1

Introduction

1.1 What we Cannot Derive

Before we talk about what we can derive from symmetry, let's clarify what we need to put into the theories by hand. First of all, there is presently no theory that is able to derive the constants of nature. These constants need to be extracted from experiments. Examples are the coupling constants of the various interactions and the masses of the elementary particles.

Besides that, there is something else we cannot explain: **The number three.** This should not be some kind of number mysticism, but we cannot explain all sorts of restrictions that are directly connected with the number three. For instance,

- there are three gauge theories[1], corresponding to the three fundamental forces described by the standard model: The electromagnetic, the weak and the strong force. These forces are described by gauge theories that correspond to the symmetry groups $U(1), SU(2)$ and $SU(3)$. Why is there no fundamental force following from $SU(4)$? Nobody knows!
- There are three lepton generations and three quark generations. Why isn't there a fourth? We only know from experiments with high accuracy that there is no fourth generation[2].
- We only include the three lowest orders in Φ in the Lagrangian (Φ^0, Φ^1, Φ^2), where Φ denotes here something generic that describes our physical system and the Lagrangian is the object we use to derive our theory from, in order to get a sensible theory describing free (=non-interacting) fields/particles.
- We only use the three lowest-dimensional representations of the double cover of the Poincaré group, which correspond to spin 0, $\frac{1}{2}$

[1] Don't worry if you don't understand some terms, like gauge theory or double cover, in this introduction. All these terms will be explained in great detail later in this book and they are included here only for completeness.

[2] For example, the element abundance in the present universe depends on the number of generations. In addition, there are strong evidence from collider experiments. (See e.g. Phys. Rev. Lett. 109, 241802) .

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and 1, respectively, to describe fundamental particles. There is no fundamental particle with spin $\frac{3}{2}$.

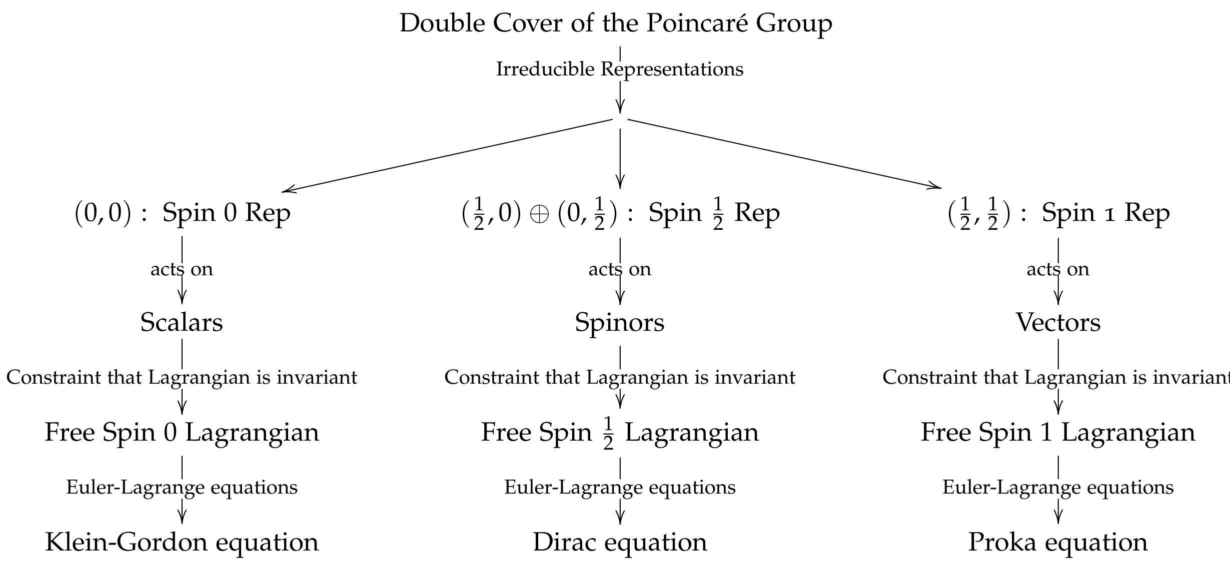
In the present theory, these things are assumptions we have to put in by hand. We know that they are correct from experiments, but there is presently no deeper principle why we have to stop after three.

In addition, there is one thing that can't be derived from symmetry, but which must be taken into account in order to get a sensible theory:

We are only allowed to include the lowest-possible, non-trivial order in the differential operator ∂_μ in the Lagrangian. For some theories these are first order derivatives ∂_μ, for other theories Lorentz invariance forbids first order derivatives and therefore second order derivatives $\partial_\mu \partial^\mu$ are the lowest-possible, non-trivial order. Otherwise, we don't get a sensible theory. Theories with higher order derivatives are unbounded from below, which means that the energy in such theories can be arbitrarily negative. Therefore states in such theories can always transition into lower energy states and are never stable.

Finally, there is another thing we cannot derive in the way we derive the other theories in this book: **gravity**. Of course there is a beautiful and correct theory of gravity, called general relativity. But this theory works quite differently than the other theories and a complete derivation lies beyond the scope of this book. Quantum gravity, as an attempt to fit gravity into the same scheme as the other theories, is still a theory under construction that no one has successfully derived. Nevertheless, some comments regarding gravity will be made in the last chapter.

1.2 Book Overview



This book uses **natural units**, which means we set the Planck constant to $\hbar = 1$ and the speed of light to $c = 1$. This is helpful for theoretical considerations, because it avoids a lot of unnecessary writing. For applications the constants need to be added again to return to standard SI units.

Our starting point will be the basic assumptions of **special relativity**. These are: The velocity of light has the same value c in all inertial frames of reference, which are frames moving with constant velocity relative to each other and physics is the same in all inertial frames of reference.

The set of all transformations permitted by these symmetry constraints is called the **Poincaré group**. To be able to utilize them, we discuss the mathematical theory that enables us to work with symmetries. This branch of mathematics is called group theory. We will derive the irreducible representations of the Poincaré group[3], which you can think of as basic building blocks of all other representations. These representations are what we use later in this text to describe particles and fields of different spin. On the one hand, spin is an abstract label for different kinds of particles/fields and on the other hand can be seen as something like internal angular momentum. We will discuss in detail how this comes about.

[3] To be technically correct: We will derive the representations of the **double-cover** of the Poincaré group instead of the Poincaré group itself. The term "double-cover" comes from the observation that the map between the double-cover of a group and the group itself maps **two** elements of the double cover to **one** element of the group. This is explained in Section 3.3.1 in detail.

Afterwards, the **Lagrangian formalism** is introduced, which makes working with symmetries in a physical context very convenient. The central object is the **Lagrangian**. Different Lagrangians describe different physical systems and we will derive several Lagrangians using symmetry considerations. In addition, the **Euler-Lagrange equations** are derived. These enable us to derive the equations of motion from a given Lagrangian. Using the irreducible representations of the Poincaré group, the fundamental equations of motion for fields and particles with different spin can be derived.

The central idea here is that the Lagrangian must be invariant (=does not change) under any transformation of the Poincaré group. This makes sure the equations of motion take the same form in all frames of reference, which we stated above as "physics is the same in all inertial frames".

Then, we will discover another symmetry of the Lagrangian for free spin $\frac{1}{2}$ fields: Invariance under $U(1)$ transformations. Similarly an **internal symmetry** for spin 1 fields can be found. Demanding **local** $U(1)$ symmetry will lead us to **coupling terms** between spin $\frac{1}{2}$ and spin 1 fields. The Lagrangian with this coupling term is the correct **Lagrangian for quantum electrodynamics**. A similar procedure for local $SU(2)$ and $SU(3)$ transformations will lead us to the correct **Lagrangian for weak and strong interactions**.

In addition, we discuss **symmetry breaking** and a special way to break symmetries called the **Higgs mechanism**. The Higgs mechanism enables us to describe particles with mass[4].

[4] Without the Higgs mechanism, terms describing mass in the Lagrangian spoil the symmetry and are therefore forbidden.

Afterwards, **Noether's theorem** is derived, which reveals a deep connection between symmetries and conserved quantities. We will utilize this connection by identifying each physical quantity with the corresponding symmetry generator. This leads us to the most important equation of quantum mechanics

$$[\hat{x}_i, \hat{p}_j] = i\delta_{ij} \tag{1.1}$$

and quantum field theory

$$[\hat{\Phi}(x), \hat{\pi}(y)] = i\delta(x-y). \tag{1.2}$$

We continue by taking the non-relativistic[5] limit of the equation of motion for spin 0 particles, called Klein-Gordon equation, which result in the famous **Schrödinger equation**. This, together with the identifications we made between physical quantities and the generators of the corresponding symmetries, is the foundation of **quantum mechanics**.

[5] Non-relativistic means that everything moves slowly compared to the speed of light and therefore especially curious features of special relativity are too small to be measurable.

Then we take a look at **free quantum field theory**, by starting with the solutions of the different equations of motion[6] and Eq. 1.2. Afterwards, we take interactions into account, by taking a closer look at the Lagrangians with coupling terms between fields of different spin. This enables us to discuss how the **probability amplitude for scattering processes** can be derived.

[6] The Klein-Gordon, Dirac, Proka and Maxwell equations.

By deriving the **Ehrenfest theorem** the connection between quantum and **classical mechanics** is revealed. Furthermore, the fundamental equations of **classical electrodynamics**, including the **Maxwell equations** and the **Lorentz force law**, are derived.

Finally, the basic structure of the modern theory of **gravity**, called **general relativity**, is briefly introduced and some remarks regarding the difficulties in the derivation of a **quantum theory of gravity** are made.

The major part of this book is about the tools we need to work with symmetries mathematically and about the derivation of what is commonly known as the **standard model**. The standard model uses quantum field theory to describe the behavior of all known elementary particles. Until the present day, all experimental predictions of the standard model have been correct. Every other theory introduced here can then be seen to follow from the standard model as a special case. For example in the limit of macroscopic objects we get classical mechanics or in the limit of elementary particles with low energy, we get quantum mechanics. For those readers who have never heard about the presently-known elementary particles and their interactions, a really quick overview is included in the next section.

1.3 Elementary Particles and Fundamental Forces

There are two major categories for elementary particles: **bosons** and **fermions**. There can be never two fermions in exactly the same state, which is known as **Pauli's exclusion principle**, but infinitely many bosons. This curious fact of nature leads to the completely different behavior of these particles:

- fermions are responsible for matter
- bosons for the forces of nature.

This means, for example, that atoms consist of fermions[7], but the electromagnetic-force is mediated by bosons. The bosons that are responsible for electromagnetic interactions are called photons. One of the most dramatic consequences of thie exclusion principle is that there is stable matter at all. If there could be infinitely many fermions in the same state, there would be no stable matter[8].

[7] Atoms consist of electrons, protons and neutrons, which are all fermions. But take note that protons and neutrons are not fundamental and consist of quarks, which are fermions, too.

[8] We will discuss this in Chapter 6

There are four presently known fundamental forces

- The electromagnetic force, which is mediated by massless **photons**.
- The weak force, which is mediated by massive $\mathbf{W}^+, \mathbf{W}^-$ and **Z-bosons**.
- The strong force, which is mediated by massless **gluons**.
- Gravity, which is (maybe) mediated by **gravitons**.

Some of the these bosons are massless and some are not and this tells us something deep about nature. We will fully understand this after setting up the appropriate framework. For the moment, just take note that each force is closely related to a symmetry. The fact that the bosons mediating the weak force are massive means the related symmetry is broken. This process of **spontaneous symmetry breaking** is responsible for the masses of **all** elementary particles. We will see later that this is possible through the coupling to another fundamental boson, the **Higgs boson**.

Fundamental particles interact via some force if they carry the corresponding **charge**[9].

- For the electromagnetic force this is the **electric charge** and consequently only electrically charged particles take part in electromagnetic interactions.
- For the weak force, the charge is called **isospin**[10]. All known fermions carry isospin and therefore interact via the weak force.
- The charge of the strong force is called **color**, because of some curious features it shares with the humanly visible colors. Don't let this name confuse you, because this charge has nothing to do with the colors you see in everyday life.

[9] All charges have a beautiful common origin that will be discussed in Chapter 7.

[10] Often the charge of the weak force carries the extra prefix "weak", i.e. is called weak isospin, because there is another concept called isospin for composite objects that interact via the strong force. Nevertheless, this is not a fundamental charge and in this book the prefix "weak" is omitted.

The fundamental fermions are divided into two subcategories: **quarks**, which are the building blocks of protons and neutrons, and **leptons**. Famous leptons are, for example, electrons and neutrinos. The difference is that quarks carry color charge and therefore take part in strong interactions, whereas leptons do not. There are three quark and lepton **generations**, which consist each of two particles:

	Generation 1	Generation 2	Generation 3	Electric charge	Isospin	Color
	Up	Charm	Top	$\frac{+2}{3}e$	$\frac{1}{2}$	✓
Quarks:	Down	Strange	Bottom	$\frac{-1}{3}e$	$\frac{-1}{2}$	✓
	Electron-Neutrino	Muon-Neutrino	Tauon-Neutrino	0	$\frac{+1}{2}$	-
Leptons:	Electron	Muon	Tauon	$-e$	$\frac{-1}{2}$	-

Different particles can be identified through **labels**. In addition to the charges and the mass there is another incredibly important label called **spin**, which can be seen as some kind of internal angular momentum. Bosons carry integer spin, whereas fermions carry half-integer spin. The fundamental fermions we listed above have spin $\frac{1}{2}$ and almost all fundamental bosons have spin 1. There is only one known fundamental particle with spin 0: the Higgs boson.

There is an anti-particle for each particle, which carries exactly the same labels with opposite sign[11]. For the electron the anti-particle is called positron, but in general there is no extra name and only a prefix "anti". For example, the antiparticle corresponding to an up-quark is called anti-up-quark. Some particles, like the photon[12] are their own anti-particle.

All these notions will be explained in more detail later in this text. Now it's time to start with the derivation of the theory that describes correctly the interplay of the different characters in this particle zoo. The first cornerstone towards this goal is Einstein's famous theory of special relativity, which is the topic of the next chapter.

[11] Maybe except for the mass label. This is currently under experimental investigation, for example at the AEGIS, the ATRAP and the ALPHA experiment, located at CERN in Geneva, Switzerland.

[12] And maybe the neutrinos, which is currently under experimental investigation in many experiments that search for a neutrinoless double-beta decay.