12

Gravity

Unfortunately, the best theory of gravity we have does not fit into the picture outlined in the rest of this book. This is one of the biggest problems of modern physics and the following paragraphs try to give you a first impression.

The modern theory of gravity is **Einstein's general relativity**. The fundamental idea is that gravity is a result of the curvature of spacetime. Mass and energy change the curvature of spacetime and in turn the changed curvature influences the movement of mass and energy. This interplay between energy and curvature is described by the famous Einstein equation

$$G_{\mu\nu} = 8\pi G T_{\mu\nu}. \qquad (12.1)$$

On the left-hand side is the Einstein tensor $G_{\mu\nu}$, which describes the curvature and on the right-hand side is the energy-momentum tensor[1] $T_{\mu\nu}$. G is the gravitation constant.

[1] Recall that the energy-momentum tensor is the quantity which is directly related to translational symmetry (Eq. 4.36).

From the idea gravity = curvature of spacetime, the derivation of the Einstein equation is, from a modern point of view, relatively straightforward[2]. Firstly, one of the most important laws of physics is the conservation of energy and momentum, which as we saw, follows directly when considering a homogeneous spacetime. In a homogeneous spacetime the laws of physics are invariant under translations in space and time and using Noether's theorem, we can derive the conservation of momentum and energy. Therefore, one of the most basic assumptions of physics is that spacetime is homogeneous and therefore energy and momentum are conserved. In mathematical terms this conservation law is expressed as (Eq. 4.36)

[2] Einstein needed 100 years ago ca. 6 years for the derivation of the correct equation. Today, with the power of hindsight we are much faster.

$$\partial^\mu T_{\mu\nu} = 0. \qquad (12.2)$$

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Next we need something to describe curvature mathematically. This is what makes general relativity computationally very demanding. Nevertheless, we already know the most important object: the metric. Recall that metrics are the mathematical objects that enable us to compute the distance between two points[3]. In a curved space the distance between two points is different than in a flat space as illustrated in Fig. 12.1. Therefore metrics will play a very important role when thinking about curvature in mathematical terms.

[3] Up to this point we only considered the Minkowski metric $\eta^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$ and the Euclidean metric $\delta^{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

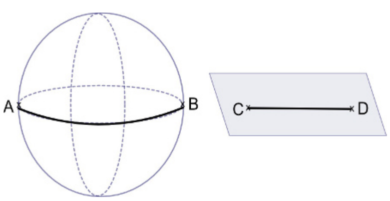

Fig. 12.1: Distance between two points in a curved and a flat space

Having talked about this, we are ready to "derive" the Einstein equation, because it turns out that there is exactly one mathematical object that we can put on the left-hand side: the Einstein tensor $G_{\mu\nu}$. The Einstein tensor is the only divergence-free[4] function of the metric $g_{\mu\nu}$ and at most its first and second partial derivative. Therefore, the Einstein tensor may be very complicated, but it's the only object we are allowed to write on the left-hand side describing curvature. This follows, because we can conclude from

$$T_{\mu\nu} = CG_{\mu\nu} \quad \text{that} \quad \partial^\mu T_{\mu\nu} = 0 \rightarrow \partial^\mu G_{\mu\nu} = 0 \tag{12.3}$$

must hold, too. The Einstein tensor is a second rank tensor[5] and has exactly this property.

[4] $\partial^\mu G_{\mu\nu} = 0$

[5] This means two indices $\mu\nu$, which is a requirement, because $T_{\mu\nu}$ on the right-hand side has two indices, too.

The Einstein tensor is defined as a sum of the Ricci Tensor $R_{\mu\nu}$ and the trace of the Ricci tensor, called Ricci scalar $R = R^\nu_\nu$

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} \tag{12.4}$$

where the Ricci Tensor $R_{\mu\nu}$ is defined in terms of the Christoffel symbols $\Gamma^\mu_{\nu\rho}$

$$R_{\alpha\beta} = \partial_\rho \Gamma^\rho_{\beta\alpha} - \partial_\beta \Gamma^\rho_{\rho\alpha} + \Gamma^\rho_{\rho\lambda}\Gamma^\lambda_{\beta\alpha} - \Gamma^\rho_{\beta\lambda}\Gamma^\lambda_{\rho\alpha} \tag{12.5}$$

and the Christoffel Symbols are defined in terms of the metric

$$\Gamma_{\alpha\beta\rho} = \frac{1}{2}\left(\frac{\partial g_{\alpha\beta}}{\partial x^\rho} + \frac{\partial g_{\alpha\rho}}{\partial x^\beta} - \frac{\partial g_{\beta\rho}}{\partial x^\alpha}\right) = \frac{1}{2}\left(\partial_\rho g_{\alpha\beta} + \partial_\beta g_{\alpha\rho} - \partial_\alpha g_{\beta\rho}\right). \tag{12.6}$$

This can be quite intimidating and shows why computations in general relativity very often need massive computational efforts.

Next we need to know how things react to such a curved spacetime. What's the path of an object from A to B in curved spacetime? The first guess is the correct one: An object follows the shortest path between two points in curved spacetime. We can start with a given distribution of energy and mass, which means some $T_{\mu\nu}$, compute the metric or Christoffel symbols with the Einstein equation and then get the trajectory through the **geodesic equation**

$$\frac{d^2x^\lambda}{dt^2} + \Gamma^\lambda_{\mu\nu}\frac{dx^\mu}{dt}\frac{dx^\nu}{dt} = 0. \tag{12.7}$$

The geodesic is the locally shortest[6] curve between two points on a manifold.

[6] This is a bit oversimplified, but the correct definition needs some terms from differential geometry we haven't introduced here.

It is interesting to note that Einstein thought about the Christoffel Symbols as the gravitational field

> If the $\Gamma^\mu_{\nu\rho}$ vanish, then the point moves uniformly in a straight line. These quantities therefore condition the deviation of the motion from uniformity. They are the components of the gravitational field.
>
> **- Albert Einstein**[7]

[7] Albert Einstein. The foundation of the general theory of relativity. 1916

We can understand what Einstein means by looking at Eq. 12.7. For $\Gamma^\mu_{\nu\rho} = 0$ the geodesic equation reduces to

$$\frac{d^2x^\lambda}{dt^2} = 0. \tag{12.8}$$

The solutions of this equation describe a straight line.

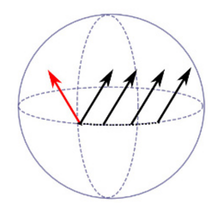

Fig. 12.2: In order to be able to compare the red arrow with the black arrow, we transport the black arrow to the location of the red arrow.

Another interesting aspect of a curved spacetime is that the notion of differentiation changes. Remember how the derivative is defined in flat space using the difference quotient

$$f'(a) = \lim_{h\to 0}\frac{f(a+h)-f(a)}{h}. \tag{12.9}$$

This definition requires that we compare the function in question at two different points. In a curved space this comparison is not as trivial as in the flat space. Take a look at Fig. 12.2. If we want to compare two vectors on a sphere, how can we make sure that the vectors are really different and the difference is not just an effect of the curved space? The answer of differential geometry is parallel transport. We have to move one vector to the location of the other one, to be able to compare them[8].

[8] In fact in differential geometry one has only local coordinate systems. The defining feature of a manifold is that it looks locally flat=Euclidean. This was discussed in Section 3.11. Therefore, the coefficients we use to describe the objects in questions are only valid in a small region of the manifold and thus we can only compare coefficients of the same coordinate system.

The derivative becomes in a curved space the **covariant derivative**[9]

$$D_b v^a \equiv \partial_b v^a + \Gamma^a{}_{bc} v^c. \tag{12.10}$$

Therefore, if we want any equation we derived so far to be valid in curved spacetime, we need to change

$$\partial_b \to D_b = \partial_b + \Gamma^a{}_{bc}. \tag{12.11}$$

[9] In this context the Christoffel symbols $\Gamma^a{}_{bc}$ are often called connection coefficients, because of their property to connect the points we want to compare.

Does this look familiar? Take a look again at Eq. 7.18. We learned in an earlier chapter that a locally $U(1)$ invariant Lagrangian for spin 0 or spin $\frac{1}{2}$ fields required a specific coupling with a spin 1 field. This specific coupling can be summarized by the prescription

$$\partial_\mu \to D_\mu \equiv \partial_\mu + ieA_\mu. \tag{12.12}$$

Remember that this wasn't just a mathematical gimmick. This prescription gives us the correct theory of electromagnetism. The same is true for weak

$$\partial_\mu \to D_\mu \equiv \partial_\mu - ig \underbrace{W_\mu}_{=W_\mu^i \sigma^i} \tag{12.13}$$

and strong interactions (see Eq. 7.165)

$$\partial_\mu \to D_\mu = \partial_\mu - ig' \underbrace{G_\mu}_{=T^C G_\mu^C} . \tag{12.14}$$

Although things look quite similar here, there is for many reasons no formulation of gravity that is compatible with the quantum description of all other forces. All other forces are described in a quantum theory and one can only make probability predictions. In contrast, general relativity is a classical theory, because particles follow defined trajectories and there is no need for probability predictions.
To make things worse, at the current time no experiment can shine any light on the interplay between those forces. The effects of gravity on elementary particles is too weak to be measured. Because of this, the standard model, which ignores gravity entirely and only takes the weak, the strong and the electromagnetic interactions into account, works very well. The effects of general relativity only become measurable with very heavy objects. For such quantum effects play no role, because massive objects consist of many, many elementary particles and all quantum effects get averaged out. We discovered in Chapter 10 that the equation of motion for the average value is just the classical one and no quantum effects are measurable. One can make a long list of things that make constructing a quantum theory of gravity so difficult, but Einstein formulated the difference between gravity and all other forces very concisely:

> ...according to the general theory of relativity, gravitation occupies an exceptional position with regard to other forces, particularly the electromagnetic forces, since the ten functions representing the gravitational field at the same time define the metrical properties of the space measured.

- Albert Einstein[10]

[10] Albert Einstein and Francis A. Davis. *The Principle of Relativity*. Dover Publications, reprint edition, 6 1952. ISBN 9780486600819

We are able do describe quantum particles in a curved space, by changing the derivative to the covariant derivative. But this is of course no dynamical theory of gravity. We could make the right-hand side of the Einstein equation quantum, if we make the usual identification with the corresponding generator, but what about the left-hand side? The Einstein tensor in terms of the Christoffel Symbols is

$$G_{\alpha\beta} = (\delta^{\gamma}_{\alpha}\delta^{\zeta}_{\beta} - \frac{1}{2}g_{\alpha\beta}g^{\gamma\zeta})(\partial_{\epsilon}\Gamma^{\epsilon}_{\gamma\zeta} - \partial_{\zeta}\Gamma^{\epsilon}_{\gamma\epsilon} + \Gamma^{\epsilon}_{\epsilon\sigma}\Gamma^{\sigma}_{\gamma\zeta} - \Gamma^{\epsilon}_{\zeta\sigma}\Gamma^{\sigma}_{\epsilon\gamma}). \quad (12.15)$$

Thus maybe we can think of the Einstein equation as the field equation[11] for $\Gamma^{\epsilon}_{\zeta\sigma}$, and the terms generated by the prescription

$$\partial_b \to D_b = \partial_b + \Gamma^a{}_{bc}$$

as the corresponding coupling between the gravitational field $\Gamma^{\epsilon}_{\zeta\sigma}$ and the other fields?

[11] Analogous to the Maxwell equation for the electromagnetic field.

Regardless of if you prefer to think of the metric or the Christoffel symbols as the gravitational field, the two or three vector indices indicate that we may need to investigate the $(1,1)$ or even higher representations, which we would call consequently spin $2, 3, \ldots$ representation of the Poincaré group. In fact most physicists believe that the boson responsible for gravitational attraction, the graviton, has spin 2.

Until the present day, there is no working[12] theory of quantum gravity and for further information have a look at the books mentioned in the next section.

[12] Many attempts result in an infinite number of infinity terms, which is quite bad for probability predictions.

Further Reading Tips

For more information about the standard theory of gravity, Einstein's general relativity, see

- **Ta-Pei Cheng - Relativity, Gravitation and Cosmology**[13] is a great, rather low-level introduction to general relativity with many very enlightening explanations. Perfect to get a quick overview.
- **A. Zee - Einstein Gravity in a Nutshell**[14], is the best book to learn about general relativity. It really starts at the beginning, avoids unnecessary, confusing mathematical tools and does a great job explaining the origin and usage of general relativity.
- **Charles W. Misner, Kip S. Thorne, John Archibald Wheeler - Gravitation**[15] is a really, really big book, but often offers in depth explanations for points that remain unclear in most other books.

[13] Ta-Pei Cheng. *Relativity, Gravitation and Cosmology: A Basic Introduction*. Oxford University Press, 2nd edition, 1 2010. ISBN 9780199573646

[14] Anthony Zee. *Einstein Gravity in a Nutshell*. Princeton University Press, 1st edition, 5 2013. ISBN 9780691145587

[15] Charles W. Misner, Kip S. Thorne, and John Archibald Wheeler. *Gravitation*. W. H. Freeman, 1st edition, 9 1973. ISBN 9780716703440

For more information about attempts to quantize gravity have a look at

- **Lee Smolin -Three Roads to Quantum Gravity**[16], which is brilliant popular science book that summarizes the various attempts to quantize gravity.
- **John C. Baez, Javier P. Muniain - Gauge Fields, Knots, and Gravity**[17] which is a magnificent book. The focus lies on introducing the mathematical tools needed to understand attempts to quantize gravity in a way that physicists understand.

[16] Lee Smolin. *Three Roads to Quantum Gravity*. Basic Books, 3 edition, 8 2017. ISBN 9780465094547

[17] John C. Baez and Javier P. Muniain. *Gauge Fields, Knots, and Gravity*. World Scientific Pub Co Inc, 1st edition, 9 1994. ISBN 9789810220341