

5

Measuring Nature

Now that we have discovered the connection between symmetries and conserved quantities, we can utilize this connection. In more technical terms, Noether's Theorem establishes a connection between the generator of a symmetry transformation and a conserved quantity. In this chapter we will utilize this connection.

Conserved quantities are what physicists commonly use to describe physical systems, because no matter how complicated the system changes, the conserved quantities stay the same. For example, to describe what is happening in an experiment physicists use the momentum, energy and angular momentum. Noether's theorem hints us towards an incredibly important idea:

We identify the quantities that we use to describe nature with the corresponding generators:

$$\text{physical quantity} \Rightarrow \text{generator of the corresponding symmetry.} \quad (5.1)$$

As we will see, this identification naturally guide us towards quantum theory. Let's make this concrete by considering a particle theory.

5.1 The Operators of Quantum Mechanics

The invariance of the Lagrangian under the action of the generator of spatial-translations leads us to the conservation of momentum. Therefore, we make the identification

$$\text{momentum } \hat{p}_i \rightarrow \text{generator of spatial-translations } -i\partial_i$$

It is conventional to use a hat: \hat{O} to denote an operator.

Analogously, the invariance under the action of the generator of time-translations leads us to the conservation of energy. Conse-

quently

$$\text{energy } \hat{E} \rightarrow \text{generator of time-translations } i\partial_0.$$

There is no symmetry corresponding to the "conservation of location" and therefore the location is not identified with a generator. We simply have¹

$$\text{location } \hat{x}_i \rightarrow x_i.$$

The physical quantities we want to use in our theory to describe nature are now given by (differential) operators. The logical next thing to ask is: What do they act on and where is the connection to things we can measure in experiments? We will discuss this in a later chapter in detail. At this point it is sufficient to note that our physical quantities, now operators, need to act on *something*. Here we want to move on with an abstract thing that the operators act on. Let's name it Ψ . We will explore later what this *something* is.

At this point, we are able to derive one of the most important² equations of quantum mechanics. As explained above, we assume that our operators act on something abstract Ψ . Then we have

$$\begin{aligned} [\hat{p}_i, \hat{x}_j]\Psi &= (\hat{p}_i\hat{x}_j - \hat{x}_j\hat{p}_i)\Psi = (-i\partial_i\hat{x}_j + \hat{x}_ji\partial_i)\Psi \\ &\underbrace{=}_{\text{product rule}} -(i\partial_i\hat{x}_j)\Psi - \hat{x}_j(i\partial_i\Psi) + \hat{x}_ji\partial_i\Psi \underbrace{=} -i\delta_{ij}\Psi. \end{aligned} \quad (5.2)$$

because $\partial_i\hat{x}_j = \frac{\partial x_j}{\partial x_i}$

This equation holds for arbitrary Ψ , because we made no assumptions about it and therefore we can write the equation without Ψ ³:

$$[\hat{p}_i, \hat{x}_j] = -i\delta_{ij}. \quad (5.3)$$

5.1.1 Spin and Angular Momentum

In the last chapter (Section 4.5.4) we saw that the conserved quantity that resulted from rotational invariance has two parts. The second part was identified with the orbital angular momentum and we therefore make the identification with the **infinite** dimensional representation of the generator

$$\text{orb. angular mom. } \hat{L}_i \rightarrow \text{gen. of rot. (inf. dim. rep.) } i\frac{1}{2}\epsilon_{ijk}(x^j\partial^k - x^k\partial^j)$$

Analogously we identify the first part, called spin, with the corresponding **finite** dimensional representation of the generators⁴

$$\text{spin } \hat{S}_i \rightarrow \text{generators of rotations (fin. dim. rep.) } S_i.$$

¹ Alternatively, we can see this by looking at the conserved quantity corresponding to invariance under boosts. Take note that we derived in Section 4.6 the conserved quantity corresponding to a non-relativistic Galilei boost, because we started with a non-relativistic Lagrangian. Nevertheless, we can do the same for the relativistic Lagrangian and end up with the conserved quantity $t\vec{p} - \vec{x}E$. The relativistic energy is given by $E = \sqrt{m^2 + p^2}$. In the non-relativistic limit $c \rightarrow \infty$ we get $E \approx m$ and therefore recover the formula we derived for a Galilei boost from the Lorentz boost conserved quantity. The conserved quantity for a particle theory is then $M_i = (tp_i - x_iE)$. The generator of boosts is (see Eq. 3.248 with $K_i = M_{0i}$) $K_i = i(x_0\partial_i - x_i\partial_0)$. Comparing the two equations, with of course $x_0 = t$, yields $M_i = (tp_i - x_iE) \leftrightarrow K_i = i(t\partial_i - x_i\partial_0)$. The identification is now, with the identifications we made earlier, straightforward. Location $\hat{x}_i \rightarrow x_i$.

² If you don't know anything about quantum mechanics, it may seem strange to you why this little equation is so important, but maybe you have heard of the Heisenberg uncertainty principle. In Section 8.3 we will take a closer look at the formalism of quantum mechanics. Then we will be able to see that this equation tells us that we aren't able to measure the momentum and the location of a particle with arbitrary precision. Our physical quantities are interpreted as measurement operators and this equation tells us that a measurement of location followed by a measurement of momentum is not the same as a measurement of momentum followed by a measurement of location.

³ If this is a new idea to you, take note that we could rewrite **every** vector equation as an equation that involves only numbers. For example Newton's second law: $\vec{F} = m\ddot{\vec{x}}$, could be written as $\vec{F}\vec{C} = m\ddot{\vec{x}}\vec{C}$, which is certainly true for any vector \vec{C} . Nevertheless, if the equation is true for any \vec{C} , writing it all the time makes little sense.

⁴ Recall that this was the part of the conserved quantity that resulted from the invariance under mixing of the field components. Hence the finite-dimensional representation.

As explained in the text below Eq. 4.30 the relation between $S_{\mu\nu}$ and the generator of rotations S_i is $S_i = \frac{1}{2}\epsilon_{ijk}S_{jk}$.

For example, when we consider a spin $\frac{1}{2}$ field, we have to use the two-dimensional representation which we derived in Section 3.7.5:

$$\hat{S}_i = i\frac{\sigma_i}{2}, \quad (5.4)$$

where once more σ_i denotes the Pauli matrices. We will return to this very interesting and very strange type of angular momentum in Section 8.5.5, after we learned how to work with the operators that we derive in this chapter. It is important to keep in mind that only the **sum** of spin and orbital angular momentum is conserved.

5.2 The Operators of Quantum Field Theory

The central objects in a field theory are, of course, fields. A field is a **function of the location in space and time**⁵ $\Phi = \Phi(x)$. Later we want to describe interactions at points in spacetime and therefore work with the densities of our dynamical variables $\pi = \pi(x)$ and not the total quantities that we get by integrating the densities over all space $\Pi = \int dx^3 \pi(x) \neq \Pi(x)$.

We discovered in the last chapter for invariance under displacements of the field itself $\Phi \rightarrow \Phi - i\epsilon$ a new conserved quantity, called conjugate momentum Π . Analogous to the identifications we made in the last section, we identify the conjugate momentum density with the corresponding generator (Eq. 4.60)

conj. mom. density $\pi(x) \rightarrow$ gen. of displ. of the field itself : $-i\frac{\partial}{\partial\Phi(x)}$.

We could now go on, like we did for a particle theory, and identify momentum, angular momentum and energy with the corresponding generators. However, we will see later that quantum field theory works a little differently and the identification we make here will prove to be sufficient.

For the same reasons discussed in the last section, we need something our operators act on. Thus again, we work again with an abstract Ψ , that we will specify in a later chapter. We are then again able to derive an incredibly important equation, this time of quantum field theory⁶

$$[\Phi(x), \pi(y)]\Psi = \left[\Phi(x), -i\frac{\partial}{\partial\Phi(y)} \right] \Psi$$

⁵ Here $x = x_0, x_1, x_2, x_3$ includes time $x_0 = t$.

⁶ For the last step we use the analogue to $\frac{\partial x_i}{\partial x_j} = \delta_{ij}$ for the delta distribution $\frac{\partial f(x_i)}{\partial f(x_j)} = \delta(x_i - x_j)$, which can be shown in a rigorous way. For some more information have a look at Appendix D.2.

$$\underbrace{= -i\Phi(x)}_{\text{product rule}} \frac{\partial \Psi}{\partial \Phi(y)} + i\Phi(x) \left(\frac{\partial \Psi}{\partial \Phi(y)} \right) + i \left(\frac{\partial \Phi(x)}{\partial \Phi(y)} \right) \Psi = i\delta(x-y)\Psi. \quad (5.5)$$

Again, the equations hold for arbitrary Ψ and we can therefore write

$$[\Phi(x), \pi(y)] = i\delta(x-y). \quad (5.6)$$

Analogously we have for more than one field component

$$[\Phi_i(x), \pi_j(y)] = i\delta(x-y)\delta_{ij}. \quad (5.7)$$

As we will see later, almost everything in quantum field theory follows from this little equation.