}(t). \tag{20.68}$$

We use the improved Euler method (Sect. 13.5)

$$\mathbf{f}(t + \Delta t) = \mathbf{f}(t) + \mathbf{v}(t + \frac{\Delta t}{2})\Delta t + O(\Delta t^3) \tag{20.69}$$

$$\mathbf{v}(t + \Delta t) = \mathbf{v}(t) + \left[\frac{c^2}{\Delta x^2} M\mathbf{f}(t + \frac{\Delta t}{2}) + \mathbf{S}(t + \frac{\Delta t}{2})\right]\Delta t + O(\Delta t^3) \tag{20.70}$$

and two different time grids

$$\mathbf{f}_n = \mathbf{f}(t_n) \quad \mathbf{S}_n = \mathbf{S}(t_n) \quad n = 0, 1 \ldots \tag{20.71}$$

$$\mathbf{f}(t_{n+1}) = \mathbf{f}(t_n) + \mathbf{v}(t_{n+1/2})\Delta t \tag{20.72}$$

$$\mathbf{v}_n = \mathbf{v}(t_{n-1/2}) \quad n = 0, 1 \ldots \tag{20.73}$$

$$\mathbf{v}(t_{n+1/2}) = \mathbf{v}(t_{n-1/2}) + \left[\frac{c^2}{\Delta x^2} M\mathbf{f}(t_n) + \mathbf{S}(t_n)\right]\Delta t. \tag{20.74}$$

We obtain a leapfrog (Fig. 20.9) like algorithm (p. 398)

$$\mathbf{v}_{n+1} = \mathbf{v}_n + \left[\frac{c^2}{\Delta x^2} M\mathbf{f}_n + \mathbf{S}_n\right]\Delta t \tag{20.75}$$

$$\mathbf{f}_{n+1} = \mathbf{f}_n + \mathbf{v}_{n+1}\Delta t \tag{20.76}$$

where the updated velocity (20.75) has to be inserted into (20.76). This can be combined into the iteration

$$\begin{pmatrix} \mathbf{f}_{n+1} \\ \mathbf{v}_{n+1} \end{pmatrix} = \begin{pmatrix} \mathbf{f}_n + \mathbf{v}_n\Delta t + \left[\frac{c^2}{\Delta x^2} M\mathbf{f}_n + \mathbf{S}_n\right]\Delta t^2 \\ \mathbf{v}_n + \left[\frac{c^2}{\Delta x^2} M\mathbf{f}_n + \mathbf{S}_n\right]\Delta t \end{pmatrix} = \begin{pmatrix} 1 + \frac{c^2\Delta t^2}{\Delta x^2} M & \Delta t \\ \frac{c^2\Delta t}{\Delta x^2} M & 1 \end{pmatrix} \begin{pmatrix} \mathbf{f}_n \\ \mathbf{v}_n \end{pmatrix} + \begin{pmatrix} \mathbf{S}_n\Delta t^2 \\ \mathbf{S}_n\Delta t \end{pmatrix}. \tag{20.77}$$

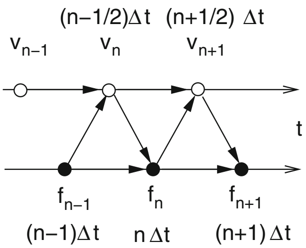

Fig. 20.9 Leapfrog method

Since the velocity appears explicitly we can easily add a velocity dependent damping like

$$-\gamma v(t_n, x_m) \tag{20.78}$$

which we approximate by

$$-\gamma v(t_n - \frac{\Delta t}{2}, x_m) \tag{20.79}$$

under the assumption of weak damping

$$\gamma \Delta t \ll 1. \tag{20.80}$$

To study the stability of this algorithm we consider the homogeneous problem with fixed boundaries. With the Courant number $\alpha = \frac{c\Delta t}{\Delta x}$ (20.77) becomes

$$\begin{pmatrix} \mathbf{f}_{n+1} \\ \mathbf{v}_{n+1} \end{pmatrix} = \begin{pmatrix} 1 + \alpha^2 M & \Delta t(1 - \gamma \Delta t) \\ \frac{\alpha^2}{\Delta t} M & 1 - \gamma \Delta t \end{pmatrix} \begin{pmatrix} \mathbf{f}_n \\ \mathbf{v}_n \end{pmatrix}. \tag{20.81}$$

Using the eigenvectors and eigenvalues of M (Sect. 10.3)

$$\lambda = -4 \sin^2 \left(\frac{k \Delta x}{2} \right) \tag{20.82}$$

we find the following equation for the eigenvalues σ:

$$\begin{aligned} (1 + \alpha^2 \lambda - \sigma) u + \Delta t (1 - \gamma \Delta t) v &= 0 \\ \alpha^2 \lambda u + \Delta t (1 - \gamma \Delta t - \sigma) v &= 0. \end{aligned} \tag{20.83}$$

Solving the second equation for u and substituting into the first equation we have

$$[(1 + \alpha^2 \lambda - \sigma) \frac{\Delta t}{-\alpha^2 \lambda} (1 - \gamma \Delta t - \sigma) + \Delta t (1 - \gamma \Delta t)] = 0 \tag{20.84}$$

hence

$$\begin{aligned} (1 + \alpha^2 \lambda - \sigma)(1 - \gamma \Delta t - \sigma) - \alpha^2 \lambda (1 - \gamma \Delta t) &= 0 \\ \sigma^2 - \sigma (2 - \gamma \Delta t + \alpha^2 \lambda) + (1 - \gamma \Delta t) &= 0 \\ \sigma = 1 - \frac{\gamma \Delta t}{2} + \frac{\alpha^2 \lambda}{2} \pm \sqrt{\left(1 - \frac{\gamma \Delta t}{2} + \frac{\alpha^2 \lambda}{2}\right)^2 - (1 - \gamma \Delta t)}. \end{aligned} \tag{20.85}$$

Instabilities are possible if the square root is real and $\sigma < -1$. ($\sigma > 1$ is not possible). This is the case for

$$-1 + \frac{\gamma \Delta t}{2} \approx -\sqrt{1 - \gamma \Delta t} < 1 - \frac{\gamma \Delta t}{2} + \frac{\alpha^2 \lambda}{2} < \sqrt{1 - \gamma \Delta t} \approx 1 - \frac{\gamma \Delta t}{2} \tag{20.86}$$

$$-2 + \gamma \Delta t < \frac{\alpha^2 \lambda}{2} < 0. \tag{20.87}$$

The right inequality is satisfied, hence it remains

$$\alpha^2 \sin^2 \left(\frac{k \Delta x}{2} \right) < 1 - \frac{\gamma \Delta t}{2}. \tag{20.88}$$

This holds for all k-values if it holds for the maximum of the sine-function

$$\alpha^2 < 1 - \frac{\gamma \Delta t}{2}. \tag{20.89}$$

This shows that inclusion of the damping term even favors instabilities.

20.7 Two Variable Method

For the 1-dimensional wave equation (20.11) there exists another possibility to reduce the order of the time derivative by splitting it up into two first order equations similar to (20.9, 20.10)

$$\frac{\partial}{\partial t} f(t, x) = c \frac{\partial}{\partial x} g(t, x) \tag{20.90}$$

$$\frac{\partial}{\partial t} g(t, x) = c \frac{\partial}{\partial x} f(t, x). \tag{20.91}$$

Several algorithms can be applied to solve these equations [254]. We discuss only methods which are second order in space and time and are rather general methods to solve partial differential equations. The boundary conditions need some special care. For closed boundaries with $f(x_0) = 0$ obviously $\frac{\partial f}{\partial t}(x_0) = 0$ whereas $\frac{\partial f}{\partial x}(x_0)$ is finite. Hence a closed boundary for $f(t, x)$ is connected with an open boundary for $g(t, x)$ with $\frac{\partial g}{\partial x}(x_0) = 0$ and vice versa. This is well known from acoustics (Fig. 20.10).

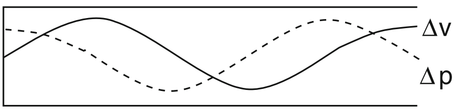

Fig. 20.10 (Standing waves in an organ pipe) At the closed (***Left***) end the amplitude of the longitudinal velocity is zero whereas the amplitudes of pressure and density changes are extremal. This is reversed at the open (***Right***) end

20.7.1 Leapfrog Scheme

We use symmetric differences (Sect. 3.2) for the first derivatives

$$\frac{f(t+\frac{\Delta t}{2},x)-f(t-\frac{\Delta t}{2},x)}{\Delta t}=c\frac{g(t,x+\frac{\Delta x}{2})-g(x-\frac{\Delta x}{2})}{\Delta x}+O(\Delta x^2,\Delta t^2) \tag{20.92}$$

$$\frac{g(t+\frac{\Delta t}{2},x)-g(t-\frac{\Delta t}{2},x)}{\Delta t}=c\frac{f(t,x+\frac{\Delta x}{2})-f(x-\frac{\Delta x}{2})}{\Delta x}+O(\Delta x^2,\Delta t^2) \tag{20.93}$$

to obtain the following scheme

$$g((t_{n+1/2},x_{m+1/2})=g(t_{n-1/2},x_{m+1/2})+\alpha\,(f(t_n,x_{m+1}))-f(t_n,x_{m-1}) \tag{20.94}$$

$$f(t_{n+1},x_m)=f(t_n,x_m)+\alpha\left(g(t_{n+1/2},x_{m+1/2})-g((t_{n+1/2},x_{m-1/2})\right). \tag{20.95}$$

Using different time grids for the two variables

$$\mathbf{f}_n=\begin{pmatrix} f_1^n \\ \vdots \\ f_M^n\end{pmatrix}=\begin{pmatrix} f(t_n,x_1) \\ \vdots \\ f(t_n,x_M)\end{pmatrix}\quad \mathbf{g}_n=\begin{pmatrix} g_1^n \\ \vdots \\ g_M^n\end{pmatrix}=\begin{pmatrix} g((t_{n-1/2},x_{1/2}) \\ \vdots \\ g(t_{n-1/2},x_{M-1/2})\end{pmatrix} \tag{20.96}$$

this translates into the algorithm (Fig. 20.11)

$$g_m^{n+1}=g_m^n+\alpha\left(f_m^n-f_{m-1}^n\right) \tag{20.97}$$

$$f_m^{n+1}=f_m^n+\alpha\left(g_{m+1}^{n+1}-g_m^{n+1}\right)=f_m^n+\alpha\left(g_{m+1}^n-g_m^n\right)+\alpha^2(f_{m+1}^n-2f_m^n+f_{m-1}^n). \tag{20.98}$$

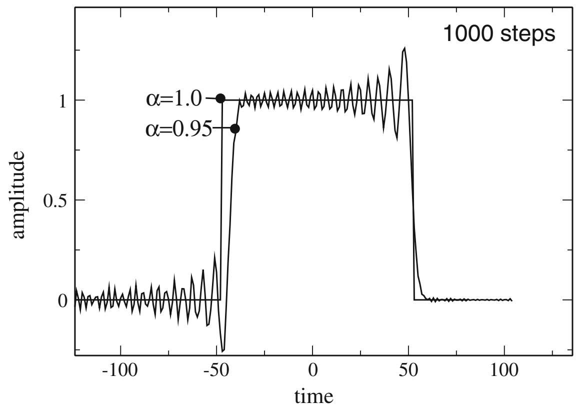

Fig. 20.11 (Simulation with the leapfrog method) A rectangular pulse is simulated with the two-variable leapfrog method. While for $\alpha = 1$ the pulse shape has not changed after 1000 steps, for smaller values the short wavelength components are lost due to dispersion

To analyze the stability we insert

$$f_m^n = u e^{an} e^{ikm\Delta x} \quad g_m^n = v e^{an} e^{ikm\Delta x} \tag{20.99}$$

and obtain the equations

$$Gv = v + \alpha u(1 - e^{-ik\Delta x}) \tag{20.100}$$

$$Gu = u + \alpha v(e^{ik\Delta x} - 1) + \alpha^2 u(2\cos k\Delta x - 2) \tag{20.101}$$

which in matrix form read

$$G\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 1 + \alpha^2(2\cos k\Delta x - 2) & \alpha(e^{ik\Delta x} - 1) \\ \alpha(1 - e^{-ik\Delta x}) & 1 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}. \tag{20.102}$$

The maximum amplification factor G is given by the largest eigenvalue, which is one of the roots of

$$(1 - 4\alpha^2 \sin^2\left(\frac{k\Delta x}{2}\right) - \sigma)(1 - \sigma) + 4\alpha^2 \sin^2\left(\frac{k\Delta x}{2}\right) = 0$$

$$(1 - \sigma + \alpha^2\lambda^2)(1 - \sigma) - \alpha^2\lambda^2 = 0 \tag{20.103}$$

$$\sigma = 1 - 2\alpha^2 \sin^2\left(\frac{k\Delta x}{2}\right) \pm \sqrt{\left(1 - 2\alpha^2 \sin^2\left(\frac{k\Delta x}{2}\right)\right)^2 - 1}. \tag{20.104}$$

The eigenvalues coincide with those of the two-step method (20.58).

20.7.2 Lax–Wendroff Scheme

The Lax–Wendroff scheme can be derived from the Taylor series expansion

$$\begin{aligned} f(t + \Delta t, x) &= f(t, x) + \frac{\partial f(t, x)}{\partial t}\Delta t + \frac{1}{2}\Delta t^2 \frac{\partial^2 f(t, x)}{\partial t^2} + \dots \\ &= f(t, x) + c\Delta t \frac{\partial g(t, x)}{\partial x} + \frac{c^2\Delta t^2}{2}\frac{\partial^2 f(t, x)}{\partial t^2} + \dots \end{aligned} \tag{20.105}$$

$$\begin{aligned} g(t + \Delta t, x) &= g(t, x) + \frac{\partial g(t, x)}{\partial t}\Delta t + \frac{1}{2}\Delta t^2 \frac{\partial^2 g(t, x)}{\partial t^2} + \dots \\ &= g(t, x) + c\Delta t \frac{\partial f(t, x)}{\partial x} + \frac{c^2\Delta t^2}{2}\frac{\partial^2 g(t, x)}{\partial t^2} + \dots . \end{aligned} \tag{20.106}$$

It uses symmetric differences on regular grids (20.45, 20.46) to obtain the iteration

$$f_m^{n+1} = f_m^n + c\Delta t \frac{g_{m+1}^n - g_{m-1}^n}{2\Delta x} + c^2 \Delta t^2 \frac{f_{m+1}^n + f_{m-1}^n - 2f_m^n}{2\Delta x^2} \tag{20.107}$$

$$g_m^{n+1} = g_m^n + c\Delta t \frac{f_{m+1}^n - f_{m-1}^n}{2\Delta x} + c^2 \Delta t^2 \frac{g_{m+1}^n + g_{m-1}^n - 2g_m^n}{2\Delta x^2} \tag{20.108}$$

$$\begin{pmatrix} \mathbf{f}^{n+1} \\ \mathbf{g}^{n+1} \end{pmatrix} = \begin{pmatrix} 1 + \frac{\alpha^2}{2} M & \frac{\alpha}{2} D \\ \frac{\alpha}{2} D & 1 + \frac{\alpha^2}{2} M \end{pmatrix} \begin{pmatrix} \mathbf{f}^n \\ \mathbf{g}^n \end{pmatrix} \tag{20.109}$$

with the tridiagonal matrix

$$D = \begin{pmatrix} 0 & 1 & & & \\ -1 & 0 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 0 & 1 \\ & & & -1 & 0 \end{pmatrix}. \tag{20.110}$$

To analyze the stability we insert

$$f_m^n = u\mathrm{e}^{an}\mathrm{e}^{\mathrm{i}km\Delta x} \quad g_m^n = v\mathrm{e}^{an}\mathrm{e}^{\mathrm{i}km\Delta x} \tag{20.111}$$

and calculate the eigenvalues (compare with 20.102) of

$$\begin{pmatrix} 1 + \alpha^2(\cos k\Delta x - 1) & \mathrm{i}\alpha \sin k\Delta x \\ \mathrm{i}\alpha \sin k\Delta x & 1 + \alpha^2(\cos k\Delta x - 1) \end{pmatrix} \tag{20.112}$$

which are given by

$$\sigma = 1 + \alpha^2(\cos k\Delta x - 1) \pm \sqrt{\alpha^2(\cos^2 k\Delta x - 1)}. \tag{20.113}$$

The root is always imaginary and

$$|\sigma|^2 = 1 + (\alpha^4 - \alpha^2)(\cos k\Delta x - 1)^2 \le 1 + 4(\alpha^4 - \alpha^2).$$

For $\alpha < 1$ we find $|\sigma| < 1$. The method is stable but there is wavelength dependent damping (Figs. 20.12 and 20.13).

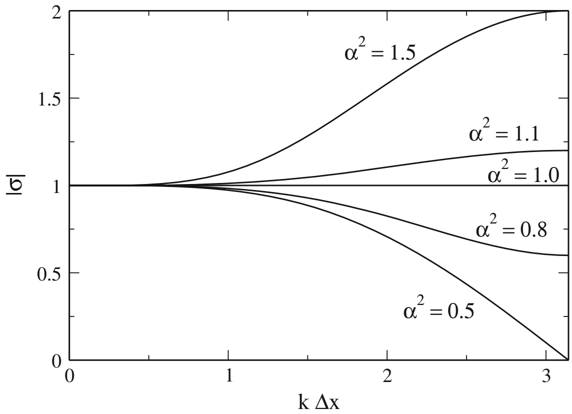

Fig. 20.12 (Stability region of the Lax–Wendroff method) Instabilities appear for $|\alpha| > 1$. In the opposite case short wavelength modes are damped

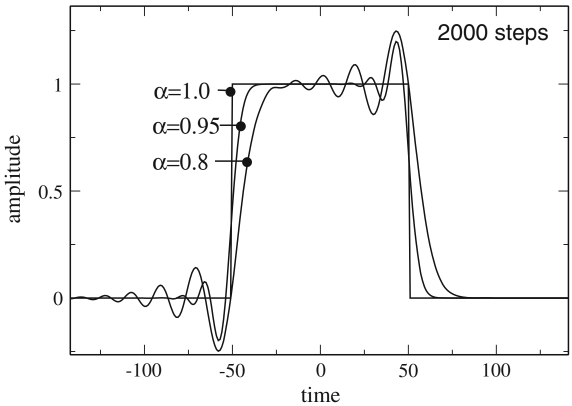

Fig. 20.13 (Simulation with the Lax–Wendroff method) A rectangular pulse is simulated with the two-variable Lax–Wendroff method. While for $\alpha = 1$ the pulse shape has not changed after 2000 steps, for smaller values the short wavelength components are lost due to dispersion and damping

20.7.3 Crank–Nicolson Scheme

This method takes the average of the explicit and implicit Euler methods

$$f(t+\Delta t) = f(t) + \frac{c}{2}\left(\frac{\partial g}{\partial x}(t,x) + \frac{\partial g}{\partial x}(t+\Delta t, x)\right)\Delta t \tag{20.114}$$

$$g(t+\Delta t) = g(t) + \frac{c}{2}\left(\frac{\partial f}{\partial x}(t,x) + \frac{\partial f}{\partial x}(t+\Delta t, x)\right)\Delta t \tag{20.115}$$

and uses symmetric differences on the regular grids (20.45, 20.46) to obtain

$$f_m^{n+1} = f_m^n + \frac{\alpha}{4}\left(g_{m+1}^n - g_{m-1}^n + g_{m+1}^{n+1} - g_{m-1}^{n+1}\right) \tag{20.116}$$

$$g_m^{n+1} = g_m^n + \frac{\alpha}{4}\left(f_{m+1}^n - f_{m-1}^n + f_{m+1}^{n+1} - f_{m-1}^{n+1}\right) \tag{20.117}$$

which reads in matrix notation

$$\begin{pmatrix} \mathbf{f}_{n+1} \\ \mathbf{g}_{n+1} \end{pmatrix} = \begin{pmatrix} 1 & \frac{\alpha}{4}D \\ \frac{\alpha}{4}D & 1 \end{pmatrix} \begin{pmatrix} \mathbf{f}_n \\ \mathbf{g}_n \end{pmatrix} + \begin{pmatrix} & \frac{\alpha}{4}D \\ \frac{\alpha}{4}D & \end{pmatrix} \begin{pmatrix} \mathbf{f}_{n+1} \\ \mathbf{g}_{n+1} \end{pmatrix}. \tag{20.118}$$

This equation can be solved formally by collecting terms at time t_{n+1}

$$\begin{pmatrix} 1 & -\frac{\alpha}{4}D \\ -\frac{\alpha}{4}D & 1 \end{pmatrix} \begin{pmatrix} \mathbf{f}_{n+1} \\ \mathbf{g}_{n+1} \end{pmatrix} = \begin{pmatrix} 1 & \frac{\alpha}{4}D \\ \frac{\alpha}{4}D & 1 \end{pmatrix} \begin{pmatrix} \mathbf{f}_n \\ \mathbf{g}_n \end{pmatrix} \tag{20.119}$$

and multiplying with the inverse matrix from left

$$\begin{pmatrix} \mathbf{f}_{n+1} \\ \mathbf{g}_{n+1} \end{pmatrix} = \begin{pmatrix} 1 & -\frac{\alpha}{4}D \\ -\frac{\alpha}{4}D & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & \frac{\alpha}{4}D \\ \frac{\alpha}{4}D & 1 \end{pmatrix} \begin{pmatrix} \mathbf{f}_n \\ \mathbf{g}_n \end{pmatrix}. \tag{20.120}$$

Now, if $\mathbf{u}$ is an eigenvector of D with purely imaginary eigenvalue λ (Sect. 10.3)

$$\begin{pmatrix} 1 & \frac{\alpha}{4}D \\ \frac{\alpha}{4}D & 1 \end{pmatrix} \begin{pmatrix} \mathbf{u} \\ \pm\mathbf{u} \end{pmatrix} = \begin{pmatrix} (1 \pm \frac{\alpha}{4}\lambda)\mathbf{u} \\ (\frac{\alpha}{4}\lambda \pm 1)\mathbf{u} \end{pmatrix} = (1 \pm \frac{\alpha}{4}\lambda) \begin{pmatrix} \mathbf{u} \\ \pm\mathbf{u} \end{pmatrix} \tag{20.121}$$

and furthermore

$$\begin{pmatrix} 1 & -\frac{\alpha}{4}D \\ -\frac{\alpha}{4}D & 1 \end{pmatrix} \begin{pmatrix} \mathbf{u} \\ \pm\mathbf{u} \end{pmatrix} = \begin{pmatrix} (1 \mp \frac{\alpha}{4}\lambda)\mathbf{u} \\ (-\frac{\alpha}{4}\lambda \pm 1)\mathbf{u} \end{pmatrix} = (1 \mp \frac{\alpha}{4}\lambda) \begin{pmatrix} \mathbf{u} \\ \pm\mathbf{u} \end{pmatrix}. \tag{20.122}$$

But, since the eigenvalue of the inverse matrix is the reciprocal of the eigenvalue, the eigenvalues of

$$T = \begin{pmatrix} 1 & -\frac{\alpha}{4}D \\ -\frac{\alpha}{4}D & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & \frac{\alpha}{4}D \\ \frac{\alpha}{4}D & 1 \end{pmatrix} \tag{20.123}$$

are given by

$$\sigma = \frac{1 \pm \frac{\alpha}{4}\lambda}{1 \mp \frac{\alpha}{4}\lambda}. \tag{20.124}$$

Since λ is imaginary, we find $|\sigma| = 1$. The Crank–Nicolson method is stable and does not show damping like the Lax–Wendroff method. However, there is considerable dispersion. Solution of the linear system (20.119) is complicated and can be replaced by an iterative predictor-corrector method. Starting from the initial guess

$$\begin{pmatrix} {}^{(0)}\mathbf{f}_{n+1} \\ {}^{(0)}\mathbf{g}_{n+1} \end{pmatrix} = \begin{pmatrix} 1 & \frac{\alpha}{2}D \\ \frac{\alpha}{2}D & 1 \end{pmatrix} \begin{pmatrix} \mathbf{f}_n \\ \mathbf{g}_n \end{pmatrix} \tag{20.125}$$

we iterate

$$\begin{pmatrix} {}^{(0)}\mathbf{f}_{n+1/2} \\ {}^{(0)}\mathbf{g}_{n+1/2} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} {}^{(0)}\mathbf{f}_{n+1} \\ {}^{(0)}\mathbf{g}_{n+1} \end{pmatrix} + \frac{1}{2}\begin{pmatrix} \mathbf{f}_{n} \\ {}^{)}\mathbf{g}_{n} \end{pmatrix} = \begin{pmatrix} 1 & \frac{\alpha}{4}D \\ \frac{\alpha}{4}D & 1 \end{pmatrix}\begin{pmatrix} \mathbf{f}_{n} \\ \mathbf{g}_{n} \end{pmatrix}$$

$$\begin{pmatrix} {}^{(1)}\mathbf{f}_{n+1} \\ {}^{(1)}\mathbf{g}_{n+1} \end{pmatrix} = \begin{pmatrix} \mathbf{f}_{n} \\ \mathbf{g}_{n} \end{pmatrix} + \begin{pmatrix} & \frac{\alpha}{2}D \\ \frac{\alpha}{2}D & \end{pmatrix}\begin{pmatrix} {}^{(0)}\mathbf{f}_{n+1/2} \\ {}^{(0)}\mathbf{g}_{n+1/2} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & \frac{\alpha}{4}D \\ \frac{\alpha}{4}D & 1 \end{pmatrix}\begin{pmatrix} \mathbf{f}_{n} \\ \mathbf{g}_{n} \end{pmatrix} + \begin{pmatrix} & \frac{\alpha}{4}D \\ \frac{\alpha}{4}D & \end{pmatrix}\begin{pmatrix} {}^{(0)}\mathbf{f}_{n+1} \\ {}^{(0)}\mathbf{g}_{n+1} \end{pmatrix} \tag{20.126}$$

$$\begin{pmatrix} {}^{(1)}\mathbf{f}_{n+1/2} \\ {}^{(1)}\mathbf{g}_{n+1/2} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \mathbf{f}_{n} \\ {}^{)}\mathbf{g}_{n} \end{pmatrix} + \frac{1}{2}\begin{pmatrix} {}^{(1)}\mathbf{f}_{n+1} \\ {}^{(1)}\mathbf{g}_{n+1} \end{pmatrix} = \begin{pmatrix} \mathbf{f}_{n} \\ \mathbf{g}_{n} \end{pmatrix} + \begin{pmatrix} & \frac{\alpha}{4}D \\ \frac{\alpha}{4}D & \end{pmatrix}\begin{pmatrix} {}^{(0)}\mathbf{f}_{n+1/2} \\ {}^{(0)}\mathbf{g}_{n+1/2} \end{pmatrix} \tag{20.127}$$

$$\begin{pmatrix} {}^{(2)}\mathbf{f}_{n+1} \\ {}^{(2)}\mathbf{g}_{n+1} \end{pmatrix} = \begin{pmatrix} \mathbf{f}_{n} \\ \mathbf{g}_{n} \end{pmatrix} + \begin{pmatrix} & \frac{\alpha}{2}D \\ \frac{\alpha}{2}D & \end{pmatrix}\begin{pmatrix} {}^{(1)}\mathbf{f}_{n+1/2} \\ {}^{(1)}\mathbf{g}_{n+1/2} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & \frac{\alpha}{4}D \\ \frac{\alpha}{4}D & 1 \end{pmatrix}\begin{pmatrix} \mathbf{f}_{n} \\ \mathbf{g}_{n} \end{pmatrix} + \begin{pmatrix} & \frac{\alpha}{4}D \\ \frac{\alpha}{4}D & \end{pmatrix}\begin{pmatrix} {}^{(1)}\mathbf{f}_{n+1} \\ {}^{(1)}\mathbf{g}_{n+1} \end{pmatrix}. \tag{20.128}$$

In principle this iteration could be repeated more times, but as Teukolsky showed [255], two iterations are optimal for hyperbolic equations like the advection or wave equation. The region of stability is reduced (Figs. 20.14 and 20.15) compared to the implicit Crank–Nicolson method. The eigenvalues are

$${}^{(0)}\sigma = 1 \pm \mathrm{i}\alpha \sin k\Delta x \quad |{}^{(0)}\sigma| > 1 \tag{20.129}$$

$${}^{(1)}\sigma = 1 \pm \mathrm{i}\alpha \sin k\Delta x - \frac{\alpha^2}{2}\sin^2 k\Delta x \quad |{}^{(1)}\sigma| > 1 \tag{20.130}$$

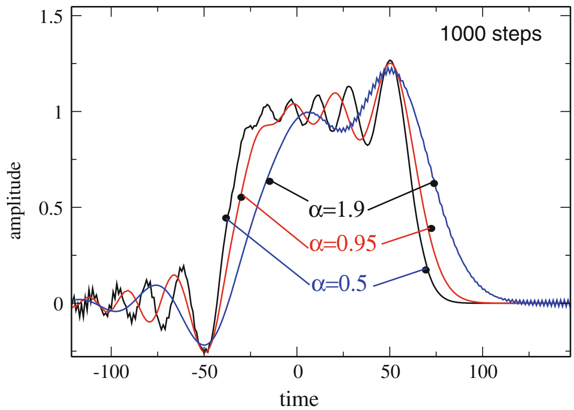

Fig. 20.14 (Simulation with the iterated Crank–Nicolson method) A rectangular pulse is simulated with the two-variable iterated Crank–Nicolson method. Only this method is stable for values $\alpha > 1$

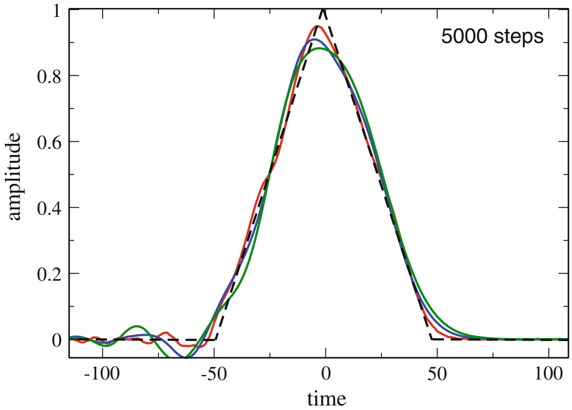

Fig. 20.15 (Simulation of a triangular pulse) A triangular pulse is simulated with different two-variable methods (*dashed curve* initial conditions, *red* leapfrog, *blue* Lax–Wendroff, *green* iterated Crank–Nicolson). This pulse contains less short wavelength components than the square pulse and shows much less deformation even after 5000 steps

$$
{}^{(2)}\sigma = 1 - \frac{\alpha^2}{2}\sin^2 k\Delta x \pm \mathrm{i}(\alpha \sin k\Delta x - \frac{\alpha^3 \sin^3 k\Delta x}{4})
$$
$$
|{}^{(2)}\sigma|^2 = 1 - \frac{\alpha^4 \sin^4 k\Delta x}{4} + \frac{\alpha^6 \sin^6 k\Delta x}{16} \le 1 \text{ for } |\alpha| \le 2. \tag{20.131}
$$

Problems

Problem 20.1 Waves on a Damped String

In this computer experiment we simulate waves on a string with a moving boundary with the method from Sect. 20.6.

- Excite the left boundary with a continuous sine function and try to generate standing waves
- Increase the velocity until instabilities appear
- Compare reflection at open and fixed right boundary
- Observe the dispersion of pulses with different shape and duration
- The velocity can be changed by a factor n (refractive index) in the region $x > 0$. Observe reflection at the boundary $x = 0$

Problem 20.2 Waves with the Fourier Transform Method

In this computer experiment we use the method from Sect. 20.3 to simulate waves on a string with fixed boundaries.

- Different initial excitations of the string can be selected
- The dispersion can be switched off by using $\omega_k = ck$ instead of the proper eigenvalues (20.31)

Problem 20.3 Two Variable Methods

In this computer experiment we simulate waves with periodic boundary conditions. Different initial values (rectangular, triangular or Gaussian pulses of different widths) and methods (leapfrog, Lax–Wendroff, iterated Crank–Nicolson) can be compared.