9

Quantum Field Theory

Summary

In this chapter the framework of quantum field theory is introduced. Starting with the equation derived in Chapter 5

$$[\Phi(x), \pi(y)] = i\delta(x - y),$$

we are able to see that the fields themselves are operators. The solutions of the equations of motion for spin 0, $\frac{1}{2}$ and 1 are written in terms of their **Fourier expansions**[1]. Using the commutation relation, cited above, we discover that the Fourier coefficients are now operators. Afterwards, we will see how these **operators**, and with them of course the fields, **create and annihilate particles**. Using the Lagrangian for the corresponding fields, we are able to derive the Hamiltonian operator representing energy.

Afterwards, we start developing **interaction theory**, which is what quantum field theory is all about. We will see that in interaction theory, the Hamiltonian is given by a linear combination of the Hamiltonian for the free field plus an **interaction Hamiltonian**. This insight can then be used in the **interaction picture**, where the time evolution of the fields is governed by the free Hamiltonian and the time evolution of the states by the interaction Hamiltonian. Using this picture, we are able to derive the probability amplitudes for scattering processes. These are denoted in the Dirac notation

$$\langle f | \hat{S} | i \rangle \, ,$$

where $\hat{S}$ denotes the operator describing the scattering process, $|i\rangle$ is the initial state and $\langle f |$ is the final state. We will discover that the operator $\hat{S}$ can be written in terms of the interaction Hamiltonian H_I:

$$\hat{S}(t, t_i) = e^{-i \int_{t_i}^{t} dt' H_I}.$$

[1] The idea behind the Fourier transform is explained in Appendix D.1.

This can't be solved and therefore we evaluate the exponential in terms of its series expansions. For most experiments the first few terms suffice to get an accurate description.

Each term in this series expansion can be interpreted physically as describing a different kind of scattering process. The interaction Hamiltonian contains linear combinations of the fields, which create and annihilate particles as mentioned above. For the first non-trivial order, we get 8 terms and we will see that the first term describes a scattering process of the form $e^- e^+ \rightarrow \gamma$. This means we start with an initial state $|e^- e^+\rangle$ consisting of an electron and a positron, which are annihilated by the field operators of the spin $\frac{1}{2}$ fields and afterwards a photon $\langle \gamma |$ is created by the photon field. The other terms result in zero when operating on this initial state $|e^- e^+\rangle$.

The next order in the series expansion consists of many, many terms and we will take a look at just one of them. Again, we start with an initial state $|e^- e^+\rangle$ and we will see that **one** term describes the process $e^- e^+ \rightarrow \gamma \rightarrow e^- e^+$, where the initial and final electron and positron have, in general, completely different momenta.

In exactly the same way, all terms can be interpreted for all interaction Hamiltonians. A pictorial way to simplify these kinds of computations are the famous Feynman diagrams. Each line and vertex in such a diagram represents a factor of the kind mentioned above.

9.1   Field Theory Identifications

In this section, we want to understand how the Lagrangians that we derived from symmetry constraints can be used in a field theoretical framework. The first step in deriving a field theory describing nature is combining the Lagrangians we found with the result from Chapter 5, specifically Eq. 5.5, which we recite here for convenience:

$$[\Phi(x), \pi(y)] = i\delta(x - y), \tag{9.1}$$

where the conjugate momentum density $\pi(y)$ is given by

$$\pi(y) = \frac{\partial \mathscr{L}}{\partial(\partial_0 \Phi(y))}. \tag{9.2}$$

9.2 Free Spin 0 Field Theory

"Every act of creation is first of all an act of destruction."

- Pablo Picasso[2]

Again, let's start with the simplest possible case: free spin 0 fields. We describe such fields with scalars, which are objects that do not change at all under Lorentz transformations[3]. We already derived in Chapter 6.2 the corresponding Lagrangian

[3] This was derived in Section 3.7.4

$$\mathscr{L} = \frac{1}{2}(\partial_\mu \Phi \partial^\mu \Phi - m^2 \Phi^2) \tag{9.3}$$

and the equation of motion, called Klein-Gordon equation

$$(\partial_\mu \partial^\mu + m^2)\Phi = 0. \tag{9.4}$$

Using the Lagrangian in Eq. 9.3, we can calculate the conjugate momentum

$$\pi(x) = \frac{\partial \mathscr{L}}{\partial(\partial_0 \Phi(x))} = \frac{\partial}{\partial(\partial_0 \Phi(x))} \frac{1}{2}(\partial_\mu \Phi(x)\partial^\mu \Phi(x) - m^2 \Phi^2(x)) = \partial_0 \Phi(x).$$

The most general solution of the Klein-Gordon equation can be written in terms of its Fourier-expansion[4]

$$\Phi(x) = \int dk^3 \frac{1}{(2\pi)^3 2\omega_k} \left(a(k)e^{-i(kx)} + b(k)e^{i(kx)} \right), \tag{9.5}$$

with $(\omega_k)^2 \equiv \vec{k}^2 + m^2$. For **real** scalar fields we can rewrite this as

$$\Phi(x) = \int dk^3 \frac{1}{(2\pi)^3 2\omega_k} \left(a(k)e^{-i(kx)} + a^\dagger(k)e^{i(kx)} \right) \tag{9.6}$$

because $c + c^\dagger = \underbrace{\text{Re}(c) + i \cdot \text{Im}(c)}_{c} + \underbrace{\text{Re}(c) - i \cdot \text{Im}(c)}_{c^\dagger} = 2\text{Re}(c)$.

Now we are having a look at the implications of Eq. 9.1, i.e. the non vanishing commutator $[\Phi(x), \pi(y)] \neq 0$. This means that $\Phi(x)$ and $\pi(y)$ cannot be ordinary functions, but must be **operators**, because ordinary functions commute: $(3+x)(7xy) = (7xy)(3+x)$. By looking at Eq. 9.6, we conclude that the Fourier-coefficients $a(k)$ and $a(k)^\dagger$ are operators, because $e^{\pm i(kx)}$ is just a complex number and complex numbers commute.

Using Eq. 9.1 we can compute[5]

$$[a(k), a^\dagger(k')] = (2\pi)^3 \delta^3(\vec{k} - \vec{k}') \tag{9.7}$$

and

$$[a(k), a(k')] = 0 \tag{9.8}$$

$$[a^\dagger(k), a^\dagger(k')] = 0.\tag{9.9}$$

Now that we know that our field itself is an operator, the logical next thing to ask is: **What does it operate on?** In a particle theory, we identify the dynamical variables as operators acting on something describing a particle (the wavefunction, an abstract Dirac vector, etc.). In a field theory we have up to now nothing to describe a particle. At this point, it is completely unclear how particles appear in a field theory. Nevertheless, let's have a look at how our field coefficents $a(k)$ and $a^\dagger(k)$ act on something abstract and by doing this, we learn how the fields act on something abstract. To get some intuition about what is going on here, let's first have a look at something we are familiar with: energy.

The energy E of a scalar field is given by[6]

$$E = \int d^3x\, T^{00}$$

$$= \int d^3x \left(\frac{\partial \mathscr{L}}{\partial(\partial_0 \Phi)} \underbrace{\frac{\partial \Phi}{\partial x_0}}_{=\partial_0 \Phi} - \mathscr{L} \right)$$

$$= \int d^3x \left((\partial_0 \Phi)^2 - \frac{1}{2}(\partial_\mu \Phi \partial^\mu \Phi - m^2 \Phi^2) \right)$$

$$\underbrace{=}_{\partial_\mu \partial^\mu = \partial_0 \partial_0 - \partial_i \partial_i} \frac{1}{2} \int d^3x \left((\partial_0 \Phi)^2 + (\partial_i \Phi)^2 + m^2 \Phi^2 \right) \tag{9.10}$$

By substituting Eq. 9.6 into Eq. 9.10 and using the commutation relations (Eq. 9.7-9.9), we can write

$$E = \frac{1}{2} \int dk^3 \frac{1}{(2\pi)^3} \omega_k \left(a^\dagger(k)a(k) + a(k)a^\dagger(k) \right)$$

$$\underbrace{=}_{\text{Eq. 9.7}} \int dk^3 \frac{1}{(2\pi)^3} \omega_k \left(a^\dagger(k)a(k) + \frac{1}{2}(2\pi)^3 \delta^3(0) \right) \tag{9.11}$$

At this point we can see that our theory explodes. The second term in the integral is infinite. We could stop at this point and say that this kind of theory does not work. Nevertheless, some brave physicists dug deeper, ignoring this infinite term and discovered a theory describing nature very accurately. There is no explanation for this and the standard way of continuing from here on is to ignore the second term. The crux here is that this term appears in the energy of every system and we are only able to measure energy differences. Therefore, this constant infinite term appears in none of our measurements.

Conventionally, the energy written as an operator is called Hamiltonian $\hat{H}$. We can compute the commutator[7] of $\hat{H}$ and the Fourier coefficents $a(k)$ and $a^\dagger(k)$. We get

$$
\begin{aligned}
[\hat{H}, a^\dagger(k')] &= \int \mathrm{d}k^3 \frac{1}{(2\pi)^3} \omega_k [a^\dagger(k)a(k), a^\dagger(k')] \\
&\underbrace{=}_{[a^\dagger(k), a^\dagger(k')]=0} \int \mathrm{d}k^3 \frac{1}{(2\pi)^3} \omega_k a^\dagger(k)[a(k), a^\dagger(k')] \\
&= \int \mathrm{d}k^3 \omega_k a^\dagger(k) \delta^3(k-k') \\
&\underbrace{=}_{\text{See Appendix D.2}} \omega_{k'} a^\dagger(k')
\end{aligned}
$$

$$(9.12)$$

and equally

$$[\hat{H}, a(k')] = -\omega_{k'} a(k') . \tag{9.13}$$

The quantum formalism works by operating with operators on something that describes the physical system[8]. Here, if we act with the energy operator, i.e. the Hamiltonian $\hat{H}$, on something abstract $|?\rangle$ describing our physical system, we get the energy of the system:

$$\hat{H} |?\rangle = E |?\rangle \tag{9.14}$$

Now we return to our starting question: **How does a field[9] act on our system?** Let's have a look at the effect of the first Fourier coefficient, now an operator, on the energy E of the system[10]:

$$
\begin{aligned}
\hat{H}\left(a(k')|?\rangle\right) &= \left(a(k')\hat{H} + \underbrace{\hat{H}a(k') - a(k')\hat{H}}_{[\hat{H}, a(k')]}\right)|?\rangle \\
&= a(k') \underbrace{\hat{H}|?\rangle}_{=E|?\rangle} + [\hat{H}, a(k')]|?\rangle \\
&= \left(a(k')E + [\hat{H}, a(k')]\right)|?\rangle \\
&\underbrace{=}_{\text{Eq. 9.13}} \left(a(k')E - \omega_{k'}a(k')\right)|?\rangle \\
&= \left(E - \omega_{k'}\right)\left(a(k')|?\rangle\right)
\end{aligned}
$$

$$(9.15)$$

and equally for the second Fourier coefficent

$$\hat{H}a^\dagger(k')|?\rangle = (E + \omega_{k'})a^\dagger(k')|?\rangle . \tag{9.16}$$

How can we interpret this? We see that $a(k')|?\rangle$ can be interpreted as a new system with energy $E - \omega_k$. To make this more concrete we define

$$|?_2\rangle \equiv a(k')|?\rangle$$

[7] It will become clear in a moment why this is useful.

[8] This was explained in Section 8.3.

[9] Remember: Field=Operator!

[10] We will do something very clever here, which was first discovered by Dirac while solving the problem of the harmonic oscillator in quantum mechanics.

with

$$\hat{H} \ket{?_2} \underbrace{=}_{\text{Using Eq. 9.15}} (E - \omega_{k'}) \ket{?_2} .$$

This suggests how we should interpret what the field does. Imagine a completely empty system $\ket{0}$, with, by definition $H \ket{0} = 0 \ket{0}$. If we now act with $a^\dagger(k')$ on $\ket{0}$, we know that this transforms our empty system into a system having energy $\omega_{k'}$

$$\hat{H} a^\dagger(k') \ket{0} \underbrace{=}_{\text{Using Eq. 9.16}} \omega_{k'} a^\dagger(k') \ket{0} . \tag{9.17}$$

We see that $a^\dagger(k')$ creates something in the completely empty system with energy $\omega_{k'}$, **which is what we call a particle with momentum k'**! If we act a second time on this system with $a^\dagger(k')$ we create a second particle with the same momentum. If we act on it with $a^\dagger(k'')$ we create a particle with momentum k'' and so on. Therefore, we call $a^\dagger(k')$ a **creation operator**. Similarly to $a^\dagger(k')$, we can interpret $a(k')$: $a(k')$ destroys or annihilates a particle of energy $\omega_{k'}$ and is therefore called **annihilation** or **destruction operator**. To make this more concrete we introduce a new notation for particle states

$$a^\dagger(k) \ket{0} \equiv \ket{1_k} \tag{9.18}$$

$$a^\dagger(k) \ket{1_k} \equiv \ket{2_k} \tag{9.19}$$

$$a^\dagger(k') \ket{2_k} \equiv \ket{2_k, 1_{k'}} \tag{9.20}$$

Take a look at the energy again:

$$E = \int dk^3 \frac{1}{(2\pi)^3} \omega_k a^\dagger(k) a(k) .$$

What happens if this operator acts on a state like $\ket{2k_1, k_2}$? The result should be

$$E = 2\omega_{k_1} + \omega_{k_2},$$

which is the energy of two particles with energy ω_{k_1} and one particle with energy ω_{k_2}. Therefore, the operator

$$N(k) \equiv a^\dagger(k) a(k) \tag{9.21}$$

appearing here is a number operator, denoted $N(k)$ that extracts the number of particles with momentum k from a state:

$$N(k) \ket{n_k, n'_{k'}, \ldots} = n_k \ket{n_k, n'_{k'}, \ldots} \tag{9.22}$$

The energy operator can then be written as

$$E = \int dk^3 \frac{1}{(2\pi)^3} \omega_k N(k).$$

Furthermore, take note that there are physical systems, where the momentum spectrum is not continuous, but discrete[11]. For such systems all integrals change to sums, for example, the energy is then of the form

$$E = \sum_k \omega_k N(k)$$

and the commutation relation changes to

$$[a(k), a^\dagger(k')] = \delta_{k,k'}. \tag{9.23}$$

Take note that quantum field theory is, like quantum mechanics, a theory making probabilistic predictions. Therefore, our states need to be normalized $\langle k, k', .. | k, k', .. \rangle \overset{!}{=} 1$, because a probability of more than $100\% = 1$ doesn't make sense. If we act with an operator like $a(k)$ on a ket, the new ket does not necessarily have unit norm[12]. Therefore, we write

$$a^\dagger(k) |n_k\rangle = C |n_k + 1\rangle, \tag{9.24}$$

where n_k denotes the number of particles with momentum k and C is some number. From this we get[13]

$$(a^\dagger(k) |n_k\rangle)^\dagger = (C |n_k + 1\rangle)^\dagger$$

$$\rightarrow \langle n_k| a(k) = \langle n_k + 1| C^\dagger. \tag{9.25}$$

We can therefore write

$$\underbrace{\langle n_k| a(k)}_{9.25} \underbrace{a^\dagger(k) |n_k\rangle}_{9.24} = \langle n_k + 1| \underbrace{C^\dagger C}_{\text{a number and no operator}} |n_k + 1\rangle = C^\dagger C \underbrace{\langle n_k + 1|n_k + 1\rangle}_{=1} \tag{9.26}$$

or using the discrete commutation relation (Eq. 9.23)

$$\langle n_k| a(k) a^\dagger(k) |n_k\rangle = \langle n_k| \left(\underbrace{a^\dagger(k) a(k)}_{=N(k)\ \text{Eq. 9.21}} + \underbrace{\delta_{k,k}}_{=1} \right) |n_k\rangle$$

$$\overset{\text{Eq. 9.22}}{=} \langle n_k| \underbrace{(n_k + 1)}_{\text{a number and no operator}} |n_k\rangle = (n_k + 1) \underbrace{\langle n_k|n_k\rangle}_{=1}. \tag{9.27}$$

Putting Eq. 9.26 and Eq. 9.27 together yields

$$C^\dagger C = n_k + 1 \rightarrow C = \sqrt{n_k + 1} \tag{9.28}$$

and we therefore have

$$a^\dagger(k) |n_k\rangle = \sqrt{n_k + 1} |n_k + 1\rangle. \tag{9.29}$$

Following the same steps we can derive

$$a(k) |n_k\rangle = \sqrt{n_k} |n_k - 1\rangle. \tag{9.30}$$

Two questions may pop up at this point.

[11] Remember the particle in a box example. It is an often used trick in quantum field theory to assume the system in question is restricted to a volume V. This results in a discrete momentum spectrum. At the end of the computation the limit $\lim_{V \to \infty}$ is taken.

[12] Exactly the same line of though was discussed in Section 3.6.1, where we derived the finite-dimensional $\mathfrak{su}(2)$ representations. There, we also introduced "ladder operators" $J_\pm$ and derived the constants that we emerge when we act with these operators on a given vector.

[13] Recall that $|n_k\rangle^\dagger = \langle n_k|$ and we have of course $(a^\dagger)^\dagger = a$.

1. What happens if we want to annihilate a particle in a completely empty system?

2. What about energy or charge conservation? How can we create something from nothing without violating conservation laws?

The conservation laws are, of course, never violated, but how this comes about will only become clear when we develop the theory further. Maybe it helps to see that at this point Richard Feynman had the same problem[14]

> I remember that when someone had started to teach me about creation and annihilation operators, that this operator creates an electron, I said, "how do you create an electron? It disagrees with the conservation of charge", and in that way, I blocked my mind from learning a very practical scheme of calculation.

Secondly, we are never able to destroy something which is not there in the first place. If we act with the destruction operator $a(k)$ on a completely empty state $|0\rangle$ we get, using Eq. 9.30

$$a(k)|0_k\rangle = \sqrt{0}\,|0_k - 1_k\rangle = 0 \tag{9.31}$$

or equally

$$a(k')|1_k\rangle = \sqrt{0}\,|1_k, 0_{k'} - 1k'\rangle = 0. \tag{9.32}$$

We can see that if we act with an annihilation operator $a(k')$ on a ket, like $|k\rangle$ that does not include a particle with this momentum k', the theory produces a zero. The creation and annihilation operators appear in the Fourier expansion of the fields, which includes an integral (or sum) over all possible momenta. Therefore, if these fields act on a ket like $|k\rangle$, only one annihilation operator will result in something non-zero. This will be of great importance when we try to describe interactions using quantum field theory.

Before we move on to interactions, we take a brief look at free spin $\frac{1}{2}$ and spin 1 fields.

9.3 Free Spin $\frac{1}{2}$ Field Theory

[15] We derived the Dirac equation in Section 6.3.

The equation of motion for free spin $\frac{1}{2}$ fields is the Dirac equation[15]

$$(i\gamma_\mu \partial^\mu - m)\Psi = 0.$$

[16] The Dirac equation is solved in Section 8.9. The general solution is then written analogous to the solution of the Klein-Gordon equation, discussed in the last section.

The general solution of the Dirac equation can be written in the form[16]

$$\Psi = \sum_r \sqrt{\frac{m}{(2\pi)^3}} \int \frac{d^3p}{\sqrt{w_p}} \left(c_r(p)u_r(p)e^{-ipx} + d_r(p)v_r(p)e^{+ipx} \right)$$
$$= \Psi^+ + \Psi^- , \tag{9.33}$$

but this time we do not restrict to real fields, because we saw in Section 6.3 that a Lorentz invariant Lagrangian needs complex spin $\frac{1}{2}$ fields. In addition, we follow the standard convention and write the solutions as

$$\Psi = \sum_r \sqrt{\frac{m}{(2\pi)^3}} \int \frac{d^3p}{\sqrt{w_p}} \left(c_r(p)u_r(p)e^{-ipx} + d_r^\dagger(p)v_r(p)e^{+ipx} \right) , \tag{9.34}$$

because in this way, $d_r^\dagger(p)$ can be seen to create an anti-particle. If we would name it $d_r(p)$ in the solution, this could lead to confusion, because for particles $c_r^\dagger(p)$ creates and $c_r(p)$ annihilates. Naming the Fourier coefficient $d_r^\dagger(p)$ instead of $d_r(p)$, leads to an analogous interpretation for anti-particles: $d_r^\dagger(p)$ creates and $d_r(p)$ annihilates.

Analogously, we have for the adjoint Dirac equation

$$(i\partial^\mu \bar\Psi \gamma_\mu + m\bar\Psi) = 0,$$

which was also derived in Section 6.3, the solution

$$\bar\Psi = \sum_r \sqrt{\frac{m}{(2\pi)^3}} \int \frac{d^3p}{\sqrt{w_p}} \left(c_r^\dagger(p)\bar u_r(p)e^{+ipx} + d_r(p)\bar v_r(p)e^{-ipx} \right)$$
$$\equiv \bar\Psi^+ + \bar\Psi^- . \tag{9.35}$$

In these solutions u_1, u_2, v_1, v_3 denote the "basis spinors" in an arbitrary frame[17]

[17] We derived these spinors in the rest frame ($p_i = 0$) in Section 8.10. The basis spinors in an arbitrary frame can be computed from the basis spinors in the rest frame by a boost transformation.

$$u_1 = \sqrt{\frac{E+m}{2m}} \begin{pmatrix} 1 \\ 0 \\ \frac{p_3}{E+m} \\ \frac{p_1+ip_2}{E+m} \end{pmatrix} \qquad u_2 = \sqrt{\frac{E+m}{2m}} \begin{pmatrix} 0 \\ 1 \\ \frac{p_1-ip_2}{E+m} \\ \frac{-p_3}{E+m} \end{pmatrix} \tag{9.36}$$

$$v_1 = \sqrt{\frac{E+m}{2m}} \begin{pmatrix} \frac{p_1-ip_2}{E+m} \\ \frac{-p_3}{E+m} \\ 0 \\ 1 \end{pmatrix} \qquad v_2 = \sqrt{\frac{E+m}{2m}} \begin{pmatrix} \frac{p_3}{E+m} \\ \frac{p_1+ip_2}{E+m} \\ 1 \\ 0 \end{pmatrix} . \tag{9.37}$$

The rest of the theory for free spin $\frac{1}{2}$ fields can be developed similarly to the scalar theory, but there is one small difference[18]. Let's try

[18] With incredible huge consequences! In fact, nothing in this universe would be stable if the spin $\frac{1}{2}$ theory would work exactly like the scalar theory.

what we get when we use the formula

$$[\Psi(x), \pi(y)] = i\delta(x - y) \tag{9.38}$$

for spin $\frac{1}{2}$ fields, too. In the general solution for the equation of motion, i.e. the Dirac equation, we have two different coefficients: $c^\dagger$ creates particles, whereas $d^\dagger$ creates anti-particles. When we now compute the Hamiltonian for a spin $\frac{1}{2}$ field, using the *commutation* relation[19], we get something of the form

$$H \sim \int c^\dagger c - d^\dagger d \, .$$

This shows that the energy of anti-particles is then *negative*. This is a serious problem because every state could forever decay to lower energy states and nothing in this universe would be stable. The Hamiltonian is not bounded from below, in contrast to the scalar case.

However, when we use instead the *anticommutation* relation

$$\{\Psi(x), \pi(y)\} = \Psi(x)\pi(y) + \pi(y)\Psi(x) = -i\delta(x - y),$$

the term involving the anti-particle creation and annihilation operators comes out positive and the problem is therefore solved.

This has very interesting consequences. For example, we then have[20]

$$\{c^\dagger(p), c^\dagger(p')\} = 0$$

from which we can conclude

$$\{c^\dagger(p), c^\dagger(p)\} = c^\dagger(p)c^\dagger(p) + c^\dagger(p)c^\dagger(p) = 2c^\dagger(p)c^\dagger(p) = 0.$$

$$\Rightarrow c^\dagger(p)c^\dagger(p) = 0 \tag{9.39}$$

The action of two equal creation operators[21] always results in a zero! This means we cannot create two equal spin $\frac{1}{2}$ particles and this is famously called **Pauli-exclusion principle**.

The other anti-commutation relations for the Fourier coefficients can be derived from the anti-commutator of the field and the conjugate momentum to be:

$$\{c_r(p), c_s^\dagger(p')\} = \delta_{rs}\delta(p - p') \qquad \{d_r(p), d_s^\dagger(p')\} = \delta_{rs}\delta(p - p') \tag{9.40}$$

and all other possible combinations equal zero. Therefore, these coefficients can be seen to have the same properties as those we derived in the last section for the spin 0 field. They create and destroy, with

[19] $[\Psi(x), \pi(y)]$
$= \Psi(x)\pi(y) - \pi(y)\Psi(x) = i\delta(x - y)$

[20] Analogous to Eq. 9.9 for scalars, but now with the anticommutator instead of the commutator $[,] \rightarrow \{,\}$.

[21] For spin 0 fields the corresponding equation isn't very surprising, because there we use the commutator: $[c^\dagger(p), c^\dagger(p)] = c^\dagger(p)c^\dagger(p) - c^\dagger(p)c^\dagger(p) = 0$.

the difference that acting twice with the same operator on a state results in a zero. The Hamiltonian for a free spin $\frac{1}{2}$ field can be derived analogous to the Hamiltonian for a free spin 0 field

$$H_{\text{free}}^{\frac{1}{2}} = \int d^3x (-i\bar{\Psi}\gamma_i\partial^i\Psi - m\bar{\Psi}\Psi) \tag{9.41}$$

or expressed in terms of the Fourier coefficients

$$\mathscr{H}_{\text{free}}^{\frac{1}{2}} = \sum_r \int d^3p \; w_p \left(c_r^\dagger(p)c_r(p) + d_r^\dagger(p)d_r(p) + \text{const} \right), \tag{9.42}$$

where again the constant term leads to an infinite term and we choose to ignore this.

9.4   Free Spin 1 Field Theory

The solution for the equation of motion for free spin 1 fields, the Proca equation[22]

$$m^2 A^\rho = \frac{1}{2}\partial_\sigma(\partial^\sigma A^\rho - \partial^\rho A^\sigma) \tag{9.43}$$

is, analogous to the spin 0 field solution, of the form

$$A_\mu = \int \frac{d^3k}{\sqrt{(2\pi)^3 2\omega_k}} \left(\epsilon_{r,\mu}(k)a_r(k)e^{-ikx} + \epsilon_{r,\mu}(k)a_r^\dagger(k)e^{ikx} \right)$$
$$\equiv A_\mu^+ + A_\mu^- \tag{9.44}$$

where $\epsilon_{r,\mu}(k)$ are **basis vectors**, called **polarization vectors**. For spin 1 fields we are again able to use the commutator instead of the anti-commutator. We are therefore able to derive that the coefficients $a_r, a_r^\dagger$ have the same properties as the coefficients for spin 0 fields.

9.5   Interacting Field Theory

The next step is to look at interactions between fields of different spin. The corresponding Lagrangians were derived in earlier chapters from Lorentz and gauge symmetry. For example, for the interaction between a massive spin $\frac{1}{2}$ field and one massless spin 1 field, we have the Lagrangian (Eq. 7.17)

$$\mathscr{L}_{\text{Dirac+Extra-Term}} = -m\bar{\Psi}\Psi + i\bar{\Psi}\gamma_\mu\partial^\mu\Psi + gA_\mu\bar{\Psi}\gamma^\mu\Psi \tag{9.45}$$

from which we derive the corresponding Hamiltonian[23]

$$H = \int d^3x \, T^{00}$$

$$= \int d^3x \left(\underbrace{\frac{\partial \mathscr{L}}{\partial(\partial_0 \Psi)}}_{=i\bar{\Psi}\gamma_0} \partial_0 \Psi - \mathscr{L} \right)$$

$$= \int d^3x \left(i\bar{\Psi}\gamma_0 \partial_0 \Psi + m\bar{\Psi}\Psi - i\bar{\Psi} \underbrace{\gamma_\mu \partial^\mu}_{=\gamma_0 \partial_0 - \gamma_i \partial_i} \Psi - g A_\mu \bar{\Psi}\gamma^\mu \Psi \right)$$

$$= \underbrace{\int d^3x (m\bar{\Psi}\Psi + i\bar{\Psi}\gamma_i \partial_i \Psi)}_{=H_{\text{free}}^{\frac{1}{2}}} - \underbrace{\int d^3x \left(g A_\mu \bar{\Psi}\gamma^\mu \Psi \right)}_{\equiv -H_I}$$

$$= H_{\text{free}}^{\frac{1}{2}} + H_I \tag{9.46}$$

9.5.1   Scatter Amplitudes

One of the main goals in quantum field theory is to compute the probability for a fixed number n of particles with defined[24] momenta $p_1, p_2, .., p_n$ to transform into a (possibly different) number of (possibly different) particles n' with momenta $q_1, q_2, .., q_{n'}$. We can write this using Dirac's notation

$$\langle q_1, q_2, .., q_{n'} | \, \hat{S} \, | p_1, p_2, .., p_n \rangle \, , \tag{9.47}$$

where $\hat{S}$ is the operator describing the scattering. We will derive how this operator looks concretely in the next section. Formulated differently, we have some particles at initial time t_i at some points in space x_i and after the interaction, i.e. at the final time t_f, (possibly different) particles at a possibly different location x_f.

Therefore, the first question we have to answer is: What can we say about the time evolution of states in quantum field theory?

9.5.2   Time Evolution of States

To answer this question we observe that the energy operator is given on the one hand by the identification with the generator of time-translations $i\partial_0$ and on the other hand by the Hamiltonian. For example, for the free spin 0 theory we derived (Eq. 9.10)

$$H_{\text{free}}^0 = \frac{1}{2} \int d^3x \left((\partial_0 \Phi)^2 + (\partial_i \Phi)^2 + m\Phi^2 \right) , \tag{9.48}$$

while for the free spin $\frac{1}{2}$ theory we have (Eq. 9.41)

$$H_{\text{free}}^{\frac{1}{2}} = \int d^3x (m\bar{\Psi}\Psi + i\bar{\Psi}\gamma_i \partial^i \Psi). \tag{9.49}$$

Both identifications are operators in a field theory, representing energy, and we therefore write

$$i\partial_0 \left| ?(t) \right\rangle = H \left| ?(t) \right\rangle, \tag{9.50}$$

which is the equation governing the time evolution of a state in quantum field theory. We can use this equation to define a time-evolution operator U that transforms the state from one point in time to another. If we choose, for brevity, the start time $t = 0$ we have[25]

$$\left| ?(t) \right\rangle = U(t) \left| ?(0) \right\rangle. \tag{9.51}$$

Putting this ansatz into Eq. 9.50 yields

$$i\partial_0 U(t) \left| ?(0) \right\rangle = HU(t) \left| ?(0) \right\rangle. \tag{9.52}$$

This equation holds for arbitrary $\left| ?(0) \right\rangle$ and therefore

$$i\partial_0 U(t) = HU(t). \tag{9.53}$$

The general solution of this equation is

$$U(t) = e^{-i \int_0^t dx_0 H} \tag{9.54}$$

as we can check

$$i\partial_0 U(t) = HU(t) \rightarrow i\partial_0 e^{-i \int_0^t dx_0 H} = He^{-i \int_0^t dx_0 H}$$
$$\rightarrow i(-iH)e^{-i \int_0^t dx_0 H} = He^{-i \int_0^t dx_0 H}$$
$$\rightarrow He^{-i \int_0^t dx_0 H} = He^{-i \int_0^t dx_0 H} \quad \checkmark \tag{9.55}$$

In experiments we never measure a ket $\left| ?(t) \right\rangle$, but always the combinations of a bra with a ket. In general, we have objects of the form

$$\left\langle f(t) \right| \hat{O} \left| i(t) \right\rangle, \tag{9.56}$$

with some operator $\hat{O}$, initial state $\left| i(t) \right\rangle$ and final state $\left\langle f(t) \right|$. Here the states evolve in time, as described by the operator U, and the operators are time independent. This somewhat arbitrary choice is called Schrödinger picture and in the following we will talk about different possible choices, i.e. different "pictures".

Writing Eq. 9.56 in terms of the time-evolution operator leads us to[26]

$$\left\langle f(0) \right| U^\dagger(t) \hat{O} U(t) \left| i(0) \right\rangle. \tag{9.57}$$

An important idea is that we can switch our perspective and say the operator $\hat{O}$ evolves in time, according to the rule $U^\dagger(t)\hat{O}U(t)$ and

[25] In general $\left| ?(t) \right\rangle = U(t - t') \left| ?(t') \right\rangle$.

[26] Remember $\left\langle f \right| = \left| f \right\rangle^\dagger$

the bra and ket are time-independent. This new perspective is called **Heisenberg picture**.

There is a similar, very useful trick that is helpful in interaction theory. The Hamiltonian is always the sum of the free Hamiltonian and the interaction Hamiltonian (Eq. 9.46)

$$H = H_{\text{free}} + H_I. \tag{9.58}$$

The trick is to use a mixture of the two perspectives introduced above. We let the states evolve according to H_I and the operators[27] according to H_{free}. This is incredibly useful, because then we can reuse all results we already derived for free fields. This type of perspective is called **interaction picture**. We define a state in the interaction picture

$$|i(t)\rangle_I \equiv U_{\text{free}}^\dagger |i(t)\rangle_S, \tag{9.59}$$

where $U_{\text{free}} = e^{-i\int_0^t dx_0 H_{\text{free}}}$ and the index S stands for **Schrödinger picture**, which is the name for the standard perspective where the states evolve according to the full Hamiltonian and the operators are time-independent.

Putting now Eq. 9.59 into Eq. 9.50 we get[28] the time-evolution equation in the interaction picture:

$$i\partial_0 |i(t)\rangle_S = H |i(t)\rangle_S$$

$$\rightarrow i\partial_0 U_{\text{free}} |i(t)\rangle_I = H U_{\text{free}} |i(t)\rangle_I$$

$$\rightarrow i\partial_0 e^{-i\int_0^t dx_0 H_{\text{free}}} |i(t)\rangle_I = (H_{\text{free}} + H_I) e^{-i\int_0^t dx_0 H_{\text{free}}} |i(t)\rangle_I$$

$$\underbrace{\rightarrow}_{\text{product rule}} H_{\text{free}} \cancel{e^{-i\int_0^t dx_0 H_{\text{free}}} |i(t)\rangle_I} + i e^{-i\int_0^t dx_0 H_{\text{free}}} \partial_0 |i(t)\rangle_I = (\cancel{H_{\text{free}}} + H_I) e^{-i\int_0^t dx_0 H_{\text{free}}} |i(t)\rangle_I$$

$$\rightarrow i e^{-i\int_0^t dx_0 H_{\text{free}}} \partial_0 |i(t)\rangle_I = H_I e^{-i\int_0^t dx_0 H_{\text{free}}} |i(t)\rangle_I$$

$$\rightarrow i\partial_0 |i(t)\rangle_I = \underbrace{e^{i\int_0^t dx_0 H_{\text{free}}} H_I e^{-i\int_0^t dx_0 H_{\text{free}}}}_{} |i(t)\rangle_I$$

$=H_I^{\text{int}}$ = the interaction Hamiltonian in the interaction picture.

$$\rightarrow i\partial_0 |i(t)\rangle_I = H_I^{\text{int}} |i(t)\rangle_I \tag{9.60}$$

We conclude that the time-evolution of the states is now indeed governed by the interaction Hamiltonian H_I^{int}. This little equation will be incredibly important for everything that follows.

Now we are able to return to the starting question: How can we compute scattering processes

$$\langle f(t_f)| \hat{S}(t_f, t_i) |i(t_i)\rangle \quad ? \tag{9.61}$$

---

[27] Recall that in quantum field theory the fields are our operators.

[28] The equation holds in this form for the Schrödinger picture. Therefore, we have to solve $|i(t)\rangle_I \equiv U_{\text{free}}^\dagger |i(t)\rangle_S$ for $|i(t)\rangle_S$. We multiply the equation with U_{free} and use $U_{\text{free}} U_{\text{free}}^\dagger = 1$. This yields $|i(t)\rangle_S = U_{\text{free}} |i(t)\rangle_I$, which we can then put into Eq. 9.50.

The operator $\hat{S}(t_f, t_i)$ transforms the initial state $|i(t_i)\rangle$ at time t_i into the final state $|\Psi(t_f)\rangle$ at time (t_f). In general, the final state is not one specific particle state, but a linear combination of many possible outcomes. If this would be not the case particle physics would be boring. An interaction would always result in one specific outcome. We are more general here and allow arbitrary linear combinations. After we specify the operator $\hat{S}$ we will see that this is actually the case.

Concretely this means that $\hat{S}$ transforms an initial state into a linear combination of final states. To avoid confusion we will call the final time in the following simply t instead of t_f. Then we have[29]

$$\hat{S}(t, t_i)\,|i(t_i)\rangle = |\Psi(t)\rangle = \sum_f \underbrace{S_{fi}}_{\text{complex numbers}} |f(t)\rangle. \qquad (9.62)$$

The multiplication with one specific $\langle f'(t)|$ from the left-hand side terminates all but one term of the sum:

$$\langle f'(t)|\sum_f S_{fi}\,|f(t)\rangle = \sum_f S_{fi}\underbrace{\langle f'(t)|f(t)\rangle}_{=\delta_{ff'}\text{ because basis states are orthogonal}} = \sum_f S_{fi}\delta_{ff'} = S_{f'i}. \qquad (9.63)$$

Therefore, the probability for this process to happen is $|S_{f'i}|^2$.

Now we specify the scatter operator $\hat{S}$. For this purpose we take a look again at the time-evolution equation that we derived above[30]

$$i\partial_t\,|\Psi(t)\rangle_I = H_I\,|\Psi(t)\rangle_I. \qquad (9.64)$$

We can rewrite this in terms of our initial state and the operator $\hat{S}$ using Eq. 9.62

$$\rightarrow i\partial_t\hat{S}(t, t_i)\,|i(t_i)\rangle_I = H_I\hat{S}\,|i(t_i)\rangle_I$$

$$\underbrace{\rightarrow}_{\text{product rule}} i(\partial_t\hat{S}(t, t_i))\,|i(t_i)\rangle_I + i\hat{S}(t, t_i)\underbrace{\partial_t\,|i(t_i)\rangle_I}_{=0\text{ because }|i(t_i)\rangle_I\text{ does not depend on }t} = H_I\hat{S}\,|i(t_i)\rangle_I.$$

Now using that this equation holds for arbitrary initial states we can write

$$i\partial_t\hat{S}(t, t_i) = H_I\hat{S} \qquad (9.65)$$

with the general solution[31]

$$\hat{S}(t, t_i) = e^{-i\int_{t_i}^t dt'\, H_I} \qquad (9.66)$$

At a first glance the problem is easy now. We know H_I and "just have to solve the integral". This gives the operator $\hat{S}(t, t_i)$ and by acting with it on our initial states we get the probability amplitudes for different processes. Unfortunately, the integral cannot be solved

and we therefore have to use an approximation method. To simplify the problem mathematically, the initial time is taken to be $t_i = -\infty$ and the final time $t = \infty$. By doing this we avoid our probability amplitudes depending on time. For example, if we scatter particles, the probability that a certain process has happened 10^{-24} seconds after the assumed "collision" is different than $2 \cdot 10^{-24}$ seconds after the "collision", because the interaction is still happening! By taking the time values to be very large we avoid these kind of complications. What we measure is usually always a result after all the interactions have happened.

9.5.3   Dyson Series

[32] This is derived in Appendix B.4.1:
$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + ...$

Because there is no analytic solution for the exponentiated integral, we expand it in a Taylor series[32]

$$\hat{S}(\infty, -\infty) = e^{-i \int_{-\infty}^{\infty} dt' H_I(t')}$$

$$= 1 - i \int_{-\infty}^{\infty} dt_1 H_I(t_1)$$

$$- \frac{1}{2!} \left(\int_{-\infty}^{\infty} dt_1 H_I(t_1) \right) \left(\int_{-\infty}^{\infty} dt_2 H_I(t_2) \right) + ... \quad (9.67)$$

This is called the **Dyson series**. We need to take a careful look at the third term. $H_I(t_1)$ and $H_I(t_2)$ are not just numeric values, but operators acting on a ket to their right. Therefore, we need to make sure that the earlier time operator acts on the ket before the later time operator. The operators need to operate on the ket in a time ordered manner. It makes no sense if we act on a state with $H_I(t = 5 \text{ s})$ and after that with $H_I(t = 2 \text{ s})$. In the series above, we need for $t_1 < t_2$

$$\left(\int_{-\infty}^{\infty} dt_2 H_I(t_2) \right) \left(\int_{-\infty}^{\infty} dt_1 H_I(t_1) \right)$$

and for $t_2 < t_1$

$$\left(\int_{-\infty}^{\infty} dt_1 H_I(t_1) \right) \left(\int_{-\infty}^{\infty} dt_2 H_I(t_2) \right)$$

For this purpose an abstract time-ordering operator T is introduced, which is defined by

$$T\{A(x)B(y)\} := \begin{array}{ll} A(t_1)B(t_2) & \text{if} \quad t_1 > t_2 \\ B(t_2)A(t_1) & \text{if} \quad t_1 < t_2. \end{array} \quad (9.68)$$

Therefore we write, giving a name to each term of the series expansion,

$$\hat{S}(\infty, -\infty) = \underbrace{1}_{S^{(0)}} \underbrace{-i \int_{-\infty}^{\infty} dt_1 H_I(t_1)}_{S^{(1)}}$$

$$\underbrace{-\frac{1}{2!} T\left\{ \left(\int_{-\infty}^{\infty} dt_1 H_I(t_1) \right) \left(\int_{-\infty}^{\infty} dt_2 H_I(t_2) \right) \right\}}_{S^{(2)}} + \dots \quad (9.69)$$

or written as a sum

$$\hat{S}(\infty, -\infty) = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} T\left\{ \left(\int_{-\infty}^{\infty} dt_1 H_I(t_1) \right) \left(\int_{-\infty}^{\infty} dt_2 H_I(t_2) \right) \dots \left(\int_{-\infty}^{\infty} dt_n H_I(t_n) \right) \right\}$$

$$= \sum_{n=0}^{\infty} S^{(n)}. \quad (9.70)$$

This is useful because H_I has a numerical factor[33] in it, the coupling constant of the corresponding interaction, i.e. $H_I \propto g$. This coupling constant is, for example, for electromagnetic interactions, smaller than one. Therefore, the second term in the expansion $S^{(2)} \propto (H_I)^2 \propto g^2$ contributes less than the first term $S^{(1)} \propto g$. The higher order terms in the expansion contribute even less: $g > g^2 > g^3 > \dots$. To describe the system in question it often suffices to evaluate the first few terms of the series expansion. Higher order terms often deliver corrections that lie outside the possibility of measurement.

Unfortunately, going further from this point needs many pages of heavy algebra. The first step is Wick's-Theorem which enables one to express the time ordering in terms of commutators. Furthermore, these commutators need to be computed, which results in the famous Feynman propagators. Nevertheless, we want to go further[34], so we are going to use these results without proofs. The interested reader is referred to the standard texts on quantum field theory[35].

[33] See Eq. 9.46.

[34] Because now, the fun is about to begin!

[35] Some recommended books will be listed in the last section of this chapter.

9.5.4 Evaluating the Series

We will now return to the example introduced at the beginning of this chapter: The interaction between a massive spin $\frac{1}{2}$ field and a massless spin 0 field. The corresponding interaction Hamiltonian is (Eq. 9.46)

$$H_I = -\int d^3x g A_\mu \bar{\Psi} \gamma^\mu \Psi.$$

As explained above, we will look at each term of the series in Eq. 9.70 individually.

The first term of the series is trivial as it is simply the identity operator

$$S^{(0)} = I \tag{9.71}$$

The second term is more exciting:

$$S^{(1)} = -i \int_{-\infty}^{\infty} dt H_I = ig \int_{-\infty}^{\infty} d^4x A_\mu \bar{\Psi} \gamma^\mu \Psi \tag{9.72}$$

which we rewrite, recalling Eq. 9.33 and Eq. 9.44, as

$$S^{(1)} = ig \int_{-\infty}^{\infty} d^4x (A_\mu^+ + A_\mu^-)(\bar{\Psi}^+ + \bar{\Psi}^-)\gamma^\mu(\Psi^+ + \Psi^-). \tag{9.73}$$

We can see this second term is actually 8 terms. Let us have a look at how one of these terms, we call it $S_1^{(1)}$, acts on a state consisting of, for example, one electron and one positron with prepared momenta $|e^+(p_1), e^-(p_2)\rangle$

$$ig \int_{-\infty}^{\infty} d^4x A_\mu^+ \bar{\Psi}^+ \gamma^\mu \Psi^+ |e^+(p_1), e^-(p_2)\rangle.$$

Ψ^+ consists of destruction[36] operators for particles for all[37] possible momenta, multiplied with constants and a term of the form e^{-ipx_1}:

$$\Psi^+ \propto \int d^3p \, c_r(p) e^{-ipx}.$$

For each momentum this destroys the ket, which means we get a zero, because we are trying to destroy something which is not there, except for $c_r(p) = c_r(p_2)$. Therefore, operating with Ψ^+ results in

$$\Psi^+ |e^+(p_1), e^-(p_2)\rangle \propto e^{-ip_2x} |e^+, 0\rangle$$

In the same way, operating with $\bar{\Psi}^+$ on the ket results in

$$\bar{\Psi}^+ e^{-ip_2x} |e^+(p_1), 0\rangle \propto e^{-ip_2x} e^{-ip_1x} |0, 0\rangle.$$

Therefore, we are left with the pure vacuum state, multiplied with lots of constants. The last term operating on the ket is A_μ^+, which creates photons of all momenta.

Qualitatively we have, when we want the contribution of this one term to the probability amplitude for the creation of a photon $\langle \gamma |$ with momentum k'

$$\langle f | S_1^{(1)} | i \rangle = \langle \gamma_{k'} | S_1^{(1)} |e^+(p_1), e^-(p_2)\rangle \tag{9.74}$$

$$= \int_{-\infty}^{\infty} d^4x \, \langle \gamma_{k'} | \sum_k \text{constant}(k) |\gamma_k\rangle \, e^{-ix(p_1+p_2-k)}$$

[36] Remember the operators with the † are those who create and the operators without are those who destroy. Ψ^+ is defined in Eq. 9.33.

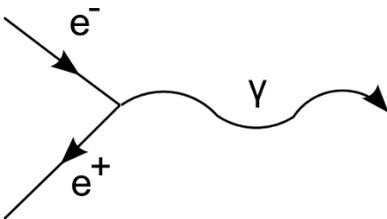[37] We integrate over all possible momenta!

Fig. 9.1: Feynman graph for the process $e^+e^- \to \gamma$

$$= \int_{-\infty}^{\infty} d^4x \sum_k \text{constant}(k) \underbrace{\langle \gamma_{k'} | \, | \gamma_k \rangle}_{=\delta_{kk'}} e^{-ix(p_1+p_2-k)}$$

$$= \int_{-\infty}^{\infty} d^4x \, \text{constant}(k') e^{-ix(p_1+p_2-k')}.$$

The integration over x results in a delta function $\delta(p_1 + p_2 - k)$ that represents 4-momentum conservation[38]. In experiments we are never able to measure or prepare a system in one defined momentum, but only in a range. Therefore, at the end of our computation, we have to integrate over the relevant momentum range.

Take note that the seven other terms contributing to $\hat{S}^{(1)}$ result in a zero, because they destroy for example, a photon, which is not there in the beginning. If we had started with particles other than an electron and a positron, for example, a photon and a positron $|\gamma, e^+\rangle$, the first term results in a zero and some other term is non-zero.

Next we take a very quick, qualitative look at the third term of $\hat{S}$:

$$S^{(2)} = -\frac{1}{2!} T \left\{ \left(\int_{-\infty}^{\infty} d^4x_1 \mathscr{H}_I(x_1) \right) \left(\int_{-\infty}^{\infty} d^4x_2 \mathscr{H}_I(x_2) \right) \right\}$$

$$= -\frac{1}{2!} T \left\{ \left(\int_{-\infty}^{\infty} d^4x_1 g A_\mu(x_1) \bar{\Psi}(x_1) \gamma^\mu \Psi(x_1) \right) \left(\int_{-\infty}^{\infty} d^4x_2 g A_\mu(x_2) \bar{\Psi}(x_2) \gamma^\mu \Psi(x_2) \right) \right\},$$

$$(9.75)$$

where the time-ordering can be rewritten using Wick's Theorem into a sum of normal-ordered, denoted $N\{\}$, terms with commutators in it. Normal ordering means, putting all creation operators to the left, and all annihilation operators to the right. For instance, $N\{aa^\dagger a^\dagger a\} = a^\dagger a^\dagger aa$. One of the terms of this sum, for example, is

$$-\frac{1}{2!} g^2 \int \int d^4x_1 d^4x_2 N \left\{ \bar{\Psi}(x_1) \gamma^\mu \Psi(x_1) [A_\mu(x_1), A_\mu(x_2)] \bar{\Psi}(x_2) \gamma^\mu \Psi(x_2) \right\}$$

where the computation of the commutator can be done using the explicit solution for A_μ and the result is called Photon propagator[39] $[A_\mu(x_1), A_\mu(x_2)] \equiv iD_\mu(x_1 - x_2)$.

From this term we get again many, many terms, because every Ψ, $\bar{\Psi}$ etc. is actually a sum of two terms, and we will take a look at just one of them. Therefore, qualitatively we have for one[40] of these many many terms, if we start again with an electron and a positron, something of the form

$$-\frac{1}{2!} g^2 \int \int d^4x_1 d^4x_2 \bar{\Psi}^-(x_1) \Psi^-(x_1) D_\mu(x_1 - x_2) \bar{\Psi}^+(x_2) \Psi^+(x_2) |e^+, e^-\rangle$$

$$(9.76)$$

[38] The 4-momentum (=energy $p_0 = E$ and ordinary momentum p_i) of e^- plus the 4-momentum of e^+ must be equal to the 4-momentum of the photon γ: $p_1 + p_2 - k \overset{!}{=} 0$. Otherwise $\delta(p_1 + p_2 - k) = 0$ as explained in Appendix D.2.

[39] The propagators are one of the most complicated things to derive in quantum field theory. For instance, considering the scalar propagator, the starting point is $i\Delta \equiv \langle 0| T\{\Phi(x)\Phi^\dagger(y)\} |0\rangle$, which creates a particle from the vacuum at y and destroys it again at x. This can be rewritten as the commutator $i\Delta = \langle 0| [\Phi^{+\dagger}(y), \Phi^-(x)] |0\rangle$ and after many pages of math we get an expression of the form $i\Delta = \frac{-i}{2\pi^3} \int \frac{d^3k}{\omega_k} e^{-ik(x-y)}$. This final expression is what we can use in computations. This is just a sketch and many things here are not completely correct. Nevertheless it should give you a rough idea of what is going on.

[40] We pick again the term not resulting in a zero for the collision of an electron with a positron.

This can be understood physically:

- The two particles we start with are destroyed at x_2 by $\bar{\Psi}^+(x_2)\Psi^+(x_2)$.

- Then the propagator creates a "virtual" photon at x_2 and propagates it to x_1 where it is destroyed.

- Finally $\bar{\Psi}^-(x_1)\Psi^-(x_1)$ again create at x_1 an electron and a positron.

We can therefore compute with this term the probability amplitude for a reaction $e^+e^- \rightarrow e^+e^-$, where of course the individual momenta of the incoming and outgoing particles can be completely different, but not their sum[41]. In the same way, all the other terms can be interpreted as some reaction between massive[42] spin $\frac{1}{2}$ and massless[43] spin 1 fields.

The probability amplitudes we get from computations like this can then be directly checked in experiments, because the probability amplitude is directly connected to a quantity that can be measured in experiments: the cross section.

The techniques outlined in this chapter can be used to derive many important results of quantum field theory. The other interaction terms we derived can be put into the interaction Hamiltonian and the corresponding probability amplitudes follow analogously. Take not that the method we discussed here, only works if the coupling constant is smaller than 1. If the coupling constant is bigger than 1, higher order terms in the sum in Eq. 9.70 are bigger than lower order terms and therefore it is by no means justified to use just the first terms of the series, to get an approximation.
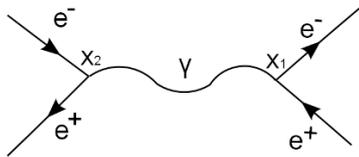
[41] This must be the case, because we have conservation of momentum.

[42] Here we looked at electrons and positrons. Other possibilities are the quarks or the other two leptons μ and τ.

[43] Here photons

Fig. 9.2: Feynman graph for the process $e^+e^- \rightarrow \gamma \rightarrow e^+e^-$

Further Reading Tips

- **Robert D. Klauber - Student Friendly Quantum Field Theory**[44] is, in my humble opinion, the best introduction to quantum field theory. All chapters are pedagogically brilliant, because the author spent a lot of time thinking about what problems someone learning quantum field theory faces.

- **Francis Halzen, Alan D. Martin - Quarks and Leptons: An Introductory Course in Modern Particle Physics**[45] is a great book which focusses on the applications of the computational schemes of quantum field theory.

- **Anthony Zee - Quantum Field Theory in a Nutshell**[46] has some brilliant and unique explanations, but some chapters are simply too short to understand as a beginner. Highly recommended *after* learning some quantum field theory from another book.

[44] Robert D. Klauber. *Student Friendly Quantum Field Theory*. Sandtrove Press, 2nd edition, 12 2013. ISBN 9780984513956

[45] Francis Halzen and Alan D. Martin. *Quarks and Leptons: An Introductory Course in Modern Particle Physics*. Wiley, 1st edition, 1 1984. ISBN 9780471887416

[46] Anthony Zee. *Quantum Field Theory in a Nutshell*. Princeton University Press, 1st edition, 3 2003. ISBN 9780691010199

- **Franz Mandl, Graham Shaw - Quantum Field Theory**[47] is a very good starting point regarding weak and strong interaction theory

- **Michele Maggiore - A Modern Introduction to Quantum Field Theory**[48] is a great introduction with a strong focus on group theoretical concepts.

- **Matthew Schwartz - Quantum Field Theory and the Standard Model**[49] offers illuminating explanations on many advanced topics.

[47] Franz Mandl and Graham Shaw. *Quantum Field Theory*. Wiley, 2nd edition, 5 2010. ISBN 9780471496847

[48] Michele Maggiore. *A Modern Introduction to Quantum Field Theory*. Oxford University Press, 1st edition, 2 2005. ISBN 9780198520740

[49] Matthew D. Schwartz. *Quantum Field Theory and the Standard Model*. Cambridge University Press, 1 edition, 12 2013. ISBN 9781107034730

9.6 Appendix: Most General Solution of the Klein-Gordon Equation

It is not too hard to find **one** solution of the Klein-Gordon equation. Certainly plane waves

$$\Phi(x) = a e^{i(px - Et)} = a e^{-i(p_\mu x^\mu)}$$

do the job, because of the energy-momentum relation of special relativity (Eq. 8.2)

$$E^2 = \vec{p}^2 + m^2 \rightarrow p_\mu p^\mu = m^2$$

as we can check:

$$
\begin{aligned}
(\partial_\mu \partial^\mu + m^2)\Phi &= (\partial_\mu \partial^\mu + m^2)e^{-i(p_\mu x^\mu)} \\
&= (-p_\mu p^\mu + m^2)e^{-i(p_\mu x^\mu)} \\
&= (-m^2 + m^2)e^{-i(p_\mu x^\mu)} \\
&= 0 \quad \checkmark
\end{aligned}
\tag{9.77}
$$

Because we differentiate the field twice the sign in the exponent does not matter. Equally

$$\Phi^\dagger(x) = a^\dagger e^{-i(px - Et)} = a^\dagger e^{i(p_\mu x^\mu)}$$

is a solution. Further solutions can be built as linear combinations. A general solution is given by superposition of all possible solutions, which can be seen as Fourier expansion[50]

$$\Phi(x) = \int \frac{\mathrm{d}p^4}{(2\pi)^4}\left(a(p)e^{-i(p_\mu x^\mu)} + a^\dagger(p)e^{i(p_\mu x^\mu)}\right)$$

Take note that we wrote $a = a(p)$ because we can have a different multiplication factor for each value of p and each term in the summation is a solution on its own.

In this context it is conventional to work with the wave number $k_i \equiv \frac{p_i}{\hbar}$ and the frequency $k_0 = w \equiv \frac{E}{\hbar}$ instead of the energy and

[50] That's were the factors of 2π come from. Another way of seeing this is demanding the solutions to form an orthonormal set: $\int dk e^{ikx} e^{-ikx'} = \int dk e^{ik(x-x')} = 2\pi\delta(x - x')$. Therefore, the factors of 2π are normalisation constants.

momentum. Because we work with $\hbar = 1$ we simply have to rename our variables to get the standard textbook expressions. Furthermore it's conventional to abbreviate $kx \equiv k_\mu x^\mu$. We therefore write

$$\Phi(x) = \int \frac{\mathrm{d}k^4}{(2\pi)^4} \left(a(k)\mathrm{e}^{-i(kx)} + a^\dagger(k)\mathrm{e}^{i(kx)} \right)$$

Take note that not all solutions of the Klein-Gordon equation are suited to describe nature, because we have the **"mass-shell" condition**

$$p_\mu p^\mu = k_\mu k^\mu = m^2 \rightarrow k_0^2 - k_i^2 \overset{!}{=} m^2 \rightarrow k^2 \overset{!}{=} m^2. \qquad (9.78)$$

Only solutions satisfying the mass-shell condition are in agreement with the relativistic energy-momentum relation (Eq. 8.2). We can build this condition into our equation, excluding all non-physical (off-shell) solutions with a delta distribution[51]

[51] This is explained in Appendix D.2.

$$\Phi(x)_{\text{physical}} = \int \frac{\mathrm{d}k^4}{(2\pi)^4} 2\pi\delta(k^2 - m^2) \left(a(k)\mathrm{e}^{-i(kx)} + a^\dagger(k)\mathrm{e}^{i(kx)} \right)$$

Besides that, only positive energy solutions are physical, because as explained earlier, otherwise the energy would be unbounded from below and nothing would be stable. We can built this constraint on our solutions into the equation by using a Heaviside function $\theta(k_0)$, being zero for $k_0 < 0$ and 1 for $k_0 \geq 0$. Our integral then reads

$$\Phi(x)_{\text{physical}} = \int \frac{1}{(2\pi)^3} \underbrace{\mathrm{d}k^4 \delta(k^2 - m^2)\theta(k_0)}_{\text{measure}} \left(a(k)\mathrm{e}^{-i(kx)} + a^\dagger(k)\mathrm{e}^{i(kx)} \right)$$

where we can rewrite the measure as follows

$$\mathrm{d}k^4 \delta(k^2 - m^2)\theta(k_0) = \mathrm{d}k^4 \delta(k_0^2 \underbrace{-\vec{k}^2 - m^2}_{\equiv -\omega_k^2 \ (\text{definition})})\theta(k_0)$$

$$= \mathrm{d}k^4 \delta(k_0^2 - \omega_k^2)\theta(k_0)$$

$$= \mathrm{d}k^4 \delta\big((k_0 - \omega_k)(k_0 + \omega_k)\big)\theta(k_0)$$

$$\underbrace{=}_{} \mathrm{d}k^4 \frac{1}{2k_0} \big(\delta(k_0 - \omega_k) + \delta(k_0 + \omega_k) \big)\theta(k_0)$$

using $\delta\big(f(x)\big) = \sum_i \frac{\delta(x-a_i)}{\left|\frac{df}{dx}(a_i)\right|}$ where a_i denotes the roots, i.e. $f(a_i)=0$, of the function $f(x)$

$$\underbrace{=}_{} \mathrm{d}k^4 \frac{1}{2k_0} \delta(k_0 - \omega_k)$$

because the argument of $\delta(k_0+\omega_k)$ never becomes zero with $k_0 \geq 0$

$$= \mathrm{d}k^3 \mathrm{d}k_0 \frac{1}{2k_0} \delta(k_0 - \omega_k)$$

$$\underbrace{=}_{\text{Integrating over } k_0} \mathrm{d}k^3 \frac{1}{2\omega_k} . \qquad (9.79)$$

So finally the general and physical solution of the Klein-Gordon equation reads

$$\Phi(x) = \int dk^3 \frac{1}{(2\pi)^3 2\omega_k} \left(a(k) e^{-i(kx)} + a^\dagger(k) e^{i(kx)} \right) . \qquad (9.80)$$