11

Electrodynamics

We already derived in Chapter 7 one of the most important equations of classical electrodynamics: the inhomogeneous Maxwell equations (Eq. 7.22)

$$\partial_\sigma(\partial^\sigma A^\rho - \partial^\rho A^\sigma) = J^\rho \qquad (11.1)$$

or in a more compact form using the electromagnetic tensor[1]

[1] $F^{\sigma\rho} \equiv (\partial^\sigma A^\rho - \partial^\rho A^\sigma)$

$$\partial_\sigma F^{\sigma\rho} = J^\rho . \qquad (11.2)$$

We discovered in Section 7.1.6 that J^ρ is a Noether current, i.e. $\partial_\rho J^\rho = 0$. In a macroscopic theory this conserved current is the electric four-current. The tensor $F^{\sigma\rho}$ is antisymmetric $F^{\sigma\rho} = -F^{\rho\sigma}$, which can be seen directly from the definition $F^{\sigma\rho} = \partial^\sigma A^\rho - \partial^\rho A^\sigma$, and has therefore 6 independent components. Three are

$$F^{i0} = \partial^i A^0 - \partial^0 A^i \qquad (11.3)$$

with $i = 1, 2, 3$ and the three other are

$$F^{ij} = \partial^i A^j - \partial^j A^i = (\delta^i_l \delta^j_m - \delta^i_m \delta^j_l)\partial^l A^m \underbrace{=}_{\text{This is a standard relation for the multiplication of two } \epsilon \text{ with one coinciding index}} \epsilon^{ijk}\epsilon^{klm}\partial^l A^m . \qquad (11.4)$$

The standard way to label these components is

$$\partial^i A^0 - \partial^0 A^i \equiv E^i \qquad (11.5)$$

$$\epsilon^{ijk}\partial^j A^k \equiv -B^i \qquad (11.6)$$

and therefore

$$F^{i0} = E^i \qquad (11.7)$$

$$F^{ij} = \epsilon^{ijk}\epsilon^{klm}\partial^l A^m = -\epsilon^{ijk} B^k . \qquad (11.8)$$

If we now rewrite the inhomogeneous Maxwell equations[2] as

[2] Plural because we have one equation for each component ρ.

J. Schwichtenberg, *Physics from Symmetry*, Undergraduate
Lecture Notes in Physics, https://doi.org/10.1007/978-3-319-66631-0_11

$$\partial_\sigma F^{\rho\sigma} = \partial_0 F^{\rho 0} - \partial_k F^{\rho k} = J^\rho \tag{11.9}$$

we have for the three spatial components[3] ($\rho \to i$)

$$\partial_0 F^{i0} - \partial_k F^{ik} = \partial_0 E^i + \epsilon^{ikl}\partial_k B^l = J^i$$

$$\to \partial_t \vec{E} + \nabla \times \vec{B} = \vec{J} \tag{11.10}$$

and for the time-component ($\rho \to 0$)

$$\partial_0 \underbrace{F^{00}}_{=0 \text{ see the definition of } F^{\rho\sigma}} - \partial_k F^{0k} \overbrace{=}^{\text{Because } F^{\mu\nu}=-F^{\nu\mu}} \partial_k F^{k0} \underbrace{=}_{\text{Eq. 11.7}} \partial_k E^k = J^0$$

$$\to \nabla \vec{E} = J^0 \,. \tag{11.11}$$

This is the form of the inhomogeneous Maxwell equations that, for example, engineers use.

[3] $\nabla \times \vec{B}$ is $\epsilon^{ikl}\partial_k B^l$ in vector notation and commonly called cross product.

11.1 The Homogeneous Maxwell Equations

It follows directly from the definition of the electromagnetic tensor $F_{\mu\nu}$ that if we multiply it with something totally antisymmetric[4]

$$\tilde{F}^{\mu\nu} = \epsilon^{\mu\nu\rho\sigma} F^{\rho\sigma} \tag{11.12}$$

the derivative ∂_μ of this new object $\tilde{F}^{\mu\nu}$, which is called the **dual electromagnetic tensor**, vanishes:

$$\partial_\mu \tilde{F}^{\mu\nu} = \partial_\mu \epsilon^{\mu\nu\rho\sigma}(\partial_\sigma A_\rho - \partial_\rho A_\sigma) = 0. \tag{11.13}$$

This follows from the fact that if we contract two symmetric indices with two antisymmetric indices, the result is always zero[5]. This can be seen, focussing for brevity on the first term, as follows

$$\begin{aligned}
\epsilon^{\mu\nu\rho\sigma}\partial_\mu\partial_\sigma A_\rho &= \frac{1}{2}(\epsilon^{\mu\nu\rho\sigma}\partial_\mu\partial_\sigma A_\rho + \epsilon^{\mu\nu\rho\sigma}\partial_\mu\partial_\sigma A_\rho) \\
&\underbrace{=}_{\text{Renaming dummy indices}} \frac{1}{2}(\epsilon^{\mu\nu\rho\sigma}\partial_\mu\partial_\sigma A_\rho + \epsilon^{\sigma\nu\rho\mu}\partial_\sigma\partial_\mu A_\rho) \\
&\underbrace{=}_{\text{Because } \epsilon^{\mu\nu\rho\sigma}=-\epsilon^{\sigma\nu\rho\mu} \text{ and } \partial_\mu\partial_\sigma=\partial_\sigma\partial_\mu} \frac{1}{2}(\epsilon^{\mu\nu\rho\sigma}\partial_\mu\partial_\sigma A_\rho - \epsilon^{\mu\nu\rho\sigma}\partial_\mu\partial_\sigma A_\rho) = 0 \qquad \checkmark
\end{aligned} \tag{11.14}$$

Equally the second term is zero. The equations[6]

$$\partial_\mu \tilde{F}^{\mu\nu} = 0 \tag{11.15}$$

are known as **homogeneous Maxwell equations** and we can see that they are a direct consequence of the definition of $F^{\mu\nu}$. In order to

[4] $\epsilon^{\mu\nu\rho\sigma}$ is the four-dimensional Levi-Civita symbol, which is defined in Appendix B.5.5.

[5] This is explained in Appendix B.5.4.

[6] Plural because we have one equation for each component $v = 0, 1, 2, 3$.

rewrite this in terms of B and E, we take a look at the component $\nu = 0$:

$$
\begin{aligned}
0 &= \partial_\mu \tilde{F}^{\mu 0} \\
&= \partial_\mu \epsilon^{\mu 0 \rho\sigma} F^{\rho\sigma} \\
&= \partial_0 \underbrace{\epsilon^{00\rho\sigma}}_{=0} F^{\rho\sigma} + \partial_i \epsilon^{i0\rho\sigma} F^{\rho\sigma} \\
&\underbrace{=}_{\text{Because } \epsilon^{i0\rho\sigma}=0 \text{ for } \rho=0 \text{ or } \sigma=0} \partial_i \epsilon^{i0jk} F^{jk} \\
&= -\partial_i \epsilon^{0ijk} F^{jk} \\
&\underbrace{=}_{\text{See Eq. 11.8}} \partial_i \underbrace{\epsilon^{0ijk}\epsilon^{ljk}}_{=2\delta_{il}} B^l \\
&= 2\partial_i \delta_{il} B^l = 2\partial_i B^i
\end{aligned}
$$

$$
\Rightarrow \partial_i B^i = 0 \quad \text{or in vector notation} \quad \nabla \vec{B} = 0 \tag{11.16}
$$

Analogously, we can take a look at the components $\nu = i$ and derive

$$
\nabla \times \vec{E} + \partial_t \vec{B} = 0 \tag{11.17}
$$

This is the conventional form of the homogeneous Maxwell equations that is used in practical applications.

11.2 The Lorentz Force

We can use the connection between quantum and classical mechanics that we discovered in the last chapter (the Ehrenfest theorem), to derive the famous **Lorentz force law**. The starting point is the equation describing a non-relativistic particle, without spin, in an external electromagnetic field, i.e. the Schrödinger equation with coupling to an external electromagnetic field[7] (Eq. 8.24):

$$
i\partial_t \Psi = \underbrace{\left(\frac{1}{2m}(\vec{p} - q\vec{A})^2 + q\Phi\right)}_{\equiv H} \Psi. \tag{11.18}
$$

If we define the momentum[8] of this system as $\vec{\Pi} = \vec{p} - q\vec{A}$, we can write the Hamiltonian:

$$
H = \frac{1}{2m}\vec{\Pi}^2 + q\Phi\,. \tag{11.19}
$$

Having defined the Hamiltonian H we are able to follow the exact same steps described in the last section and arrive at Eq. 10.1. But

[7] This equation can be derived from the Klein Gordon equation with coupling to an external electromagnetic field, which we derived from the Lagrangian describing spin 0 particles which are coupled to a massless spin 1 field, i.e. the photon field. Therefore, the real starting point is once more Lorentz and gauge symmetry, which we used to derive the corresponding Lagrangian. We use here the notation $A_0 \equiv \Phi$.

[8] This is actually the momentum of the system following from invariance under translations using the Noether theorem $\frac{\partial L}{\partial \dot{x}} = \Pi$.

this time we can't neglect the partial time-derivative term, because the operator we are going to look at is time dependent:

$$\frac{d}{dt}\langle\hat{O}\rangle = \frac{1}{i}\langle[O,H]\rangle + \langle\frac{\partial O}{\partial t}\rangle$$

$$\rightarrow \frac{d}{dt}\langle\vec{\Pi}\rangle = \frac{1}{i}\langle[\vec{\Pi},H]\rangle + \langle\frac{\partial\vec{\Pi}}{\partial t}\rangle \,.$$

We can see that $\frac{\partial\vec{\Pi}}{\partial t} \neq 0$ because A can change with time. If we now put the explicit form of H (Eq. 11.19) into the equation, we get

$$\rightarrow \frac{d}{dt}\langle\vec{\Pi}\rangle = \frac{1}{i}\langle[\vec{\Pi},\frac{1}{2m}\vec{\Pi}^2 + q\Phi]\rangle + \langle\frac{\partial\vec{\Pi}}{\partial t}\rangle$$

$$\rightarrow \frac{d}{dt}\langle\vec{\Pi}\rangle = \frac{1}{i}\langle[\vec{\Pi},\frac{1}{2m}\vec{\Pi}^2]\rangle + \underbrace{\frac{1}{i}\langle[\vec{\Pi},q\Phi]\rangle}_{=\langle q\nabla\Phi\rangle} + \langle\frac{\partial\vec{\Pi}}{\partial t}\rangle.$$

In the last step we use that $[\vec{\Pi},q\Phi]$ can be computed analogous to $[\hat{p},V]$, which we considered in the last chapter, because $[A,\Phi] = 0$. The next task is to compute $[(\vec{\Pi})^2,\vec{\Pi}]$, which is non-trivial, because the components Π_i do not commute. Instead, we have

$$[\Pi_i,\Pi_j] = -\frac{q}{i}\left(\frac{\partial A_j}{\partial x_i} - \frac{\partial A_i}{\partial x_j}\right) = -\frac{q}{i}\epsilon_{ijk}\underbrace{B_k}_{=\epsilon_{klm}\frac{\partial}{\partial x_l}A_m} \tag{11.20}$$

with the usual definition of the magnetic field $B = \nabla \times A$ written in index form. If we now define the speed[9] of our particle as $\vec{v} \equiv \frac{\vec{\Pi}}{m}$ we arrive at

$$\frac{1}{2m}[\vec{\Pi}^2,\vec{\Pi}] = \frac{q}{2i}(v\times B - B\times v) = \frac{q}{i}(\vec{v}\times\vec{B})$$

[9] The momentum divided by the mass of the particle: $p = mv$

Then we can write the classical equation of motion as

$$\frac{d}{dt}\langle\vec{\Pi}\rangle = \langle q\nabla\Phi\rangle - \langle q(v\times B)\rangle + \underbrace{\langle\frac{\partial\vec{\Pi}}{\partial t}\rangle}_{=-q\frac{\partial\vec{A}}{\partial t}}$$

$$\frac{d}{dt}\langle\vec{\Pi}\rangle = -q\langle(v\times B)\rangle + q\underbrace{\langle\nabla\Phi - \frac{\partial\vec{A}}{\partial t}\rangle}_{=\langle E\rangle \text{ see Eq. 11.5}} \,.$$

We finally get

$$\frac{d}{dt}\langle\vec{\Pi}\rangle \equiv F_{\text{Lorentz}} = -q(\langle(v\times B)\rangle + \langle E\rangle). \tag{11.21}$$

This is the equation of motion that describes the classical trajectory of a particle in an external electromagnetic field.

11.3 Coulomb Potential

We learned in Chapter 7 that our Lagrangians are invariant under internal transformations. We can use this freedom to simplify computations, i.e. we transform the field in question with an internal transformation such that the computation becomes especially simple. This is allowed because the physics we describe with the field and the transformed field are the same, as long as we stick to the gauge transformations that leave the Lagrangian invariant. An often used choice is gauging the photon field A^μ such that[10] $\partial_\mu A^\mu = 0$. This is called the Lorenz gauge. Using this gauge simplifies the inhomogeneous Maxwell equations to

$$\partial_\sigma(\partial^\sigma A^\rho - \partial^\rho A^\sigma) = \partial_\sigma \partial^\sigma A^\rho = J^\rho \tag{11.22}$$

We now have a look at the physical situation where a fixed, static charge is located in a spherically symmetric region around the origin of our coordinate system. We want to describe physics in the outside region, which means in the region without a source in it, i.e. $J^\rho = 0$. Therefore the Maxwell equations are in this region

$$\partial_\sigma \partial^\sigma A^\rho = \partial_0 \partial_0 A^\rho - \partial_i \partial_i A^\rho = 0. \tag{11.23}$$

We now use that we are considering a static ($\partial_0 A^\rho = 0$), spherically symmetric system, by rewriting the equation using spherical coordinates[11]. Then we can neglect all terms[12] but the term involving the ∂_r derivative. This yields

$$\rightarrow \frac{\partial^2}{\partial r^2}(rA^\mu) = 0. \tag{11.24}$$

The general solution of this equation is

$$A^\mu = \epsilon^\mu \frac{C}{r} + \epsilon^\mu D \tag{11.25}$$

with some constant four-vector ϵ^μ and constants C and D. The field A^μ must vanish at infinity and therefore $D = 0$. For the zeroth component[13] of A^μ this is the famous Coulomb potential

$$A^0 = \Phi = \frac{C}{r} = \frac{Ze}{r}, \tag{11.26}$$

where Z is an integer and e the electric charge of an electron. The reason for writing the constant C like this is that the electric charge is quantized in terms of multiples of the electron charge. In the standard model there is no satisfying explanation for this curious fact of nature.

[10] It can be shown explicitly that there is such a gauge choice. Further details can be found in the standard textbooks about electrodynamics.

[11] Instead of using x, y, z to determine the position of some objects, it's possible to use two angels θ, ϕ and the distance from the origin r. Then we have $\partial_i \partial_i A^\rho = \frac{\partial^2}{\partial r^2}(rA^\rho) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial A^\rho}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 A^\rho}{\partial\phi^2}$. This is especially useful when considering spherically symmetric systems, because these do only depend on r.

[12] For a spherically symmetric field, we have $\partial_\theta A = \partial_\phi A = 0$.

[13] In a lengthy computation using symmetry considerations it can be shown that all other components vanish.

Further Reading Tips

- **Richard P. Feynman - The Feynman Lectures on Physics Volume 2**[14] is a great book to start learning electrodynamics.
- **David J. Griffiths - Introduction to Electrodynamics**[15] is another great book to learn more about the concepts of electrodynamics.

[14] Richard P. Feynman, Robert B. Leighton, and Matthew Sands. *The Feynman Lectures on Physics: Volume 2*. Addison-Wesley, 1st edition, 2 1977. ISBN 9780201021172

[15] David J. Griffiths. *Introduction to Electrodynamics*. Addison-Wesley, 4th edition, 10 2012. ISBN 9780321856562