19

Scattering Theory

19.1. Introduction

One of the best ways to understand the structure of particles and the forces between them is to scatter them off each other. This is particularly true at the quantum level where the systems cannot be seen in the literal sense and must be probed by indirect means. The scattering process gives us information about the projectile, the target, and the forces between them. A natural way to proceed (when possible) is to consider cases where two of these are known and learn about the third. Consider, for example, experiments at the Stanford Linear Accelerator Center in which high-energy photons were used to bombard static neutrons. The structure of the photon and its coupling to matter are well understood—the photon is a point particle to an excellent approximation and couples to electric charge in a way we have studied in some detail. It therefore serves as an excellent probe of the neutron. For instance, the very fact that the neutron, which is electrically neutral, interacts with the photon tells us that the neutron is built out of charged constituents (whose total charge add up to zero). These scattering experiments also revealed that the neutron's constituents have spin $\frac{1}{2}$, and fractional charges ($\frac{2}{3}e$, $-\frac{1}{3}e$), a picture that had been arrived at from another independent line of reasoning. Furthermore they also indicated that the interaction between these constituents (called quarks) gets very weak as they get close. This information has allowed us to choose, from innumerable possible models of the interquark force, one that is now considered most likely to succeed, and goes by the name of quantum chromodynamics (QCD), a subject that is being vigorously investigated by many particle physicists today.

Scattering theory is a very extensive subject and this chapter aims at giving you just the flavor of the basic ideas. For more information, you must consult books devoted to this subject.‡

A general scattering event is of the form

$$a(\alpha)+b(\beta)+\cdots\rightarrow f(\gamma)+g(\delta)+\cdots$$

where $\{a, b, \ldots\}$ are particle names and $\{\alpha, \beta, \gamma, \ldots\}$ are the kinematical variables

‡ See, for example, the excellent book by J. R. Taylor, *Scattering Theory*, Wiley, New York (1971). Any details omitted here due to lack of space may be found there.

specifying their states, such as momentum, spin, etc. We are concerned only with nonrelativistic, elastic scattering of structureless spinless particles.

In the next three sections, we deal with a formalism that describes a single particle scattering from a potential $V(\mathbf{r})$. As it stands, the formalism describes a particle colliding with an immobile target whose only role is to provide the potential. (This picture provides a good approximation to processes where a light particle collides with a very heavy one, say an α particle colliding with a heavy nucleus.) In Section 19.6 we see how, upon proper interpretation, the same formalism describes two-body collisions in the CM frame. In that section we will also see how the description of the scattering process in the CM frame can be translated to another frame, called the lab frame, where the target is initially at rest. It is important to know how to pass from one frame to the other, since theoretical calculations are most easily done in the CM frame, whereas most experiments are done in the lab frame.

19.2. Recapitulation of One-Dimensional Scattering and Overview

Although we are concerned here with scattering in three dimensions, we begin by recalling one-dimensional scattering, for it shares many common features with its three-dimensional counterpart. The practical question one asks is the following: If a beam of nearly monoenergetic particles with mean momenta $\langle P\rangle = \hbar k_0$ are incident from the far left $(x\to-\infty)$ on a potential $V(x)$ which tends to zero as $|x|\to\infty$, what fraction T will get transmitted and what fraction R will get reflected?‡ It is not *a priori* obvious that the above question can be answered, since the mean momentum does not specify the quantum states of the incoming particles. But it turns out that if the individual momentum space wave functions are sharply peaked at $\hbar k_0$, the reflection and transmission probabilities depend only on k_0 and not on the detailed shapes of the wave functions. Thus it is possible to calculate $R(k_0)$ and $T(k_0)$ that apply to every particle in the beam. Let us recall some of the details.

(1) We start with some wave packet, say a Gaussian, with $\langle P\rangle = \hbar k_0$ and $\langle X\rangle \to -\infty$.

(2) We expand this packet in terms of the eigenfunctions ψ_k of $H=T+V$ with coefficients $a(k)$. The functions ψ_k have the following property:

$$\psi_k \xrightarrow[x\to-\infty]{} A\,e^{-ikx}+B\,e^{ikx}$$
$$\xrightarrow[x\to\infty]{} C\,e^{ikx} \tag{19.2.1}$$

In other words, the asymptotic form of ψ_k contains an incident wave $A\,e^{ikx}$ and a reflected wave $B\,e^{-ikx}$ as $x\to-\infty$, and just a transmitted wave $C\,e^{ikx}$ as $x\to\infty$. Although the most general solution also contains a $D\,e^{-ikx}$ piece as $x\to\infty$, we set

‡ In general, the particle can come in from the far right as well. Also $V(x)$ need not tend to zero at both ends, but to constants V_+ and V_- as $x\to\pm\infty$. We assume $V_+=V_-=0$ for simplicity. We also assume $|xV(x)|\to 0$ as $|x|\to\infty$, so that the particle is asymptotically free $(\psi\sim e^{\pm ikx})$.

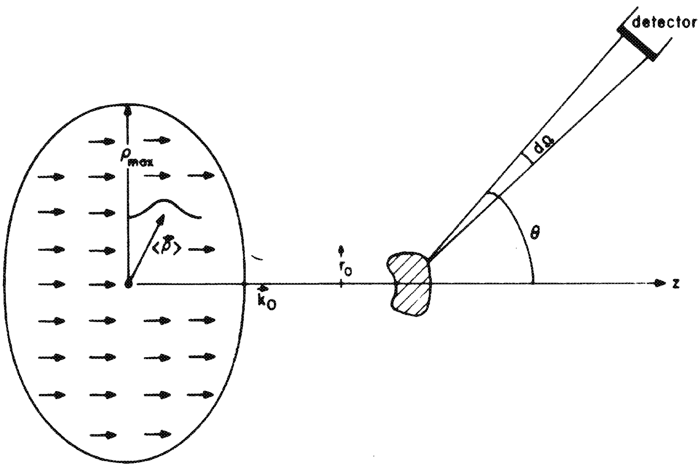

Figure 19.1. A schematic description of scattering. The incident particles, shown by arrows, are really described by wave packets (only one is shown) with mean momentum $\langle \mathbf{P} \rangle = \langle \hbar \mathbf{k}_0 \rangle$ and mean impact parameter $\langle \boldsymbol{\rho} \rangle$ uniformly distributed in the $\boldsymbol{\rho}$-plane out to $\boldsymbol{\rho}_{max} \gg r_0$, the range of the potential. The shaded region near the origin stands for the domain where the potential is effective. The detector catches all particles that emerge in the cone of opening angle $d\Omega$. The beam is assumed to be coming in along the z axis.

$D=0$ on physical grounds: the incident wave $A\, e^{ikx}$ can only produce a right-going wave as $x \to \infty$.

(3) We propagate the wave packet in time by attaching to the expansion coefficients $a(k)$ the time dependence $e^{-iEt/\hbar}$, where $E = \hbar^2 k^2 / 2\mu$. We examine the resulting solution as $t \to \infty$ and identify the reflected and transmitted packets. From the norms of these we get R and T respectively.

(4) We find at this stage that if the incident packet is sharply peaked in momentum space at $\hbar k_0$, R and T depend only on k_0 and not on the detailed shape of the wave function. Thus the answer to the question raised at the outset is that a fraction $R(k_0)$ of the incident particles will get reflected and a fraction $T(k_0)$ will get transmitted.

(5) Having done all this hard work, we find at the end that the same result could have been obtained by considering just one eigenfunction ψ_{k_0} and taking the ratios of the transmitted and reflected current densities to the incident current density.

The scattering problem in three dimensions has many similarities with its one-dimensional counterpart and also several differences that inevitably accompany the increase in dimensionality. First of all, the incident particles (coming out of the accelerator) are characterized, not by just the mean momentum $\langle \mathbf{P} \rangle = \hbar \mathbf{k}_0$, but also by the fact that they are uniformly distributed in the *impact parameter* $\boldsymbol{\rho}$, which is the coordinate in the plane perpendicular to $\mathbf{k}_0$ (Fig. 19.1). The distribution is of course not uniform out to $\rho \to \infty$, but only up to $\rho_{max} \gg r_0$, where r_0, the *range of the potential*, is the distance scale beyond which the potential is negligible. [For instance, if $V(r) = e^{-r^2/a^2}$, the range $r_0 \cong a$.] The problem is to calculate the rate at which particles get scattered into a far away detector that subtends a solid angle $d\Omega$ in the direction (θ, ϕ) measured relative to the beam direction (Fig. 19.1). To be precise, one wants the *differential cross section* $d\sigma/d\Omega$ defined as follows:

$$\frac{d\sigma(\theta,\phi)}{d\Omega}d\Omega=\frac{\text{number of particles scattered into } d\Omega/\text{sec}}{\text{number incident/sec/area in the } \boldsymbol{\rho} \text{ plane}} \tag{19.2.2}$$

The calculation of $d\sigma/d\Omega$ proceeds as follows.‡

(1) One takes some initial wave packet with mean momentum $\langle \mathbf{P}\rangle=\hbar\mathbf{k}_0$ and mean impact parameter $\langle\boldsymbol{\rho}\rangle$. The mean coordinate in the beam direction is not relevant, as long as it is far away from the origin.

(2) One expands the wave packet in terms of the eigenfunctions $\psi_{\mathbf{k}}$ of $H=T+V$ which are of the form

$$\psi_{\mathbf{k}}=\psi_{\text{inc}}+\psi_{\text{sc}} \tag{19.2.3}$$

where ψ_{inc} is the incident wave $e^{i\mathbf{k}\cdot\mathbf{r}}$ and ψ_{sc} is the scattered wave. One takes only those solutions in which ψ_{sc} is purely outgoing. We shall have more to say about ψ_{sc} in a moment.

(3) One propagates the wave packet by attaching the time-dependence factor $e^{-iEt/\hbar}$ $(E=\hbar^2k^2/2\mu)$ to each coefficient $a(\mathbf{k})$ in the expansion.

(4) One identifies the scattered wave as $t\to\infty$, and calculates the probability current density associated with it. One integrates the total flow of probability into the cone $d\Omega$ at (θ,ϕ). This gives the probability that the incident particle goes into the detector at (θ,ϕ). One finds that if the momentum space wave function of the incident wave packet is sharply peaked at $\langle\mathbf{P}\rangle=\hbar\mathbf{k}_0$, the probability of going into $d\Omega$ depends only on $\hbar\mathbf{k}_0$ and $\langle\boldsymbol{\rho}\rangle$. Call this probability $P(\boldsymbol{\rho},\mathbf{k}_0\to d\Omega)$.

(5) One considers next a beam of particle with $\eta(\boldsymbol{\rho})$ particles per second per unit area in the $\boldsymbol{\rho}$ plane. The number scattering into $d\Omega$ per second is

$$\eta(d\Omega)=\int P(\boldsymbol{\rho},\mathbf{k}_0\to d\Omega)\,\eta(\boldsymbol{\rho})\,d^2\boldsymbol{\rho} \tag{19.2.4}$$

Since in the experiment $\eta(\boldsymbol{\rho})=\eta$, a constant, we have from Eq. (192.2)

$$\frac{d\sigma}{d\Omega}d\Omega=\frac{\eta(d\Omega)}{\eta}=\int P(\boldsymbol{\rho},\mathbf{k}_0\to d\Omega)\,d^2\boldsymbol{\rho} \tag{19.2.5}$$

(6) After all this work is done one finds that $d\sigma/d\Omega$ could have been calculated from considering just the static solution $\psi_{\mathbf{k}_0}$ and computing in the limit $r\to\infty$, the ratio of the probability flow per second into $d\Omega$ associated with ψ_{sc}, to the incident probability current density associated with $e^{i\mathbf{k}_0\cdot\mathbf{r}}$. The reason the time-dependent picture reduces to the time-independent picture is the same as in one dimension: as we broaden the incident wave packet more and more in coordinate space, the incident and scattered waves begin to coexist in a steady-state configuration, $\psi_{\mathbf{k}_0}$. What about

‡ We do not consider the details here, for they are quite similar to the one-dimensional case. The few differences alone are discussed. See Taylor's book for the details.

the average over $\langle \mathbf{p} \rangle$? This is no longer necessary, since the incident packet is now a plane wave $e^{i\mathbf{k}_0\cdot\mathbf{r}}$ which is already uniform in $\mathbf{p}$.‡

Let us consider some details of extracting $d\sigma/d\Omega$ from $\psi_{\mathbf{k}_0}$. Choosing the z axis parallel to $\mathbf{k}_0$ and dropping the subscript 0, we obtain

$$\psi_{\mathbf{k}} = e^{ikz} + \psi_{sc}(r, \theta, \phi) \tag{19.2.6}$$

where θ and ϕ are defined in Fig. 19.1. Although the detailed form of ψ_{sc} depends on the potential, we know that far from the origin it satisfies the free-particle equation [assuming $rV(r) \to 0$ as $r \to \infty$].

$$(\nabla^2 + k^2)\psi_{sc} = 0 \qquad (r \to \infty) \tag{19.2.7}$$

and is purely outgoing.

Recalling the general solution to the free-particle equation (in a region that excludes the origin) we get

$$\psi_{sc} \underset{r\to\infty}{\longrightarrow} \sum_l \sum_m (A_l j_l(kr) + B_l n_l(kr)) Y_l^m(\theta, \phi) \tag{19.2.8}$$

Notice that we do not exclude the Neumann functions because they are perfectly well behaved as $r \to \infty$. Since

$$\begin{aligned} j_l(kr) &\underset{r\to\infty}{\longrightarrow} \sin(kr - l\pi/2)/(kr) \\ n_l(kr) &\underset{r\to\infty}{\longrightarrow} -\cos(kr - l\pi/2)/(kr) \end{aligned} \tag{19.2.9}$$

it must be that $A_l/B_l = -i$, so that we get a purely outgoing wave e^{ikr}/kr. With this condition, the asymptotic form of the scattered wave is

$$\psi_{sc} \underset{r\to\infty}{\longrightarrow} \frac{e^{ikr}}{kr} \sum_l \sum_m (-i)^l(-B_l) Y_l^m(\theta, \phi) \tag{19.2.10}$$

or

$$\psi_{sc} \underset{r\to\infty}{\longrightarrow} \frac{e^{ikr}}{r} f(\theta, \phi)\S \tag{19.2.11}$$

and

$$\psi_{\mathbf{k}} \underset{r\to\infty}{\longrightarrow} e^{ikz} + f(\theta, \phi)\frac{e^{ikr}}{r} \tag{19.2.12}$$

where f is called the *scattering amplitude*.

‡ Let us note, as we did in one dimension, that a wave packet does not simply become a plane wave as we broaden it, for the former has norm unity and the latter has norm $\delta^3(0)$. So it is assumed that as the packet is broadened, its norm is steadily increased in such a way that we end up with a plane wave. In any case, the overall norm has no significance.

§ Actually f also depends on k; this dependence is not shown explicitly.

To get the differential cross section, we need the ratio of the probability flowing into $d\Omega$ per second to the incident current density. So what are $\mathbf{j}_{sc}$ and $\mathbf{j}_{inc}$, the incident and scattered current densities? Though we have repeatedly spoken of these quantities, they are not well defined unless we invoke further physical ideas. This is because there is only one current density $\mathbf{j}$ associated with $\psi_{\mathbf{k}}$ and it is *quadratic* in $\psi_{\mathbf{k}}$. So $\mathbf{j}$ is not just a sum of two pieces, one due to e^{ikz} and one due to ψ_{sc}; there are cross terms.‡ We get around this problem as follows. We note that as $r\to\infty$, ψ_{sc} is negligible compared to e^{ikz} because of the $1/r$ factor. So we calculate the incident current due to e^{ikz} to be

$$|j_{inc}| = \left|\frac{\hbar}{2\mu i}(e^{-ikz}\nabla e^{ikz} - e^{ikz}\nabla e^{-ikz})\right|$$
$$= \frac{\hbar k}{\mu} \tag{19.2.13}$$

We cannot use this trick to calculate $\mathbf{j}_{sc}$ into $d\Omega$ because ψ_{sc} never dominates over e^{ikz}. So we use another trick. We say that e^{ikz} is really an abstraction for a wave that is limited in the transverse direction by some $\rho_{max}(\gg r_0)$. Thus in any realistic description, only ψ_{sc} will survive as $r\to\infty$ for $\theta\neq 0$.§ (For a given ρ_{max}, the incident wave is present only for $\delta\theta \lesssim \rho_{max}/r$. We can make $\delta\theta$ arbitrarily small by increasing the r at which the detector is located.) With this in mind we calculate (for $\theta\neq 0$)

$$\mathbf{j}_{sc} = \frac{\hbar}{2\mu i}(\psi_{sc}^*\nabla\psi_{sc} - \psi_{sc}\nabla\psi_{sc}^*) \tag{19.2.14}$$

Now

$$\nabla = \mathbf{e}_r\frac{\partial}{\partial r} + \mathbf{e}_\theta\frac{\partial}{r\,\partial\theta} + \mathbf{e}_\phi\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi} \tag{19.2.15}$$

The last two pieces in ∇ are irrelevant as $r\to\infty$. When the first acts on the asymptotic ψ_{sc},

$$\frac{\partial}{\partial r}f(\theta,\phi)\frac{e^{ikr}}{r} = f(\theta,\phi)ik\frac{e^{ikr}}{r} + O\left(\frac{1}{r^2}\right)$$

so that

$$\mathbf{j}_{sc} = \frac{\mathbf{e}_r}{r^2}|f|^2\frac{\hbar k}{\mu} \tag{19.2.16}$$

‡ We did not have to worry about this in one dimension because j due to $A\,e^{ikx} + B\,e^{-ikx}$ is $(\hbar k/\mu)(|A|^2 - |B|^2) = j_{inc} + j_{ref}$ with no cross terms.

§ In fact, only in this more realistic picture is it sensible to say that the particles entering the detectors at $\theta\neq 0$ are scattered (and not unscattered incident) particles. At $\theta = 0$, there is no way (operationally) to separate the incident and scattered particles. To compare theory with experiment, one extracts $f(\theta=0)$ by extrapolating $f(\theta)$ from $\theta\neq 0$.

Probability flows into $d\Omega$ at the rate

$$R(d\Omega)=\mathbf{j}_{sc}\cdot\mathbf{e}_r r^2 d\Omega$$
$$=|f|^2\frac{\hbar k}{\mu}d\Omega \qquad (19.2.17)$$

Since it arrives at the rate

$$j_{inc}=\hbar k/\mu \ \text{sec}^{-1}\ \text{area}^{-1}$$
$$\frac{d\sigma}{d\Omega}d\Omega=\frac{R(d\Omega)}{j_{inc}}=|f|^2 d\Omega$$

so that finally

$$\frac{d\sigma}{d\Omega}=|f(\theta,\phi)|^2 \qquad (19.2.18)$$

Thus, in the time-independent picture, the calculation of $d\sigma/d\Omega$ reduces to the calculation of $f(\theta, \phi)$.

After this general discussion, we turn to specific calculations. In the next section the calculation of $d\sigma/d\Omega$ is carried out in the time-dependent picture *to first order*. In Section 4, we calculate $d\sigma/d\Omega$ to first order in the time-independent picture. (The two results agree, of course.) In Section 5, we go beyond perturbation theory and discuss some general features of f for spherically symmetric potentials. Two-particle scattering is discussed in Section 6.

19.3. The Born Approximation (Time-Dependent Description)

Consider an initial wave packet that is so broad that it can be approximated by a plane wave $|\mathbf{p}_i\rangle$. Its fate after scattering is determined by the propagator $U(t_f\to\infty, t_i\to-\infty)$, that is, by the operator

$$S=\lim_{\substack{t_f\to\infty\\ t_i\to-\infty}} U(t_f, t_i)$$

which is called the *S matrix*. The probability of the particle entering the detector in the direction (θ, ϕ) with opening angle $d\Omega$ is the probability that the final momentum $\mathbf{p}_f$ lies in a cone of opening angle $d\Omega$ in the direction (θ, ϕ):

$$P(\mathbf{p}_i\to d\Omega)=\sum_{\mathbf{p}_f \text{ in } d\Omega}|\langle\mathbf{p}_f|S|\mathbf{p}_i\rangle|^2$$

If we evaluate S or U to first order, treating V as a perturbation, the problem reduces to the use of Fermi's Golden Rule, which tells us that the transition rate is

$$R_{i\to d\Omega}=\frac{dP(\mathbf{p}_i\to d\Omega)}{dt}$$

$$=\frac{2\pi}{\hbar}\left[\int_0^\infty |\langle\mathbf{p}_f|V|\mathbf{p}_i\rangle|^2\delta\left(\frac{p_f^2}{2\mu}-\frac{p_i^2}{2\mu}\right)p_f^2\,dp_f\right]d\Omega \qquad (19.3.1)$$

$$=\frac{2\pi}{\hbar}|\langle\mathbf{p}_f|V|p_i\rangle|^2\mu\mathbf{p}_i\,d\Omega \qquad (19.3.2)$$

(Hereafter $p_f=p_i=p=\hbar k$ is understood.) This transition rate is just the rate of the flow of probability into $d\Omega$. Since the probability comes in at a rate [recall $j=\rho v|\mathbf{p}_i\rangle\to(2\pi\hbar)^{-3/2}e^{i\mathbf{p}_i\cdot\mathbf{r}/\hbar}$]

$$j_{\text{inc}}=\frac{\hbar k}{\mu}\left(\frac{1}{2\pi\hbar}\right)^3 \qquad (19.3.3)$$

in the direction $\mathbf{p}_i$, the differential cross section, which measures the rate at which probability is intercepted (and channeled off to $d\Omega$), is

$$\frac{d\sigma}{d\Omega}d\Omega=\frac{R_{i\to d\Omega}}{j_{\text{inc}}}=(2\pi)^4\mu^2\hbar^2|\langle\mathbf{p}_j|V|\mathbf{p}_i\rangle|^2\,d\Omega$$

$$\frac{d\sigma}{d\Omega}=\left|\frac{\mu}{2\pi\hbar^2}\int e^{-i\mathbf{q}\cdot\mathbf{r}'}V(\mathbf{r}')\,d^3\mathbf{r}'\right|^2 \qquad (19.3.4)$$

where

$$\hbar\mathbf{q}=\mathbf{p}_f-\mathbf{p}_i \qquad (19.3.5)$$

is the *momentum transferred* to the particle. For later reference note that

$$|\mathbf{q}|^2=|\mathbf{k}_f-\mathbf{k}_i|^2=2k^2(1-\cos\theta)=4k^2\sin^2(\theta/2) \qquad (19.3.6)$$

Thus the dependence of $d\sigma/d\Omega$ on the incident energy and the scattering angle is through the combination $|\mathbf{q}|\equiv q=2k\sin(\theta/2)$.

By comparing Eqs. (19.3.4) and (19.2.18) we can get $f(\theta)$, up to a phase factor of unit modulus (relative to the incident wave). We shall see later that this factor is -1. So,

$$f(\theta,\phi)=\frac{-\mu}{2\pi\hbar^2}\int e^{-i\mathbf{q}\cdot\mathbf{r}'}V(\mathbf{r}')\,d^3\mathbf{r}' \qquad (19.3.7)$$

Thus, in this *Born approximation, $f(\theta,\phi)=f(\mathbf{q})$ is just the Fourier transform of the potential with respect to momentum transfer (up to a constant factor).*

Hereafter we focus on potentials that are spherically symmetric: $V(\mathbf{r})=V(r)$. In this case, we can choose the z' direction parallel to $\mathbf{q}$ in the $d^3\mathbf{r}'$ integration, so that

$$\begin{aligned} f(\theta,\phi)&=\frac{-\mu}{2\pi\hbar^2}\int e^{-iqr'\cos\theta'}V(r')\,d(\cos\theta')\,d\phi'r'^2\,dr' \\ &=\frac{-2\mu}{\hbar^2}\int\frac{\sin qr'}{q}V(r')r'\,dr' \\ &=f(\theta) \end{aligned} \tag{19.3.8}$$

That f should be independent of ϕ in this case could have been anticipated. The incident wave e^{ikz} is insensitive to a change in ϕ, i.e., to rotations around the z axis. The potential, being spherically symmetric, also knows nothing about ϕ. It follows that f cannot pick up any dependence on ϕ. In the language of angular momentum, the incident wave has no l_z and this feature is preserved in the scattering. Consequently the scattered wave must also have no l_z, i.e., be independent of ϕ.

Let us calculate $f(\theta)$ for the *Yukawa potential*

$$V(r)=\frac{g\,e^{-\mu_0 r}}{r} \tag{19.3.9}$$

From Eq. (19.3.8),

$$\begin{aligned} f(\theta)&=-\frac{2\mu g}{\hbar^2 q}\int_0^\infty\frac{e^{iqr'}-e^{-iqr'}}{2i}e^{-\mu_0 r'}\,dr' \\ &=\frac{-2\mu g}{\hbar^2(\mu_0^2+q^2)} \end{aligned} \tag{19.3.10}$$

$$\frac{d\sigma}{d\Omega}=\frac{4\mu^2g^2}{\hbar^4[\mu_0^2+4k^2\sin^2(\theta/2)]^2} \tag{19.3.11}$$

If we now set $g=Ze^2$, $\mu_0=0$, we get the cross section for *Coulomb scattering* of a particle of charge e on a potential $\phi=Ze/r$ (or $V=Ze^2/r$):

$$\begin{aligned} \left.\frac{d\sigma}{d\Omega}\right|_{\text{Coulomb}}&=\frac{\mu^2(Ze^2)^2}{4p^4\sin^4(\theta/2)} \\ &=\frac{(Ze^2)^2}{16E^2\sin^4(\theta/2)} \end{aligned} \tag{19.3.12}$$

where $E=p^2/2\mu$ is the kinetic energy of the incident particle. This answer happens to be exact quantum mechanically as well as classically. (It was calculated classically by Rutherford and is called the *Rutherford cross section*.) Although we managed to get the right $d\sigma/d\Omega$ by taking the $\mu_0\to 0$ limit of the Yukawa potential calculation, there are some fine points to note. First of all, the Coulomb potential cannot be handled by the formulation we have developed, since the potential does not vanish faster than r^{-1}. In other words, the asymptotic form

$$\psi \xrightarrow[r\to\infty]{} e^{ikz} + f(\theta, \phi) \frac{e^{ikr}}{r}$$

is not applicable here since the particle is never free from the influence of the potential. (This manifests itself in the fact that the total cross section is infinite: if we try to integrate $d\sigma/d\Omega$ over θ, the integral diverges as $\int d\theta/\theta^3$ as $\theta\to 0$.) It is, however, possible to define a scattering amplitude $f_c(\theta)$ in the following sense. One finds that as $r\to\infty$, there are positive energy eigensolutions to the Coulomb Hamiltonian of the form‡

$$\psi \xrightarrow[r\to\infty]{} \widetilde{e^{ikz}} + f_c(\theta) \left(\widetilde{\frac{e^{ikr}}{r}}\right) \tag{19.3.13}$$

where the tilde tells us that these are not actually plane or spherical waves, but rather these objects modified by the long-range Coulomb force. For example

$$\widetilde{\frac{e^{ikr}}{r}} = \frac{e^{i(kr-\gamma \ln kr)}}{r} \tag{19.3.14}$$

$$\gamma = \frac{Ze^2\mu}{\hbar^2 k} \tag{19.3.15}$$

is the distorted spherical wave, familiar to us from Section 12.6. By comparing the ratio of flux into $d\Omega$ to flux coming in (due to these distorted waves) one finds that

$$\frac{d\sigma}{d\Omega} = |f_c|^2$$

where

$$f_c(\theta) = -\frac{\gamma}{2k(\sin\theta/2)^2} \exp(-i\gamma \ln \sin^2\theta/2 + \text{const}) \tag{19.3.16}$$

and where the constant is *purely imaginary*. Comparing this to the Yukawa amplitude, Eq. (19.3.10), after setting $\mu_0 = 0$, $g = Ze^2$, we find agreement up to the exponential phase factor. This difference does not show up in $d\sigma/d\Omega$, but will show up when we consider identical-particle scattering later in this chapter.

‡ See A. Messiah, *Quantum Mechanics*, Wiley, New York (1966), page 422.

*Exercise 19.3.1.** Show that

$$\sigma_{\text{Yukawa}} = 16\pi r_0^2 \left(\frac{g\mu r_0}{\hbar^2}\right)^2 \frac{1}{1+4k^2 r_0^2}$$

where $r_0 = 1\mu_0$ is the range. Compare σ to the geometrical cross section associated with this range.

Exercise 19.3.2* (1) Show that if $V(r) = -V_0\theta(r_0 - r)$,

$$\frac{d\sigma}{d\Omega} = 4r_0^2 \left(\frac{\mu V_0 r_0^2}{\hbar^2}\right)^2 \frac{(\sin qr_0 - qr_0 \cos qr_0)^2}{(qr_0)^6}$$

(2) Show that as $kr_0 \to 0$, the scattering becomes isotropic and

$$\sigma \cong \frac{16\pi r_0^2}{9} \left(\frac{\mu V_0 r_0^2}{\hbar^2}\right)^2$$

*Exercise 19.3.3.** Show that for the *Gaussian potential*, $V(r) = V_0 e^{-r^2/r_0^2}$,

$$\frac{d\sigma}{d\Omega} = \frac{\pi r_0^2}{4} \left(\frac{\mu V_0 r_0^2}{\hbar^2}\right)^2 e^{-q^2 r_0^2/2}$$

$$\sigma = \frac{\pi^2}{2k^2} \left(\frac{\mu V_0 r_0^2}{\hbar^2}\right)^2 (1 - e^{-2k^2 r_0^2})$$

[Hint: Since $q^2 = 2k^2(1 - \cos\theta)$, $d(\cos\theta) = -d(q^2)/2k^2$.]

Let us end this section by examining some general properties of $f(\theta)$. We see from Eq. (19.3.7) that at *low energies* ($k \to 0$), $q = 2k \sin(\theta/2) \to 0$ and

$$f(\theta) \sim -\frac{\mu}{2\pi\hbar^2} \int V(\mathbf{r}')\, d^3\mathbf{r}'$$

$$\cong -\frac{\mu V_0 r_0^3}{\hbar^2} \qquad (19.3.17)$$

where V_0 is some effective height of V, and r_0 is some effective range. At *high energies*, the exponential factor $e^{-iqr'\cos\theta'}$ oscillates rapidly. This means that the scattered waves coming from different points $\mathbf{r}'$ add with essentially random phases, except in the small range where the phase is stationary:

$$qr' \cos\theta' \lesssim \pi$$

$$2k \sin(\theta/2) r_0 \lesssim \pi \qquad (\text{since } r' \cos\theta' \cong r_0)$$

$$k\theta r_0 \lesssim \pi \quad (\sin\theta/2 \simeq \theta/2)$$

Thus the scattering amplitude is appreciable only in a small forward cone of angle (dropping constants of order unity)

$$\theta \lesssim \frac{1}{kr_0} \tag{19.3.18}$$

These arguments assume $V(r')$ is regular near $r'=0$. But in some singular cases $[V \propto (r')^{-3}$, say] the r' integral is dominated by small r' and $kr' \cos \theta'$ is not necessarily a large phase. Both the Yukawa and Gaussian potential (Exercise 19.3.3) are free of such pathologies and exhibit this forward peak at high energies.

Exercise 19.3.4. Verify the above claim for the Gaussian potential.

When can we trust the Born approximation? Since we treated the potential as a perturbation, our guess would be that it is reliable at high energies. We shall see in the next section that this is indeed correct, but that the Born approximation can also work at low energies provided a more stringent condition is satisfied.

19.4. Born Again (The Time-Independent Description)

In this approach, the central problem is to find solutions to the full Schrödinger equation

$$(\nabla^2 + k^2)\psi_{\mathbf{k}} = \frac{2\mu}{\hbar^2} V\psi_{\mathbf{k}} \tag{19.4.1}$$

of the form

$$\psi_{\mathbf{k}} = e^{i\mathbf{k}\cdot\mathbf{r}} + \psi_{sc} \tag{19.4.2a}$$

where

$$\psi_{sc} \xrightarrow[r\to\infty]{} f(\theta, \phi)\frac{e^{ikr}}{r} \tag{19.4.2b}$$

In the above, θ and ϕ are measured relative to $\mathbf{k}$, chosen along the z axis (Fig. 19.1). One approaches the problem as follows. One finds a *Green's function* $G^0(\mathbf{r}, \mathbf{r}')$ which satisfies

$$(\nabla^2 + k^2)G^0(\mathbf{r}, \mathbf{r}') = \delta^3(\mathbf{r} - \mathbf{r}') \tag{19.4.3}$$

in terms of which the *formal* general solution to Eq. (19.4.1) is

$$\psi_{\mathbf{k}}(\mathbf{r})=\psi^0(\mathbf{r})+\frac{2\mu}{\hbar^2}\int G^0(\mathbf{r},\mathbf{r}')V(\mathbf{r}')\psi_{\mathbf{k}}(\mathbf{r}')\,d^3\mathbf{r}' \tag{19.4.4}$$

where $\psi^0(\mathbf{r})$ is an arbitrary free-particle solution of energy $\hbar^2k^2/2\mu$:

$$(\nabla^2+k^2)\psi^0=0 \tag{19.4.5}$$

We will soon nail down ψ^0 using the boundary conditions.

Applying ∇^2+k^2 to both sides of Eq. (19.4.4) one may easily verify that $\psi_{\mathbf{k}}$ indeed is a solution to Eq. (19.4.1). The idea here is quite similar to what is employed in solving Poisson's equation for the electrostatic potential in terms of the charge density ρ:

$$\nabla^2\phi=-4\pi\rho$$

One first finds G, the response to a point charge at $\mathbf{r}'$:

$$\nabla^2G=-4\pi\delta^3(\mathbf{r}-\mathbf{r}')$$

Exercise 12.6.4 tells us that

$$G(\mathbf{r},\mathbf{r}')=G(\mathbf{r}-\mathbf{r}')=\frac{1}{|\mathbf{r}-\mathbf{r}'|}$$

One then views ρ as a superposition of point charges and, since Poisson's equation is linear, obtains ϕ as the sum of ϕ's produced by these charges:

$$\phi(\mathbf{r})=\int G(\mathbf{r}-\mathbf{r}')\rho(\mathbf{r}')\,d^3\mathbf{r}'=\int\frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\,d^3\mathbf{r}'$$

(By acting on both sides with ∇^2 and using $\nabla^2G=-4\pi\delta^3$, you may verify that ϕ satisfies Poisson's equation.)

One can add to this $\phi(\mathbf{r})$ any ϕ^0 that satisfies $\nabla^2\phi^0=0$. Using the boundary condition $\phi=0$ when $\rho=0$, we get rid of ϕ^0.

In the scattering problem we pretend that the right-hand side of Eq. (19.4.1) is some given source and write Eq. (19.4.4) for $\psi_{\mathbf{k}}$ in terms of the Green's function. The only catch is that the source for $\psi_{\mathbf{k}}$ is $\psi_{\mathbf{k}}$ itself. Thus Eq. (19.4.4) is really not a solution, but an integral equation for $\psi_{\mathbf{k}}$. The motivation for converting the differential equation to an integral equation is similar to that in the case of $U_I(t,t_0)$: to obtain a perturbative expansion for $\psi_{\mathbf{k}}$ in powers of V. *To zeroth order in V*, Eq. (19.4.2a) tells us that $\psi_{\mathbf{k}}$ is $e^{i\mathbf{k}\cdot\mathbf{r}}$, since there is no scattered wave if V is neglected; whereas Eq. (19.4.4) tells us that $\psi_{\mathbf{k}}=\psi^0$, since the integral over $\mathbf{r}'$ has an explicit power of V in it while ψ^0 has no dependence on V [since it is the solution to Eq. (19.4.5)]. We are thus able to nail down the arbitrary function ψ^0 in Eq. (19.4.4):

$$\psi^0=e^{i\mathbf{k}\cdot\mathbf{r}} \tag{19.4.6}$$

and conclude that in the present scattering problem

$$\psi_{\mathbf{k}}=e^{i\mathbf{k}\cdot\mathbf{r}}+\frac{2\mu}{\hbar^2}\int G^0(\mathbf{r},\mathbf{r}')V(\mathbf{r}')\psi_{\mathbf{k}}(\mathbf{r}')\,d^3\mathbf{r}' \qquad (19.4.7)$$

Upon comparing this to Eq. (19.4.2a) we see that we are associating the second piece with the scattered wave. For consistency of interpretation, it must contain purely outgoing waves at spatial infinity. Since $G^0(\mathbf{r},\mathbf{r}')$ is the scattered wave produced by a point source at $\mathbf{r}'$, it is necessary that $G^0(\mathbf{r},\mathbf{r}')$ be purely outgoing asymptotically. This is an additional physical constraint on G^0 over and above Eq. (19.4.3). As we shall see, this constraint, together with Eq. (19.4.3), will determine G^0 for us uniquely.

Imagine that we have found this G^0. We are now in a position to obtain a perturbative solution for $\psi_{\mathbf{k}}$ starting with Eq. (19.4.7). To zeroth order we have seen that $\psi_{\mathbf{k}}=e^{i\mathbf{k}\cdot\mathbf{r}}$. To go to first order, we feed the zeroth-order $\psi_{\mathbf{k}}$ into the right-hand side and obtain

$$\psi_{\mathbf{k}}=e^{i\mathbf{k}\cdot\mathbf{r}}+\frac{2\mu}{\hbar^2}\int G^0(\mathbf{r},\mathbf{r}')V(\mathbf{r}')\,e^{i\mathbf{k}\cdot\mathbf{r}'}\,d^3\mathbf{r}'+O(V^2) \qquad (19.4.8)$$

If we feed this first-order result back into the right-hand side of Eq. (19.4.7), we get (in symbolic form) the result good to second order:

$$\psi_{\mathbf{k}}=\psi^0+\frac{2\mu}{\hbar^2}G^0V\psi^0+\left(\frac{2\mu}{\hbar^2}\right)^2G^0VG^0V\psi^0+O(V^3)$$

and so on.

Let us now turn to the determination of G^0, starting with Eq. (19.4.3):

$$(\nabla^2+k^2)G^0(\mathbf{r},\mathbf{r}')=\delta^3(\mathbf{r}-\mathbf{r}')$$

We note that this equation does not have a unique solution, since, given any solution, we can get another by adding to it a function η^0 that obeys the homogeneous equation

$$(\nabla^2+k^2)\eta^0=0$$

Conversely, any two columns G^0 and $G^{0\prime}$ can differ only by some η^0. So we will first find the simplest G^0 we can, and then add whatever η^0 it takes to make the sum purely outgoing.

Since (∇^2+k^2) and $\delta^3(\mathbf{r}-\mathbf{r}')$ are invariant under the overall translation of $\mathbf{r}$ *and* $\mathbf{r}'$, we know the equation admits translationally invariant solutions‡:

$$G^0(\mathbf{r},\mathbf{r}')=G^0(\mathbf{r}-\mathbf{r}')$$

‡ Note that if an equation has some symmetry, like rotational invariance, it means only that rotationally invariant solutions exist, and not that all solutions are rotationally invariant. For example, the hydrogen atom Hamiltonian is rotationally invariant, but the eigenfunctions are not in general. But there are some (with $l=m=0$) which are.

Replace $\mathbf{r}-\mathbf{r}'$ by $\mathbf{r}$ for convenience. [Once we find $G^0(\mathbf{r})$, we can replace $\mathbf{r}$ by $\mathbf{r}-\mathbf{r}'$.] So we want to solve

$$(\nabla^2+k^2)G^0(\mathbf{r})=\delta^3(\mathbf{r}) \tag{19.4.9}$$

For similar reasons as above, we look for a rotationally invariant solution

$$G^0(\mathbf{r})=G^0(r)$$

Writing

$$G^0(r)=\frac{U(r)}{r}$$

we find that *for* $r\neq 0$, $U(r)$ satisfies

$$\frac{d^2U}{dr^2}+k^2U=0$$

the general solution to which is

$$U(r)=A\,e^{ikr}+B\,e^{-ikr}$$

or

$$G^0(r)=\frac{A\,e^{ikr}}{r}+\frac{B\,e^{-ikr}}{r} \tag{19.4.10}$$

where A and B are arbitrary constants at this point. Since we want G^0 to be purely outgoing we set $B=0$:

$$G^0(r)=\frac{A\,e^{ikr}}{r} \tag{19.4.11}$$

We find A by calculating $(\nabla^2+k^2)G^0(r)$ as $r\to 0$‡

$$(\nabla^2+k^2)G^0(r)\xrightarrow[r\to 0]{}-4\pi A\delta^3(\mathbf{r}) \tag{19.4.12}$$

‡ We use $\nabla^2(\psi\chi)=\psi\nabla^2\chi+\chi\nabla^2\psi+2\nabla\psi\cdot\nabla\chi$ and $\nabla^2=r^{-2}(\partial/\partial r)r^2\,\partial/\partial r$ on a function of r alone.

which gives us

$$G^0(r) = -\frac{e^{ikr}}{4\pi r} \tag{19.4.13}$$

We cannot add any η^0 to this solution, without destroying its purely outgoing nature, since the general form of the free-particle solution, regular in all space, is

$$\eta^0(\mathbf{r}) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} C_{lm} j_l(kr) Y_l^m(\theta, \phi) \tag{19.4.14}$$

and since, as $r \to \infty$, the spherical Bessel functions are made up of incoming and outgoing waves of equal amplitude

$$j_l(kr) \xrightarrow[r\to\infty]{} \frac{\sin(kr - l\pi/2)}{kr} = \frac{e^{i(kr-l\pi/2)} - e^{-i(kr-l\pi/2)}}{2ikr} \tag{19.4.15}$$

Let us now feed

$$G^0(\mathbf{r}, \mathbf{r}') = G^0(\mathbf{r} - \mathbf{r}') = -\frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|} \tag{19.4.16}$$

into Eq. (19.4.7) to obtain

$$\begin{aligned} \psi_{\mathbf{k}} &= e^{i\mathbf{k}\cdot\mathbf{r}} - \frac{2\mu}{4\pi\hbar^2} \int \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} V(\mathbf{r}')\psi_{\mathbf{k}}(\mathbf{r}')\, d^3\mathbf{r}' \\ &= e^{i\mathbf{k}\cdot\mathbf{r}} + \psi_{\text{sc}} \end{aligned} \tag{19.4.17}$$

Let us now verify that as $r \to \infty$, ψ_{sc} has the desired form $f(\theta, \phi)\, e^{ikr}/r$. Our first instinct may be to approximate as follows:

$$\frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} \cong \frac{e^{ikr}}{r}$$

in the $\mathbf{r}'$ integral since $\mathbf{r}'$ is confined to $|\mathbf{r}'| \lesssim r_0$ (the range), whereas $r \to \infty$. That this is wrong is clear from the fact that if we do so, the corresponding f has no θ or ϕ dependence. Let us be more careful. We first approximate

$$\begin{aligned} |\mathbf{r}-\mathbf{r}'| &= (r^2 + r'^2 - 2\mathbf{r}\cdot\mathbf{r}')^{1/2} \\ &= r\left[1 + \left(\frac{r'}{r}\right)^2 - 2\frac{\mathbf{r}\cdot\mathbf{r}'}{r^2}\right]^{1/2} = \end{aligned}$$

$$= r\left(1 - 2\frac{\mathbf{r}\cdot\mathbf{r}'}{r^2}\right)^{1/2} + O\left[\left(\frac{r'}{r}\right)^2\right] r$$

$$\cong r\left(1 - \frac{\mathbf{r}\cdot\mathbf{r}'}{r^2}\right) \tag{19.4.18}$$

We have thrown away the term quadratic in (r'/r) and used the approximation $(1+x)^n \cong 1 + nx$ for small x. So

$$\frac{1}{|\mathbf{r}-\mathbf{r}'|} = \frac{1}{r[1-(\mathbf{r}\cdot\mathbf{r}')/r^2]} \cong \frac{1}{r}\left(1 + \frac{\mathbf{r}\cdot\mathbf{r}'}{r^2}\right) \tag{19.4.19}$$

Whereas replacing $|\mathbf{r}-\mathbf{r}'|^{-1}$ in the integral leads to errors which vanish as $r\to\infty$, this is not so for the factor $e^{ik|\mathbf{r}-\mathbf{r}'|}$. We have

$$\begin{aligned} k|\mathbf{r}-\mathbf{r}'| &= kr\left(1 - \frac{\mathbf{r}\cdot\mathbf{r}'}{r^2}\right) \\ &= kr - k\hat{r}\cdot\mathbf{r}' \\ &= kr - \mathbf{k}_f\cdot\mathbf{r}' \end{aligned} \tag{19.4.20}$$

where $\mathbf{k}_f$ is the wave vector of the detected particle: it has the same magnitude (k) as the incident particle and points in the direction ($\hat{r}$) of observation (Fig. 19.2). Consequently

$$\frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} \cong \frac{e^{ikr}}{r} e^{-i\mathbf{k}_f\cdot\mathbf{r}'} \tag{19.4.21}$$

and

$$\psi_{\mathbf{k}} \xrightarrow[r\to\infty]{} e^{i\mathbf{k}\cdot\mathbf{r}} - \frac{e^{ikr}}{r}\frac{2\mu}{4\pi\hbar^2}\int e^{-i\mathbf{k}_f\cdot\mathbf{r}'} V(\mathbf{r}')\psi_{\mathbf{k}}(\mathbf{r}')\, d^3\mathbf{r}' \tag{19.4.22}$$

Thus the solution we have found has the desired form as $r\to\infty$. Equation (19.4.22) of course does not determine $f(\theta, \phi)$ since $\psi_{\mathbf{k}}$ is present in the $\mathbf{r}'$ integration.

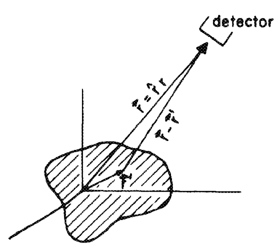

Figure 19.2. The particle is observed at the point $\mathbf{r}$. The $\mathbf{r}'$ integration is restricted to the shaded region which symbolizes the range of the potential.

However, to any desired order this $\psi_{\mathbf{k}}$ can be replaced by the *calculable* lower-order approximation. In particular, to first order,

$$f(\theta, \phi) = -\frac{2\mu}{4\pi\hbar^2} \int e^{-i\mathbf{k}_f\cdot\mathbf{r}'} V(\mathbf{r}')\, e^{i\mathbf{k}_i\cdot\mathbf{r}'}\, d^3\mathbf{r}' \tag{19.4.23}$$

where we have added a subscript i to $\mathbf{k}$ to remind us that it is the initial or incident wave vector. We recognize $f(\theta, \phi)$ to be just the Born approximation calculated in the last section [Eq. (19.3.7)]. The phase factor -1, relative to the incident wave was simply assumed there. The agreement between the time-dependent and time-independent calculations of f persists to all orders in the perturbation expansion.

There is another way (involving Cauchy's theorem) to solve

$$(\nabla^2 + k^2)G^0(\mathbf{r}) = \delta^3(\mathbf{r}) \tag{19.4.24}$$

Fourier transforming both sides, we get

$$\left(\frac{1}{2\pi}\right)^{3/2} \int e^{-i\mathbf{q}\cdot\mathbf{r}}(\nabla^2 + k^2)G^0(\mathbf{r})\, d^3\mathbf{r} = \left(\frac{1}{2\pi}\right)^{3/2} \tag{19.4.25}$$

If we let ∇^2 act to the left (remember it is Hermitian) we get

$$(k^2 - q^2)\left(\frac{1}{2\pi}\right)^{3/2} \int e^{-i\mathbf{q}\cdot\mathbf{r}} G^0(\mathbf{r})\, d^3\mathbf{r} = \left(\frac{1}{2\pi}\right)^{3/2} \tag{19.4.26}$$

$$(k^2 - q^2)G^0(\mathbf{q}) = \left(\frac{1}{2\pi}\right)^{3/2} \tag{19.4.27}$$

As always, going to momentum space has reduced the differential equation to an algebraic equation. The solution is

$$G^0(\mathbf{q}) = \frac{1}{(2\pi)^{3/2}(k^2 - q^2)} \tag{19.4.28}$$

except at the point $q = k$ where $G^0(\mathbf{q})$ diverges. The reason for this divergence is the following. Equation (19.4.24) is the coordinate space version of the abstract equation

$$(D^2 + k^2)G^0 = I \tag{19.4.29}$$

where

$$D^2 = D_x^2 + D_y^2 + D_z^2 \tag{19.4.30}$$

(D_x is just the x derivative operator D introduced in Section 1.10, and D_y and D_z are y and z derivative operators.) Thus G^0 is the inverse of (D^2+k^2):

$$G^0=(D^2+k^2)^{-1} \tag{19.4.31}$$

Now, we know that we cannot invert an operator that has a vanishing determinant or equivalently (for a Hermitian operator, since it can be diagonalized) a zero eigenvalue. The operator (D^2+k^2) has a zero eigenvalue since

$$(\nabla^2+k^2)\psi=0 \tag{19.4.32}$$

has nontrivial (plane wave) solutions. We therefore consider a slightly different operator, $D^2+k^2+i\varepsilon$, where ε is positive and infinitesimal. This too has a zero eigenvalue, but the corresponding eigenfunctions are plane waves of *complex wave number*. Such functions are not part of the space we are restricting ourselves to, namely, the space of functions normalized to unity or the Dirac delta function.‡ Thus $D^2+k^2+i\varepsilon$ may be inverted *within* the *physical Hilbert space*. Let us call the corresponding Green's function G_ε^0. At the end of the calculation we will send ε to zero.§

Clearly

$$G_\varepsilon^0(\mathbf{q})=\frac{1}{(2\pi)^{3/2}}\frac{1}{k^2+i\varepsilon-q^2} \tag{19.4.33}$$

The coordinate space function is given by the inverse transform:

$$G_\varepsilon^0(\mathbf{r})=\frac{1}{(2\pi)^3}\int\frac{e^{i\mathbf{q}\cdot\mathbf{r}}}{k^2+i\varepsilon-q^2}\,d^3\mathbf{q} \tag{19.4.34}$$

We choose the q_z axis parallel to $\mathbf{r}$. If θ and ϕ are the angles in $\mathbf{q}$ space,

$$G_\varepsilon^0(\mathbf{r})=\frac{1}{8\pi^3}\int\frac{e^{iqr\cos\theta}}{k^2+i\varepsilon-q^2}\,d(\cos\theta)\,d\phi q^2\,dq$$

$$=\frac{1}{4\pi^2}\int_0^\infty\frac{e^{iqr}-e^{-iqr}}{iqr}\,\frac{q^2\,dq}{k^2+i\varepsilon-q^2} \tag{19.4.35a}$$

$$=\frac{1}{4\pi^2}\int_{-\infty}^\infty\frac{e^{iqr}}{iqr}\,\frac{q^2\,dq}{k^2+i\varepsilon-q^2} \tag{19.4.35b}$$

$$=\frac{-i}{4\pi^2 r}\int_{-\infty}^\infty\frac{e^{iqr}q\,dq}{k^2+i\varepsilon-q^2} \tag{19.4.35c}$$

‡ Recall from Section 1.10 that if k is complex, the norm diverges exponentially.

§ This is called the "$i\varepsilon$ prescription." Throughout the analysis ε will be considered only to first order.

[In going from (19.4.35a) to (19.4.35b) above, we changed q to $-q$ in the e^{-iqr} piece.]

We proceed to evaluate the above integral by means of Cauchy's residue theorem, which states that for any analytic function $f(z)$ of a complex variable z,

$$\oint f(z)\, dz = 2\pi i \sum_j R(z_j) \tag{19.4.36}$$

where $\oint$ denotes integration around a closed contour in the complex z plane and $R(z_j)$ is the residue of the pole at the point z_j lying inside the contour.‡

Let us view q as a complex variable which happens to be taking only real values ($-\infty$ to $+\infty$) in Eq. (19.4.35).

We are trying to evaluate the integral of the function

$$w(q) = \frac{-i}{4\pi^2 r} \frac{e^{iqr} q}{k^2 + i\varepsilon - q^2} \tag{19.4.37}$$

along the real axis from $-\infty$ to $+\infty$.

This function has poles where

$$k^2 + i\varepsilon - q^2 = 0$$

or (to first order in ε),

$$(k+q+i\eta)(k-q+i\eta) = 0 \qquad (\eta \cong \varepsilon/2k) \tag{19.4.38}$$

These poles are shown in Fig. 19.3.

We are not yet ready to use Cauchy's theorem because we do not have a closed contour. Let us now close the contour via a large semicircle C_ρ whose radius $\rho \to \infty$. Now we can use Cauchy's theorem, but haven't we changed the quantity we wanted to calculate? No, because C_ρ does not contribute to the integral as $\rho \to \infty$. To see this, let us write $q = \rho\, e^{i\theta}$ on C_ρ. Then

$$w(q) \xrightarrow[\rho \to \infty]{} (\text{const}) \frac{e^{iqr}}{q} \tag{19.4.39}$$

and

‡ Recall that if

$$f(z) \xrightarrow[z \to z_j]{} \frac{R(z_j)}{z - z_j}$$

then

$$R(z_j) = \lim_{z \to z_j} f(z)(z - z_j)$$

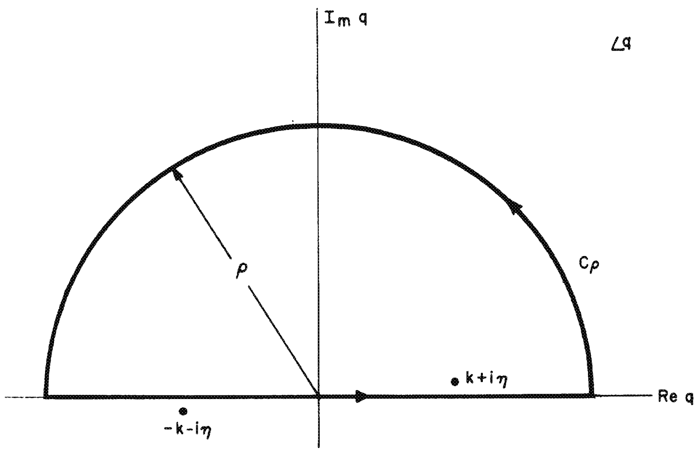

Figure 19.3. The poles of the function $w(q)$ in the complex q plane. We want the integral along the real axis from $-\infty$ to $+\infty$. We add to it the contribution along C_ρ (which vanishes as ρ tends to ∞) in order to close the contour of integration and to use Cauchy's theorem.

$$\int_{C_\rho} w(q)\, dq \sim \int_{C_\rho} e^{iqr} \frac{dq}{q} = \int_0^\pi e^{i\rho r(\cos\theta + i\sin\theta)} i\, d\theta \qquad (19.4.40)$$

Except for the tiny region near $\theta = 0$ (which contributes only an infinitesimal amount) the integral vanishes since $e^{-\rho r \sin\theta} \to 0$ as $\rho \to \infty$. We now use Cauchy's theorem. The only pole enclosed is at $q = k + i\eta$. The residue here is

$$R(k+i\eta) = \lim_{q \to k+i\eta} (q-k-i\eta) w(q) = \frac{i}{8\pi^2 r} e^{i(k+i\eta)r} \qquad (19.4.41)$$

and

$$G^0(\mathbf{r}) = \lim_{\eta \to 0} 2\pi i R = -\frac{e^{ikr}}{4\pi r} \qquad (19.4.42)$$

Notice that although the $i\varepsilon$ ($\varepsilon > 0$) prescription happens to give the right answer here, there are other ways to evaluate the integral, which may be appropriate in other contexts. For example if we choose $\varepsilon < 0$, we get a purely incoming wave, since η changes sign and the pole near $q \cong -k$ gets into the contour.

Validity of the Born Approximation

Since in the Born approximation one replaces $\psi_\mathbf{k} = e^{i\mathbf{k}\cdot\mathbf{r}'} + \psi_{sc}$ by just $e^{i\mathbf{k}\cdot\mathbf{r}'}$ in the right-hand side of the integral Eq. (19.4.17), it is a good approximation only if $|\psi_{sc}| \ll |e^{i\mathbf{k}\cdot\mathbf{r}'}|$ in the region $|\mathbf{r}'| \lesssim r_0$. Since we expect ψ_{sc} to be largest near the origin, let us perform a comparison there, using Eq. (19.4.17) itself to evaluate $\psi_{sc}(0)$:

$$\frac{|\psi_{sc}(0)|}{|e^{ikz}(0)|}=|\psi_{sc}(0)|=\left|\frac{2\mu}{4\pi\hbar^2}\int\frac{e^{ikr'}}{r'}V(\mathbf{r}')\,e^{-i\mathbf{k}_i\cdot\mathbf{r}'}\,d^3\mathbf{r}'\right| \tag{19.4.43}$$

Let us assume $V(\mathbf{r})=V(r)$. In this case a rough criterion for the validity of the Born approximation is

$$\frac{2\mu}{\hbar^2 k}\left|\int e^{ikr'}\sin kr'\,V(r')\,dr'\right|\ll 1 \tag{19.4.44}$$

Exercise 19.4.1. Derive the inequality (19.4.44).

At low energies, $kr'\to 0$, $e^{ikr'}\to 1$, $\sin kr'\to kr'$ and we get the condition

$$\frac{2\mu}{\hbar^2}\left|\int r'V(r')\,dr'\right|\ll 1 \tag{19.4.45}$$

If $V(r)$ has an effective depth (or height) V_0 and range r_0, the condition becomes (dropping constants of order unity)

$$\frac{\mu V_0 r_0^2}{\hbar^2}\ll 1 \tag{19.4.46}$$

The low energy condition may be written as

$$V_0\ll\frac{\hbar^2}{\mu r_0^2}$$

Now a particle confined to a well of dimension r_0 must have a momentum of order $\hbar/r_0$ and a kinetic energy of order $\hbar^2/\mu r_0^2$. The above inequality says that if the Born approximation is to work at low energies, the potential must be too shallow to bind a particle confined to a region of size r_0.

At high energies when $kr_0\gg 1$ let us write inside the integral in Eq. (19.4.44)

$$e^{ikr'}\sin kr'=\frac{e^{2ikr'}-1}{2i}$$

and drop the exponential which will be oscillating too rapidly within the range of the potential and keep just the -1 part to get the following condition:

$$\frac{\mu}{\hbar^2 k}\left|\int V(r')\,dr'\right|\ll 1 \tag{19.4.47}$$

which could be rewritten as

$$\frac{\mu V_0 r_0^2}{\hbar^2} \ll k r_0 \tag{19.4.48}$$

We found that the Born approximation can be good even at low energies if the inequality (19.4.46) is satisfied. In fact, if it is, the Born approximation is good at all energies, i.e., Eq. (19.4.48) is automatically satisfied.

19.5. The Partial Wave Expansion

We have noted that if $V(\mathbf{r}) = V(r)$, $f(\theta, \phi) = f(\theta)$. Actually f is also a function of the energy $E = \hbar^2 k^2 / 2\mu$, though this dependence was never displayed explicitly. Since any function of θ can be expanded in terms of the Legendre polynomials

$$P_l(\cos\theta) = \left(\frac{4\pi}{2l+1}\right)^{1/2} Y_l^0 \tag{19.5.1}$$

we can expand $f(\theta, k)$ in terms of $P_l(\cos\theta)$ with k-dependent coefficients:

$$f(\theta, k) = \sum_{l=0}^{\infty} (2l+1) a_l(k) P_l(\cos\theta) \tag{19.5.2}$$

One calls $a_l(k)$ the *lth partial wave amplitude*. It has the following significance. The incident plane wave e^{ikz} is composed of states of all angular momenta [from Eq. (12.6.41)]:

$$e^{ikz} = e^{ikr\cos\theta} = \sum_{l=0}^{\infty} i^l (2l+1) j_l(kr) P_l(\cos\theta) \tag{19.5.3}$$

Since the potential conserves angular momentum, each angular momentum component scatters independently. The amplitude a_l is a measure of the scattering in the angular momentum l sector.

As it stands, the expansion in Eq. (19.5.2) has not done anything for us: we have traded one function of the two variables (θ and k) for an infinite number of functions $a_l(k)$ of one variable k. What makes the expansion useful is that at low energies, only the first few $a_l(k)$ are appreciably different from zero. In this case, one manages to describe the scattering in terms of just a few functions $a_0, a_1, \ldots$ of one variable. The following heuristic argument (corroborated by explicit calculations) is usually given to explain why the scattering is restricted to a few low l values at low k. Coming out of the accelerator is a uniform beam of particles moving along the z axis. All particles in a cylinder of radius ρ and thickness $d\rho$ (ρ is the impact parameter) have angular momentum

$$\hbar l \simeq \hbar k \rho \tag{19.5.4}$$

If the potential has range r_0, particles with $\rho > r_0$ will "miss" the target. Thus there will be scattering only up to

$$l_{\max} = k\rho_{\max} \cong kr_0 \tag{19.5.5}$$

[Conversely, by measuring $l_{\max}$ (from the angular dependence of f) we can deduce the range of the potential.]

*Exercise 19.5.1.** Show that for a 100-MeV (kinetic energy) neutron incident on a fixed nucleus, $l_{\max} \cong 2$. (Hint: The range of the nuclear force is roughly a Fermi $= 10^{-5}$ Å. Also $\hbar c \cong 200$ MeV F is a more useful mnemonic for nuclear physics.)

Given a potential $V(r)$, how does one calculate $a_l(k)$ in terms of it? In other words, how is a_l related to the solution to the Schrödinger equation for angular momentum l? We begin by considering a free particle. Using

$$j_l(kr) \underset{r\to\infty}{\longrightarrow} \frac{\sin(kr - l\pi/2)}{kr} \tag{19.5.6}$$

we get, from Eq. (19.5.3),

$$e^{ikz} \underset{r\to\infty}{\longrightarrow} \frac{1}{2ik} \sum_{l=0}^{\infty} i^l (2l+1)\left(\frac{e^{i(kr-l\pi/2)}}{r} - \frac{e^{-i(kr-l\pi/2)}}{r}\right) P_l(\cos\theta) \tag{19.5.7}$$

$$= \frac{1}{2ik} \sum_{l=0}^{\infty} (2l+1)\left(\frac{e^{ikr}}{r} - \frac{e^{-(ikr-l\pi)}}{r}\right) P_l(\cos\theta) \tag{19.5.8}$$

upon using $i = e^{i\pi/2}$. Thus at each angular momentum we have incoming and outgoing waves of the same amplitude. (Their phases differ by $l\pi$ because the repulsive centrifugal barrier potential is present at $l \neq 0$ even for a free particle.) The probability currents associated with the two waves are equal and opposite.‡ This equality is expected since in this steady state there should be no net probability flux flowing into the origin or coming out of it. (This balance should occur separately for each l, since scattering in each l is independent due to angular momentum conservation.)

What happens if we turn on a potential? As $r\to\infty$, the radial wave functions must reduce to the free-particle wave function, although there can be a *phase shift* $\delta_l(k)$ due to the potential:

$$R_l(r) = \frac{U_l(r)}{r} \underset{r\to\infty}{\longrightarrow} \frac{A_l \sin[kr - l\pi/2 + \delta_l(k)]}{r} \tag{19.5.9}$$

‡ Once again, can we speak of the current associated with a given l and also with the incoming and outgoing waves at a given l? Yes. If we calculate the total $\mathbf{j}$ (which will have only a radial part as $r\to\infty$) and integrate over all angles, the orthogonality of P_l's will eliminate all interference terms between different l's. There will also be no interference between the incoming and outgoing waves. [See footnote related to Eq. (19.2.13).]

where A_l is some constant. So

$$\psi_{\mathbf{k}}(\mathbf{r}) \xrightarrow[r\to\infty]{} \sum_{l=0}^{\infty} A_l \frac{(e^{i(kr-l\pi/2+\delta_l)} - e^{-i(kr-l\pi/2+\delta_l)})P_l(\cos\theta)}{r} \tag{19.5.10}$$

To find A_l, we note that since $V(r)$ produces only an outgoing wave, the *incoming* waves must be the same for $\psi_{\mathbf{k}}$ and the plane wave $e^{i\mathbf{k}\cdot\mathbf{r}} = e^{ikz}$. Comparing the coefficients of e^{-ikr}/r in Eqs. (19.5.8) and (19.5.10), we get

$$A_l = \frac{2l+1}{2ik} e^{i(l\pi/2+\delta_l)} \tag{19.5.11}$$

Feeding this into Eq. (19.5.10) we get

$$\psi_{\mathbf{k}}(\mathbf{r}) \xrightarrow[r\to\infty]{} \frac{1}{2ikr} \sum_{l=0}^{\infty} (2l+1)[e^{ikr} e^{2i\delta_l} - e^{-i(kr-l\pi)}]P_l(\cos\theta) \tag{19.5.12}$$

$$= e^{ikz} + \left[\sum_{l=0}^{\infty} (2l+1)\left(\frac{e^{2i\delta_l}-1}{2ik}\right)P_l(\cos\theta)\right]\frac{e^{ikr}}{r} \tag{19.5.13}$$

Comparing this to Eq. (19.5.2) we get

$$a_l(k) = \frac{e^{2i\delta_l}-1}{2ik} \tag{19.5.14}$$

Thus, to calculate $a_l(k)$, one must calculate the phase shift δ_l in the asymptotic wave function.

A comparison of Eqs. (19.5.12) and (19.5.8) tells us that the effect of the potential is to attach a phase factor $e^{2i\delta_l}$ to the outgoing wave. This factor does not change the probability current associated with it and the balance between the total incoming and outgoing currents is preserved. This does not mean there is no scattering, since the angular distribution is altered by this phase shift.

One calls

$$S_l(k) = e^{2i\delta_l(k)} \tag{19.5.15}$$

the *partial wave S matrix element* or the S matrix for angular momentum l. Recall that the S matrix is just the $t\to\infty$ limit of $U(t, -t)$. It is therefore a function of the Hamiltonian. Since in this problem $\mathbf{L}$ is conserved, S (like H) will be diagonal in the common eigenbasis of energy ($E=\hbar^2k^2/2\mu$), angular momentum (l), and z component of angular momentum ($m=0$). Since S is unitary (for U is), its eigenvalues $S_l(k)$ must be of the form $e^{i\theta}$ and here $\theta = 2\delta_l$. If we go to some other basis, say the $|\mathbf{p}\rangle$ basis, $\langle\mathbf{p}'|S|\mathbf{p}\rangle$ will still be elements of a unitary matrix, but no longer diagonal, for $\mathbf{p}$ is not conserved in the scattering process.

If we rewrite $a_l(k)$ as

$$a_l(k)=\frac{e^{2i\delta_l}-1}{2ik}=\frac{e^{i\delta_l}\sin\delta_l}{k} \tag{19.5.16}$$

we get

$$f(\theta)=\frac{1}{k}\sum_{l=0}^{\infty}(2l+1)\,e^{i\delta_l}\sin\delta_l P_l(\cos\theta) \tag{19.5.17}$$

The total cross section

$$\sigma=\int |f|^2\,d\Omega$$

is given by

$$\sigma=\frac{4\pi}{k^2}\sum_{l=0}^{\infty}(2l+1)\sin^2\delta_l \tag{19.5.18}$$

upon using the orthogonality relations for the Legendre polynomials

$$\int P_l(\cos\theta)P_{l'}(\cos\theta)\,d(\cos\theta)=\frac{2}{2l+1}\,\delta_{ll'}$$

Note that σ is a sum of partial cross sections at each l:

$$\sigma=\sum_{l=0}^{\infty}\sigma_l,\qquad \sigma_l=\frac{4\pi}{k^2}(2l+1)\sin^2\delta_l \tag{19.5.19}$$

Each σ_l has an upper bound $\sigma_l^{\max}$, called the *unitarity bound*

$$\sigma_l<\sigma_l^{\max}=\frac{4\pi}{k^2}(2l+1) \tag{19.5.20}$$

The bound is saturated when $\delta_l=n\pi/2$, n odd.

Comparing Eqs. (19.5.17) and (19.5.18) and using $P_l(\cos\theta)=1$ at $\theta=0$, we get

$$\sigma=\frac{4\pi}{k}\operatorname{Im} f(0) \tag{19.5.21}$$

This is called the *optical theorem*. It is not too surprising that there exists a relation between the total cross section and the forward amplitude, for the following reason. The incident plane wave brings in some current density in the z direction. Some of it gets scattered into the various directions. This must reflect itself in the form of a decrease in current density behind the target, i.e., in the $\theta=0$ direction. The decrease can only occur because the incident plane wave and the scattered wave in the forward direction interfere destructively. It is of course not obvious why just the imaginary part of $f(0)$ is relevant or where the factor $4\pi/k$ comes from. To find out, you must do Exercise 19.5.6.

A Model Calculation of δ_l: The Hard Sphere

Consider a hard sphere, which is represented by

$$\begin{aligned} V(r) &= \infty, \qquad r<r_0 \\ &= 0, \qquad r>r_0 \end{aligned} \tag{19.5.22}$$

We now proceed to solve the radial Schrödinger equation, look at the solution as $r\to\infty$, and identify the phase shift. Clearly the (unnormalized) radial function $R_l(r)$ vanishes inside $r\le r_0$. Outside, it is given by the free-particle function:

$$R_l(r)=A_l j_l(kr)+B_l n_l(kr) \tag{19.5.23}$$

(We keep the n_l function since it is regular for $r>0$.) The coefficients A_l and B_l must be chosen such that

$$R_l(r_0)=0 \tag{19.5.24}$$

to ensure the continuity of the wave function at $r=r_0$. Thus

$$\frac{B_l}{A_l}=-\frac{j_l(kr_0)}{n_l(kr_0)} \tag{19.5.25}$$

From Eq. (12.6.32), which gives the asymptotic form of j_l and n_l,

$$\begin{aligned} R_l(r) &\xrightarrow[r\to\infty]{} \frac{1}{kr}[A_l\sin(kr-l\pi/2)-B_l\cos(kr-l\pi/2)] \\ &= \frac{(A_l^2+B_l^2)^{1/2}}{kr}\left[\sin\left(kr-\frac{l\pi}{2}+\delta_l\right)\right] \end{aligned} \tag{19.5.26}$$

where

$$\delta_l=\tan^{-1}\left(\frac{-B_l}{A_l}\right)=\tan^{-1}\left[\frac{j_l(kr_0)}{n_l(kr_0)}\right] \tag{19.5.27}$$

For instance [from Eq. (12.6.31)]

$$\begin{aligned}\delta_0 &= \tan^{-1}\left[\frac{\sin(kr_0)/kr_0}{-\cos(kr_0)/kr_0}\right] \\ &= -\tan^{-1}\tan(kr_0) \\ &= -kr_0\end{aligned} \tag{19.5.28}$$

It is easy to understand the result: the hard sphere has pushed out the wave function, forcing it to start its sinusoidal oscillations at $r=r_0$ instead of $r=0$. In general, repulsive potentials give negative phase shifts (since they slow down the particle and reduce the phase shift per unit length) while attractive potentials give positive phase shifts (for the opposite reason). This correspondence is of course true only if δ is small, since δ is defined only modulo π. For instance, if the phase shift $kr_0=\pi$, a_0 vanishes and s-wave scattering does not expose the hard sphere centered at the origin.

Consider the hard sphere phase shift as $k\to 0$. Using

$$j_l(x) \xrightarrow[x\to 0]{} x^l/(2l+1)!!$$

$$n_l(x) \xrightarrow[x\to 0]{} -x^{-(l+1)}(2l-1)!!$$

we get

$$\tan\delta_l \underset{k\to 0}{\cong} \delta_l \propto (kr_0)^{2l+1} \tag{19.5.29}$$

This agrees with the intuitive expectation that at low energies there should be negligible scattering in the high angular momentum states. The above $(kr_0)^{2l+1}$ dependence of δ_l at low energies is true for any reasonable potential, with r_0 being some length scale characterizing the range. [Since there is no hard and fast definition of range, we can *define* the range of any potential to be the r_0 that appears in Eq. (19.5.29).] Notice that although $\delta_0 \propto k^1$, the partial cross section does not vanish because $\sigma_0 \propto k^{-2}\sin^2\delta_l \sim k^{-2}\delta_l^2 \nrightarrow 0$, as $k\to 0$.

Resonances

The partial cross section σ_l is generally very small at low energies since $\delta_l \propto (k)^{2l+1}$ as $k\to 0$. But it sometimes happens that δ_l rises very rapidly from 0 to π [or more generally, from $n\pi$ to $(n+1)\pi$] in a very small range of k or E. In this region, near $k=k_0$ or $E=E_0$, we may describe δ_l by

$$\delta_l = \delta_b + \tan^{-1}\left(\frac{\Gamma/2}{E_0-E}\right) \tag{19.5.30}$$

where δ_b is some *background phase* ($\cong n\pi$) that varies very little. The corresponding cross section, neglecting δ_b, is

$$\sigma_l = \frac{4\pi}{k^2}(2l+1)\sin^2\delta_l$$

$$\underset{E\cong E_0}{=} \frac{4\pi}{k^2}(2l+1)\frac{(\Gamma/2)^2}{(E_0-E)^2+(\Gamma/2)^2} \tag{19.5.31}$$

σ_l is described by a bell-shaped curve, called the *Breit–Wigner* form, with a maximum height $\sigma_l^{\max}$ (the unitarity bound) and a half-width $\Gamma/2$. This phenomenon is called a *resonance*.

In Eq. (19.5.31) for σ_l, valid only near E_0, we have treated Γ as a constant. Its k dependence may be deduced by noting that as $k\to 0$, we have [from Eq. (19.5.29)],

$$\sigma_l \sim \frac{1}{k^2}\sin^2\delta_l \cong \frac{1}{k^2}\delta_l^2 \cong \frac{(kr_0)^{4l+2}}{k^2}$$

which implies

$$\Gamma/2 = (kr_0)^{2l+1}\gamma \tag{19.5.32}$$

where γ is some constant with dimensions of energy. Thus the expression for σ_l that is valid over a wider range is

$$\sigma_l = \frac{4\pi}{k^2}(2l+1)\frac{[\gamma(kr_0)^{2l+1}]^2}{(E-E_0)^2+[\gamma(kr_0)^{2l+1}]^2} \tag{19.5.33}$$

For any $l \neq 0$, σ_l is damped in the entire low-energy region by the net k^{4l} factor, except near E_0, where a similar factor in the denominator neutralizes it. Clearly, as l goes up, the resonances get sharper. The situation at $l=0$ (where σ_0 starts out nonzero at $k=0$) depends on the potential. More on this later.

We would now like to gain some insight into the dynamics of resonances. We ask what exactly is going on at a resonance, in terms of the underlying Schrödinger equation. We choose to analyze the problem through the S matrix. Near a resonance we have

$$S_l(k) = e^{2i\delta_l} = \frac{e^{i\delta_l}}{e^{-i\delta_l}} = \frac{1+i\tan\delta_l}{1-i\tan\delta_l} = \frac{E-E_0-i\Gamma/2}{E-E_0+i\Gamma/2} \tag{19.5.34}$$

Although k and E are real in any experiment (and in our analysis so far), let us think of $S_l(k)$ as a function of complex E or k. Then we find that the resonance corresponds to a pole in S_l at a complex point,

$$E = E_0 - i\Gamma/2 \tag{19.5.35}$$

or

$$k = k_0 - i\eta/2 \tag{19.5.36}$$

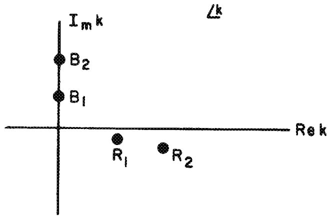

Figure 19.4. Some of the singularities of $S_l(k)$ in the complex k plane. The dots on the positive imaginary axis stand for bound state poles and the dots below the real axis stand for resonance poles. The physical or experimentally accessible region is along the real axis, where S_l has the form $e^{2i\delta_l}$.

where $E_0 = \hbar^2 k_0^2/2\mu$ and $\Gamma = \eta\hbar^2 k_0/\mu$ (for small η and Γ). Since Γ and η are small, the pole is very close to the real axis, which is why we trust the form of S_l that is valid near the point $E = E_0$ on the real axis.

What is the implication of the statement that the resonance corresponds to a (nearby) pole in $S_l(k)$? To find out, we take a new look at bound states in terms of the S matrix. Recall that for k real and positive, if

$$R_{kl}(r) \underset{r\to\infty}{\longrightarrow} \frac{A\,e^{ikr}}{r} + \frac{B\,e^{-ikr}}{r} \tag{19.5.37}$$

then [from Eqs. (19.5.9) and (19.5.10) or Eq. (19.5.12)],

$$e^{2i\delta_l} = S_l(k) = \frac{A}{B} = \frac{\text{outgoing wave amplitude}}{\text{incoming wave amplitude}} \tag{19.5.38}$$

(up to a constant factor i^{2l}). We now define $S_l(k)$ for complex k as follows: solve the radial equation with k set equal to a complex number, find $R(r\to\infty)$, and take the ratio A/B. Consider now the case $k = i\kappa\,(\kappa > 0)$, which corresponds to E real and negative. Here we will find

$$R_{kl}(r) \underset{r\to\infty}{\longrightarrow} \frac{A\,e^{-\kappa r}}{r} + \frac{B\,e^{\kappa r}}{r} \tag{19.5.39}$$

Whereas $S_l(k = i\kappa)$ is well defined, the corresponding R_{kl} does not interest us, since it is not normalizable. But recall that for some special values of k, R_{kl} is exponentially damped and describes the wave function of a bound state. These bound states correspond to k such that $B = 0$, or $S_l(k) = \infty$. Thus poles of $S_l(k)$ at $k = i\kappa$ correspond to bound states.

So a resonance, which is a pole at $k = k_0 - i\eta$ must also be some kind of bound state. (See Fig. 19.4 for poles of the S matrix.) We next argue heuristically as follows.‡ Since the bound state at $E = E_B$ (a negative number) has the time dependence

$$e^{-iE_B t/\hbar}$$

‡ This result may be established rigorously.

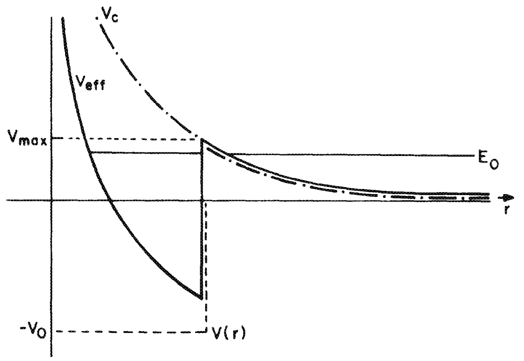

Figure 19.5. A typical potential that can sustain resonances. The centrifugal repulsion V_c (dot–dash line) plus the actual attractive potential (dotted line) gives the effective potential V_{eff} (solid line). The figure shows an example where there would have been a bound state at E_0 but for tunneling. But because of tunneling the particle can leak out, and by the same token, a particle can come from outside with positive energy E_0, form a metastable bound state (with a lifetime inversely proportional to the tunneling probability), and then escape. This is called resonance.

the resonance must have a time dependence

$$e^{-i(E_0-i\Gamma/2)t/\hbar}=e^{-iE_0t/\hbar}\,e^{-\Gamma t/2\hbar}$$

This describes a state of positive energy E_0, but whose norm falls exponentially with a half-life $t\sim\hbar/\Gamma$. Thus, a resonance, corresponding to a pole at $E=E_0-i\Gamma/2$, describes a *metastable* bound state of energy E_0 and lifetime $t=\hbar/\Gamma$.‡

So we must next understand how a positive-energy particle manages to form a metastable bound state. Consider the case where $V(r)$ is attractive, say a square well of depth V_0 and range r_0. The potential appearing in the radial equation is $V_{eff}=V+V_c$, where V_c is the centrifugal repulsion (Fig. 19.5). The main point is that V_{eff} is attractive at short distances and repulsive at long distances. Consider now a particle with energy $E_0<V_{max}$, such that if tunneling is ignored, the particle can form a bound state inside the attractive region, i.e., we can fit in an integral number of half-wavelengths. But tunneling is of course present and the particle can escape to infinity as a free particle of energy E_0. Conversely, a free particle of energy E_0 shot into the potential can penetrate the barrier and form a metastable bound state and leak out again. This is when we say resonance is formed. This picture also explains why the resonances get narrower as l increases: as l increases, V_c grows, tunneling is suppressed more, and the lifetime of the metastable state grows. We can also see why $l=0$ is different: there is no repulsive barrier due to V_c. If $V=V_{eff}$ is purely attractive, only genuine (negative energy) bound states are possible. The closest thing to a resonance is the formation of a bound state near zero energy (Exercise 19.5.4). If, however, V itself has the form of V_{eff} in Fig. 19.5, resonances are possible.

*Exercise 19.5.2.** Derive Eq. (19.5.18) and provide the missing steps leading to the optical theorem, Eq. (19.5.21).

‡ The energy is not strictly E_0 because the uncertainty principle does not allow us to define a precise energy for a state of finite lifetime. E_0 is the mean energy.

Exercise 19.5.3. (1) Show that $\sigma_0 \to 4\pi r_0^2$ for a hard sphere as $k \to 0$.

(2) Consider the other extreme of kr_0 very large. From Eq. (19.5.27) and the asymptotic forms of j_l and n_l show that

$$\sin^2 \delta_l \xrightarrow[kr_0 \to \infty]{} \sin^2(kr_0 - l\pi/2)$$

so that

$$\sigma = \sum_{l=0}^{l_{\max}=kr_0} \sigma_l \cong \frac{4\pi}{k^2} \int_0^{kr_0} (2l) \sin^2 \delta_l \, dl$$
$$\cong 2\pi r_0^2$$

if we approximate the sum over l by an integral, $2l+1$ by $2l$, and the oscillating function $\sin^2 \delta$ by its mean value of $1/2$.

*Exercise 19.5.4.** Show that the s-wave phase shift for a square well of depth V_0 and range r_0 is

$$\delta_0 = -kr_0 + \tan^{-1}\left(\frac{k}{k'} \tan k'r_0\right)$$

where k' and k are the wave numbers inside and outside the well. For k small, kr_0 is some small number and we ignore it. Let us see what happens to δ_0 as we vary the depth of the well, i.e., change k'. Show that whenever $k' \simeq k'_n = (2n+1)\pi/2r_0$, δ_0 takes on the resonant form Eq. (19.5.30) with $\Gamma/2 = \hbar^2 k_n/\mu r_0$, where k_n is the value of k when $k' = k'_n$. Starting with a well that is too shallow to have any bound state, show k'_1 corresponds to the well developing its first bound state, at zero energy. (See Exercise 12.6.9.) (Note: A zero-energy bound state corresponds to $k=0$.) As the well is deepened further, this level moves down, and soon, at k'_2, another zero-energy bound state is formed, and so on.

Exercise 19.5.5. Show that even if a potential absorbs particles, we can describe it by

$$S_l(k) = \eta_l(k)\, e^{2i\delta_l}$$

where $\eta(<1)$, is called the *inelasticity factor*.

(1) By considering probability currents, show that

$$\sigma_{\text{inel}} = \frac{\pi}{k^2} \sum_{l=0}^{\infty} (2l+1)[1-\eta_l^2]$$

$$\sigma_{\text{el}} = \frac{\pi}{k^2} \sum_{l=0}^{\infty} (2l+1)(1+\eta_l^2 - 2\eta_l \cos 2\delta_l)$$

and that once again

$$\sigma_{\text{tot}} = \frac{4\pi}{k} \operatorname{Im} f(0)$$

(2) Consider a "black disk" which absorbs everything for $r \le r_0$ and is ineffective beyond. Idealize it by $\eta = 0$ for $l \le kr_0$; $\eta = 1$, $\delta = 0$ for $l > kr_0$. Show that $\sigma_{el} = \sigma_{inel} \simeq \pi r_0^2$. Replace the sum by an integral and assume $kr_0 \gg 1$. (See Exercise 19.5.3.) Why is σ_{inel} always accompanied by σ_{el}?

Exercise 19.5.6. (The Optical Theorem). (1) Show that the radial component of the current density due to interference between the incident and scattered waves is

$$j_r^{\text{int}} \underset{r\to\infty}{\sim} \left(\frac{\hbar k}{\mu}\right)\frac{1}{r}\,\text{Im}[i\,e^{ikr(\cos\theta - 1)}f^*(\theta)\cos\theta + i\,e^{ikr(1-\cos\theta)}f(\theta)]$$

(2) Argue that as long as $\theta \neq 0$, the average of j_r^{int} over any small solid angle is zero because $r \to \infty$. [Assume $f(\theta)$ is a smooth function.]

(3) Integrate j_r^{int} over a tiny cone in the forward direction and show that (see hint)

$$\int_{\text{forward cone}} j_r^{\text{int}} r^2\, d\Omega = -\left(\frac{\hbar k}{\mu}\right)\frac{4\pi}{k}\,\text{Im}\, f(0)$$

Thus, if we integrate the total current in the region behind the target, we find that the interference term (important only in the near-forward direction, behind the target) produces a depletion of particles, casting a "shadow." The total number of particles (per second) missing in the shadow region is given by the above expression for the integrated flux. Equating this loss to the product of the incident flux $\hbar k/\mu$ and the cross section σ, we regain the optical theorem. (Hint: Since θ is small, set $\sin\theta \simeq \theta$, $\cos\theta = 1$ or $1 - \theta^2/2$ using the judgment. In evaluating the upper limit in the θ integration, use the idea introduced in Chapter 1, namely, that the limit of a function that oscillates as its argument approaches infinity is equal to its average value.)

19.6. Two-Particle Scattering

In this section we will see how the differential cross section for two-body scattering may be extracted from the solution of the Schrödinger equation for the relative coordinate with a potential $V(\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2)$. Let us begin by considering the total and differential cross sections for two-body scattering. Let σ be the total cross section for the scattering of the two particles. Imagine a beam of projectiles with density ρ_1 and *magnitude* of velocity v_1 colliding *head on* with the beam of targets with parameters ρ_2 and v_2. How many collisions will there be per second? We know that if there is only one target and it is at rest,

$$\begin{aligned}\text{No. of collisions/sec} &= \sigma \times \text{incident projectiles/sec/area}\\ &= \sigma\rho_1 v_1 \qquad (19.6.1)\end{aligned}$$

Here we modify this result to take into account that (1) there are ρ_2 targets per *unit volume* (ρ_2 is assumed so small that the targets scatter independently of each other), and (2) the targets are moving toward the projectiles at a relative velocity $v_{\text{rel}} = v_1 + v_2$. Consequently we have

$$\text{No. of collisions/sec/volume of interaction} = \sigma\rho_1(v_1+v_2)\rho_2$$
$$= \sigma\rho_1\rho_2 v_{\text{rel}} \tag{19.6.2}$$

Note that σ is the same for all observers moving along the beam–target axis.

What about the differential cross section? It *will* depend on the frame. In the lab frame, where the target is initially at rest, we define, in analogy with Eq. (19.6.2),

$$\text{No. of projectiles scattered into } d(\cos\theta_L)\, d\phi_L/\text{sec/vol}$$
$$= \frac{d\sigma}{d\Omega_L}\, d\Omega_L\, \rho_1\, \rho_2 v_{\text{rel}} \tag{19.6.3}$$

Here v_{rel} is just the projectile velocity and θ_L and ϕ_L are angles in the lab frame measured relative to the projectile direction. (We can also define a $d\sigma/d\Omega_L$ in terms of how many *target* particles are scattered into $d\Omega_L$, but it would not be an independent quantity since momentum conservation will fix the fate of the target, given the fate of the projectile.) The only other frame we consider is the CM frame, where $(d\sigma/d\Omega)\, d\Omega$ is defined as in Eq. (19.6.3).‡ We relate $d\sigma/d\Omega$ to $d\sigma/d\Omega_L$ by the following argument. Imagine a detector in the lab frame at (θ_L, ϕ_L) which subtends an angle $d\Omega_L$. The number of counts it registers is an absolute, frame-independent quantity, although its orientation and acceptance angle $d\Omega$ may vary from frame to frame. (For example, a particle coming at right angles to the beam axis in the lab frame will be tilted forward in a frame moving backward.) So we deduce the following equality from Eq. (19.6.2) after noting the frame invariance of $\rho_1\rho_2 v_{\text{rel}}$:

$$\frac{d\sigma}{d\Omega_L}\, d\Omega_L = \frac{d\sigma}{d\Omega}\, d\Omega \tag{19.6.4}$$

or

$$\frac{d\sigma}{d\Omega_L} = \frac{d\sigma}{d\Omega}\,\frac{d\Omega}{d\Omega_L} \tag{19.6.5}$$

We will consider first the calculation of $d\sigma/d\Omega$, and then $d\Omega/d\Omega_L$.

Let us represent the state of the two colliding particles, long before they begin to interact, by the product wave function (in some general frame):

$$\psi_{\text{inc}} = e^{i\mathbf{k}_1\cdot\mathbf{r}_1}\, e^{i\mathbf{k}_2\cdot\mathbf{r}_2} \tag{19.6.6}$$

‡ The CM variables will carry no subscripts.

We should remember that these plane waves are idealized forms of broad wave packets. Assuming both are moving along the z axis,

$$\begin{aligned}\psi_{\text{inc}} &= e^{ik_1z_1}\,e^{ik_2z_2}\\ &= \exp\left[i(k_1+k_2)\left(\frac{z_1+z_2}{2}\right)\right]\exp\left[i\left(\frac{k_1-k_2}{2}\right)(z_1-z_2)\right]\\ &= \psi_{\text{inc}}^{\text{CM}}(z_{\text{CM}})\psi_{\text{inc}}^{\text{rel}}(z) \qquad (19.6.7)\end{aligned}$$

Since the potential affects only the relative coordinate, the plane wave describes the CM completely; there is no scattering for the CM as a whole. On the other hand, $\psi_{\text{inc}}^{\text{rel}}(z)$ will develop a scattered wave and become

$$\begin{aligned}\psi(z) &= e^{ikz}+\psi_{\text{sc}}(\mathbf{r})\\ &\xrightarrow[r\to\infty]{} e^{ikz}+f(\theta,\phi)\,e^{ikr}/r \qquad (19.6.8)\end{aligned}$$

where we have dropped the superscript "rel," since the argument z makes it obvious, and set $(k_1-k_2)/2$ equal to k. Thus the static solution for the entire system is

$$\begin{aligned}\psi_{\text{system}}(\mathbf{r}_1,\mathbf{r}_2) &= \psi^{\text{CM}}(z_{\text{CM}})[e^{ikz}+\psi_{\text{sc}}(\mathbf{r})]\\ &\xrightarrow[r\to\infty]{} \psi^{\text{CM}}(z_{\text{CM}})[e^{ikz}+f(\theta,\phi)\,e^{ikr}/r] \qquad (19.6.9)\end{aligned}$$

If we go to the CM frame, $\psi^{\text{CM}}(z_{\text{CM}})=e^{i(k_1+k_2)\cdot z_{\text{CM}}}=1$, since $k_1+k_2=0$ defines this frame. So we can forget all about the CM coordinate. The scattering in the CM frame is depicted in Fig. 19.6. The classical trajectories are not to be taken literally;

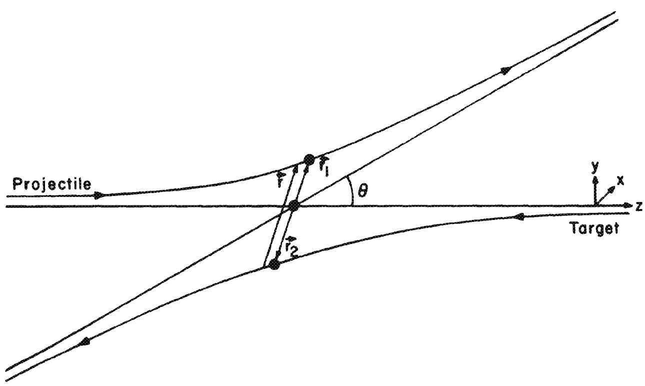

Figure 19.6. Two-body scattering in the CM frame. The projectile and target coordinates are $\mathbf{r}_1$ and $\mathbf{r}_2$, respectively. The relative coordinate $\mathbf{r}=\mathbf{r}_1-\mathbf{r}_2$ is slightly displaced in the figure for clarity. Since $\mathbf{r}_1$ and $\mathbf{r}$ are always parallel, the probability that the projectile scatters into $d\Omega$ is the same as the probability that the fictitious particle described by $\mathbf{r}$ scatters into $d\Omega$. To find the latter, we must solve the Schrödinger equation for $\mathbf{r}$.

they merely define the relative coordinate $\mathbf{r}$ and the individual coordinates $\mathbf{r}_1$(projectile) and $\mathbf{r}_2$(target).

What we want is the rate at which the projectile scatters into $d\Omega$. But since $\mathbf{r}_1$ is parallel to $\mathbf{r}$, this equals the rate at which the fictitious particle described by $\mathbf{r}$ scatters into solid angle $d\Omega$. We find this rate by solving the Schrödinger equation for the relative coordinate. Having done so, and having found

$$\psi(\mathbf{r}) \xrightarrow[r\to\infty]{} e^{ikz} + f(\theta, \phi)\frac{e^{ikr}}{r} \qquad (19.6.10)$$

we recall from Eq. (19.2.17) the rate of scattering into $d\Omega$:

$$R_{i\to d\Omega} = |f(\theta, \phi)|^2 \frac{\hbar k}{\mu} d\Omega \qquad (19.6.11)$$

Note that is the rate *per unit volume* of target–beam interaction, since the probability density for the CM is unity. To extract $d\sigma/d\Omega$ from $R_{i\to d\Omega}$ above we turn to Eq. (19.6.3) which defines $d\sigma/d\Omega$ (upon dropping the subscript L). Since the definition makes sense only for a flux of wave packets and since we are dealing with plane waves here, we replace the number scattered into $d\Omega$ per second by the probability flowing into $d\Omega$ per second, and the particle densities ρ_1 and ρ_2 by probability densities of the colliding beams. Since the colliding beams ($e^{ikz} = e^{ik(z_1 - z_2)} = e^{ikz_1} \cdot e^{-ikz_2}$) are plane waves of unit modulus, $\rho_1 = \rho_2 = 1$. How about $v_{\rm rel}$? Remember that in the CM frame

$$m_1 v_1 = m_2 v_2$$

so

$$v_{\rm rel} = v_1 + v_2 = v_1\left(1 + \frac{m_1}{m_2}\right) = v_1\left(\frac{m_2 + m_1}{m_2}\right) = m_1 v_1\left(\frac{m_2 + m_1}{m_1 m_2}\right)$$

$$= \hbar k\left(\frac{m_2 + m_1}{m_1 m_2}\right) = \frac{\hbar k}{\mu} \qquad (19.6.12)$$

So

$$\frac{d\sigma}{d\Omega} d\Omega = \frac{R_{i\to d\Omega}}{\rho_1 \rho_2 (v_1 + v_2)} = \frac{|f|^2(\hbar k/\mu)}{\hbar k/\mu} d\Omega$$

or

$$\frac{d\sigma}{d\Omega} = |f|^2 \qquad (19.6.13)$$

Thus the $d\sigma/d\Omega$ we calculated in the previous sections for a single particle scattering off a potential $V(\mathbf{r})$ can also be interpreted as the CM cross section for two bodies interacting via a potential $V(\mathbf{r}=\mathbf{r}_1-\mathbf{r}_2)$.

Passage to the Lab Frame

We now consider the passage to the lab frame, i.e., the calculation of $d\Omega/d\Omega_L$. We discuss the equal mass case, leaving the unequal mass case as an exercise. Figure 19.7a shows the particles coming in with momenta p and $-p$ along the z axis in the CM frame. If $\mathbf{p}'$ is the final momentum of the projectile,

$$\tan\theta=\frac{(p_x'^2+p_y'^2)^{1/2}}{p_z'}\equiv\frac{p_\perp'}{p_z'} \tag{19.6.14a}$$

$$\tan\phi=p_y'/p_x' \tag{19.6.14b}$$

To go to the lab frame we must move leftward at a speed p/m. In this frame, all momenta get an increment in the z *direction* (only) equal to p. (Thus T, the target, will be at rest before collision.) The scattering angles in the lab frame are given by

$$\tan\theta_L=p_\perp'/(p_z'+p) \tag{19.6.15a}$$

$$\tan\phi_L=p_y'/p_x' \tag{19.6.15b}$$

Comparing Eqs. (19.6.14) and (19.6.15) we get

$$\phi_L=\phi \tag{19.6.16}$$

$$\tan\theta_L=\frac{p_\perp'}{p_z'+p}=\frac{p_\perp'/p}{p_z'/p+1}=\frac{\sin\theta}{\cos\theta+1}$$
$$=\tan(\theta/2)\quad(\text{using }|\mathbf{p}'|=p)$$

So

$$\theta_L=\theta/2 \tag{19.6.17}$$

One consequence of the result is that $\theta_L\leq\pi/2$. Given Eqs. (19.6.16) and (19.6.17) it is a simple matter to relate $d\sigma/d\Omega$ to $d\sigma/d\Omega_L$.

*Exercise 19.6.1.** (1) Starting with Eqs. (19.6.16) and (19.6.17), show that the relation between $d\sigma/d\Omega$ and $d\sigma/d\Omega_L$ is

$$\left.\frac{d\sigma}{d\Omega_L}\right|_{\theta_0}=\left.\frac{d\sigma}{d\Omega}\right|_{2\theta_0}4\cos\theta_0$$

(2) Show that $\theta_L\leq\pi/2$ by using just energy and momentum conservation.

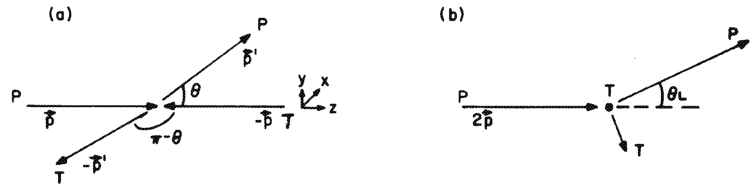

Figure 19.7. (a) Collision of two equal masses in the CM frame. The labels P and T refer to projectile and target. The angle ϕ equals $\frac{1}{2}\pi$ in the figure. (b) The same collision in the lab frame (where T is initially at rest).

(3) For unequal mass scattering, show that

$$\tan \theta_L = \frac{\sin \theta}{\cos \theta + (m_1/m_2)}$$

where m_2 is the target mass.

Scattering of Identical Particles

Consider the scattering of two identical spin-zero bosons in their CM frame. We must describe them by a symmetrized wave function. Under the exchange $\mathbf{r}_1 \leftrightarrow \mathbf{r}_2$; $\mathbf{r}_{CM} = (\mathbf{r}_1 + \mathbf{r}_2)/2$ is invariant while $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$ changes sign. So $\psi^{CM}(\mathbf{r}_{CM})$ is automatically symmetric. We must symmetrize $\psi(\mathbf{r})$ by hand:

$$\psi_{sym}(\mathbf{r}) \xrightarrow[r\to\infty]{} (e^{ikz} + e^{-ikz}) + [f(\theta, \phi) + f(\pi - \theta, \phi + \pi)]\, e^{ikr}/r \tag{19.6.18}$$

We have used the fact that under $\mathbf{r} \to -\mathbf{r}$, $\theta \to \pi - \theta$ and $\theta \to \phi + \pi$. The scattering amplitude is thus

$$f_{sym}(\theta, \phi) = f(\theta, \phi) + f(\pi - \theta, \phi + \pi) \tag{19.6.19}$$

Note that f_{sym} is consistent with the fact that since the particles are identical, one cannot say which one scattered into (θ, ϕ) and which one into $(\pi - \theta, \phi + \pi)$ (Fig. 19.7). The differential cross section is

$$\begin{aligned}\frac{d\sigma}{d\Omega} &= |f(\theta, \phi) + f(\pi - \theta, \phi + \pi)|^2 \\ &= |f(\theta, \phi)|^2 + |f(\pi - \theta, \phi + \pi)|^2 + 2\,\mathrm{Re}[f(\theta, \phi) f^*(\pi - \theta, \phi + \pi)]\end{aligned} \tag{19.6.20}$$

The first two terms are what we would get if we had two distinguishable particles and asked for the rate at which one or the other comes into $d\Omega$. The third term gives the usual quantum mechanical interference that accompanies identical particles. There are two features worth noting about Eq. (19.6.20):

(1) To find σ, we must integrate over only 2π radians and not 4π radians (if not, we will count each *distinguishable* event twice).

(2) Recall that when we obtained the Rutherford cross section by taking the $\mu_0 \to 0$ limit of the Yukawa cross section, we got the right answer although $f(\theta)$ was not right: it did not contain the exponential phase factor that comes from a careful treatment of the Coulomb potential [see Eq. (19.3.16) and the sentences following it.] When we consider the Coulomb scattering of identical bosons (of charge e, say) the interference terms expose the inadequacy of the $\mu_0 \to 0$ approach. The correct cross section is‡

$$\frac{d\sigma}{d\Omega} = \left(\frac{e^2}{4E}\right)^2 \left[\frac{1}{\sin^4 \theta/2} + \frac{1}{\cos^4 \theta/2} + \frac{2\cos(\gamma \ln \tan^2 \theta/2)}{\sin^2 \theta/2 \cos^2 \theta/2}\right] \tag{19.6.21}$$

whereas the $\mu_0 \to 0$ trick would not have given the $\cos(\gamma \ln \tan^2 \theta/2)$ factor. (The classical Rutherford treatment would not give the third term at all. Notice, however, that as $\hbar \to 0$, it oscillates wildly and averages to zero over any realistic detector.)

Consider now the scattering of two identical spin-1/2 fermions, say electrons. Let us assume that the spin variables are spectators, except for their role in the statistics: in the triplet state the spatial function is antisymmetric, while in the singlet it is symmetric. If the electrons are assumed to come in with random values of s_z, the triplet is three times as likely as the singlet and the average cross section will be

$$\frac{d\sigma}{d\Omega} = \frac{3}{4} |f(\theta, \phi) - f(\pi - \theta, \phi + \pi)|^2 + \frac{1}{4} |f(\theta, \phi) + f(\pi - \theta, \phi + \pi)|^2 \tag{19.6.22}$$

For Coulomb scattering of electrons this becomes

$$\frac{d\sigma}{d\Omega} = \left(\frac{e^2}{4E}\right)^2 \left[\frac{1}{\sin^4 \theta/2} + \frac{1}{\cos^4 \theta/2} - \frac{\cos(\gamma \ln \tan^2 \theta/2)}{\sin^2 \theta/2 \cos^2 \theta/2}\right] \tag{19.6.23}$$

Exercise 19.6.2. Derive Eq. (19.6.21) using Eq. (19.3.16) for $f_c(\theta)$.

Exercise 19.6.3. Assuming $f = f(\theta)$ show that $(d\sigma/d\Omega)_{\pi/2} = 0$ for fermions in the triplet state.

‡ $\gamma = e^2\mu/\hbar^2 k$ here.