10

Systems with N Degrees of Freedom

10.1. N Particles in One Dimension

So far, we have restricted our attention (apart from minor digressions) to a system with one degree of freedom, namely, a single particle in one dimension. We now consider the quantum mechanics of systems with N degrees of freedom. The increase in degrees of freedom may be due to an increase in the number of particles, number of spatial dimensions, or both. In this section we consider N particles in one dimension, and start with the case $N=2$.

The Two-Particle Hilbert Space

Consider two particles described classically by (x_1, p_1) and (x_2, p_2). The rule for quantizing this system [Postulate II, Eq. (7.4.39)] is to promote these variables to quantum operators (X_1, P_1) and (X_2, P_2) obeying the canonical commutation relations:

$$[X_i, P_j] = i\hbar\{x_i, p_j\} = i\hbar\delta_{ij} \qquad (i=1, 2) \tag{10.1.1a}$$

$$[X_i, X_j] = i\hbar\{x_i, x_j\} = 0 \tag{10.1.1b}$$

$$[P_i, P_j] = i\hbar\{p_i, p_j\} = 0 \tag{10.1.1c}$$

It might be occasionally possible (as it was in the case of the oscillator) to extract all the physics given just the canonical commutators. In practice one works in a basis, usually the coordinate basis. This basis consists of the kets $|x_1x_2\rangle$ which are simultaneous eigenkets of the commuting operators X_1 and X_2:

$$\begin{aligned} X_1|x_1x_2\rangle &= x_1|x_1x_2\rangle \\ X_2|x_1x_2\rangle &= x_2|x_1x_2\rangle \end{aligned} \tag{10.1.2}$$

and are normalized as‡

$$\langle x_1'x_2'|x_1x_2\rangle = \delta(x_1'-x_1)\delta(x_2'-x_2) \tag{10.1.3}$$

In this basis

$$\begin{aligned} |\psi\rangle &\to \langle x_1x_2|\psi\rangle = \psi(x_1, x_2) \\ X_i &\to x_i \\ P_i &\to -i\hbar\frac{\partial}{\partial x_i} \end{aligned} \tag{10.1.4}$$

We may interpret

$$P(x_1, x_2) = |\langle x_1x_2|\psi\rangle|^2 \tag{10.1.5}$$

as the absolute probability density for catching particle 1 near x_1 and particle 2 near x_2, provided we normalize $|\psi\rangle$ to unity

$$1 = \langle\psi|\psi\rangle = \int |\langle x_1x_2|\psi\rangle|^2\, dx_1\, dx_2 = \int P(x_1, x_2)\, dx_1\, dx_2 \tag{10.1.6}$$

There are other bases possible besides $|x_1x_2\rangle$. There is, for example, the momentum basis, consisting of the simultaneous eigenkets $|p_1p_2\rangle$ of P_1 and P_2. More generally, we can use the simultaneous eigenkets $|\omega_1\omega_2\rangle$ of two commuting operators§ $\Omega_1(X_1, P_1)$ and $\Omega_2(X_2, P_2)$ to define the Ω basis. We denote by $\mathbb{V}_{1\otimes 2}$ the two-particle Hilbert space spanned by any of these bases.

$\mathbb{V}_{1\otimes 2}$ As a Direct Product Space

There is another way to arrive at the space $\mathbb{V}_{1\otimes 2}$, and that is to build it out of two one-particle spaces. Consider a system of two particles described classically by (x_1, p_1) and (x_2, p_2). If we want the quantum theory of just particle 1, we define operators X_1 and P_1 obeying

$$[X_1, P_1] = i\hbar I \tag{10.1.7}$$

The eigenvectors $|x_1\rangle$ of X_1 form a complete (coordinate) basis for the Hilbert space

‡ Note that we denote the bra corresponding to $|x_1'x_2'\rangle$ as $\langle x_1'x_2'|$.
§ Note that any function of X_1 and P_1 commutes with any function of X_2 and P_2.

$\mathbb{V}_1$ of particle 1. Other bases, such as $|p_1\rangle$ of P_1 or in general, $|\omega_1\rangle$ of $\Omega_1(X_1, P_1)$ are also possible. Since the operators X_1, P_1, Ω_1, etc., act on $\mathbb{V}_1$, let us append a superscript (1) to all of them. Thus Eq. (10.1.7) reads

$$[X_1^{(1)}, P_1^{(1)}] = i\hbar I^{(1)} \tag{10.1.8a}$$

where $I^{(1)}$ is the identity operator on $\mathbb{V}_1$. A similar picture holds for particle 2, and in particular,

$$[X_2^{(2)}, P_2^{(2)}] = i\hbar I^{(2)} \tag{10.1.8b}$$

Let us now turn our attention to the two-particle system. What will be the coordinate basis for this system? Previously we assigned to every possible outcome x_1 of a position measurement a vector $|x_1\rangle$ in $\mathbb{V}_1$ and likewise for particle 2. Now a position measurement will yield a pair of numbers (x_1, x_2). Since after the measurement particle 1 will be in state $|x_1\rangle$ and particle 2 in $|x_2\rangle$, let us denote the corresponding ket by $|x_1\rangle\otimes|x_2\rangle$:

$$|x_1\rangle\otimes|x_2\rangle \leftrightarrow \begin{cases}\text{particle 1 at } x_1\\ \text{particle 2 at } x_2\end{cases} \tag{10.1.9}$$

Note that $|x_1\rangle\otimes|x_2\rangle$ is a new object, quite unlike the inner product $\langle\psi_1|\psi_2\rangle$ or the outer product $|\psi_1\rangle\langle\psi_2|$ both of which involve *two vectors from the same space*. The product $|x_1\rangle\otimes|x_2\rangle$, called the *direct product*, is the product of *vectors from two different spaces*. The direct product is a linear operation:

$$(\alpha|x_1\rangle + \alpha'|x_1'\rangle)\otimes(\beta|x_2\rangle) = \alpha\beta|x_1\rangle\otimes|x_2\rangle + \alpha'\beta|x_1'\rangle\otimes|x_2\rangle \tag{10.1.10}$$

The set of all vectors of the form $|x_1\rangle\otimes|x_2\rangle$ forms the basis for a space which we call $\mathbb{V}_1\otimes\mathbb{V}_2$, and refer to as the *direct product of the spaces* $\mathbb{V}_1$ *and* $\mathbb{V}_2$. The dimensionality (number of possible basis vectors) of $\mathbb{V}_1\otimes\mathbb{V}_2$ is the product of the dimensionality of $\mathbb{V}_1$ and the dimensionality of $\mathbb{V}_2$. Although all the dimensionalities are infinite here, the statement makes heuristic sense: to each basis vector $|x_1\rangle$ of $\mathbb{V}_1$ and $|x_2\rangle$ of $\mathbb{V}_2$, there is one and only one basis vector $|x_1\rangle\otimes|x_2\rangle$ of $\mathbb{V}_1\otimes\mathbb{V}_2$. This should be compared to the direct sum (Section 1.4):

$$\mathbb{V}_{1\oplus 2} = \mathbb{V}_1\oplus\mathbb{V}_2$$

in which case the dimensionalities of $\mathbb{V}_1$ and $\mathbb{V}_2$ add (assuming the vectors of $\mathbb{V}_1$ are linearly independent of those of $\mathbb{V}_2$).

The coordinate basis, $|x_1\rangle\otimes|x_2\rangle$, is just one possibility; we can use the momentum basis $|p_1\rangle\otimes|p_2\rangle$, or, more generally, $|\omega_1\rangle\otimes|\omega_2\rangle$. *Although these vectors span* $\mathbb{V}_1\otimes\mathbb{V}_2$, *not every element of* $\mathbb{V}_1\otimes\mathbb{V}_2$ *is a direct product.* For instance

$$|\psi\rangle = |x_1'\rangle\otimes|x_2'\rangle + |x_1''\rangle\otimes|x_2''\rangle$$

cannot be written as

$$|\psi\rangle = |\psi_1\rangle \otimes |\psi_2\rangle$$

where $|\psi_1\rangle$ and $|\psi_2\rangle$ are elements of $\mathbb{V}_1$ and $\mathbb{V}_2$, respectively.

The inner product of $|x_1\rangle \otimes |x_2\rangle$ and $|x_1'\rangle \otimes |x_2'\rangle$ is

$$\begin{aligned}(\langle x_1'| \otimes \langle x_2'|)(|x_1\rangle \otimes |x_2\rangle) &= \langle x_1'|x_1\rangle\langle x_2'|x_2\rangle \\ &= \delta(x_1'-x_1)\delta(x_2'-x_2) \qquad (10.1.11)\end{aligned}$$

Since any vector in $\mathbb{V}_1 \otimes \mathbb{V}_2$ can be expressed in terms of the $|x_1\rangle \otimes |x_2\rangle$ basis, this defines the inner product between any two vectors in $\mathbb{V}_1 \otimes \mathbb{V}_2$.

It is intuitively clear that when two particles are amalgamated to form a single system, the position and momentum operators of each particle, $X_1^{(1)}$, $P_1^{(1)}$ and $X_2^{(2)}$, $P_2^{(2)}$, which acted on $\mathbb{V}_1$ and $\mathbb{V}_2$, respectively, must have counterparts in $\mathbb{V}_1 \otimes \mathbb{V}_2$ and have the same interpretation. Let us denote by $X_1^{(1)\otimes(2)}$ the counterpart of $X_1^{(1)}$, and refer to it also as the "X operator of particle 1." Let us define its action on $\mathbb{V}_1 \otimes \mathbb{V}_2$. Since the vectors $|x_1\rangle \otimes |x_2\rangle$ span the space, it suffices to define its action on these. Now the ket $|x_1\rangle \otimes |x_2\rangle$ denotes a state in which particle 1 is at x_1. Thus it must be an eigenket of $X_1^{(1)\otimes(2)}$ with eigenvalue x_1:

$$X_1^{(1)\otimes(2)}|x_1\rangle \otimes |x_2\rangle = x_1|x_1\rangle \otimes |x_2\rangle \qquad (10.1.12)$$

Note that $X_1^{(1)\otimes(2)}$ does not really care about the second ket $|x_2\rangle$, i.e., it acts trivially (as the identity) on $|x_2\rangle$ and acts on $|x_1\rangle$ just as $X_1^{(1)}$ did. In other words

$$X_1^{(1)\otimes(2)}|x_1\rangle \otimes |x_2\rangle = |X_1^{(1)}x_1\rangle \otimes |I^{(2)}x_2\rangle \qquad (10.1.13)$$

Let us define a *direct product of two operators*, $\Gamma_1^{(1)}$ and $\Lambda_2^{(2)}$ (denoted by $\Gamma_1^{(1)} \otimes \Lambda_2^{(2)}$), whose action on a direct product ket $|\omega_1\rangle \otimes |\omega_2\rangle$ is

$$(\Gamma_1^{(1)} \otimes \Lambda_2^{(2)})|\omega_1\rangle \otimes |\omega_2\rangle = |\Gamma_1^{(1)}\omega_1\rangle \otimes |\Lambda_2^{(2)}\omega_2\rangle \qquad (10.1.14)$$

In this notation, we may write $X_1^{(1)\otimes(2)}$, in view of Eq. (10.1.13), as

$$X_1^{(1)\otimes(2)} = X_1^{(1)} \otimes I^{(2)} \qquad (10.1.15)$$

We can similarly promote $P_2^{(2)}$, say, from $\mathbb{V}_2$ to $\mathbb{V}_1 \otimes \mathbb{V}_2$ by defining the momentum operator for particle 2, $P_2^{(1)\otimes(2)}$, as

$$P_2^{(1)\otimes(2)} = I^{(1)} \otimes P_2^{(2)} \qquad (10.1.16)$$

The following properties of direct products of operators may be verified (say by acting on the basis vectors $|x_1\rangle \otimes |x_2\rangle$):

*Exercise 10.1.1.** Show the following:

(1) $$[\Omega_1^{(1)}\otimes I^{(2)}, I^{(1)}\otimes\Lambda_2^{(2)}]=0 \text{ for any } \Omega_1^{(1)} \text{ and } \Lambda_2^{(2)} \qquad (10.1.17a)$$

(operators of particle 1 commute with those of particle 2).

(2) $$(\Omega_1^{(1)}\otimes\Gamma_2^{(2)})(\theta_1^{(1)}\otimes\Lambda_2^{(2)})=(\Omega\theta)_1^{(1)}\otimes(\Gamma\Lambda)_2^{(2)} \qquad (10.1.17b)$$

(3) If

$$[\Omega_1^{(1)}, \Lambda_1^{(1)}]=\Gamma_1^{(1)}$$

then

$$[\Omega_1^{(1)\otimes(2)}, \Lambda_1^{(1)\otimes(2)}]=\Gamma_1^{(1)}\otimes I^{(2)} \qquad (10.1.17c)$$

and similarly with $1\to 2$.

(4) $$(\Omega_1^{(1)\otimes(2)}+\Omega_2^{(1)\otimes(2)})^2=(\Omega_1^2)^{(1)}\otimes I^{(2)}+I^{(1)}\otimes(\Omega_2^2)^{(2)}+2\Omega_1^{(1)}\otimes\Omega_2^{(2)} \qquad (10.1.17d)$$

The notion of direct products of vectors and operators is no doubt a difficult one, with no simple analogs in elementary vector analysis. The following exercise should give you some valuable experience. It is recommended that you reread the preceding discussion after working on the exercise.

*Exercise 10.1.2.** Imagine a fictitious world in which the single-particle Hilbert space is two-dimensional. Let us denote the basis vectors by $|+\rangle$ and $|-\rangle$. Let

$$\sigma_1^{(1)}=\begin{array}{c} \\ + \\ - \end{array}\!\!\begin{array}{c} \begin{array}{cc} + & - \end{array} \\ \begin{bmatrix} a & b \\ c & d \end{bmatrix}\end{array} \quad\text{and}\quad \sigma_2^{(2)}=\begin{array}{c} \\ + \\ - \end{array}\!\!\begin{array}{c} \begin{array}{cc} + & - \end{array} \\ \begin{bmatrix} e & f \\ g & h \end{bmatrix}\end{array}$$

be operators in $\mathbb{V}_1$ and $\mathbb{V}_2$, respectively (the $\pm$ signs label the basis vectors. Thus $b=\langle+|\sigma_1^{(1)}|-\rangle$ etc.) The space $\mathbb{V}_1\otimes\mathbb{V}_2$ is spanned by four vectors $|+\rangle\otimes|+\rangle$, $|+\rangle\otimes|-\rangle$, $|-\rangle\otimes|+\rangle$, $|-\rangle\otimes|-\rangle$. Show (using the method of images or otherwise) that

(1) $$\sigma_1^{(1)\otimes(2)}=\sigma_1^{(1)}\otimes I^{(2)}=\begin{array}{c} \\ ++ \\ +- \\ -+ \\ -- \end{array}\!\!\begin{array}{c} \begin{array}{cccc} ++ & +- & -+ & -- \end{array} \\ \begin{bmatrix} a & 0 & b & 0 \\ 0 & a & 0 & b \\ c & 0 & d & 0 \\ 0 & c & 0 & d \end{bmatrix}\end{array}$$

(Recall that $\langle\alpha|\otimes\langle\beta|$ is the bra corresponding to $|\alpha\rangle\otimes|\beta\rangle$.)

(2) $$\sigma_2^{(1)\otimes(2)}=\begin{bmatrix} e & f & 0 & 0 \\ g & h & 0 & 0 \\ 0 & 0 & e & f \\ 0 & 0 & g & h \end{bmatrix}$$

(3) $$(\sigma_1\sigma_2)^{(1)\otimes(2)}=\sigma_1^{(1)}\otimes\sigma_2^{(2)}=\begin{bmatrix} ae & af & be & bf \\ ag & ah & bg & bh \\ ce & cf & de & df \\ cg & ch & dg & dh \end{bmatrix}$$

Do part (3) in two ways, by taking the matrix product of $\sigma_1^{(1)\otimes(2)}$ and $\sigma_2^{(1)\otimes(2)}$ and by directly computing the matrix elements of $\sigma_1^{(1)}\otimes\sigma_2^{(2)}$.

From Eqs. (10.1.17a) and (10.1.17c) it follows that the commutation relations between the position and momentum operators on $\mathbb{V}_1\otimes\mathbb{V}_2$ are

$$\begin{aligned} &[X_i^{(1)\otimes(2)}, P_j^{(1)\otimes(2)}]=i\hbar\delta_{ij}I^{(1)}\otimes I^{(2)}=i\hbar\delta_{ij}I^{(1)\otimes(2)} \\ &[X_i^{(1)\otimes(2)}, X_j^{(1)\otimes(2)}]=[P_i^{(1)\otimes(2)}, P_j^{(1)\otimes(2)}]=0 \qquad i,j=1,2 \end{aligned} \tag{10.1.18}$$

Now we are ready to assert something that may have been apparent all along: the space $\mathbb{V}_1\otimes\mathbb{V}_2$ is just $\mathbb{V}_{1\otimes 2}$, $|x_1\rangle\otimes|x_2\rangle$ is just $|x_1x_2\rangle$, and $X_1^{(1)\otimes(2)}$ is just X_1, etc. Notice first that both spaces have the same dimensionality: the vectors $|x_1x_2\rangle$ and $|x_1\rangle\otimes|x_2\rangle$ are both in one-to-one correspondence with points in the x_1-x_2 plane. Notice next that the two sets of operators $X_1, \ldots, P_2$ and $X_1^{(1)\otimes(2)}, \ldots, P_2^{(1)\otimes(2)}$ have the same connotation and commutation rules [Eqs. (10.1.1) and (10.1.18)]. Since X and P are *defined* by their commutators we can make the identification

$$\begin{aligned} X_i^{(1)\otimes(2)}&=X_i \\ P_i^{(1)\otimes(2)}&=P_i \end{aligned} \tag{10.1.19a}$$

We can also identify the simultaneous eigenkets of the position operators (since they are nondegenerate):

$$|x_1\rangle\otimes|x_2\rangle=|x_1x_2\rangle \tag{10.1.19b}$$

In the future, we shall use the more compact symbols occurring on the right-hand side of Eqs. (10.1.19). We will, however, return to the concept of direct products of vectors and operators on and off and occasionally use the symbols on the left-hand side. Although the succinct notation suppresses the label $(1\otimes 2)$ of the space on which the operators act, it should be clear from the context. Consider, for example, the CM kinetic energy operator of the two-particle system:

$$T_{\mathrm{CM}}=\frac{P_{\mathrm{CM}}^2}{2(m_1+m_2)}=\frac{P_{\mathrm{CM}}^2}{2M}=\frac{(P_1+P_2)^2}{2M}=\frac{P_1^2+P_2^2+2P_1P_2}{2M}$$

which really means

$$\begin{aligned}2MT_{\mathrm{CM}}^{(1)\otimes(2)}&=(P_1^2)^{(1)\otimes(2)}+(P_2^2)^{(1)\otimes(2)}+2P_1^{(1)\otimes(2)}\cdot P_2^{(1)\otimes(2)}\\&=(P_1^{(1)}\otimes I^{(2)})^2+(I^{(1)}\otimes P_2^{(2)})^2+2P_1^{(1)}\otimes P_2^{(2)}\end{aligned}$$

The Direct Product Revisited

Since the notion of a direct product space is so important, we revisit the formation of $\mathbb{V}_{1\otimes 2}$ as a direct product of $\mathbb{V}_1$ and $\mathbb{V}_2$, but this time in the coordinate basis instead of in the abstract. Let $\Omega_1^{(1)}$ be an operator on $\mathbb{V}_1$ whose nondegenerate eigenfunctions $\psi_{\omega_1}(x_1)\equiv\omega_1(x_1)$ form a complete basis. Similarly let $\omega_2(x_2)$ form a basis for $\mathbb{V}_2$. Consider now a function $\psi(x_1, x_2)$, which represents the abstract ket $|\psi\rangle$ from $\mathbb{V}_{1\otimes 2}$. If we keep x_1 fixed at some value, say $\bar{x}_1$, then ψ becomes a function of x_2 alone and may be expanded as

$$\psi(\bar{x}_1, x_2)=\sum_{\omega_2} C_{\omega_2}(\bar{x}_1)\omega_2(x_2) \tag{10.1.20}$$

Notice that the coefficients of the expansion depend on the value of $\bar{x}_1$. We now expand the function $C_{\omega_2}(\bar{x}_1)$ in the basis $\omega_1(\bar{x}_1)$:

$$C_{\omega_2}(\bar{x}_1)=\sum_{\omega_1} C_{\omega_1,\omega_2}\omega_1(\bar{x}_1) \tag{10.1.21}$$

Feeding this back to the first expansion and dropping the bar on $\bar{x}_1$ we get

$$\psi(x_1, x_2)=\sum_{\omega_1}\sum_{\omega_2} C_{\omega_1,\omega_2}\omega_1(x_1)\omega_2(x_2) \tag{10.1.22a}$$

What does this expansion of an arbitrary $\psi(x_1, x_2)$ in terms of $\omega_1(x_1)\times\omega_2(x_2)$ imply? Equation (10.1.22a) is the coordinate space version of the abstract result

$$|\psi\rangle=\sum_{\omega_1}\sum_{\omega_2} C_{\omega_1,\omega_2}|\omega_1\rangle\otimes|\omega_2\rangle \tag{10.1.22b}$$

which means $\mathbb{V}_{1\otimes 2}=\mathbb{V}_1\otimes\mathbb{V}_2$, for $|\psi\rangle$ belongs to $\mathbb{V}_{1\otimes 2}$ and $|\omega_1\rangle\otimes|\omega_2\rangle$ spans $\mathbb{V}_1\otimes\mathbb{V}_2$. If we choose $\Omega=X$, we get the familiar basis $|x_1\rangle\otimes|x_2\rangle$. By dotting both sides of Eq. (10.1.22b) with these basis vectors we regain Eq. (10.1.22a). (In the coordinate basis, the direct product of the kets $|\omega_1\rangle$ and $|\omega_2\rangle$ becomes just the ordinary product of the corresponding wave functions.)

Consider next the operators. The momentum operator on $\mathbb{V}_1$, which used to be $-i\hbar\, d/dx_1$ now becomes $-i\hbar\,\partial/\partial x_1$, where the partial derivative symbol tells us it operates on x_1 as before and leaves x_2 alone. This is the coordinate space version of $P^{(1)\otimes(2)}=P_1^{(1)}\otimes I^{(2)}$. You are encouraged to pursue this analysis further.

Evolution of the Two-Particle State Vector

The state vector of the system is an element of $\mathbb{V}_{1\otimes 2}$. It evolves in time according to the equation

$$i\hbar|\dot{\psi}\rangle=\left[\frac{P_1^2}{2m_1}+\frac{P_2^2}{2m_2}+V(X_1, X_2)\right]|\psi\rangle=H|\psi\rangle \tag{10.1.23}$$

There are two classes of problems.

Class A: H is separable, i.e.,

$$H=\frac{P_1^2}{2m_1}+V_1(X_1)+\frac{P_2^2}{2m_2}+V_2(X_2)=H_1+H_2 \tag{10.1.24}$$

Class B: H is not separable, i.e.,

$$V(X_1, X_2)\neq V_1(X_1)+V_2(X_2)$$

and

$$H\neq H_1+H_2 \tag{10.1.25}$$

Class A corresponds to two particles interacting with external potentials V_1 and V_2 but not with each other, while in class B there is no such restriction. We now examine these two classes.

Class A: Separable Hamiltonians. Classically, the decomposition

$$\mathscr{H}=\mathscr{H}_1(x_1, p_1)+\mathscr{H}_2(x_2, p_2)$$

means that the two particles evolve independently of each other. In particular, their energies are *separately* conserved and the total energy E is E_1+E_2. Let us see these results reappear in quantum theory. For a stationary state,

$$|\psi(t)\rangle=|E\rangle\, e^{-iEt/\hbar} \tag{10.1.26}$$

Eq. (10.1.23) becomes

$$[H_1(X_1, P_1)+H_2(X_2, P_2)]|E\rangle=E|E\rangle \tag{10.1.27}$$

Since $[H_1, H_2]=0$ [Eq. (10.1.17a)] we can find their simultaneous eigenstates, which are none other than $|E_1\rangle\otimes|E_2\rangle=|E_1E_2\rangle$, where $|E_1\rangle$ and $|E_2\rangle$ are solutions to

$$H_1^{(1)}|E_1\rangle=E_1|E_1\rangle \tag{10.1.28a}$$

and

$$H_2^{(2)}|E_2\rangle=E_2|E_2\rangle \tag{10.1.28b}$$

It should be clear that the state $|E_1\rangle\otimes|E_2\rangle$ corresponds to particle 1 being in the energy eigenstate $|E_1\rangle$ and particle 2 being in the energy eigenstate $|E_2\rangle$. Clearly

$$H|E\rangle=(H_1+H_2)|E_1\rangle\otimes|E_2\rangle=(E_1+E_2)|E_1\rangle\otimes|E_2\rangle=(E_1+E_2)|E\rangle$$

so that

$$E=E_1+E_2 \tag{10.1.28c}$$

(The basis $|E_1\rangle\otimes|E_2\rangle$ is what we would get if in forming basis vectors of the direct product $\mathbb{V}_1\otimes\mathbb{V}_2$, we took the energy eigenvalues from each space, instead of, say, the position eigenvectors.) Finally, feeding $|E\rangle=|E_1\rangle\otimes|E_2\rangle$, $E=E_1+E_2$ into Eq. (10.1.26) we get

$$|\psi(t)\rangle=|E_1\rangle\, e^{-iE_1t/\hbar}\otimes|E_2\rangle\, e^{-iE_2t/\hbar} \tag{10.1.29}$$

It is worth rederiving Eqs. (10.1.28) and (10.1.29) in the coordinate basis to illustrate a useful technique that you will find in other textbooks. By projecting the eigenvalue Eq. (10.1.27) on this basis, and making the usual operator substitutions, Eq. (10.1.4), we obtain

$$\left[\frac{-\hbar^2}{2m_1}\frac{\partial^2}{\partial x_1^2}+V_1(x_1)-\frac{\hbar^2}{2m_2}\frac{\partial^2}{\partial x_2^2}+V_2(x_2)\right]\psi_E(x_1, x_2)=E\psi_E(x_1, x_2)$$

where

$$\psi_E(x_1, x_2)=\langle x_1x_2|E\rangle \tag{10.1.30}$$

We solve the equation by the method of *separation of variables*. We assume

$$\psi_E(x_1, x_2)=\psi_{E_1}(x_1)\psi_{E_2}(x_2) \tag{10.1.31}$$

The subscripts E_1 and E_2 have no specific interpretation yet and merely serve as labels. Feeding this *ansatz* into Eq. (10.1.30) and *then* dividing both sides by

$\psi_{E_1}(x_1)\psi_{E_2}(x_2)$ we get

$$\frac{1}{\psi_{E_1}(x_1)}\left[\frac{-\hbar^2}{2m_1}\frac{\partial^2}{\partial x_1^2}+V_1(x_1)\right]\psi_{E_1}(x_1)$$
$$+\frac{1}{\psi_{E_2}(x_2)}\left[\frac{-\hbar^2}{2m_2}\frac{\partial^2}{\partial x_2^2}+V_2(x_2)\right]\psi_{E_2}(x_2)=E \tag{10.1.32}$$

This equation says that a function of x_1 alone, plus one of x_2 alone, equals a constant E. Since x_1 and x_2, and hence the two functions, may be varied independently, it follows that each function separately equals a constant. We will call these constants E_1 and E_2. Thus Eq. (10.1.32) breaks down into three equations:

$$\begin{aligned}\frac{1}{\psi_{E_1}(x_1)}\left[\frac{-\hbar^2}{2m_1}\frac{\partial^2}{\partial x_1^2}+V_1(x_1)\right]\psi_{E_1}(x_1)&=E_1\\ \frac{1}{\psi_{E_2}(x_2)}\left[\frac{-\hbar^2}{2m_2}\frac{\partial^2}{\partial x_2^2}+V_2(x_2)\right]\psi_{E_2}(x_2)&=E_2\\ E_1+E_2&=E\end{aligned} \tag{10.1.33}$$

Consequently

$$\begin{aligned}\psi_E(x_1,x_2,t)&=\psi_E(x_1,x_2)\,e^{-iEt/\hbar}\\ &=\psi_{E_1}(x_1)\,e^{-iE_1t/\hbar}\psi_{E_2}(x_2)\,e^{-iE_2t/\hbar}\end{aligned} \tag{10.1.34}$$

where ψ_{E_1} and ψ_{E_2} are eigenfunctions of the one-particle Schrödinger equation with eigenvalues E_1 and E_2, respectively. We recognize Eqs. (10.1.33) and (10.1.34) to be the projections of Eqs. (10.1.28) and (10.1.29) on $|x_1x_2\rangle=|x_1\rangle\otimes|x_2\rangle$.

Case B: Two Interacting Particles. Consider next the more general problem of two interacting particles with

$$\mathscr{H}=\frac{p_1^2}{2m_1}+\frac{p_2^2}{2m_2}+V(x_1,x_2) \tag{10.1.35}$$

where

$$V(x_1,x_2)\neq V_1(x_1)+V(x_2)$$

Generally this cannot be reduced to two independent single-particle problems. If, however,

$$V(x_1,x_2)=V(x_1-x_2) \tag{10.1.36}$$

which describes two particles responding to each other but nothing external, one can always, by employing the CM coordinate

$$x_{\rm CM}=\frac{m_1x_1+m_2x_2}{m_1+m_2} \tag{10.1.37a}$$

and the relative coordinate

$$x=x_1-x_2 \tag{10.1.37b}$$

reduce the problem to that of two independent fictitious particles: one, the CM, which is free, has mass $M=m_1+m_2$ and momentum

$$p_{\rm CM}=M\dot{x}_{\rm CM}=m_1\dot{x}_1+m_2\dot{x}_2$$

and another, with the reduced mass $\mu=m_1m_2/(m_1+m_2)$, momentum $p=\mu\dot{x}$, moving under the influence of $V(x)$:

$$\begin{aligned}\mathscr{H}(x_1,p_1;x_2,p_2)&\to\mathscr{H}(x_{\rm CM},p_{\rm CM};x,p)\\&=\mathscr{H}_{\rm CM}+\mathscr{H}_{\rm relative}=\frac{p_{\rm CM}^2}{2M}+\frac{p^2}{2\mu}+V(x)\end{aligned} \tag{10.1.38}$$

which is just the result from Exercise 2.5.4 modified to one dimension. Since the new variables are also canonical (Exercise 2.7.6) and Cartesian, the quantization condition is just

$$[X_{\rm CM},P_{\rm CM}]=i\hbar \tag{10.1.39a}$$

$$[X,P]=i\hbar \tag{10.1.39b}$$

and all other commutators zero. In the quantum theory,

$$H=\frac{P_{\rm CM}^2}{2M}+\frac{P^2}{2\mu}+V(X) \tag{10.1.40}$$

and the eigenfunctions of H factorize:

$$\begin{aligned}\psi_E(x_{\rm CM},x)&=\frac{e^{ip_{\rm CM}\cdot x_{\rm CM}/\hbar}}{(2\pi\hbar)^{1/2}}\cdot\psi_{E_{\rm rel}}(x)\\E&=\frac{p_{\rm CM}^2}{2M}+E_{\rm rel}\end{aligned} \tag{10.1.41}$$

The real dynamics is contained in $\psi_{E_{\rm rel}}(x)$ which is the energy eigenfunction for a particle of mass μ in a potential $V(x)$. Since the CM drifts along as a free particle, one usually chooses to study the problem in the CM frame. In this case $E_{\rm CM}=$

$p_{\mathrm{CM}}^2/2M$ drops out of the energy, and the plane wave factor in ψ representing CM motion becomes a constant. In short, one can forget all about the CM in the quantum theory just as in the classical theory.

N Particles in One Dimension

All the results but one generalize from $N=2$ to arbitrary N. The only exception is the result from the last subsection: for $N>2$, one generally cannot, by using CM and relative coordinates (or other sets of coordinates) reduce the problem to N independent one-particle problems. There are a few exceptions, the most familiar ones being Hamiltonians quadratic in the coordinates and momenta which may be reduced to a sum over oscillator Hamiltonians by the use of normal coordinates. In such cases the oscillators become independent and their energies add both in the classical and quantum cases. This result (with respect to the quantum oscillators) was assumed in the discussion on specific heats in Chapter 7.

*Exercise 10.1.3.** Consider the Hamiltonian of the coupled mass system:

$$\mathscr{H}=\frac{p_1^2}{2m}+\frac{p_2^2}{2m}+\frac{1}{2}m\omega^2[x_1^2+x_2^2+(x_1-x_2)^2]$$

We know from Example 1.8.6 that $\mathscr{H}$ can be decoupled if we use normal coordinates

$$x_{\mathrm{I,II}}=\frac{x_1\pm x_2}{2^{1/2}}$$

and the corresponding momenta

$$p_{\mathrm{I,II}}=\frac{p_1\pm p_2}{2^{1/2}}$$

(1) Rewrite $\mathscr{H}$ in terms of normal coordinates. Verify that the normal coordinates are also canonical, i.e., that

$$\{x_i, p_j\}=\delta_{ij}\text{ etc.};\qquad i,j=\mathrm{I, II}$$

Now quantize the system, promoting these variables to operators obeying

$$[X_i, P_j]=i\hbar\delta_{ij}\text{ etc.};\qquad i,j=\mathrm{I, II}$$

Write the eigenvalue equation for H in the simultaneous eigenbasis of X_{I} and X_{II}.

(2) Quantize the system directly, by promoting x_1, x_2, p_1, and p_2 to quantum operators. Write the eigenvalue equation for H in the simultaneous eigenbasis of X_1 and X_2. Now change from x_1, x_2 (and of course $\partial/\partial x_1$, $\partial/\partial x_2$) to x_{I}, x_{II} (and $\partial/\partial x_{\mathrm{I}}$, $\partial/\partial x_{\mathrm{II}}$) *in the differential equation*. You should end up with the result from part (1).

In general, one can change coordinates and then quantize or first quantize and then change variables in the differential equation, if the change of coordinates is canonical. (We are assuming that all the variables are Cartesian. As mentioned earlier in the book, if one wants to employ non-Cartesian coordinates, it is best to first quantize the Cartesian coordinates and then change variables in the differential equation.)

10.2. More Particles in More Dimensions

Mathematically, the problem of a single particle in two dimensions (in terms of Cartesian coordinates) is equivalent to that of two particles in one dimension. It is, however, convenient to use a different notation in the two cases. We will denote the two Cartesian coordinates of the single particle by x and y rather than x_1 and x_2. Likewise the momenta will be denoted by p_x and p_y. The quantum operators will be called X and Y; and P_x, and P_y, their common eigenkets $|xy\rangle$, $|p_xp_y\rangle$, respectively, and so on. The generalization to three dimensions is obvious. We will also write a position eigenket as $|\mathbf{r}\rangle$ and the orthonormality relation $\langle xyz|x'y'z'\rangle = \delta(x-x')\delta(y-y')\delta(z-z')$ as $\langle \mathbf{r}|\mathbf{r}'\rangle = \delta^3(\mathbf{r}-\mathbf{r}')$. The same goes for the momentum eigenkets $|\mathbf{p}\rangle$ also. When several particles labeled by numbers $1, \ldots, N$ are involved, this extra label will also be used. Thus $|\mathbf{p}_1\mathbf{p}_2\rangle$ will represent a two-particle state in which particle 1 has momentum $\mathbf{p}_1$ and particle 2 has momentum $\mathbf{p}_2$ and so on.

*Exercise 10.2.1** (Particle in a Three-Dimensional Box). Recall that a particle in a one-dimensional box extending from $x=0$ to L is confined to the region $0 \le x \le L$; its wave function vanishes at the edges $x=0$ and L and beyond (Exercise 5.2.5). Consider now a particle confined in a three-dimensional cubic box of volume L^3. Choosing as the origin one of its corners, and the x, y, and z axes along the three edges meeting there, show that the normalized energy eigenfunctions are

$$\psi_E(x, y, z) = \left(\frac{2}{L}\right)^{1/2} \sin\left(\frac{n_x \pi x}{L}\right)\left(\frac{2}{L}\right)^{1/2} \sin\left(\frac{n_y \pi y}{L}\right)\left(\frac{2}{L}\right)^{1/2} \sin\left(\frac{n_z \pi z}{L}\right)$$

where

$$E = \frac{\hbar^2\pi^2}{2ML^2}(n_x^2 + n_y^2 + n_z^2)$$

and n_i are positive integers.

*Exercise 10.2.2.** Quantize the two-dimensional oscillator for which

$$\mathscr{H} = \frac{p_x^2 + p_y^2}{2m} + \frac{1}{2} m\omega_x^2 x^2 + \frac{1}{2} m\omega_y^2 y^2$$

(1) Show that the allowed energies are

$$E = (n_x + 1/2)\hbar\omega_x + (n_y + 1/2)\hbar\omega_y, \qquad n_x, n_y = 0, 1, 2, \ldots$$

(2) Write down the corresponding wave functions in terms of single oscillator wave functions. Verify that they have definite parity (even/odd) number $x \to -x$, $y \to -y$ and that the parity depends only on $n = n_x + n_y$.

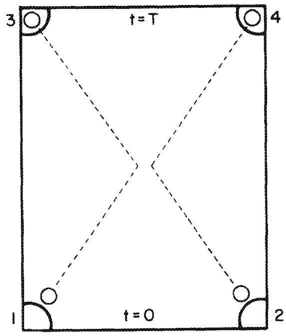

Figure 10.1. Two identical billiard balls start near holes 1 and 2 and end up in holes 3 and 4, respectively, as predicted by P_1. The prediction of P_2, that they would end up in holes 4 and 3, respectively, is wrong, even though the two final configurations would be indistinguishable to an observer who walks in at $t=T$.

(3) Consider next the *isotropic* oscillator ($\omega_x=\omega_y$). Write *explicit*, normalized eigenfunctions of the first three *states* (that is, for the cases $n=0$ and 1). Reexpress your results in terms of polar coordinates ρ and ϕ (for later use). Show that the degeneracy of a level with $E=(n+1)\hbar\omega$ is $(n+1)$.

*Exercise 10.2.3.** Quantize the three-dimensional *isotropic oscillator* for which

$$\mathscr{H}=\frac{p_x^2+p_y^2+p_z^2}{2m}+\frac{1}{2}m\omega^2(x^2+y^2+z^2)$$

(1) Show that $E=(n+3/2)\hbar\omega$; $n=n_x+n_y+n_z$; $n_x, n_y, n_z=0, 1, 2, \ldots$.

(2) Write the corresponding eigenfunctions in terms of single-oscillator wave functions and verify that the parity of the level with a given n is $(-1)^n$. Reexpress the first four states in terms of spherical coordinates. Show that the degeneracy of a level with energy $E=(n+3/2)\hbar\omega$ is $(n+1)(n+2)/2$.

10.3. Identical Particles

The formalism developed above, when properly applied to a system containing identical particles, leads to some very surprising results. We shall say two particles are identical if they are exact replicas of each other in every respect—there should be no experiment that detects any intrinsic‡ difference between them. Although the definition of identical particles is the same classically and quantum mechanically, the implications are different in the two cases.

The Classical Case

Let us first orient ourselves by recapitulating the situation in classical physics. Imagine a billiard table with four holes, numbered 1 through 4 (Fig. 10.1). Near holes 1 and 2 rest two identical billiard balls. Let us call these balls 1 and 2. The difference between the labels reflects not any intrinsic difference in the balls (for they are identical) but rather a difference in their environments, namely, the holes near which they find themselves.

‡ By intrinsic I mean properties inherent to the particle, such as its charge or mass and not its location or momentum.

Now it follows from the definition of identity, that if these two balls are exchanged, the resulting configuration would appear exactly the same. Nonetheless these two configurations are treated as distinct in classical physics. In order for this distinction to be meaningful, there must exist some experiments in which these two configurations are inequivalent. We will now discuss one such experiment.

Imagine that at time $t=0$, two players propel the balls toward the center of the table. At once two physicists P_1 and P_2 take the initial-value data and make the following predictions:

$$P_1: \quad \left.\begin{array}{l}\text{ball 1 goes to hole 3}\\ \text{ball 2 goes to hole 4}\end{array}\right\} \quad \text{at } t=\text{T}$$

$$P_2: \quad \left.\begin{array}{l}\text{ball 1 goes to hole 4}\\ \text{ball 2 goes to hole 3}\end{array}\right\} \quad \text{at } t=T$$

Say at time T we find that ball 1 ends up in hole 3 and ball 2 in hole 4. We declare that P_1 is correct and P_2 is wrong. Now, the configurations predicted by them for $t=T$ differ only by the exchange of two identical particles. If seen in isolation they would appear identical: an observer who walks in just at $t=T$ and is given the predictions of P_1 and P_2 will conclude that both are right. What do we know about the balls (that allows us to make a distinction between them and hence the two outcomes), that the newcomer does not? The answer of course is—their histories. Although both balls appear identical to the newcomer, we are able to trace the ball in hole 3 back to the vicinity of hole 1 and the one in hole 4 back to hole 2. Similarly at $t=0$, the two balls which seemed identical to us would be distinguishable to someone who had been following them from an earlier period. Now of course it is not really necessary that either we or any other observer be actually present in order for this distinction to exist. One imagines in classical physics the fictitious observer who sees everything and disturbs nothing; if he can make the distinction, the distinction exists.

To summarize, it is possible in classical mechanics to distinguish between identical particles by following their nonidentical trajectories (without disturbing them in any way). Consequently two configurations related by exchanging the identical particles are physically nonequivalent.

An immediate consequence of the above reasoning, and one that will play a dominant role in what follows, is that in quantum theory, which completely outlaws the notion of continuous trajectories for the particles, there exists no physical basis for distinguishing between identical particles. Consequently two configurations related by the exchange of identical particles must be treated as one and the same configuration and described by the same state vector. We now proceed to deduce the consequences of this restriction.

Two-Particle Systems—Symmetric and Antisymmetric States

Suppose we have a system of two *distinguishable* particles 1 and 2 and a position measurement on the system shows particle 1 to be at $x=a$ and particle 2 to be at

$x=b$. We write the state just after measurement as

$$|\psi\rangle = |x_1 = a, x_2 = b\rangle = |ab\rangle \tag{10.3.1}$$

where we are adopting the convention that the state of particle 1 is described by the first label (a) and that of particle 2 by the second label (b). Since the particles are distinguishable, the state obtained by exchanging them is distinguishable from the above. It is given by

$$|\psi\rangle = |ba\rangle$$

and corresponds to having found particle 1 at b and particle 2 at a.

Suppose we repeat the experiment with two *identical* particles and catch one at $x=a$ and the other at $x=b$. Is the state vector just after measurement $|ab\rangle$ or $|ba\rangle$? The answer is, neither. We have seen that in quantum theory two configurations related by the exchange of identical particles must be viewed as one and the same and be described by the same state vector. Since $|\psi\rangle$ and $\alpha|\psi\rangle$ are physically equivalent, we require that $|\psi(a, b)\rangle$, the state vector just after the measurement, satisfy the constraint

$$|\psi(a, b)\rangle = \alpha|\psi(b, a)\rangle \tag{10.3.2}$$

where α is any complex number. Since under the exchange

$$|ab\rangle \leftrightarrow |ba\rangle$$

and the two vectors are not multiples of each other‡ (i.e., are physically distinct) neither is acceptable. The problem is that our position measurement yields not an ordered pair of numbers (as in the distinguishable particle case) but just a pair of numbers: to assign them to the particles in a definite way is to go beyond what is physically meaningful in quantum theory. What our measurement *does* permit us to conclude is that the state vector is an eigenstate of X_1+X_2 with eigenvalue $a+b$, the sum of the eigenvalue being insensitive to how the values a and b are assigned to the particles. In other words, given an *unordered* pair of numbers a and b we can still define a unique sum (but not difference). Now, there are just two product vectors, $|ab\rangle$ and $|ba\rangle$ with this eigenvalue, and the state vector lies somewhere in the two-dimensional degenerate (with respect to X_1+X_2) eigenspace spanned by them. Let $|\psi(a, b)\rangle = \beta|ab\rangle + \gamma|ba\rangle$ be the allowed vector. If we impose the constraint Eq. (10.3.2):

$$\beta|ab\rangle + \gamma|ba\rangle = \alpha[\beta|ba\rangle + \gamma|ab\rangle]$$

we find, upon equating the coefficients of $|ab\rangle$ and $|ba\rangle$ that

$$\beta = \alpha\gamma, \qquad \gamma = \alpha\beta$$

‡ We are assuming $a \neq b$. If $a=b$, the state is acceptable, but the choice we are agonizing over does not arise.

so that

$$\alpha = \pm 1 \tag{10.3.3}$$

It is now easy to construct the allowed state vectors. They are

$$|ab, S\rangle = |ab\rangle + |ba\rangle \tag{10.3.4}$$

called the *symmetric* state vector ($\alpha = 1$) and

$$|ab, A\rangle = |ab\rangle - |ba\rangle \tag{10.3.5}$$

called the *antisymmetric* state vector ($\alpha = -1$). (These are *unnormalized* vectors. Their normalization will be taken up shortly.)

More generally, if some variable Ω is measured and the values ω_1 and ω_2 are obtained, the state vector immediately following the measurement is either $|\omega_1\omega_2, S\rangle$ or $|\omega_1\omega_2, A\rangle$.‡ Although we have made a lot of progress in nailing down the state vector corresponding to the measurement, we have still to find a way to choose between these two alternatives.

Bosons and Fermions

Although both S and A states seem physically acceptable (in that they respect the indistinguishability of the particles) we can go a step further and make the following assertion:

A given species of particles must choose once and for all between S and A states.

Suppose the contrary were true, and the Hilbert space of two identical particles contained both S and A vectors. Then the space also contains linear combinations such as

$$|\psi\rangle = \alpha|\omega_1\omega_2, S\rangle + \beta|\omega_1'\omega_2', A\rangle$$

which are neither symmetric nor antisymmetric. So we rule out this possibility.

Nature seems to respect the constraints we have deduced. Particles such as the pion, photon, and graviton are *always* found in symmetric states and are called *bosons*, and particles such as the electron, proton, and neutron are *always* found in antisymmetric states and are called *fermions*.

Thus if we catch two identical bosons, one at $x=a$ and the other at $x=b$, the state vector immediately following the measurement is

$$\begin{aligned}|\psi\rangle &= |x_1=a, x_2=b\rangle + |x_1=b, x_2=a\rangle \\ &= |ab\rangle + |ba\rangle = |ab, S\rangle\end{aligned}$$

‡ We are assuming Ω is nondegenerate. If not, let ω represent the eigenvalues of a complete set of commuting operators.

Had the particles been fermions, the state vector after the measurement would have been

$$|\psi\rangle = |x_1 = a, x_2 = b\rangle - |x_1 = b, x_2 = a\rangle = |ab\rangle - |ba\rangle$$
$$= |ab, A\rangle$$

Note that although we still use the labels x_1 and x_2, we do not attach them to the particles in any particular way. Thus having caught the bosons at $x=a$ and $x=b$, we need not agonize over whether $x_1 = a$ and $x_2 = b$ or vice versa. Either choice leads to the same $|\psi\rangle$ for bosons, and to state vectors differing only by an overall sign for fermions.

We are now in a position to deduce a fundamental property of fermions, which results from the antisymmetry of their state vectors. Consider a two-fermion state

$$|\omega_1\omega_2, A\rangle = |\omega_1\omega_2\rangle - |\omega_2\omega_1\rangle$$

Let us now set $\omega_1 = \omega_2 = \omega$. We find

$$|\omega\omega, A\rangle = |\omega\omega\rangle - |\omega\omega\rangle = 0 \tag{10.3.6}$$

This is the celebrated *Pauli exclusion principle: Two identical fermions cannot be in the same quantum state.* This principle has profound consequences—in statistical mechanics, in understanding chemical properties of atoms, in nuclear theory, astrophysics, etc. We will have occasion to return to it often.

With this important derivation out of our way, let us address a question that may have plagued you: our analysis has only told us that a given type of particle, say a pion, has to be either a boson or a fermion, but does not say which one. There are two ways to the answer. The first is by further cerebration, to be specific, within the framework of quantum field theory, which relates the spin of the particle to its "statistics"—which is the term physicists use to refer to its bosonic or fermionic nature. Since the relevant arguments are beyond the scope of this text I merely quote the results here. Recall that the spin of the particle is its internal angular momentum. The *magnitude* of spin happens to be an invariant for a particle (and thus serves as a label, like its mass or charge) and can have only one of the following values: 0, $\hbar/2$, $\hbar$, $3\hbar/2$, $2\hbar$, ... The spin statistics theorem, provable in quantum field theory, asserts that particles with (magnitude of spin) equal to an even multiple of $\hbar/2$ are bosons, and those with spin equal to an odd multiple of $\hbar/2$ are fermions. However, this connection, proven in three dimensions, does not apply to one dimension, where it is not possible to define spin or any form of angular momentum. (This should be clear classically.) Thus the only way to find if a particle in one dimension is a boson or fermion is to determine the symmetry of the wave function experimentally. This is the second method, to be discussed in a moment.

Before going on to this second method, let us note that the requirement that the state vector of two identical particles be symmetric or antisymmetric (under the exchange of the quantum numbers labeling them) applies in three dimensions as well, as will be clear by going through the arguments in one dimension. The only difference will be the increase in the number of labels. For example, the position eigenket of a spin-zero boson will be labeled by three numbers x, y, and z. For fermions, which have spin at least equal to $\hbar/2$, the states will be labeled by the orientation of the spin as well as the orbital labels that describe spinless bosons.‡ We shall consider just spin-$\frac{1}{2}$ particles, for which this label can take only two values, call them $+$ and $-$ or spin up and down (the meaning of these terms will be clear later). If we denote by ω all the orbital labels and by s the spin label, the state vector of the fermion that is antisymmetric under the exchange of the particles, i.e., under the exchange of all the labels, will be of the form

$$|\omega_1 s_1, \omega_2 s_2, A\rangle = |\omega_1 s_1, \omega_2 s_2\rangle - |\omega_2 s_2, \omega_1 s_1\rangle \tag{10.3.7}$$

We see that the state vector vanishes if

$$\omega_1 = \omega_2 \qquad \text{and} \qquad s_1 = s_2 \tag{10.3.8}$$

Thus we find once again that two fermions cannot be in the same quantum state, but we mean by a quantum state a state of definite ω and s. Thus two electrons can be in the same orbital state if their spin orientations are different.

We now turn to the second way of finding the statistics of a given species of particles, the method that works in one or three dimensions, because it appeals to a simple experiment which determines whether the two-particle state vector is symmetric or antisymmetric for the given species. As a prelude to the discussion of such an experiment, let us study in some detail the Hilbert space of bosons and fermions.

Bosonic and Fermionic Hilbert Spaces

We have seen that two identical bosons will always have symmetric state vectors and two identical fermions will always have antisymmetric state vectors. Let us call the Hilbert space of symmetric bosonic vectors $\mathbb{V}_S$ and the Hilbert space of the antisymmetric fermionic vectors $\mathbb{V}_A$. We first examine the relation between these two spaces on the one hand and the direct product space $\mathbb{V}_{1\otimes 2}$ on the other.

The space $\mathbb{V}_{1\otimes 2}$ consists of all vectors of the form $|\omega_1\omega_2\rangle = |\omega_1\rangle \otimes |\omega_2\rangle$. To each pair of vectors $|\omega_1 = a, \omega_2 = b\rangle$ and $|\omega_1 = b, \omega_2 = a\rangle$ there is one (unnormalized) bosonic vector $|\omega_1 = a, \omega_2 = b\rangle + |\omega_1 = b, \omega_2 = a\rangle$ and one fermionic vector $|\omega_1 = a, \omega_2 = b\rangle - |\omega_1 = a, \omega_2 = b\rangle$. If $a = b$; the vector $|\omega_1 = a, \omega_2 = a\rangle$ is already symmetric and we may take it to be the bosonic vector. There is no corresponding fermionic vector (the Pauli principle). Thus $\mathbb{V}_{1\otimes 2}$ has just enough basis vectors to form one bosonic Hilbert space and one fermionic Hilbert space. We express this relation as

$$\mathbb{V}_{1\otimes 2} = \mathbb{V}_S \oplus \mathbb{V}_A \tag{10.3.9}$$

‡ Since spin has no classical counterpart, the operator representing it is not a function of the coordinate and momentum operators and it commutes with any orbital operator Ω. Thus spin may be specified simultaneously with the orbital variables.

with $\mathbb{V}_S$ getting slightly more than half the dimensionality of $\mathbb{V}_{1\otimes 2}$.‡ Our analysis has shown that at any given time, the state of two bosons is an element of $\mathbb{V}_S$ and that of two fermions an element of $\mathbb{V}_A$. It can also be shown that a system that starts out in $\mathbb{V}_S(\mathbb{V}_A)$ remains in $\mathbb{V}_S(\mathbb{V}_A)$ (see Exercise 10.3.5). *Thus in studying two identical particles we need only consider* $\mathbb{V}_S$ *or* $\mathbb{V}_A$. It is however convenient, for bookkeeping purposes, to view $\mathbb{V}_S$ and $\mathbb{V}_A$ as subspaces of $\mathbb{V}_{1\otimes 2}$ and the elements of $\mathbb{V}_S$ or $\mathbb{V}_A$ as elements also of $\mathbb{V}_{1\otimes 2}$.

Let us now consider the normalization of the vectors in $\mathbb{V}_S$. Consider first the eigenkets $|\omega_1\omega_2, S\rangle$ corresponding to a variable Ω with discrete eigenvalues. The unnormalized state vector is

$$|\omega_1\omega_2, S\rangle = |\omega_1\omega_2\rangle + |\omega_2\omega_1\rangle$$

Since $|\omega_1\omega_2\rangle$ and $|\omega_2\omega_1\rangle$ are orthonormal states in $\mathbb{V}_{1\otimes 2}$, the normalization factor is just $2^{-1/2}$, i.e.,

$$|\omega_1\omega_2, S\rangle = 2^{-1/2}[|\omega_1\omega_2\rangle + |\omega_2\omega_1\rangle] \tag{10.3.10a}$$

is a normalized eigenvector. You may readily check that $\langle\omega_1\omega_2, S|\omega_1\omega_2, S\rangle = 1$. The preceding discussion assumes $\omega_1 \neq \omega_2$. If $\omega_1 = \omega_2 = \omega$ the product ket $|\omega\omega\rangle$ is itself both symmetric and normalized and we choose

$$|\omega\omega, S\rangle = |\omega\omega\rangle \tag{10.3.10b}$$

Any vector $|\psi_S\rangle$ in $\mathbb{V}_S$ may be expanded in terms of this Ω basis. As usual we identify

$$P_S(\omega_1, \omega_2) = |\langle\omega_1\omega_2, S|\psi_S\rangle|^2 \tag{10.3.11}$$

as the *absolute* probability of finding the particles in state $|\omega_1\omega_2, S\rangle$ when an Ω measurement is made on a system in state $|\psi_S\rangle$. The normalization condition of $|\psi_S\rangle$ and $P_S(\omega_1, \omega_2)$ may be written as

$$\begin{aligned} 1 = \langle\psi_S|\psi_S\rangle &= \sum_{\text{dist}} |\langle\omega_1\omega_2, S|\psi_S\rangle|^2 \\ &= \sum_{\text{dist}} P_S(\omega_1, \omega_2) \end{aligned} \tag{10.3.12a}$$

where $\sum_{\text{dist}}$ denotes a sum over all physically *distinct* states. If ω_1 and ω_2 take values between $\omega_{\min}$ and $\omega_{\max}$, then

$$\sum_{\text{dist}} = \sum_{\omega_2=\omega_{\min}}^{\omega_{\max}} \sum_{\omega_1=\omega_{\min}}^{\omega_2} \tag{10.3.12b}$$

In this manner we avoid counting *both* $|\omega_1\omega_2, S\rangle$ *and* $|\omega_2\omega_1, S\rangle$, which are physically equivalent. Another way is to count them both and then divide by 2.

‡ Since every element of $\mathbb{V}_S$ is perpendicular to every element of $\mathbb{V}_A$ (you should check this) the dimensionality of $\mathbb{V}_{1\otimes 2}$ equals the sum of the dimensionalities of $\mathbb{V}_S$ and $\mathbb{V}_A$.

What if we want the absolute probability density for some continuous variable such as X? In this case we must take the projection of $|\psi_S\rangle$ on the normalized position eigenket:

$$|x_1x_2, S\rangle = 2^{-1/2}[|x_1x_2\rangle + |x_2x_1\rangle] \tag{10.3.13}$$

to obtain

$$P_S(x_1, x_2) = |\langle x_1x_2, S|\psi_S\rangle|^2 \tag{10.3.14}$$

The normalization condition for $P_S(x_1, x_2)$ and $|\psi_S\rangle$ is

$$1 = \iint P_S(x_1, x_2)\frac{dx_1\,dx_2}{2} = \iint |\langle x_1x_2, S|\psi_S\rangle|^2 \frac{dx_1\,dx_2}{2} \tag{10.3.15}$$

where the factor $1/2$ makes up for the double counting done by the $dx_1\,dx_2$ integration.‡ In this case it is convenient to define the wave function as

$$\psi_S(x_1, x_2) = 2^{-1/2}\langle x_1x_2, S|\psi_S\rangle \tag{10.3.16}$$

so that the normalization of ψ_S is

$$1 = \iint |\psi_S(x_1, x_2)|^2\,dx_1\,dx_2 \tag{10.3.17}$$

However, in this case

$$P_S(x_1, x_2) = 2|\psi_S(x_1, x_2)|^2 \tag{10.3.18}$$

due to the rescaling. Now, note that

$$\begin{aligned}\psi_S(x_1, x_2) &= \frac{1}{2^{1/2}}\langle x_1x_2, S|\psi_S\rangle = \frac{1}{2}[\langle x_1x_2|\psi_S\rangle + \langle x_2x_1|\psi_S\rangle] \\ &= \langle x_1x_2|\psi_S\rangle \end{aligned} \tag{10.3.19}$$

where we have exploited the fact that $|\psi_S\rangle$ is symmetrized between the particles and has the same inner product with $\langle x_1x_2|$ and $\langle x_2x_1|$. Consequently, the normalization

‡ The points $x_1 = x_2 = x$ pose some subtle questions both with respect to the factor $1/2$ and the normalization of the kets $|xx, S\rangle$. We do not get into these since the points on the line $x_1 = x_2 = x$ make only an infinitesimal contribution to the integration in the $x_1 - x_2$ plane (of any smooth function). In the following discussion you may assume that quantities such as $P_S(x, x)$, $\psi_S(x, x)$ are all given by the limits $x_1 \to x_2 \to x$ of $P_S(x_1, x_2)$, $\psi_S(x_1, x_2)$, etc.

condition Eq. (10.3.17) becomes

$$1=\langle\psi_S|\psi_S\rangle=\iint|\psi_S|^2\,dx_1\,dx_2=\iint\langle\psi_S|x_1x_2\rangle\langle x_1x_2|\psi_S\rangle\,dx_1\,dx_2$$

which makes sense, as $|\psi_S\rangle$ is an element of $\mathbb{V}_{1\otimes 2}$ as well. Note, however, that the kets $|x_1x_2\rangle$ enter the definition of the wave function Eq. (10.3.19), and the normalization integral above, only as bookkeeping devices. They are not elements of $\mathbb{V}_S$ and the inner product $\langle x_1x_2|\psi\rangle$ would be of no interest to us, were it not for the fact that the quantity that *is* of physical interest $\langle x_1x_2, S|\psi_S\rangle$, is related to it by just a scale factor of $2^{1/2}$. Let us now consider a concrete example. We measure the energy of two noninteracting bosons in a box extending from $x=0$ to $x=L$ and find them to be in the quantum states $n=3$ and $n=4$. The normalized state vector just after measurement is then

$$|\psi_S\rangle=\frac{|3,4\rangle+|4,3\rangle}{2^{1/2}} \tag{10.3.20}$$

in obvious notation. The wave function is

$$\begin{aligned}
\psi_S(x_1,x_2)&=2^{-1/2}\langle x_1x_2, S|\psi_S\rangle\\
&=\frac{1}{2}\left(\langle x_1x_2|+\langle x_2x_1|\right)\left(\frac{|3,4\rangle+|4,3\rangle}{2^{1/2}}\right)\\
&=\frac{1}{2(2^{1/2})}\left[\langle x_1x_2|3,4\rangle+\langle x_1x_2|4,3\rangle+\langle x_2x_1|3,4\rangle+\langle x_2x_1|4,3\rangle\right]\\
&=\frac{1}{2(2^{1/2})}\left[\psi_3(x_1)\psi_4(x_2)+\psi_4(x_1)\psi_3(x_2)+\psi_3(x_2)\psi_4(x_1)\right.\\
&\quad\left.+\psi_4(x_2)\psi_3(x_1)\right]\\
&=2^{-1/2}\left[\psi_3(x_1)\psi_4(x_2)+\psi_4(x_1)\psi_3(x_2)\right]\\
&=\langle x_1x_2|\psi_S\rangle
\end{aligned} \tag{10.3.21a}$$

where in all of the above,

$$\psi_n(x)=\left(\frac{2}{L}\right)^{1/2}\sin\left(\frac{n\pi x}{L}\right) \tag{10.3.21b}$$

These considerations apply with obvious modifications to the fermionic space $\mathbb{V}_A$. The basis vectors are of the form

$$|\omega_1\omega_2, A\rangle=2^{-1/2}\left[|\omega_1\omega_2\rangle-|\omega_2\omega_1\rangle\right] \tag{10.3.22}$$

(The case $\omega_1 = \omega_2$ does not arise here.) The wave function is once again

$$\begin{aligned}\psi_A(x_1, x_2) &= 2^{-1/2}\langle x_1 x_2, A | \psi_A\rangle \\ &= \langle x_1 x_2 | \psi_A\rangle\end{aligned} \tag{10.3.23}$$

and as in the bosonic case

$$P_A(x_1, x_2) = 2|\psi_A(x_1, x_2)|^2 \tag{10.3.24}$$

The normalization condition is

$$1 = \iint P_A(x_1, x_2)\frac{dx_1\, dx_2}{2} = \iint |\psi_A(x_1, x_2)|^2\, dx_1\, dx_2 \tag{10.3.25}$$

Returning to our example of two particles in a box, if we had obtained the values $n=3$ and $n=4$, then the state just after measurement would have been

$$|\psi_A\rangle = \frac{|3, 4\rangle - |4, 3\rangle}{2^{1/2}} \tag{10.3.26}$$

(We may equally well choose

$$|\psi_A\rangle = \frac{|4, 3\rangle - |3, 4\rangle}{2^{1/2}}$$

which makes no physical difference). The corresponding wave function may be written in the form of a determinant:

$$\begin{aligned}\psi_A(x_1, x_2) = \langle x_1 x_2 | \psi_A\rangle &= 2^{-1/2}[\psi_3(x_1)\psi_4(x_2) - \psi_4(x_1)\psi_3(x_2)] \\ &= 2^{-1/2}\begin{vmatrix}\psi_3(x_1) & \psi_4(x_1) \\ \psi_3(x_2) & \psi_4(x_2)\end{vmatrix}\end{aligned} \tag{10.3.27}$$

Had we been considering the state $|\omega_1\omega_2, A\rangle$ [Eq. (10.3.22)],‡

$$\psi_A(x_1, x_2) = 2^{-1/2}\begin{vmatrix}\psi_{\omega_1}(x_1) & \psi_{\omega_2}(x_1) \\ \psi_{\omega_1}(x_2) & \psi_{\omega_2}(x_2)\end{vmatrix} \tag{10.3.28}$$

Determination of Particle Statistics

We are finally ready to answer the old question: how does one determine empirically the statistics of a given species, i.e., whether it is a boson or fermion, without turning to the spin statistics theorem? For concreteness, let us say we have two identical noninteracting pions and wish to find out if they are bosons or fermions.

‡ The determinantal form of ψ_A makes it clear that ψ_A vanishes if $x_1 = x_2$ or $\omega_1 = \omega_2$.

We proceed as follows. We put them in a one-dimensional box‡ and make an energy measurement. Say we find one in the state $n=3$ and the other in the state $n=4$. The probability distribution in x space would be, depending on their statistics,

$$\begin{aligned}P_{S/A}(x_1, x_2) &= 2|\psi_{S/A}(x_1, x_2)|^2\\ &= 2|2^{-1/2}[\psi_3(x_1)\psi_4(x_2) \pm \psi_4(x_1)\psi_3(x_2)]|^2\\ &= |\psi_3(x_1)|^2|\psi_4(x_2)|^2 + |\psi_4(x_1)|^2|\psi_3(x_2)|^2\\ &\quad \pm [\psi_3^*(x_1)\psi_4(x_1)\psi_4^*(x_2)\psi_3(x_2) + \psi_4^*(x_1)\psi_3(x_1)\psi_3^*(x_2)\psi_4(x_2)]\end{aligned} \tag{10.3.29}$$

Compare this situation with two particles carrying labels 1 and 2, but otherwise identical,§ with particle 1 in state 3 and described by a probability distribution $|\psi_3(x)|^2$, and particle 2 in state 4 and described by the probability distribution $|\psi_4(x)|^2$. In this case, the first term represents the probability that particle 1 is at x_1 and particle 2 is at x_2, while the second gives the probability for the exchanged event. The sum of these two terms then gives $P_D(x_1, x_2)$, the probability for finding one at x_1 and the other at x_2, with no regard paid to their labels. (The subscript D denotes distinguishable.) The next two terms, called interference terms, remind us that there is more to identical particles in quantum theory than just their identical characteristics: they have no separate identities. Had they separate identities (as in the classical case) and we were just indifferent to which one arrives at x_1 and which one at x_2, we would get just the first two terms. There is a parallel between this situation and the double-slit experiment, where the probabilities for finding a particle at a given point x on the screen with both slits open was not the sum of the probabilities with either slit open. In both cases, the interference terms arise, because in quantum theory, when an event can take place in two (or more) indistinguishable ways, we add the corresponding amplitudes and not the corresponding probabilities.

Just as we were not allowed then to assign a definite trajectory to the particle (through slits 1 or 2), we are not allowed now to assign definite labels to the two particles.

The interference terms tell us if the pions are bosons or fermions. The difference between the two cases is most dramatic as $x_1 \to x_2 \to x$:

$$P_A(x_1 \to x, x_2 \to x) \to 0 \text{ (Pauli principle applied to state } |x\rangle) \tag{10.3.30}$$

whereas

$$P_S(x_1 \to x, x_2 \to x) = 2[|\psi_3(x)|^2|\psi_4(x)|^2 + |\psi_4(x)|^2|\psi_3(x)|^2] \tag{10.3.31}$$

which is twice as big as $P_D(x_1 \to x, x_2 \to x)$, the probability density for two distinct label carrying (but otherwise identical) particles, whose labels are disregarded in the position measurement.

One refers to the tendency of fermions to avoid each other (i.e., avoid the state $x_1 = x_2 = x$) as obeying "Fermi–Dirac statistics" and the tendency of bosons to

‡ We do this to simplify the argument. The basic idea works just as well in three dimensions.

§ The label can, for example, be the electric charge.

conglomerate as "obeying Bose–Einstein statistics," after the physicists who first explored the consequences of the antisymmetrization and symmetrization requirements on the statistical mechanics of an ensemble of fermions and bosons, respectively. (This is the reason for referring to the bosonic/fermionic nature of a particle as its statistics.)

Given the striking difference in the two distributions, we can readily imagine deciding (once and for all) whether pions are bosons or fermions by preparing an ensemble of systems (with particles in $n=3$ and 4) and measuring $P(x_1, x_2)$.

Note that $P(x_1, x_2)$ helps us decide not only whether the particles are bosons or fermions, but also whether they are identical in the first place. In other words, if particles that we think are identical differ with respect to some label that we are not aware of, the nature of the interference term will betray this fact. Imagine, for example, two bosons, call them K and $\bar{K}$, which are identical with respect to mass and charge, but different with respect to a quantum number called "hypercharge." Let us assume we are ignorant of hypercharge. In preparing an ensemble that we think contains N identical pairs, we will actually be including some (K, K) pairs, some $(\bar{K}, \bar{K})$ pairs. If we now make measurements on the ensemble and extract the distribution $P(x_1, x_2)$ (once again ignoring the hypercharge), we will find the interference term has the + sign but is not as big as it should be. If the ensemble contained only identical bosons, $P(x, x)$ should be twice as big as $P_D(x, x)$, which describes label-carrying particles; if we get a ratio less than 2, we know the ensemble is contaminated by label-carrying particles which produce no interference terms.

From the above discussions, it is also clear that one cannot hastily conclude, upon catching two electrons in the same orbital state in three dimensions that they are not fermions. In this case, the label we are ignoring is the spin orientation s. As mentioned earlier on, s can have only two values, call them + and −. If we assume that s never changes (during the course of the experiment) it can serve as a particle label that survives with time. If $s=+$ for one electron and − for the other, they are like two distinct particles and *can* be in the same orbital state. The safe thing to do here is once again to work with an ensemble rather than an isolated measurement. Since we are ignorant of spin, our ensemble will contain $(+, +)$ pairs, $(-, -)$ pairs, and $(+, -)$ pairs. The $(+, +)$ and $(-, -)$ pairs are identical fermions and will produce a negative interference term, while the $(+, -)$ pairs will not. Thus we will find $P(\mathbf{r}, \mathbf{r})$ is smaller than $P_D(\mathbf{r}, \mathbf{r})$ describing labeled particles, but not zero. This will tell us that our ensemble has identical fermion pairs contaminated by pairs of distinguishable particles. It will then be up to us to find the nature of the hidden degree of freedom which provides the distinction.

Systems of N Identical Particles

The case $N=2$ lacks one feature that is found at larger N. We illustrate it by considering the case of three identical particles in a box. Let us say that an energy measurement shows the quantum numbers of the particles to be n_1, n_2, and n_3. Since the particles are identical, all we can conclude from this observation is that the total energy is

$$E=\left(\frac{\hbar^2\pi^2}{2mL^2}\right)(n_1^2+n_2^2+n_3^2)$$

Now there are $3! =$ six product states with this energy: $|n_1n_2n_3\rangle$, $|n_1n_3n_2\rangle$, $|n_2n_3n_1\rangle$, $|n_2n_1n_3\rangle$, $|n_3n_2n_1\rangle$, and $|n_3n_1n_2\rangle$. The physical states are elements of the six-dimensional eigenspace spanned by these vectors and distinguished by the property that under the exchange of *any* two particle labels, the state vector changes only by a factor α. Since double exchange of the same two labels is equivalent to no exchange, we conclude as before that $\alpha = \pm 1$. There are only two states with this property:

$$|n_1n_2n_3, S\rangle = \frac{1}{(3!)^{1/2}}[|n_1n_2n_3\rangle + |n_1n_3n_2\rangle + |n_2n_3n_1\rangle + |n_2n_1n_3\rangle + |n_3n_2n_1\rangle + |n_3n_1n_2\rangle] \tag{10.3.32}$$

called the totally symmetric state,‡ for which $\alpha = +1$ for all three possible exchanges ($1 \leftrightarrow 2$, $2 \leftrightarrow 3$, $1 \leftrightarrow 3$); and

$$|n_1n_2n_3, A\rangle = \frac{1}{(3!)^{1/2}}[|n_1n_2n_3\rangle - |n_1n_3n_2\rangle + |n_2n_3n_1\rangle - |n_2n_1n_3\rangle + |n_3n_1n_2\rangle - |n_3n_2n_1\rangle] \tag{10.3.33}$$

called the totally antisymmetric state, for which $\alpha = -1$ for all three possible exchanges.

Bosons will always pick the S states and fermions, the A states. It follows that no two fermions can be in the same state.

As in the $N=2$ case, the wave function in the X basis is

$$\psi_{S/A}(x_1, x_2, x_3) = (3!)^{-1/2}\langle x_1x_2x_3, S/A|\psi_{S/A}\rangle = \langle x_1x_2x_3|\psi_{S/A}\rangle \tag{10.3.34}$$

and

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |\psi_{S/A}|^2\, dx_1\, dx_2\, dx_3 = 1$$

For instance, the wave function associated with $|n_1n_2n_3, S/A\rangle$, Eqs. (10.3.33) and (10.3.34), is

$$\begin{aligned}\psi_{n_1n_2n_3}&(x_1, x_2, x_3, S/A)\\ &= (3!)^{-1/2}[\psi_{n_1}(x_1)\psi_{n_2}(x_2)\psi_{n_3}(x_3) \pm \psi_{n_1}(x_1)\psi_{n_3}(x_2)\psi_{n_2}(x_3)\\ &\quad + \psi_{n_2}(x_1)\psi_{n_3}(x_2)\psi_{n_1}(x_3) \pm \psi_{n_2}(x_1)\psi_{n_1}(x_2)\psi_{n_3}(x_3)\\ &\quad + \psi_{n_3}(x_1)\psi_{n_1}(x_2)\psi_{n_2}(x_3) \pm \psi_{n_3}(x_1)\psi_{n_2}(x_2)\psi_{n_1}(x_3)]\end{aligned} \tag{10.3.35}$$

‡ The normalization factor $(3!)^{-1/2}$ is correct only if all three n's are different. If, for example, $n_1 = n_2 = n_3 = n$, then the product state $|nnn\rangle$ is normalized and symmetric and can be used as the S state. A similar question does not arise for the fermion state due to the Pauli principle.

The fermion wave function may once again be written as a determinant:

$$\psi_{n_1n_2n_3}(x_1, x_2, x_3, A) = \frac{1}{(3!)^{1/2}} \begin{vmatrix} \psi_{n_1}(x_1) & \psi_{n_2}(x_1) & \psi_{n_3}(x_1) \\ \psi_{n_1}(x_2) & \psi_{n_2}(x_2) & \psi_{n_3}(x_2) \\ \psi_{n_1}(x_3) & \psi_{n_2}(x_3) & \psi_{n_3}(x_3) \end{vmatrix} \tag{10.3.36}$$

Using the properties of the determinant, one easily sees that ψ vanishes if two of the x's or n's coincide. All these results generalize directly to any higher N.

Two questions may bother you at this point.

Question I. Consider the case $N=3$. There are three possible exchanges here: $(1 \leftrightarrow 2)$, $(1 \leftrightarrow 3)$, and $(2 \leftrightarrow 3)$. The S states pick up a factor $\alpha = +1$ for all three exchanges, while the A states pick up $\alpha = -1$ for all three exchanges. What about states for which some of the α's are $+1$ and the others -1? Such states do not exist. You may verify this by exhaustion: take the 3! product vectors and try to form such a linear combination. Since a general proof for this case and all N involves group theory, we will not discuss it here. Note that since we get only two acceptable vectors for every $N!$ product vectors, the direct product space for $N \geq 3$ is bigger (in dimensionality) than $\mathbb{V}_S \oplus \mathbb{V}_A$.

Question II. We have tacitly assumed that if *two* identical particles of a given species always pick the S (or A) state, so will three or more, i.e., we have extended our definition of bosons and fermions from $N=2$ to all N. What if two pions always pick the S state while *three* always pick the A state? While intuition revolts at such a possibility, it still needs to be formally ruled out. We do so at the end of the next subsection.

When Can We Ignore Symmetrization and Antisymmetrization?

A basic assumption physicists make before they can make any headway is that they can single out some part of the universe (the system) and study it in isolation from the rest. While no system is truly isolated, one can often get close to this ideal. For instance, when we study the oscillations of a mass coupled to a spring, we ignore the gravitational pull of Pluto.

Classically, the isolation of the system is expressed by the separability of the Hamiltonian of the universe:

$$\mathscr{H}_{\text{universe}} = \mathscr{H}_{\text{sys}} + \mathscr{H}_{\text{rest}} \tag{10.3.37}$$

where $\mathscr{H}_{\text{sys}}$ is a function of the system coordinates and momenta alone. It follows that the time evolution of the system's p's and q's are independent of what is going on in the rest of the universe. In our example, this separability is ruined (to give just one example) by the gravitational interaction between the mass and Pluto, which depends on their relative separation. If we neglect this absurdly small effect (and other such effects) we obtain separability to an excellent approximation.

Quantum mechanically, separability of H leads to the factorization of the wave function of the universe:

$$\psi_{\text{universe}} = \psi_{\text{sys}} \cdot \psi_{\text{rest}} \tag{10.3.38}$$

where ψ_{sys} is a function only of system coordinates, collectively referred to as x_s. Thus if we want the probability that the system has a certain coordinate x_s, and do not care about the rest, we find (symbolically)

$$\begin{aligned} P(x_s) &= \int |\psi_{\text{universe}}(x_s, x_{\text{rest}})|^2 \, dx_{\text{rest}} \\ &= |\psi_{\text{sys}}(x_s)|^2 \int |\psi(x_{\text{rest}})|^2 \, dx_{\text{rest}} \\ &= |\psi_{\text{sys}}(x_s)|^2 \end{aligned} \tag{10.3.39}$$

We could have obtained this result by simply ignoring ψ_{rest} from the outset.

Things get complicated when the system and the "rest" contain identical particles. Even if there is no interaction between the system and the rest, i.e., the Hamiltonian is separable, product states are not allowed and only S or A states must be used. Once the state vector fails to factorize, we will no longer have

$$P(x_s, x_{\text{rest}}) = P(x_s)P(x_{\text{rest}}) \tag{10.3.40}$$

(i.e., the systems will not be statistically independent), and we can not integrate out $P(x_{\text{rest}})$ and regain $P(x_s)$.

Now it seems reasonable that at least in certain cases it should be possible to get away with the product state and ignore the symmetrization or antisymmetrization conditions.

Suppose, for example, that at $t=0$, we find one pion in the ground state of an oscillator potential centered around a point on earth and another pion in the same state, but on the moon. It seems reasonable that we can give the particles the labels "earth pion" and "moon pion," which will survive with time. Although we cannot follow their trajectories, we can follow their wave functions: we know the first wave function is a Gaussian $G_E(x_E)$ centered at a point in the lab on earth and that the second is a Gaussian $G_M(x_M)$ centered at a point on the moon. If we catch a pion somewhere on earth at time t, the theory tells us that it is almost certainly the "earth pion" and that the chances of its being the "moon pion" are absurdly small. Thus the uncertainty in the position of each pion is compensated by a separation that is much larger. (Even in classical mechanics, it is not necessary to know the trajectories exactly to follow the particles; the band of uncertainty about each trajectory has merely to be much thinner than the minimum separation between the particles during their encounter.) We therefore believe that if we assumed

$$\psi(x_E, x_M) = G_E(x_E)G_M(x_M) \tag{10.3.41}$$

we should be making an error that is as negligible as is the chance of finding the earth pion on the moon and vice versa. Given this product form, the person on earth can compute the probability for finding the earth pion at some x by integrating out the moon pion:

$$\begin{aligned} P(x_E) &= |G_E(x_E)|^2 \int |G_M(x_M)|^2 \, dx_M \\ &= |G_E(x_E)|^2 \end{aligned} \tag{10.3.42}$$

Likewise the person on the moon, who does not care about (i.e., sums over) the earth pion will obtain

$$P(x_M) = |G_M(x_M)|^2 \tag{10.3.43}$$

Let us now verify that if we took a properly symmetrized wave function it leads to essentially the same predictions (with negligible differences).

Let us start with

$$\psi_S(x_1, x_2) = 2^{-1/2}[G_E(x_1)G_M(x_2) + G_M(x_1)G_E(x_2)] \tag{10.3.44}$$

We use the labels x_1 and x_2 rather than x_E and x_M to emphasize that the pions are indeed being treated as indistinguishable. Now, the probability (density) of finding one particle near x_1 and one near x_2 is

$$\begin{aligned} P(x_1, x_2) = 2|\psi|^2 &= |G_E(x_1)|^2|G_M(x_2)|^2 + |G_M(x_1)|^2|G_E(x_2)|^2 \\ &+ G_E^*(x_1)G_M(x_1)G_M^*(x_2)G_E(x_2) \\ &+ G_M^*(x_1)G_E(x_1)G_E^*(x_2)G_M(x_2) \end{aligned} \tag{10.3.45}$$

Let us ask for the probability of finding one particle near some point x_E on the earth, with no regard to the other. This is given by setting *either* one of the variables (say x_1) equal to x_E and integrating out the other [since $P(x_1, x_2) = P(x_2, x_1)$]. There is no need to divide by 2 in doing this integration (why?). We get

$$\begin{aligned} P(x_E) &= |G_E(x_E)|^2 \int |G_M(x_2)|^2 \, dx_2 + |G_M(x_E)|^2 \int |G_E(x_2)|^2 \, dx_2 \\ &+ G_E^*(x_E)G_M(x_E) \int G_M^*(x_2)G_E(x_2) \, dx_2 \\ &+ G_M^*(x_E)G_E(x_E) \int G_E^*(x_2)G_M(x_2) \, dx_2 \end{aligned} \tag{10.3.46}$$

The first term is what we would get if we begin with a product wave function Eq. (10.3.41) and integrate out x_M. The other three terms are negligible since G_M is peaked on the moon and is utterly negligible at a point x_E on the earth. Similarly if we asked for $P(x_M)$, where x_M is a point on the moon, we will again get $|G_M(x_M)|^2$.

The labels "earth pion" and "moon pion" were useful only because the two Gaussians remained well separated for all times (being stationary states). If the two Gaussians had not been bound by the oscillating wells, and were wave packets drifting toward each other, the labeling (and the factorized wave function) would have become invalid when the Gaussians begin to have a significant overlap. The point is that at the start of any experiment, one can always assign the particles some labels. These labels acquire a physical significance only if they survive for some time. Labels like "a particle of mass m and charge $+1$" survive forever, while the longevity of a label like "earth pion" is controlled by whether or not some other pion is in the vicinity.

A dramatic illustration of this point is provided by the following example. At $t=0$ we catch two pions, one at $x=a$ and the other at $x=b$. We can give them the labels a and b since the two delta functions do not overlap even if a and b are in the same room. We may describe the initial state by a product wave function. But this labeling is quite useless, since after the passage of an infinitesimal period of time, the delta functions spread out completely: the probability distributions become constants. You may verify this by examining $|U(x, t; a, 0)|^2$ (the "fate" of the delta function)‡ or by noting that $\Delta P=\infty$ for a delta function (the particle has all possible velocities from 0 to ∞) and which, therefore, spreads out in no time.

All these considerations apply with no modification to two fermions: the two cases differ in the sign of the interference term, which is irrelevant to these considerations.

What if there are three pions, two on earth and one on the moon? Since the two on the earth (assuming that their wave functions appreciably overlap) *can* be confused with each other, we must symmetrize between them, and the total wave function will be, in obvious notation,

$$\psi(x_{E_1}, x_{E_2}, x_M)=\psi_S(x_{E_1}, x_{E_2})\cdot\psi(x_M) \tag{10.3.47}$$

The extension of this result to more particles and to fermions is obvious.

At this point the answer to Question II raised at the end of the last subsection becomes apparent. Suppose three-pion systems picked the A state while two-pion systems picked the S state. Let two of the three pions be on earth and the third one on the moon. Then, by assumption, the following function should provide an excellent approximation:

$$\psi(x_{E_1}, x_{E_2}, x_M)=\psi_A(x_{E_1}, x_{E_2})\psi(x_M) \tag{10.3.48}$$

If we integrate over the moon pion we get

$$P(x_{E_1}, x_{E_2})=2|\psi_A(x_{E_1}, x_{E_2})|^2 \tag{10.3.49}$$

We are thus led to conclude that two pions on earth will have a probability distribution corresponding to two fermions if there is a third pion on the moon and a distribution expected to two bosons if there is not a third one on the moon. Such

‡ It is being assumed that the particles are free.

absurd conclusions are averted only if the statistics depend on the species and not the number of particles.

A word of caution before we conclude this long discussion. If two particles have nonoverlapping wave functions in x space, then it is only in x space that a product wave function provides a good approximation to the exact symmetrized wave function, which in our example was

$$\psi_S(x_1, x_2) = 2^{-1/2}[G_E(x_1)G_M(x_2) + G_M(x_1)G_E(x_2)] \tag{10.3.50}$$

The formal reason is that for any choice of the arguments x_1 and x_2, only one or the other of the two terms in the right-hand side is important. (For example, if x_1 is on the earth and x_2 is on the moon, only the first piece is important.) Physically it is because the chance of finding one pion in the territory of the other is negligible and interference effects can be ignored.

If, however, we wish to switch to another basis, say the P basis, we must consider the Fourier transform of the symmetric function ψ_S and not the product, so that we end up with a symmetrized wave function in p space. The physical reason for this is that the two pions have the same momentum distributions—with $\langle P \rangle = 0$ and identical Gaussian fluctuations about this mean—since the momentum content of the oscillator is independent of its location. Consequently, there are no grounds in P space for distinguishing between them. Thus when a momentum measurement (which says nothing about the positions) yields two numbers, we cannot assign them to the pions in a unique way. Formally, symmetrization is important because the p-space wave functions of the pions overlap strongly and there exist values for the two momenta (both $\simeq 0$) for which both terms in the symmetric wave function are significant.

By the same token, if there are two particles with nonoverlapping wave functions in p space, we may describe the system by a product wave function in this space (using labels like "fast" and "slow" instead of "earth" and "moon" to distinguish between them), but not in another space where the distinction between them is absent. It should be clear that these arguments apply not just to X or P but to any arbitrary variable Ω.

*Exercise 10.3.1.** Two identical bosons are found to be in states $|\phi\rangle$ and $|\psi\rangle$. Write down the normalized state vector describing the system when $\langle \phi | \psi \rangle \neq 0$.

*Exercise 10.3.2.** When an energy measurement is made on a system of three bosons in a box, the n values obtained were 3, 3, and 4. Write down a symmetrized, normalized state vector.

*Exercise 10.3.3.** Imagine a situation in which there are three particles and only three states a, b, and c available to them. Show that the total number of allowed, distinct configurations for this system is

(1) 27 if they are labeled
(2) 10 if they are bosons
(3) 1 if they are fermions

*Exercise 10.3.4.** Two identical particles of mass m are in a one-dimensional box of length L. Energy measurement of the system yields the value $E_{sys}=\hbar^2\pi^2/mL^2$. Write down the state vector of the system. Repeat for $E_{sys}=5\hbar^2\pi^2/2mL^2$. (There are two possible vectors in this case.) You are not told if they are bosons or fermions. You may assume that the only degrees of freedom are orbital.

*Exercise 10.3.5.** Consider the *exchange operator* P_{12} whose action on the X basis is

$$P_{12}|x_1, x_2\rangle = |x_2, x_1\rangle$$

(1) Show that P_{12} has eigenvalues ± 1. (It is Hermitian and unitary.)

(2) Show that its action on the basis ket $|\omega_1, \omega_2\rangle$ is also to exchange the labels 1 and 2, and hence that $\mathbb{V}_{S,A}$ are its eigenspaces with eigenvalues ± 1.

(3) Show that $P_{12}X_1P_{12}=X_2$, $P_{12}X_2P_{12}=X_1$ and similarly for P_1 and P_2. Then show that $P_{12}\Omega(X_1, P_1; X_2, P_2)P_{12}=\Omega(X_2, P_2; X_1, P_1)$. [Consider the action on $|x_1, x_2\rangle$ or $|p_1, p_2\rangle$. As for the functions of X and P, assume they are given by power series and consider any term in the series. If you need help, peek into the discussion leading to Eq. (11.2.22).]

(4) Show that the Hamiltonian and propagator for two *identical* particles are left unaffected under $H \to P_{12}HP_{12}$ and $U \to P_{12}UP_{12}$. Given this, show that any eigenstate of P_{12} continues to remain an eigenstate with the same eigenvalue as time passes, i.e., elements of $\mathbb{V}_{S/A}$ never leave the symmetric or antisymmetric subspaces they start in.

*Exercise 10.3.6.** Consider a composite object such as the hydrogen atom. Will it behave as a boson or fermion? Argue in general that objects containing an even/odd number of fermions will behave as bosons/fermions.