10

Classical Mechanics

In this chapter we want to explore the connection between quantum and classical mechanics. We will see that the time derivative of the expectation value for the momentum operator gives us exactly Newton's second law, which is one of the foundations of classical mechanics.

Starting with the expectation value for an operator (Eq. 8.14)

$$\langle \hat{O} \rangle = \int d^3x \Psi^\star \hat{O} \Psi$$

and the Schrödinger equation for a particle in an external potential (Eq. 8.23)

$$(i\frac{d}{dt} - \frac{\nabla^2}{2m})\Psi - V\Psi = 0$$

$$\rightarrow i\frac{d}{dt}\Psi = \underbrace{\left(\frac{\nabla^2}{2m} + V\right)}_{=:H}\Psi$$

$$\rightarrow \frac{d}{dt}\Psi = \frac{1}{i}H\Psi$$

$$\underbrace{\rightarrow}_{\text{Because } H^\dagger = H} \frac{d}{dt}\Psi^\star = -\frac{1}{i}\underbrace{\Psi^\star}_{\text{Here } \Psi^\dagger = \Psi^\star} H \,.$$

Taking the time derivative of the expectation value yields

$$\frac{d}{dt}\langle \hat{O} \rangle = \int d^3x \left(\left(\frac{d}{dt}\Psi^\star \right) \hat{O}\Psi + \Psi^\star \left(\frac{d}{dt}\hat{O} \right) \Psi + \Psi^\star \hat{O} \left(\frac{d}{dt}\Psi \right) \right) .$$

We use now $\frac{d}{dt}\hat{O} = 0$, which is true for most operators. For example, for $\hat{O} = \hat{\vec{p}} = -i\vec{\nabla} \neq \hat{O}(t)$. In addition, we use the Schrödinger equation to rewrite the time derivatives of the wave function and its

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complex conjugate. This yields

$$\begin{aligned}\frac{d}{dt}\langle\hat{O}\rangle &= \int d^3x \left(\left(-\frac{1}{i}\Psi^\star H\right)\hat{O}\Psi + \Psi^\star\hat{O}\left(\frac{1}{i}H\Psi\right)\right)\\ &= \frac{1}{i}\int d^3x\big(-\Psi^\star H\hat{O}\Psi + \Psi^\star\hat{O}H\Psi\big)\\ &= \frac{1}{i}\int d^3x\Psi^\star[\hat{O},H]\Psi\\ &= \frac{1}{i}\langle[\hat{O},H]\rangle, \end{aligned} \tag{10.1}$$

which is known as **Ehrenfest theorem**. If we specify $\hat{O} = \hat{p}$ and use $H = \frac{\hat{p}^2}{2m} + V$, we get

$$\begin{aligned}\frac{d}{dt}\langle\hat{p}\rangle &= \frac{1}{i}\langle[\hat{p},H]\rangle\\ &= \frac{1}{i}\langle[\hat{p},\frac{\hat{p}^2}{2m}+V]\rangle\\ &= \frac{1}{i}\langle\underbrace{[\hat{p},\frac{\hat{p}^2}{2m}]}_{=0}+[\hat{p},V]\rangle\\ &= \frac{1}{i}\langle[\hat{p},V]\rangle\\ &= \frac{1}{i}\int d^3x\Psi^\star[\hat{p},V]\Psi\\ &= \frac{1}{i}\int d^3x\Psi^\star\hat{p}V\Psi - \frac{1}{i}\int d^3x\Psi^\star V\hat{p}\Psi\\ &= \frac{1}{i}\int d^3x\Psi^\star(-i\nabla)V\Psi - \frac{1}{i}\int d^3x\Psi^\star V(-i\nabla)\Psi\\ &\underbrace{=}_{\text{Product rule}} -\int d^3x\Psi^\star(\nabla V)\Psi - \int d^3x\Psi^\star V\nabla\Psi + \int d^3x\Psi^\star V\nabla\Psi\\ &= -\int d^3x\Psi^\star(\nabla V)\Psi\\ &= \langle-\nabla V\rangle = \langle F\rangle. \end{aligned} \tag{10.2}$$

In words this means that the time derivative (of the expectation value) of the momentum equals the (expectation value of the) negative gradient of the potential, which is known as force. This is exactly **Newton's second law**[1]. This equation can be used to compute the trajectories of macroscopic objects. Historically the force laws were deduced phenomenologically from experiments. All forces acting on an object were added linearly on the right-hand side of the equation. By using the phenomenologically deduced momentum $p_{\text{mak}} = mv$, we can write the left-hand side as $\frac{d}{dt}p_{\text{mak}} = \frac{d}{dt}mv$, which equals for an object with constant mass $m\frac{d}{dt}v$. The velocity is the time-derivative of the location[2] and we therefore have

$$m\frac{d^2}{dt^2}x = F_1 + F_2 + ... \tag{10.3}$$

[1] Recall that we used this equation, without a derivation, to illustrate the conserved quantities following from Noether's theorem in Section 4.5. Here we deliver the derivation, as promised.

[2] In other words, the velocity $v = \frac{d}{dt}x(t) = \dot{x}(t)$ is the change-rate of the position of the object and equally the acceleration $a\frac{d}{dt}\frac{d}{dt}x(t) = \frac{d}{dt}v = \ddot{x}(t)$ is the change-rate of the velocity.

This is the differential equation one must solve for $x = x(t)$ to get the trajectory of the object in question. An example for such a classical force will be derived in the next chapter.

10.1 Relativistic Mechanics

Using the Lagrangian formalism, we can look at classical mechanics from quite a different perspective. We need an equation that describes the motion of individual particles. As always in this book, we assume that we can derive the correct equation if we minimize *something*. We already discussed at the beginning of Chapter 4 that this *something* must be invariant under all Lorentz transformations, because otherwise we don't get the same equations of motion in all frames of reference.

Luckily, we already know something that is invariant under all Lorentz transformations: the invariant of special relativity, which was derived in Section 2.1:

$$(ds)^2 = (cd\tau)^2 = (cdt)^2 - (dx)^2 - (dy)^2 - (dz)^2, \qquad (10.4)$$

where τ is the proper time as explained in Section 2.2. Equally the square root of this is invariant and therefore the simplest possible thing we can minimize is

$$S = \int C d\tau \qquad (10.5)$$

with some constant C and

$$d\tau = \frac{1}{c}\sqrt{(cdt)^2 - (dx)^2 - (dy)^2 - (dz)^2}\,. \qquad (10.6)$$

The correct constant turns out to be $C = -mc^2$ and therefore we need to minimize

$$S = -mc^2 \int d\tau. \qquad (10.7)$$

For brevity we will restrict the following discussion to one dimension. Then we can write

$$\begin{aligned} d\tau &= \frac{1}{c}\sqrt{(cdt)^2 - (dx)^2} = \frac{1}{c}\sqrt{(cdt)^2\left(1 - \frac{(dx)^2}{c^2(dt)^2}\right)} \\ &= \frac{1}{c}(cdt)\sqrt{1 - \frac{1}{c^2}\left(\frac{dx}{dt}\right)^2} \underbrace{=}_{\frac{dx}{dt} = \dot{x} \text{ is the velocity of the particle in question}} dt\sqrt{1 - \frac{\dot{x}^2}{c^2}}. \end{aligned} \qquad (10.8)$$

Putting this into Eq. 10.7 yields

$$S = \int \underbrace{-mc^2\sqrt{1-\frac{\dot{x}^2}{c^2}}}_{\equiv \mathcal{L}}\, dt. \tag{10.9}$$

As always, we can find the extremal action by putting the Lagrangian into the Euler-Lagrange equation (Eq. 4.7):

$$\begin{aligned}
0 &= \frac{\partial \mathcal{L}}{\partial x} - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) \\
\rightarrow 0 &= \underbrace{\frac{\partial}{\partial x}\left(-mc^2\sqrt{1-\frac{\dot{x}^2}{c^2}}\right)}_{=0} - \frac{d}{dt}\left(\frac{\partial}{\partial \dot{x}}\left(-mc^2\sqrt{1-\frac{\dot{x}^2}{c^2}}\right)\right) \\
\rightarrow 0 &= c^2\frac{d}{dt}\left(\frac{-m\frac{\dot{x}}{c^2}}{\sqrt{1-\frac{\dot{x}^2}{c^2}}}\right) \\
\rightarrow 0 &= \frac{d}{dt}\left(\frac{m\dot{x}}{\sqrt{1-\frac{\dot{x}^2}{c^2}}}\right).
\end{aligned} \tag{10.10}$$

This is the correct relativistic equation for a *free* particle. If the particle moves in an external potential $V(x)$, we must simply add this potential to the Lagrangian[3]

[3] We add $-V(x)$ instead of $V(x)$, because then we get in the formula for the total energy of the system, which we can compute using Noether's theorem, the potential energy term $+V(x)$.

$$\mathcal{L} = -mc^2\sqrt{1-\frac{(\dot{x})^2}{c^2}} - V(x)\,. \tag{10.11}$$

For this Lagrangian the Euler-Lagrange equation yields

$$\frac{d}{dt}\left(\frac{m\dot{x}}{\sqrt{1-\frac{\dot{x}^2}{c^2}}}\right) = -\frac{dV}{dx} \equiv F. \tag{10.12}$$

Take note that in the non-relativistic limit ($\dot{x} \ll c$) we have $\sqrt{1-\frac{(\dot{x})^2}{c^2}} \approx 1$ and the equation is exactly the equation of motion we derived in the last section (Eq. 10.3).

10.2 The Lagrangian of Non-Relativistic Mechanics

It is instructive to have a look at the non-relativistic limit of the Lagrangian we derived in the last section (Eq. 10.9). The "non-relativistic limit" means, we consider a situation where the particle moves slowly compared with the speed of light: $\dot{x} \ll c$. We can then use the Taylor formula for the Lagrangian[4]

[4] See Appendix B.3

$$-mc^2\sqrt{1-\frac{\dot{x}^2}{c^2}} = -mc^2\left(1-\frac{1}{2}\frac{\dot{x}^2}{c^2}+\dots\right). \tag{10.13}$$

In the limit $\dot{x} \ll c$ we can neglect higher order terms and the Lagrangian reads

$$\mathcal{L} = -mc^2 + \frac{1}{2}m\dot{x}^2 - V(x)\,. \tag{10.14}$$

We already know that a constant like $-mc^2$ has no influence on the equation of motion and therefore the **Lagrangian for non-relativistic mechanics** reads

$$\mathcal{L} = \frac{1}{2}m\dot{x}^2 - V(x). \tag{10.15}$$

Without the external potential, i.e. $V(x) = 0$, this is exactly the Lagrangian we used in section 4.5.1 to illustrate the conserved quantities that follow from Noether's theorem. Putting this Lagrangian into the Euler-Lagrange equation (Eq. 4.7) yields

$$\begin{aligned}
\frac{\partial\mathcal{L}}{\partial x} - \frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{x}}\right) &= 0\\
\rightarrow \frac{\partial}{\partial x}\left(\frac{1}{2}m\dot{x}^2 - V(x)\right) - \frac{d}{dt}\left(\frac{\partial}{\partial\dot{x}}\left(\frac{1}{2}m\dot{x}^2 - V(x)\right)\right) &= 0\\
\rightarrow -\frac{\partial}{\partial x}V(x) - \frac{d}{dt}m\dot{x} &= 0\\
\rightarrow \frac{d}{dt}m\dot{x} &= -\frac{\partial}{\partial x}V(x)
\end{aligned} \tag{10.16}$$

This is once more exactly the equation of motion we derived at the beginning of this chapter (Eq. 10.3).