

All Is Not Well with Classical Mechanics

It was mentioned in the Prelude that as we keep expanding our domain of observations we must constantly check to see if the existing laws of physics continue to explain the new phenomena, and that, if they do not, we must try to find new laws that do. In this chapter you will get acquainted with experiments that betray the inadequacy of the classical scheme. The experiments to be described were never performed exactly as described here, but they contain the essential features of the actual experiments that were performed (in the first quarter of this century) with none of their inessential complications.

3.1. Particles and Waves in Classical Physics

There exist in classical physics two distinct entities: particles and waves. We have studied the particles in some detail in the last chapter and may summarize their essential features as follows. Particles are localized bundles of energy and momentum. They are described at any instant by the state parameters q and \dot{q} (or q and p). These parameters evolve in time according to some equations of motion. Given the initial values $q(t_i)$ and $\dot{q}(t_i)$ at time t_i , the trajectory $q(t)$ may be deduced for all future times from the equations of motion. A wave, in contrast, is a disturbance spread over space. It is described by a wave function $\psi(\mathbf{r}, t)$ which characterizes the disturbance at the point \mathbf{r} at time t .

In the case of sound waves, ψ is the excess air pressure above the normal, while in the case of electromagnetic waves, ψ can be any component of the electric field vector \mathbf{E} . The analogs of q and \dot{q} for a wave are ψ and $\dot{\psi}$ at each point \mathbf{r} , assuming ψ obeys a second-order wave equation in time, such as

$$\nabla^2 \psi = \frac{1}{c^2} \frac{\partial^2 \psi}{\partial t^2}$$

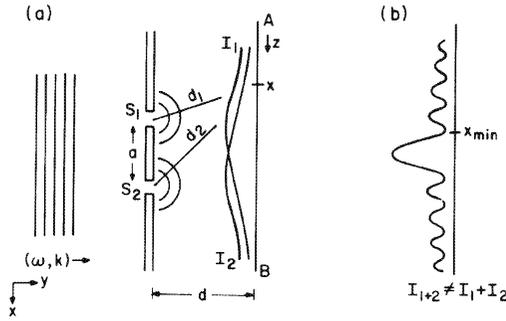


Figure 3.1. (a) When a wave $\psi = e^{i(ky - \omega t)}$ is incident on the screen with either slit S_1 or S_2 open, the intensity patterns I_1 and I_2 , respectively, are measured by the row of detectors on AB . (b) With both slits open, the pattern I_{1+2} is observed. Note that $I_{1+2} \neq I_1 + I_2$. This is called interference.

which describes waves propagating at the speed of light, c . Given $\psi(\mathbf{r}, 0)$ and $\dot{\psi}(\mathbf{r}, 0)$ one can get the wave function $\psi(\mathbf{r}, t)$ for all future times by solving the wave equation.

Of special interest to us are waves that are periodic in space and time, called *plane waves*. In one dimension, the plane wave may be written as

$$\psi(x, t) = A \exp \left[i \left(\frac{2\pi}{\lambda} x - \frac{2\pi}{T} t \right) \right] \equiv A \exp[i\phi] \tag{3.1.1}$$

At some given time t , the wave is periodic in space with a period λ , called its *wavelength*, and likewise at a given point x , it is periodic in time, repeating itself every T seconds, T being called the *time period*. We will often use, instead of λ and T , the related quantities $k = 2\pi/\lambda$ called the *wave number* and $\omega = 2\pi/T$ called the (*angular*) *frequency*. In terms of the phase ϕ in Eq. (3.1.1), k measures the phase change per unit length at any fixed time t , while ω measures the phase change per unit time at any fixed point x . This wave travels at a speed $v = \omega/k$. To check this claim, note that if we start out at a point where $\phi = 0$ and move along x at a rate $x = (\omega/k)t$, ϕ remains zero. The overall scale A up front is called the *amplitude*. For any wave, the intensity is defined to be $I = |\psi|^2$. For a plane wave this is a constant equal to $|A|^2$. If ψ describes an electromagnetic wave, the intensity is a measure of the energy and momentum carried by the wave. [Since the electromagnetic field is real, only the real part of ψ describes it. However, time averages of the energy and momentum flow are still proportional to the intensity (as defined above) in the case of plane waves.]

Plane waves in three dimension are written as

$$\psi(\mathbf{r}, t) = A e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)}, \quad \omega = |\mathbf{k}|v \tag{3.1.2}$$

where each component k_i gives the phase changes per unit length along the i th axis. One calls \mathbf{k} the *wave vector*.[‡]

3.2. An Experiment with Waves and Particles (Classical)

Waves exhibit a phenomenon called *interference*, which is peculiar to them and is not exhibited by particles described by classical mechanics. This phenomenon is illustrated by the following experiment (Fig. 3.1a). Let a wave $\psi = A e^{i(ky - \omega t)}$ be

[‡] Unfortunately we also use \mathbf{k} to denote the unit vector along the z axis. It should be clear from the context what it stands for.

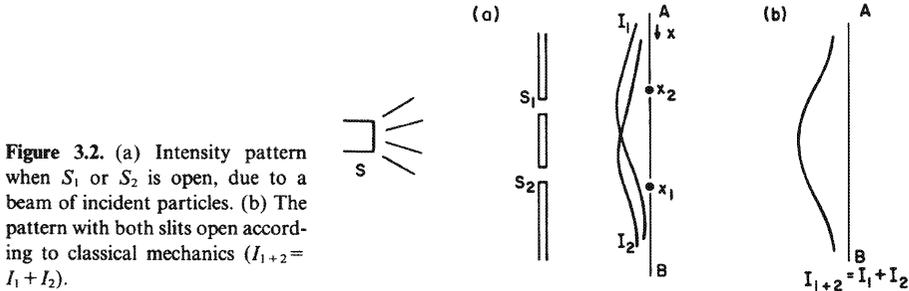


Figure 3.2. (a) Intensity pattern when S_1 or S_2 is open, due to a beam of incident particles. (b) The pattern with both slits open according to classical mechanics ($I_{1+2} = I_1 + I_2$).

incident normally on a screen with slits S_1 and S_2 , which are a distance a apart. At a distance d parallel to it is a row of detectors that measures the intensity as a function of the position x measured along AB .

If we first keep only S_1 open, the incident wave will come out of S_1 and propagate radially outward. One may think of S_1 as the virtual source of this wave ψ_1 , which has the same frequency and wavelength as the incident wave. The intensity pattern $I_1 = |\psi_1|^2$ is registered by the detectors. Similarly if S_2 is open instead of S_1 , the wave ψ_2 produces the pattern $I_2 = |\psi_2|^2$. In both cases the arrival of energy at the detectors is a smooth function of x and t .

Now if both S_1 and S_2 are opened, both waves ψ_1 and ψ_2 are present and produce an intensity pattern $I_{1+2} = |\psi_1 + \psi_2|^2$.

The interesting thing is that $I_{1+2} \neq I_1 + I_2$, but rather the interference pattern shown in Fig. 3.1b. The ups and downs are due to the fact that the waves ψ_1 and ψ_2 have to travel different distances d_1 and d_2 to arrive at some given x (see Fig. 3.1a) and thus are not always in step. In particular, the maxima correspond to the case $d_2 - d_1 = n\lambda$ (n is an integer), when the waves arrive exactly in step, and the minima correspond to the case $d_2 - d_1 = (2n + 1)\lambda/2$, when the waves are exactly out of step. In terms of the phases ϕ_1 and ϕ_2 , $\phi_2(x) - \phi_1(x) = 2n\pi$ at a maximum and $\phi_2(x) - \phi_1(x) = (2n + 1)\pi$ at a minimum. One can easily show that the spacing Δx between two adjacent maxima is $\Delta x = \lambda d/a$.

The feature to take special note of is that if x_{\min} is an interference minimum, there is more energy flowing into x_{\min} with just one slit open than with both. In other words, the opening of an extra slit can actually reduce the energy flow into x_{\min} .

Consider next the experiment with particles (Fig. 3.2a). The source of the incident plane waves is replaced by a source of particles that shoots them toward the screen with varying directions but fixed energy. Let the line AB be filled with an array of particle detectors. Let us define the intensity $I(x)$ to be the number of particles arriving per second at any given x . The patterns with S_1 or S_2 open are shown in (Fig. 3.2a). These look very much like the corresponding patterns for the wave. The only difference will be that the particles arrive not continuously, but in a staccato fashion, each particle triggering a counter at some single point x at the time of arrival. Although this fact may be obscured if the beam is dense, it can be easily detected as the incident flux is reduced.

What if both S_1 and S_2 are opened? Classical mechanics has an unambiguous prediction: $I_{1+2} = I_1 + I_2$. The reasoning is as follows: each particle travels along a definite trajectory that passes via S_1 or S_2 to the destination x . To a particle headed

for S_1 , it is immaterial whether S_2 is open or closed. Being localized in space it has no way of even knowing if S_2 is open or closed, and thus cannot respond to it in any way. Thus the number coming via S_1 to x is independent of whether S_2 is open or not and vice versa. It follows that $I_{1+2} = I_1 + I_2$ (Fig. 3.2b).

The following objection may be raised: although particles heading for S_1 are not aware that S_2 is open, they certainly can be deflected by those coming out of S_2 , if, for instance, the former are heading for x_1 and the latter for x_2 (see Fig. 3.1a).

This objection can be silenced by sending in one particle at a time. A given particle will of course not produce a pattern like I_1 or I_2 by itself, it will go to some point x . If, however, we make a histogram, the envelope of this histogram, after many counts, will define the smooth functions I_1 , I_2 , and I_{1+2} . Now the conclusion $I_{1+2} = I_1 + I_2$ is inevitable.

This is what classical physics predicts particles and waves will do in the double-slit experiment.

3.3. The Double-Slit Experiment with Light

Consider now what happens when we perform the following experiment to check the classical physics notion that light is an electromagnetic wave phenomenon.

We set up the double slit as in Fig. 3.1a, with a row of light-sensitive meters along AB and send a beam $\psi = A e^{i(kx - \omega t)}$ in a direction perpendicular to the screen. (Strictly speaking, the electromagnetic wave must be characterized by giving the orientation of the \mathbf{E} and \mathbf{B} vectors in addition to ω and k . However, for a plane wave, \mathbf{B} is uniquely fixed by \mathbf{E} . If we further assume \mathbf{E} is polarized perpendicular to the page, this polarization is unaffected by the double slit. We can therefore suppress the explicit reference to this constant vector and represent the field as a scalar function ψ .) We find that with the slits open one at a time we get patterns I_1 and I_2 , and with both slits open we get the interference pattern I_{1+2} as in Figs. 3.1a and 3.1b. (The interference pattern is of course what convinced classical physicists that light was a wave phenomenon.) The energy arrives at the detectors smoothly and continuously as befitting a wave.

Say we repeat the experiment with a change that is expected (in classical physics) to produce no qualitative effects. We start with S_1 open and cut down the intensity. A very strange thing happens. We find that the energy is not arriving continuously, but in sudden bursts, a burst here, a burst there, etc. We now cut down the intensity further so that only one detector gets activated at a given time and there is enough of a gap, say a millisecond, between counts. As each burst occurs at some x , we record it and plot a histogram. With enough data, the envelope of the histogram becomes, of course, the pattern I_1 . We have made an important discovery: light energy is not continuous—it comes in bundles. This discrete nature is obscured in intense beams, for the bundles come in so fast and all over the line AB , that the energy flow seems continuous in space and time.

We pursue our study of these bundles, called photons, in some detail and find the following properties:

1. Each bundle carries the same energy E .
2. Each bundle carries the same momentum p .

3. $E = pc$. From the famous equation $E^2 = p^2c^2 + m^2c^4$, we deduce that these bundles are particles of zero mass.
4. If we vary the frequency of the light source we discover that

$$E = \hbar\omega \quad (3.3.1)$$

$$p = \hbar k \quad (3.3.2)$$

where $\hbar = h/2\pi$ is a constant. The constant h is called *Planck's constant*, and has the dimensions of erg sec, which is the same as that of action and angular momentum. Its value is

$$\frac{h}{2\pi} = \hbar \simeq 10^{-27} \text{ erg sec} \quad (3.3.3)$$

For those interested in history, the actual experiment that revealed the granular nature of light is called the *photoelectric effect*. The correct explanation of this experiment, in terms of photons, was given by Einstein in 1905.

That light is made of particles will, of course, surprise classical physicists but will not imply the end of classical physics, for physicists are used to the idea that phenomena that seem continuous at first sight may in reality be discrete. They will cheerfully plunge into the study of the dynamics of the photons, trying to find the equations of motion for its trajectory and so on. What really undermines classical physics is the fact that if we now open both slits, still keeping the intensity so low that only one photon is in the experimental region at a given time, and watch the histogram take shape, we won't find that I_{1+2} equals $I_1 + I_2$ as would be expected of particles, but is instead an interference pattern characteristic of wave number k . This result completely rules out the possibility that photons move in well-defined trajectories like the particles of classical mechanics—for if this were true, a photon going in via S_1 should be insensitive to whether S_2 is open or not (and vice versa), and the result $I_{1+2} = I_1 + I_2$ is inescapable! To say this another way, consider a point x_{\min} which is an interference minimum. More photons arrive here with either S_1 or S_2 open than with both open. If photons followed definite trajectories, it is incomprehensible how opening an extra pathway can *reduce* the number coming to x_{\min} . Since we are doing the experiment with one photon at a time, one cannot even raise the improbable hypothesis that photons coming out of S_1 collide with those coming out of S_2 to modify (miraculously) the smooth pattern $I_1 + I_2$ into the wiggly interference pattern.

From these facts Born drew the following conclusion: with *each* photon is associated a wave ψ , called the *probability amplitude* or simply *amplitude*, whose modulus squared $|\psi(x)|^2$ gives the probability of finding the particle at x . [Strictly speaking, we must not refer to $|\psi(x)|^2$ as the probability for a given x , but rather as the probability density at x since x is a continuous variable. These subtleties can, however, wait.] The entire experiment may be understood in terms of this hypothesis as follows. Every incoming photon of energy E and momentum p has a wave function ψ associated with it, which is a plane wave with $\omega = E/\hbar$ and $k = p/\hbar$. This wave interferes with itself and forms the oscillating pattern $|\psi(x)|^2$ along AB , which gives

the probability that the given photon will arrive at x . A given photon of course arrives at some definite x and does not reveal the probability distribution. If, however, we wait till several photons, all described by the same ψ , have arrived, the number at any x will become proportional to the probability function $|\psi(x)|^2$. Likewise, if an intense (macroscopic) monochromatic beam is incident, many photons, all described by the same wave and hence the same probability distribution, arrive at the same time and all along the line AB . The intensity distribution then assumes the shape of the probability distribution right away and the energy flow seems continuous and in agreement with the predictions of classical electromagnetic theory.

The main point to note, besides the probability interpretation, is that a wave is associated not with a beam of photons, but with *each* photon. If the beam is monochromatic, every photon is given by the same ψ and the same probability distribution. A large ensemble of such photons will reproduce the phenomena expected of a classical electromagnetic wave ψ and the probabilistic aspect will be hidden.

3.4. Matter Waves (de Broglie Waves)

That light, which one thought was a pure wave phenomenon, should consist of photons, prompted de Broglie to conjecture that entities like the electron, generally believed to be particles, should exhibit wavelike behavior. More specifically, he conjectured, in analogy with photons, that particles of momentum p will produce an interference pattern corresponding to a wave number $k = p/\hbar$ in the double-slit experiment. This prediction was verified for electrons by Davisson and Germer, shortly thereafter. It is now widely accepted that all particles are described by probability amplitudes $\psi(x)$, and that the assumption that they move in definite trajectories is ruled out by experiment.

But what about common sense, which says that billiard balls and baseballs travel along definite trajectories? How did classical mechanics survive for three centuries? The answer is that the wave nature of matter is not apparent for macroscopic phenomena since \hbar is so small. The precise meaning of this explanation will become clear only after we fully master quantum mechanics. Nonetheless, the following example should be instructive. Suppose we do the double-slit experiment with pellets of mass 1 g, moving at 1 cm/sec. The wavelength associated with these particles is

$$\lambda = \frac{2\pi}{k} = \frac{h}{p} \simeq 10^{-26} \text{ cm}$$

which is 10^{-13} times smaller than the radius of the proton! For any reasonable values of the parameters a and d (see Fig. 3.1b), the interference pattern would be so dense in x that our instruments will only measure the smooth average, which will obey $I_{1+2} = I_1 + I_2$ as predicted classically.

3.5. Conclusions

The main objective of this chapter was to expose the inadequacy of classical physics in explaining certain phenomena and, incidentally, to get a glimpse of what

the new (quantum) physics ought to look like. We found that entities such as the electron are particles in the classical sense in that when detected they seem to carry all their energy, momentum, charge, etc. in localized form; and at the same time they are not particlelike in that assuming they move along definite trajectories leads to conflict with experiment. It appears that each particle has associated with it a wave function $\psi(x, t)$, such that $|\psi(x, t)|^2$ gives the probability of finding it at a point x at time t . This is called *wave-particle duality*.

The dynamics of the particle is then the dynamics of this function $\psi(x, t)$ or, if we think of functions as vectors in an infinite-dimensional space, of the ket $|\psi(t)\rangle$. In the next chapter the postulates of quantum theory will define the dynamics in terms of $|\psi(t)\rangle$. The postulates, which specify what sort of information is contained in $|\psi(t)\rangle$ and how $|\psi(t)\rangle$ evolves with time, summarize the results of the double-slit experiment and *many others not mentioned here*. The double-slit experiment was described here to expose the inadequacy of classical physics and not to summarize the entire body of experimental results from which all the postulates could be inferred. Fortunately, the double-slit experiment contains most of the central features of the theory, so that when the postulates are encountered in the next chapter, they will appear highly plausible.