Chapter 9
z-Test

Many research questions involve testing differences between two population proportions (percentages). For example, Is there a significant difference between the proportion of girls and boys who smoke cigarettes in high school?, Is there a significant difference in the proportion of foreign and domestic automobile sales?, or Is there a significant difference in the proportion of girls and boys passing the Iowa Test of Basic Skills? These research questions involve testing the differences in population proportions between two independent groups. Other types of research questions can involve differences in population proportions between related or dependent groups. For example, Is there a significant difference in the proportion of adults smoking cigarettes before and after attending a stop smoking clinic?, Is there a significant difference in the proportion of foreign automobiles sold in the U.S. between years 1999 and 2000?, or Is there a significant difference in the proportion of girls passing the Iowa Test of Basic Skills between the years 1980 and 1990? Research questions involving differences in independent and dependent population proportions can be tested using a **z-test** statistic. Unfortunately, these types of tests are not available in most statistical packages, and therefore you will need to use a calculator or spreadsheet program to conduct the test.

Independent Samples

A practical example using the hypothesis testing approach will help to illustrate the z-test for differences in proportions between two independent groups. The research question for our example will be: Do a greater proportion of high school students in the population smoke cigarettes in urban rather than rural cities? This is a directional research question. A step-by-step outline will be followed to test this research question.

Step 1. State the directional research question in a statistical hypothesis format.

H_O: $P_1 \leq P_2$ (or $P_1 - P_2 \leq 0$)
H_A: $P_1 > P_2$ (or $P_1 - P_2 > 0$)

R. Schumacker and S. Tomek, *Understanding Statistics Using R*,
DOI 10.1007/978-1-4614-6227-9_9,

Notice that the "H" stands for hypothesis with the subscripts "0" for the null statistical hypothesis and "A" for the alternative statistical hypothesis. The alternative statistical hypothesis is stated to reflect the research question. In this example, the alternative statistical hypothesis indicates the directional nature of the research question. Also, P_1 is the population proportion of high school students who smoke in an urban city and P_2 is the population proportion of high school students who smoke cigarettes in a rural city.

Step 2. Determine the criteria for rejecting the null hypothesis and accepting the alternative hypothesis.

Given $\alpha = 0.01$, we select the corresponding z value from Table A1 (see Appendix) which is the closest to 1% of the area under the normal curve. If our computed z-test statistic is greater than this tabled z-value, we would reject the null hypothesis in favor of the alternative hypothesis. This establishes our region of rejection, R, or the probability area under the normal curve where differences in sample proportions are unlikely to occur by random chance.

R: $z > 2.33$

Notice that in Table A1 (see Appendix), the first column indicates $z = 2.3$ with the other column indicating the 3 in the hundredths decimal place. Also, notice that the percentage = 0.4901 is the closest to 49%, which leaves 1% area under the normal curve. In Table A1, only one-half of the normal curve is represented, so 50% is automatically added to the 49% to get 99%, which reflects the one-tail probability for the directional alternative statistical hypothesis.

Step 3. Collect the sample data and compute the z-test statistic.

A random sample of 20% of all high school students from both an urban and a rural city was selected. In the urban city, 20,000 high school students were sampled with 25% smoking cigarettes ($n_1 = 5{,}000$). In the rural city, 1,000 high school students were sampled with 15% smoking cigarettes ($n_2 = 150$). The proportions were:

$P_1 = 0.25$ (25% of the boys in the sample of high school students smoke cigarettes)
$P_2 = 0.15$ (15% of the girls in the sample of high school students smoke cigarettes)

The standard deviation of the sampling distribution of the differences in independent sample proportions is called the standard error of the difference between independent sample proportions. This value is needed to compute the z-test statistic. The formula is:

$$S_{P_1-P_2} = \sqrt{\frac{pq}{N}}$$

where:
$p = (n_1 + n_2 / N) = (5{,}000 + 150 / 21{,}000) = 0.245$
$q = 1 - p = 1 - 0.245 = 0.755$
n_1 = number in first sample = 5,000
n_2 = number in second sample = 150
N = total sample size taken = (20,000 + 1,000) = 21,000

$$S_{P_1-P_2} = \sqrt{\frac{0.245(0.755)}{21000}} = 0.003$$

The z-test can now be computed as:

$$z = \frac{P_1 - P_2}{S_{P_1-P_2}} = \frac{0.25 - 0.15}{0.003} = \frac{0.10}{0.003} = 33.33$$

We have learned from previous chapters that to test a statistical hypothesis, we need to know the sampling distribution of the statistic. The sampling distribution of the difference between two independent proportions is normally distributed when sample sizes are greater than five. Thus, we can use the normal distribution z statistic to test our statistical hypothesis.

Step 4. Compute the confidence interval around the z-test statistic.

A confidence interval is computed by using the percent difference between the two independent groups ($P_1 - P_2 = 0.10$), the tabled z-value corresponding to a given alpha level for a two-tailed region of region ($z = 2.58$), and the standard deviation of the sampling distribution or standard error of the test statistic ($S_{P1-P2} = 0.003$).

$CI_{99} = 0.10 +/- (2.58)(0.003)$
$CI_{99} = 0.10 +/- (0.008)$
$CI_{99} = (0.092, 0.108)$

Notice that the tabled z-value selected for determining the confidence interval around the computed z-test statistic is not the same because the confidence interval is based on a two-tailed interpretation. The alternative statistical hypothesis was one-tail because of the directional nature of the research question.

Step 5. Interpret the z-test statistic results.

Our interpretation is based upon a test of the null hypothesis and a 99% confidence interval around the computed z-test statistic. Since the computed $z = 33.33$ is greater than the tabled z-value, $z = 2.33$ at the 0.01 level of significance, we reject the null statistical hypothesis in favor of the alternative statistical hypothesis. The probability that the observed difference in the sample proportions of 10% would have occurred by chance is less than 0.01. We can therefore conclude that the urban city had a greater percentage of high school students smoking cigarettes than the rural city. The TYPE I error was set at 0.01, so we can be fairly confident in our interpretation.

The confidence interval was computed as 0.092 to 0.108, indicating that we can be 99% confident that this interval contains the difference between the population proportions from which the samples were taken. Moreover, the narrowness of the confidence interval gives us some idea of how much the difference in independent sample proportions might vary from random sample to random sample. Consequently, we can feel fairly confident that a 9% (0.092) to 11% (0.108) difference would exist between urban and rural city high school students smoking cigarettes upon repeated sampling of the population.

Dependent Samples

The null hypothesis that there is no difference between two population proportions can also be tested for dependent samples using the z-test statistic. The research design would involve obtaining percentages from the same sample or group twice. The research design would therefore have paired observations. Some examples of when this occurs would be:

1. Test differences in proportions of agreement in a group before and after a discussion of the death penalty.
2. Test differences in percent passing for students who take two similar tests.
3. Test differences in the proportion of employees who support a retirement plan and the proportion that support a company daycare.

The research design involves studying the impact of diversity training on the proportion of company employees who would favor hiring foreign workers. Before and after diversity training, employees were asked whether or not they were in favor of the company hiring foreign workers. The research question could be stated as: Are the proportions of company employees who favor hiring foreign workers the same before and after diversity training? This is a non-directional research question. A step-by-step approach to hypothesis testing will be used.

Step 1. State the non-directional research question in a statistical hypothesis format.

H_O: $P_1 = P_2$ (or $P_1 - P_2 = 0$)
H_A: $P_1 \neq P_2$ (or $P_1 - P_2 \neq 0$)

Notice in this example that we are interested in testing the null hypothesis of no difference between the population proportions against the non-directional alternative hypothesis, which indicates that the proportions are different. The alternative statistical hypothesis is stated to reflect the non-directional research question. Also, P_1 is the proportion of employees in favor before diversity training and P_2 is the proportion of employees in favor after diversity training.

Step 2. Determine the criteria for rejecting the null hypothesis and accepting the alternative hypothesis.

Given $\alpha = 0.05$, we select the corresponding z-value from Table A1 which is the closest to 5% of the area under the normal curve (2.5% in each tail of the normal curve). If our computed z-test statistic is greater than this tabled z-value, we would reject the null hypothesis and accept the alternative hypothesis. This establishes our region of rejection, R, or the probability areas under the normal curve where differences in sample proportions are unlikely to occur by random chance.

R: $z \pm 1.96$

Notice that in Table A1 (see Appendix), the first column indicates $z = 1.9$ with the other column indicating the 6 in the hundredths decimal place. Also, notice that the percentage = 0.4750 indicates 0.025 probability area under the normal curve in only one tail. In Table A1, only one-half of the normal curve is represented, but 0.025 in both tails of the normal curve would equal 5%. If we add 0.4750 + 0.4750, it would equal 0.95 or 95%, which indicates the remaining percent under the normal curve. The region of rejection indicates two z-values, + 1.96 and −1.96, for rejecting the null hypothesis, which reflects testing a non-directional research question.

Step 3. Collect the sample data and compute the z-test statistic.

A random sample of 100 employees from a high-tech company were interviewed before and after a diversity training session and asked whether or not they favored the company hiring foreign workers. Their sample responses were as follows:

		After Diversity Training		
Before Diversity Training		No	Yes	
	Yes	10 (0.10)	20 (0.20)	30 (0.30)
	No	50 (0.50)	20 (0.20)	70 (0.70)
		60 (0.60)	40 (0.40)	100 Total

The sample data indicated the following proportions:

P_1 = proportion in favor before diversity training = 0.30 or 30%
P_2 = proportion in favor after diversity training = 0.40 or 40%

Notice the order of the data entry in the cells of this table. This was done so that certain cells indicate disagreement or dissimilar responses before and after diversity training.

The standard deviation of the sampling distribution of the differences in dependent sample proportions is called the standard error of the difference between dependent sample proportions. This value is needed to compute the z-test statistic. The formula is:

$$S_{P_1-P_2} = \sqrt{\frac{p_{11}+p_{22}}{N}}$$

where:
p_{11} = percent change from before to after training (yes → no) = 0.10
p_{12} = percent change from before to after training (no → yes) = 0.20
N = total sample size = 100

$$S_{P_1-P_2} = \sqrt{\frac{0.10+0.20}{100}} = 0.055$$

The z-test can now be computed as:

$$z = \frac{P_1 - P_2}{S_{P_1-P_2}} = \frac{0.30-0.40}{0.055} = \frac{-0.10}{0.055} = -1.82$$

We have learned from previous chapters that to test a statistical hypothesis, we need to know the sampling distribution of the statistic. The sampling distribution of the difference between two dependent proportions is normally distributed when the sum of the sample sizes in the diagonal cells are greater than ten. Thus, we can use the normal distribution z-statistic to test our statistical hypothesis.

Step 4. Compute the confidence interval around the z-test statistic.

A confidence interval is computed by using the percent difference between the two independent groups ($P_1 - P_2 = 0.10$), the tabled z-value corresponding to a given alpha level for a two-tailed region of region ($z = \pm 1.96$), and the standard deviation of the sampling distribution or standard error of the test statistic ($S_{P1-P2} = 0.055$).

$CI_{99} = 0.10 \pm (1.96)(0.055)$
$CI_{99} = 0.10 \pm (0.108)$
$CI_{99} = (-0.008, 0.208)$

Notice that the tabled z-value selected for determining the confidence interval around the computed z-test statistic is the same because the confidence interval is also based on a two-tailed interpretation. Also, the null hypothesized parameter of zero (no difference in proportions) is contained in the confidence interval, which is consistent with not rejecting the null hypothesis.

Step 5. Interpret the z-test statistic results.

Our interpretation is once again based upon a test of the null hypothesis, but this time using a 95% confidence interval around the computed z-test statistic because of the non-directional nature of the research question. Since the computed $z = -1.82$ is less than the tabled z-value, $z = -1.96$ at the 0.05 level of significance, we retain the null statistical hypothesis. The probability that the observed difference in the sample proportions of 10% would have occurred by chance is greater than 0.05. We therefore cannot conclude that the percent of company employees in favor of hiring foreign workers was different before and after diversity training. The TYPE I error was set at 0.05, so we can be fairly confident in our interpretation.

The confidence interval was computed from −0.008 to 0.208, indicating that we can be 95% confident that this interval contains the null hypothesis parameter of zero difference between the population proportions from which the sample was taken. Moreover, the spread in the confidence interval gives us some idea of how much the difference in the dependent sample proportions might vary from random sample to random sample. Consequently, we should be sensitive to a research design factor that may have impacted the statistical test, which is the duration and intensity of the diversity training. Obviously a 1-h training session involving watching a short slide presentation might have less of an impact on employees than a 6-week training session involving role modeling with foreign workers on the job.

A z-test statistic can be used to answer research questions concerning differences in proportions between independent samples or groups, as well as, differences in proportions between dependent samples or groups. A statistically significant difference in proportions indicates that, beyond a random chance level, two groups differ in their proportions. A confidence interval around the z-test statistic indicates the amount of difference in the group proportions one can expect upon repeated sampling. The confidence interval captures the null hypothesis parameter of zero when proportions are not statistically significantly different.

ZTEST R Programs

The ZTEST-IND program inputs a critical z-value based on the alpha level and directional nature of the test. Next, *size1* and *size2* are set to the respective sample sizes and *num1* and *num2* are set to the number of positive cases in each sample. After this, the proportion of positive cases for each sample are computed as well as the overall *p* and *q*, which represent the overall proportion of positive cases and negative cases, respectively. Next, the standard error of the difference(*s*) is computed as well as the z-statistic. The individual proportions of positive cases, the critical z-value, and the standard error of the difference are then used to calculate the confidence interval around the proportion differences and all relevant values are printed.

The ZTEST-DEP program inputs a critical z-value, but then assigns numbers to the variables *num11*, *num12*, *num21*, and *num22*, where *num11* and *num22* represent changes from one outcome to the other and *num12* and *num21* represent cases where the outcome was the same. The total number of outcomes are calculated by adding the cells and then the proportion of changes of both types (*num11* and *num22*) are calculated. The proportion of cases with a positive outcome from each group or occasion are calculated next. The standard error of the difference is determined using the proportion of changed outcomes and the total number of outcomes. The z-statistic is determined from the difference in positive outcomes between the two groups or occasions divided by the standard error of the difference. Finally, the upper and lower bounds of a confidence interval around the difference in dependent proportions is calculated and all relevant values are output.

ZTEST-IND Program Output

```
Sample1=20000 Sample2=1000
N1=5000       N2=150

Z critical=1.96

Difference in proportions=0.1
Standard Error of Diff=0.003

z Statistic=33.3333
Confidence Intervals=( 0.0941 , 0.1059 )
```

ZTEST-DEP Program Output

```
Z Critical value=1.96

First %=0.1 Second %=0.2
Difference in proportions=-0.1

z Statistic=-1.8248
Standard Error of Diff=0.0548

Confidence Interval=( -0.0074 , 0.2074 )
```

z Exercises

1. Run ZTEST-IND for the independent sample proportions in the following example.
   The research question is: Do Democrats and Republicans differ in their percent agreement on handgun control? Test the research question at the 0.01 level of significance. The following sample data was collected:

   Democrats: 50,000 with 24,000 (0.48) or 48% in favor of handgun control
   Republicans: 50,000 with 12,000 (0.24) or 24% in favor of handgun control

   a. What is the percent difference between the two independent groups?_______
   b. What is the standard error of the difference between the independent percents?________________
   c. What is the z-test statistic value?___________
   d. What are the confidence interval values? (_____ , ______)
   e. What decision is made based on the z-test and confidence interval?
      Retain Null ____ Reject Null____
   f. What percent of the time would you expect the null hypothesis to be rejected by mistake?
   g. What is the name of this type of error? _______________________

2. Run ZTEST-DEP for the dependent sample proportions in the following example.
   The research question, Is the proportion of students passing the first exam the same as the proportion of students passing the second exam?, will be tested at the 0.05 level of significance. The following sample data was collected (fill in the missing information):

		Pass Second Exam		
Pass First Exam		No	Yes	
	Yes	20 ( )	50 ( )	___ ( )
	No	15 ( )	15 ( )	___ ( )
		__ ( )	__ ( )	100

   a. What is the percent passing difference between the exams? ____
   b. What is the standard error of the difference between the dependent percents?
   c. What is the z-test statistic value? ___________
   d. What are the confidence interval values? (_____ , ______)
   e. What decision is made based on the z-test and confidence interval?
      Retain Null ____ Reject Null____
   f. What percent of the time would you expect the null hypothesis to be rejected by mistake? _________
   g. What is the name of this type of error? _______________________

True or False Questions

z-Test

T F a. A z-test statistic is used *only* with independent sample percents.

T F b. The null hypothesis in the z-test corresponds to no difference in the proportions of either independent or dependent proportions.

T F c. The z-test statistic will be large if there are large differences between the sample proportions relative to the standard error.

T F d. A Type I error may occur when using the *z* test statistic.

T F e. The z-test statistic can be negative or positive in value.

T F f. The z-test statistic can *only* be used with directional hypothesis testing.

T F g. The tabled z-value used in forming the confidence interval around the z-test statistic is always based upon two tails under the normal curve.