Challenging Students and Solving Problems with Basic Tools, Testing Students' Attitudes

5[5]

Continuous effort—not strength or intelligence—is the key to unlocking our potential.

Winston Churchill

Satisfaction lies in the effort, not in the attainment, full effort is full victory.

Mahatma Gandhi

You do not develop courage by being happy in your relationships every day. You develop it by surviving difficult times and challenging adversity.

Epicurus

Opportunities to find deeper powers within ourselves come when life seems most challenging.

Joseph Campbell

5.1 Introduction

Why include this chapter? Why these kinds of problems? We, the authors, consider of utmost importance that engineering students be challenged in their abilities. We understand that at first they will feel unable to handle many of these problems, especially for their level of difficulty. They may feel frustrated, but that is not our intention, and in the end, we hope and expect that they will feel excited and eager to learn. We want to show them that they have the capacity to deal with difficult problems. We hope to motivate them to understand, to learn, and, especially, to have a solid attitude.

As will quickly be appreciated, the skills required to solve the problems presented in this chapter are mainly the basic subjects of mathematics, chemistry, biology, and physics. They do not go beyond the subjects learned (or, rather, what should have been learned) in high school. Some of these problems are very basic, but they require the wit, ingenuity, and, especially, methodology that a good student of chemical or bioprocess engineering must have. The key to solving these problems is first the attitude and then the method. Many problems might seem like brainteasers, which are tricky and not easy to solve, but each of them can be solved to the extent that the student applies a method and then expresses the statement in terms of mathematical equations.

In this chapter, the ultimate goal is to teach students to express in a mathematically sound way, various everyday situations that, more than likely, they have faced and will likely encounter even more when they start their professional careers. With patience, effort, and discipline in following the advice proffered here, the student will develop expertise in problem solving. It is critical and

R. Simpson and S.K. Sastry, *Chemical and Bioprocess Engineering: Fundamental Concepts for First-Year Students*, DOI 10.1007/978-1-4614-9126-2_5,

fundamental that the student be able to interpret and "translate" statements and express them in a mathematical model in order to solve them. As stated previously, this mathematical representation will be simple to solve because the main goal, at this stage, is to familiarize students with the method and mathematical formulation of problems. In parallel and subsequent courses on chemical or bioprocess engineering, the student will be enrolled in varied and advanced courses in mathematics, chemistry, biology, and physics, and then he or she will be capable not only of formulating intricate problems, but also solving them. For now we are specifically concerned with empowering students with the ability to formulate problems mathematically.

In the near future, freshmen will be challenged in higher-level courses, and during their professional careers, they will appreciate, with greater objectivity, the significance and importance of being able to easily set up and resolve the various problems they will face.

In this chapter, we include different types of exercises that will familiarize students with engineering problems. More importantly, however, the problem sets will prepare students to formulate mathematically more complex problems. The type of problems presented here may seem, somehow, disconnected from chemical and bioprocess engineering, but their importance lies primarily in preparing students to raise and formulate problems in mathematical terms.

5.2 Strategy and Method for Solving Problems

The most important aspect of problem solving is having a strategy and a method, i.e., a set of rules that guarantees the optimal decision and procedure. In this case, these will be mathematical rules and procedures for finding the true solution. But remember, at this stage, the most important aspect is not the final solution, but the correct mathematical formulation. In this chapter, we will show how problems that might seem very difficult are easily and simply solved. To this end, it is vital to follow, in an almost dogmatic (strict) attitude, the strategy and method that will be explained subsequently in detail.

Figure 5.1 outlines a general strategy and approach to address these problems. As you will see, Fig. 5.1 may seem very detailed and often unnecessary for certain problems, and undeniably it is. However, as will be seen throughout the chapter, it is vital in solving most problems and also ensures high efficiency in solving all kinds of problems. A similar strategy will be used to solve most of the problems presented in the book and will possibly be of help in most problems in advanced courses in engineering and other subjects that might be encountered during your professional career.

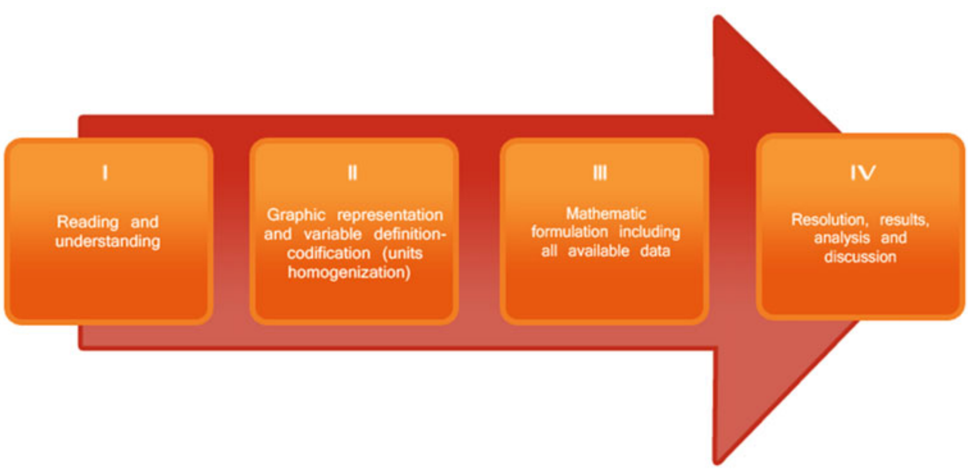

Fig. 5.1 A general strategy and approach to solve problems

Depending on the problem (referring, mainly, to this chapter), some steps can be omitted, e.g., the graphical representation.

Although some of the proposed steps are self-explanatory, an explanation of each step is included as follows.

Step I

Reading and understanding

This first step is critical. Sometimes with long statements you get confused. Although you understand the question, it might be hard to formulate the equations. As long as you have identified your unknowns, and coded accordingly, a good idea is to go phrase by phrase translating them into mathematical expressions.

Step II

Graphical representation and variable definition and codification.

Normally a good scheme or graphical representation lets us have a clear picture of the problem. In addition, we need to identify all the unknowns and then assign each one an adequate code to define each variable. For example, if in a given problem an airplane's speed is unknown, an appropriate coding to define the speed of the airplane could be V_a.

Step III

Mathematical formulation including all available data.

Once all variables have been identified and defined, it is necessary to determine which equations can be formulated. To pose the problem, the number of variables should be equal to the number of equations that can be formulated. In Chap. 7, when we deal with problems related to material balance, we will analyze and extend the concept of degrees of freedom (DF). For now DF is defined as follows:
DF $=$ Number of variables $-$ Number of equations.
If DF > 0, then the problem is underspecified.
If DF $= 0$, then the problem is correctly specified, and you can proceed to solve it.
If DF < 0, then the problem is overspecified.

In this chapter, we will be dealing with problems that are correctly specified. As presented in Chap. 7, we will further analyze and detail the concept of degrees of freedom in material balance problems.

Step IV

Resolution, results, analysis, and discussion.

After solving a problem, it is highly advisable to test the result in order to understand whether it is an expected and reasonable result and, ideally, to verify whether it is the true solution to the problem.

Before moving on to Sect. 5.3, we cordially invite you to solve the following warm-up examples.

Warm-Up Example 1

Classical problem (2 or 10^+?). Rodrigo has a shoe store, and his friend Andrew, in the same shopping center, has a sports shop. One day, early in the morning, a well-known fraudster decides to buy a pair of shoes. The shoes have a retail value of $180, and the scammer, making mischief, paid with a counterfeit $200 (two $100 bills). At that time in the morning, Rodrigo has no change and so goes to his friend Andrew for change. Andrew calmly changes his $200 and gives Rodrigo 20 $10 bills. Rodrigo returns to his store to give the customer the pair of shoes and his change, $20. The next day, his friend Andrew discovers the scam and goes to Rodrigo's shoe store to get his $200 back. Rodrigo was deceived and gives Andrew his $200 back and in return receives the counterfeit money.

What was Rodrigo's loss from this scam? The answer is at the end of the chapter.

Please stop and attempt to solve the problem!
Did you look at the answer? Surprised? We would like to stress that the problem can be easily solved by fourth and fifth grade elementary school students, but the shocking reality is that when university students, not only freshmen, are given the problem, normally most, if not all, are unable to. Why? As we stressed in the preface of the book, questioning and understanding are very important.

The problem looks so easy, and it actually is, and so it is too tempting to solve it mentally. Perhaps no one got out pen and paper to solve it. This was your big mistake; you clearly underestimated the problems level of difficulty and did not follow any method or strategy to solve it. Do not feel embarrassed. Almost every adult, including engineers, master's degree holders, and even Ph.D.s, fail to solve this problem, but please learn the lesson!

All problems deserve our attention and reflection before giving our final answer, so always follow a method.

Warm-Up Example 2
Monty Hall problem [10+]. Marilyn vos Savant is a well-known American columnist who became famous for being recognized by the *Guinness Book of Records* as the most intelligent person with an intelligence quotient (IQ) of 228. Since 1986, she has been publishing a Sunday column in *Parade* magazine in which her readers ask questions on very diverse topics. As stated by the magazine, her column has elicited responses from the elementary school to the postgraduate level (Ph.D.).

In 1990, Marilyn vos Savant received a letter from a Craig F. Whitaker (Columbia, MD) with the following problem:

Suppose you are on a game show and you are given the choice of three doors. Behind one door is a car, behind the other two, goats. You pick a door, say number **1**, and the host, who knows what is behind each door, opens one of the doors with a goat, say number **3**, and then says to you, "Would you like to pick door number **2**?" Is it to your advantage to switch your choice of doors? What would you do?

This problem is well known (in different versions) and very interesting. We invite you to think and then compare your answer with the one that follows. Because we consider it very important, please take your time and try to solve this problem. We insist that before continuing, you should attempt to solve it and compare your answer with the answer given below. Why? Because most students will not be able to solve the problem, and there is a reason for that. Then, when you read the answer, we will show you why this happens and, most importantly, start to prepare you with a method to efficiently approach a variety of problems.

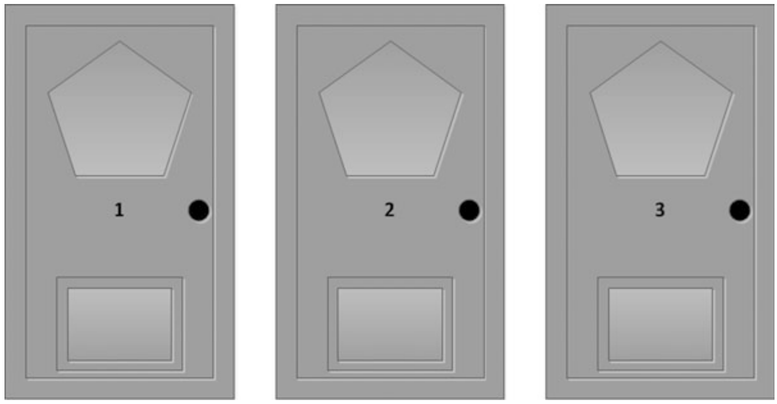

Fig. 5.2 Three doors of the Monty Hall problem

Please stop and attempt to solve the problem!

Solution of Warm-Up Example 2

First, we will give you the correct answer, and then discuss why this problem caused and continues to cause so much controversy.

Yes, it is better to change doors. If you do, your probability of winning will be 2/3, but if you do not change it, that probability will be 1/3. Confused?

As shown and illustrated below, this problem generated much controversy. *Parade* magazine received over 10,000 letters saying, often in rude terms, that Marilyn vos Savant was wrong. Believe it or not, among the detractors were several respected professional mathematicians.

Being faithful to what we have preached in this chapter, to solve this problem we will follow a method. First, make a sketch of the situation (Fig. 5.2).

Second, to address the problem, it is important to first remember what we understand by probability:

$$\text{Probability} = \text{Number of hits/total number of events.}$$

So, what we will do is investigate both cases: if I change doors and if I do not change doors. Considering all the possibilities (total number of events), we will calculate both probabilities and see which one is better. In the table below, we will assume that the car is behind door number 1 and the goats are behind doors 2 and 3 (obviously, we do not know). Then we will analyze all the possibilities, i.e., if we choose door number 1, door number 2, or door number 3, and see what happens in each case when we change doors (or do not change doors). The first column indicates our choice, while columns 2–4 show our choice in boldface.

Our Guess	Door 1	Door 2	Door 3	The Show-Man opens door[a]	Switching to	Switching door	Non switching doors
Door 1	**CAR**	Goat	Goat	3	From 1 to 2	Lose	Win
Door 2	Car	**GOAT**	Goat	3	From 2 to 1	Win	Lose
Door 3	Car	Goat	**GOAT**	2	From 3 to 1	Win	Lose
Probability of wining						**2/3**	**1/3**

[a]The Show host always chooses a door with a goat, e.g. when you chose door 3 his only chance is door 2

As depicted in the table, analyzing all the possibilities, it is clear that it is better to change doors after the host has opened a door with a goat. Strictly speaking, it is not a difficult problem because the number of ways to analyze it is rather small, just three.

In our view, the difficulty is that when faced with a problem like warm-up example 1, most people are wrongly tempted to analyze and solve it mentally and not take the problem "seriously." The strong and potent message here is that one should always consider problems as a worthwhile challenge and solve it using some method. As we have shown, if you analyze all the possibilities (just three), it is easy to see that it is better to change doors!

You can visit the following Web site for more details on the controversy surrounding the correct answer given by Marilyn vos Savant:
http://marilynvossavant.com/game-show-problem/.

What follows are some of the letters sent to *Parade* magazine:

Since you seem to enjoy coming straight to the point, I'll do the same. You blew it! Let me explain. If one door is shown to be a loser, that information changes the probability of either remaining choice, neither of which has any reason to be more likely, to 1/2. As a professional mathematician, I'm very concerned with the general public's lack of mathematical skills. Please help by confessing your error and in the future being more careful.

Robert Sachs, Ph.D., George Mason University

You blew it, and you blew it big! Since you seem to have difficulty grasping the basic principle at work here, I'll explain. After the host reveals a goat, you now have a one-in-two chance of being correct. Whether you change your selection or not, the odds are the same. There is enough mathematical illiteracy in this country, and we don't need the world's highest IQ propagating more. Shame!

Scott Smith, Ph.D., University of Florida

Your answer to the question is in error. But if it is any consolation, many of my academic colleagues have also been stumped by this problem.

Barry Pasternack, Ph.D., California Faculty Association

You're in error, but Albert Einstein earned a dearer place in the hearts of people after he admitted his errors.

Frank Rose, Ph.D., University of Michigan

I have been a faithful reader of your column, and I have not, until now, had any reason to doubt you. However, in this matter (for which I do have expertise), your answer is clearly at odds with the truth.

James Rauff, Ph.D., Millikin University

May I suggest that you obtain and refer to a standard textbook on probability before you try to answer a question of this type again?

Charles Reid, Ph.D., University of Florida

THE SHOE AND THE MARILYN VOS SAVANT PROBLEMS HAVE BEEN INCLUDED TO SHOW THE REAL IMPORTANCE OF APPLYING A METHOD WHEN SOLVING PROBLEMS. THEREFORE, WE SUGGEST THAT YOU TAKE A MINUTE AND READ THE FOLLOWING MESSAGE.

ATTENTION

Before turning to the solved problems section, we urge students to observe the following recommendation. Read the statement of each problem solved and try to solve it without examining the proposed solution. You will find the problems hard, but follow step by step the solution given. Following this procedure, you will learn to approach all the problems with a resolute attitude. And never forget the methodology.

5.3 Solved Problems

5.3.1 Rate

Although it might seem strange, in this section we have grouped problems that appear radically different from each other. For example, what do velocity, watches, work, candles, and fluid flow problems have in common? They can all be solved following a common conceptualization. Here the common conceptualization is the rate, in one case that of fluid flow (flow), in others it is the rate at which some work is done, and so on.

From a methodological point of view, we will first solve some problems from each class in order of increasing level of difficulty. The idea is to demonstrate that with an appropriate method, even the most difficult problems can be solved without much trouble. To be consistent with our predicament, for both solved and proposed problems, we have considered a variety of problems, from very basic to very difficult. We hope that after completing this section, you will agree with us that, yes, you can! You will also discover that the key to success is the strategy and method. Initially, perhaps always, it will be critical to work hard, but you will be rewarded. Here, our goal is to get you to the point where you can finally say to yourself, "Yes I can! and so can everyone else!"

As was mentioned in To Our Students, Colleagues and Tutors, each problem will be marked by level of difficulty. For example, the first solved problem has a difficulty of 7 out of 10; therefore, we added [7] to the name of the problem.

VELOCITY

1. **Travel to New York [7]**. Two cars are traveling from Boston to New York. The first car departs ***a*** hours before and arrives ***a*** hours later than the second car going to New York. If the first car takes ***b*** hours for the journey at a speed of V_1 km/h, what is the speed of the second car in kilometers per hour?

Solution

Step I

Reading and Understanding. Although the problem is simple, it involves some challenges because many students are not used to solving such problems with nonnumeric information.

The question involves expressing the speed of car 2 as a function of known data, such as the speed of the first car (V_1 km/h) and the travel time of the first car (***b*** h), and considering the time that it departs before and comes after car 2 (***a*** hours). If we designate the second cars speed V_2, we arrive at an expression such as this:

With V_2 a function of V_1, ***a*** and ***b*** can be expressed mathematically as: $V_2 = f(V_1,a,b)$

Step II

Graphical representation and variable definition and codification (Fig. 5.3)

V_1: Car 1 s speed.

V_2: Car 2 s speed.

Step III

Mathematical formulation including all available data. In this problem, it is possible to formulate two equations, one for the speed of each car. In this case, we do not have degrees of freedom (DF = 0), and the problem can be solved.

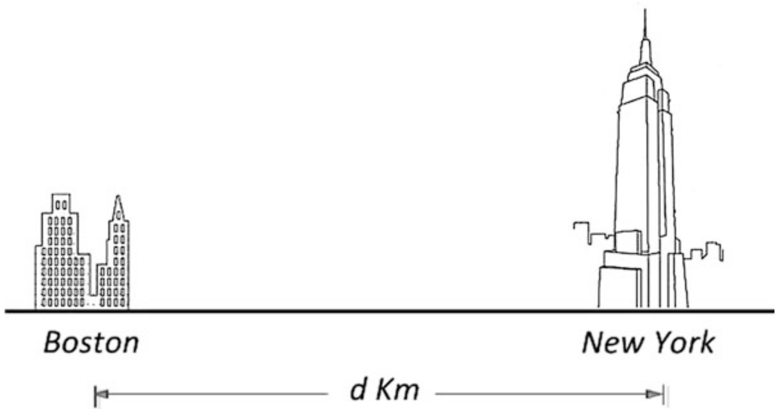

Fig. 5.3 Graphic representation of cars travelling from Boston to New York

Car 1
Given that the distance traveled by the car is d km and that distance is covered in $\boldsymbol{b}$ hours, then

$$V_1 = \frac{d}{b};\ \ V_1 b = d \tag{5.1}$$

Car 2
Given that the distance traveled by the car is $\boldsymbol{d}$ km and that distance is covered in $\boldsymbol{b} - \boldsymbol{2a}$ hours (departed $\boldsymbol{a}$ hours after and arrived $\boldsymbol{a}$ hours earlier), then

$$V_2 = \frac{d}{b - 2a}; V_2(b - 2a) = d \tag{5.2}$$

Step IV
Resolution, results, analysis, and discussion. From equations (5.1) and (5.2) we have

$$V_2(b - 2a) = V_1 b; \text{then,} \qquad V_2 = \frac{V_1 b}{b - 2a} \tag{5.3}$$

meaning: $V_2 = f(V_1, a, b)$.

An important part of solving a problem is analyzing the results. Since car 2 takes less time than car 1 to complete its trip, V_2 should be expected to be greater than V_1. Looking at (5.3) we see that $V_2 > V_1$; this is because the factor $(\boldsymbol{b} - \mathbf{2}(\boldsymbol{a}))$ is less than $\boldsymbol{b}$. Furthermore, for example, when the value of $\boldsymbol{a}$ tends to zero, i.e., when both cars depart and arrive together, in (5.3) it is observed, correctly, that the value V_2 is equal to V_1. This analysis shows us that the result might be correct.

2. Usain Bolt [3$^+$]. As a humble amateur, you are capable of running the 100 m dash in a respectable 11.3 s. In your dreams, you are pitted against Usain Bolt, owner of the world record in the 100 m dash—an astonishing 9.58 s. How many meters from the finish line would you be at the moment Usain Bolt completes his sprint in the 100 m dash?

Solution
Step I
Reading and Understanding. Reading carefully, the question is closely related to how many meters you are able to run in 9.58 s. Then to calculate the distance that you are running in 9.58, we first need to calculate your speed.

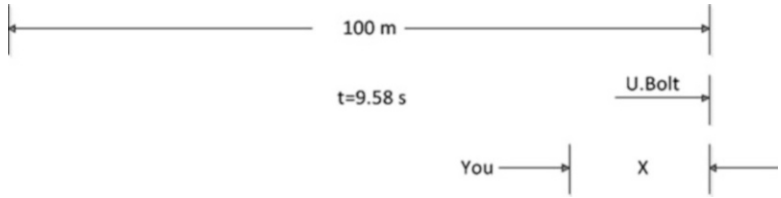

Fig. 5.4 Graphic representation of the running of your dreams

Step II

Graphical representation and variable definition and codification. We can make a simple sketch to represent what is happening in your dreams (Fig. 5.4).

v: Your speed in the 100 m dash (we assume that you run at a constant speed for the whole race. Do you think this is a reasonable assumption? If not, why not?)

d: Distance that you will cover in 9.58 s

Steps III and IV

Mathematical formulation including all available data and resolution, results, analysis, and discussion. We have the data to calculate your speed and then the distance that you will cover in 9.58 s (DF = 0).

First, we calculate your speed; remember that you run 100 m in 11.3 s; therefore,

$$v = \frac{100 \text{ m}}{11.3 \text{ s}} \cong 8.85 \text{ m/s}$$

Then in 9.58 s you will cover

$$d = v \times t = 8.85 \text{ m/s} \times 9.58 \text{ s} \cong 84.8 \text{ m; therefore, } x = 100 - 84.8 = 15.2 \text{ m.}$$

At the moment when Usain Bolt is crossing the finish line, you have covered approximately 84.8 m. Then you will be slightly more than 15 m from the finish line (15.2 m). Not too bad for an amateur!

3. Shoplifter [10^+]. A shoplifter is about to escape after robbing a grocery store on some street corner when an employee sets off the alarm.

At that time, a policeman who is four blocks away hears the alarm and runs toward the supermarket (assume that one block is 125 m and the police hears the alarm instantly, i.e., do not consider the time it takes for the sound to travel from the store to the policeman four blocks away).

When the police arrives at the grocery store, the employees tell him that the shoplifter is already two blocks away and getting into a taxi. The policeman increases the speed of his car by 20 % to overtake the shoplifter.

The shoplifter goes several blocks in the taxi when he has to stop at a red light. He estimates that the taxi will be stopped long enough for the police to catch up to him, so he gets out and takes off on foot. His speed this time is 20 % faster than when he was making his escape from the supermarket, but the policeman catches up with him 10 s after he gets out of the taxi.

As the shoplifter is getting out of the taxi, the distance between him and the police is one block. It is known that if the taxi had not stopped, the policeman would have caught up to the shoplifter in 10 min (from the time he got in the taxi).

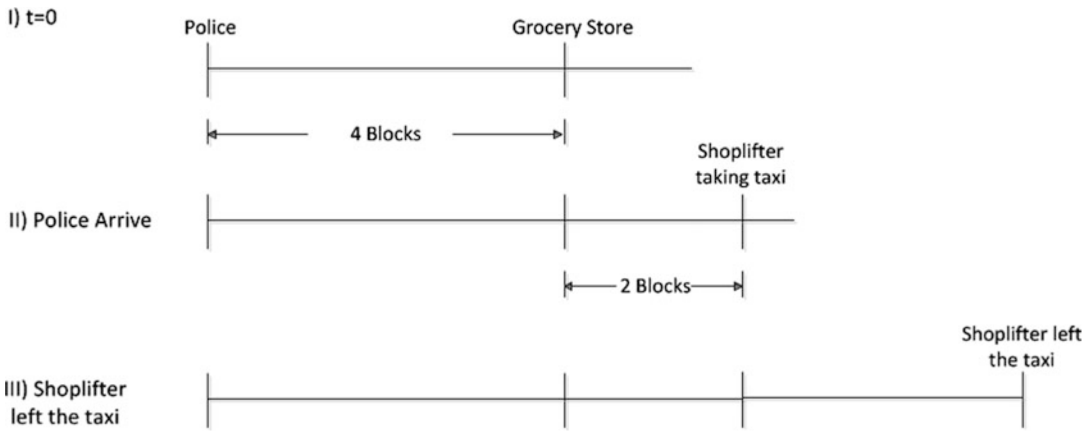

Fig. 5.5 Graphic representation of the shoplifter escape

What is the speed of the shoplifter, the policeman, and the taxi?

Solution

Step I

Reading and understanding. This is a classic example where the hard part of the problem is formulating it in mathematical terms. We need to translate step by step this intricate statement into equations. Here, a careful reading with the help of a good sketch will help us formulate the required equations. In addition, you must always be very tidy!

Step II

Graphical representation and variable definition and codification.

We have three variables, and these are and coded as follows:

V_S: speed of shoplifter m/min
V_P: speed of policeman m/min
V_T: speed of taxi m/min

Step III

Mathematical formulation including all available data. We have three unknowns, and with the information provided we can formulate three equations, then DF = 0.

Looking at the sketch (Fig. 5.5) and reading the problem statement, when the policeman arrives at the supermarket, the shoplifter is two blocks away. Meanwhile, the policeman runs four blocks to get to the supermarket, while at the same time the shoplifter runs two blocks away. Therefore, the policeman's speed is double the speed of the shoplifter. Mathematically, we express this as

$$V_P = 2 \times V_S. \tag{5.4}$$

Then the shoplifter takes a taxi and the policeman increases his speed by 20 %, so now his speed is $1.2 \times V_P$. After several blocks, the taxi has to stop at a red light, so the shoplifter gets out and makes a run for it, but this time at a speed that is 20 % faster than before. Now his speed is $1.2 \times V_S$.

In addition, when the shoplifter gets out of the taxi, the policeman is just one block away. The policeman takes 10 s to reach the shoplifter, and the equation for this situation should look like (Fig. 5.5) this:

$$\textbf{Distance run by the police} = \textbf{Distance run by the shoplifter} + \textbf{1 block } (\textbf{125 } [\textbf{m}])$$

First, distance is the product of speed and time. If we write an equation, we must remember that both speeds, that of the policeman and that of the shoplifter, were increased by 20 %, thus

$$1.2 \times V_{\mathrm{P}} \times \frac{10}{60} = 1.2 \times V_{\mathrm{S}} \times \frac{10}{60} + 125. \tag{5.5}$$

Given that the speed is in meters per minute, the time should be in minutes! This is why we put 10/60 min (10 s) in (5.5).

Finally, if the taxi had not stopped at the red light, the policeman would have caught the shoplifter in 10 min from the time the shoplifter got into the taxi. At that point, the policeman was two blocks away. The equation for this situation should look like this (Fig. 5.5):

$$\textbf{Distance run by the policeman} = \textbf{Distance run by the shoplifter in the taxi} + \textbf{2 blocks } (\textbf{250 } [\textbf{m}])$$

$$1.2 \times V_{\mathrm{P}} \times 10 = V_{\mathrm{T}} \times 10 + 250. \tag{5.6}$$

In this case, the time is in minutes.

Step IV

Resolution, results, analysis, and discussion. As shown in the mathematical formulation, we have three equations and three unknowns. From (5.4) and (5.5), we obtain

$$V_{\mathrm{P}} = 1,250 \ [\mathrm{m/\,min}]$$

$$V_{\mathrm{S}} = 625 \ [\mathrm{m/\,min}]$$

And replacing V_{P} in (5.6) we obtain

$$V_{\mathrm{T}} = 1,475 \ [\mathrm{m/\,min}]$$

Although it should be evident, it is important to notice that the speed of the policeman must be greater than the speed of the shoplifter. If not, why would the shoplifter take a taxi? In addition, although subtle, it is logical to expect that the speed of the taxi was greater than the speed of the policeman. Remember, when the policeman becomes aware that the shoplifter is taking a taxi, he speeds up by 20 %.

This specific problem shows again the power of following a method for solving problems. Read and understand, represent things graphically, and so on.

WATCHES

4. Travel to San Francisco [6]. You set out by car from Portland, Oregon, to San Francisco, California, at 10 a.m. Your friend Peter leaves a little later (but before 11 a.m.), when the minute and hour hands overlap. If both arrive at the same time in San Francisco, how much less time did Peter take for the trip?

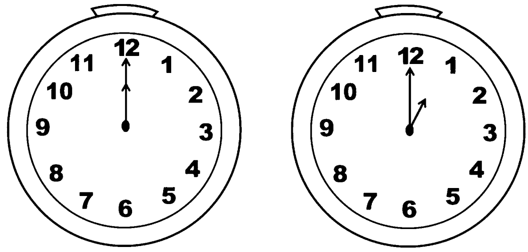

Fig. 5.6 Clocks at 12:00 AM and another clock at 1:00 PM

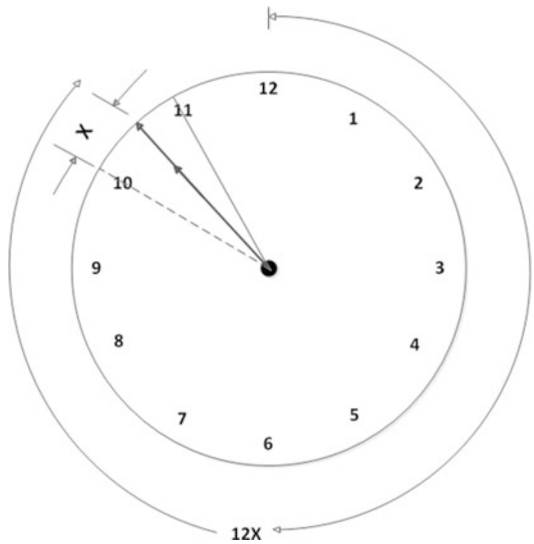

Fig. 5.7 The clock indicates Peter's departure, after 10:00 AM but before 11:00 AM

<u>**Solution**</u>

Step I

Reading and understanding. Unlike the first problem (Travel to New York), in this case, the data are numeric, but the difficulty is that we have to relate the speed ratio between the hour hand speed and the minute hand speed. A simple observation (Fig. 5.6) shows a clock at 12 noon and another clock at 1 p.m. We can infer that within 1 h the minute hand will have made a full turn (60 min) and the hour hand will have advanced from 12 to 1 p.m., that is, 5 min. From this we can deduce that the minute hand moves 12 times faster than the hour hand (60/5).

Step II

Graphical representation and variable definition and codification. A clock is shown schematically at 10 a.m. and another that indicates Peter's departure, after 10 a.m. but before 11 a. m., and with the minute and hour hands overlapping.

$12x$: minutes advanced by the minute hand

x: "minutes" advanced by the hour hand

Step III

Mathematical formulation of problem including all available data. As shown in Fig. 5.7, we have called x the "minutes" advanced by the hour hand and $12x$ the minutes moved by the minute hand

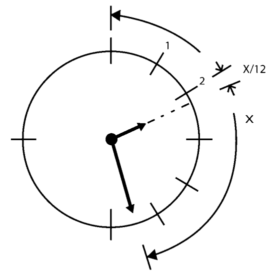

Fig. 5.8 Starting time of the exam of Introduction to Chemical and Bioprocess Engineering

(remember that the minute hand moves 12 times faster than the hour hand). Therefore, if we consider that from 12 to 10 there are 50 min, then we can formulate the following equation:

$$50 + \mathbf{x} = 12\mathbf{x}.\text{Solving for } x, \text{we have } \mathbf{x} = 50/11; \text{approximately } 4 \text{ min } 33 \text{ s}.$$

So Peter set out at 10 h 54 min 33 s. Given that they arrived at the same time, he used 54 min 33 less time than you to make the trip from Portland to San Francisco.

Step IV

Results, analysis, and discussion. The key to solving this problem, as in many engineering problems, is to build an adequate graphical representation of the situation. In this particular case, the graphical representation facilitates our understanding of why the speed of the minute hand is 12 times faster than the hour hand speed. With this background and understanding, most problems involving time will be easy to solve.

5. How long was the exam? [7]. An exam in your Introduction to Chemical and Bioprocess Engineering class starts when the hour hand of the clock is between 2 and 3 p.m. and the minute hand is between 5 and 6. The exam ends when the positions of the hour and minute hands are reversed. How much time do you have for the Introduction to Chemical and Bioprocess Engineering exam?

Solution

Step I

Reading and understanding. As was deduced in the previous problem (Travel to San Francisco), the minute hand moves 12 times faster than the hour hand. With the information provided in the problem statement, we can calculate when the exam started and when it finished. Then, it is straightforward to calculate the amount of time for the exam.

Step II

Graphical representation and variable definition and codification. Figures 5.8 and 5.9 show schematically and graphically when the exam starts and finishes.

As depicted in Fig. 5.8 X represents the minutes after 2 p.m. (when starting the exam), and in Fig. 5.9 Y represents the minutes after 5 p.m. (when finishing the exam).

X: minutes after 2 p.m.

Y: minutes after 5 p.m.

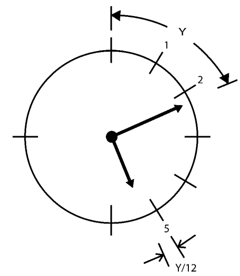

Fig. 5.9 Finishing time of the exam of Introduction to Chemical and Bioprocess Engineering

Step III

Mathematical formulation of problem. We have two unknowns, and we can formulate two equations, then DF = 0. Observing Figs. 5.8 and 5.9 simultaneously, and considering that the hour and minute hands are reversed, we can write the following equations:

$$\frac{Y}{12} + 25 = x, \tag{5.7}$$

$$\frac{X}{12} + 10 = y. \tag{5.8}$$

Step IV

Results, analysis, and discussion. Solving (5.7) and (5.8) we obtain

$X = 26$

$Y = 12$

So the exam started at 2 h 26 min and finished at 5 h 12 min. Therefore, the exam lasted 2 h 46 min.

WORK

6. **Overtime [6]**. A worker is able to complete a job in two shifts of 8 h. Due to an emergency, his boss wants him to complete the work as quickly as possible, and therefore the worker has to work without a break. In this case, during the first 10 h, the worker works like he normally does, but after that, because he tired, he works at 75 % of his normal rate. How many hours will it take him to finish the job?

Solution

Step I

Reading and understanding. For this type of problem it is necessary to define the concept of worker speed. It will be defined as follows:

$v = \frac{T}{t}$, where v is the work speed, T is the work done (completed work), and t is the time to complete the entire job.

Step II

Graphical representation and variable definition and codification. Although in most cases it is useful to have a graphical representation, in this case it is unnecessary.

v: work speed

T: completed work

t: time

Step III

Mathematical formulation including all available data. There is one unknown variable, and so one equation can be formulated (DF = 0). We are told that the worker can complete the job in two shifts of 8 h, i.e., 16 h; therefore, the work speed can be represented by

$$v = \frac{T}{16}. \tag{5.9}$$

Working continuously, the employee works the first 10 h at this rate and the remaining time at 75 % of his normal rate. Therefore, part of the work will be completed in the first 10 h (10v) and the rest of the work ($t \times 0.75v$) will be fully completed (T); then

$$10v + 0.75vt = T. \tag{5.10}$$

Step IV

Resolution, results, analysis, and discussion. Substituting (5.9) into (5.10) we obtain

$$10\frac{T}{16} + 0.75\frac{T}{16}t = T.$$

Simplifying T, we obtain $t = 8$ h, and so the work done by the worker without a break is completed in 18 h (10 + 8).

The result is logical, given that part of the work was performed at a slower speed. In the first case, the work was finished in 16 h in two shifts, and in the second case in 18 continuous hours. Problems like this could perhaps be solved in a simpler way, but it is vital to stress that the method is fundamental, especially for more complex situations. Recall the lesson of the shoe problem. In many of the problems proposed in this chapter, it is necessary, perhaps essential, to continue using the proposed method. We emphasize that, however basic a problem may be, the method for dealing with it is of utmost importance.

7. Apprentices and masters [10^+]. A group of apprentices and masters is building a large wall made of 50,000 bricks. They complete the job in 25 days. The apprentices work 2 h more than the masters per day. The boss pays per brick, and at the end of the day an apprentice earns three-quarters of what a master earns. If the masters work alone, but with the working hours per day of an apprentice, they will complete the big wall in 50 days. The boss pays $12,000 for the completed job, and a master earns $96 a day. Calculate (a) the number of masters, (b) the number of apprentices, (c) the working speed of a master, (d) the working speed of an apprentice, (e) the daily working hours of masters, and (f) the daily working hours of apprentices.

Solution

Step I

Reading and understanding. As in the previous problem (Overtime), the key is to correctly define work speed, in this case number of bricks per hour, then formulate all the equations from the problem statement.

Step II

Graphical representation and variable definition and codification. In this case it is not necessary to have a graphical or schematic representation. To code and define the unknowns, we will follow the order of the questions:

N_M: number of masters
N_A: number of apprentices
V_M: working speed of a master bricks/h
V_A: working speed of an apprentice bricks/h
t_M: daily working hours of a master h/day
t_A: daily working hours of an apprentice h/day

Step III
Mathematical formulation including all available data. We have six unknowns, and after a careful reading of the problem statement, we can write five equations. At first glance, the problem seems to have no solution because we need six equations. But if we assume that the problem has been correctly stated, then we need to further analyze the situation in step IV: Resolution, results, analysis, and discussion.

Taking into account that for the time being, the problem cannot be solved, we will write all the equations that can be inferred from the problem statement.

From "*A group of apprentices and masters are building a large wall made of 50,000 bricks and are able to complete the job in 25 days*" we can write

$$25 \times N_M \times V_M \times t_M + 25 \times N_A \times V_A \times t_A = 50,000. \tag{5.11}$$

To understand (5.11), remember that V_M is the work speed [bricks/h] per master, which, multiplied by t_M [h/day], gives us the number of bricks per day for masters. Then multiplying this result by the number of masters and the number of days that masters work, we obtain the total number of bricks laid by the masters. In the same way, the second term of the equation calculates the total number of bricks laid by the apprentices. Adding these two numbers, we arrive at the completed job: 50,000 bricks.

Given that the apprentices work 2 h more per day than the masters, we obtain

$$t_M + 2 = t_A. \tag{5.12}$$

From *"The boss pays per brick and at the end of the day an apprentice gains three-quarters of what a master earns"* we can write the following:

First, the boss is paying \$12,000 for the completed job (50,000 bricks) and \$0.24 per brick; therefore,

$$0.24 \times V_A \times t_A = \frac{3}{4} \times 0.24 \times V_M \times t_M. \tag{5.13}$$

Masters working alone, but with the daily working hours of an apprentice, would complete the wall in 50 days; thus,

$$50 \times N_M \times V_M \times t_A = 50,000. \tag{5.14}$$

Finally, a master earns \$96 a day, so

$$0.24 \times V_M \times t_M = 96. \tag{5.15}$$

Step IV

Resolution, results, analysis, and discussion. Assuming that the problem has a solution, we need to think about it. Clearly, from the problem statement we can formulate five equations (5.11–5.15). We need to be confident that we have already written all the possible equations from the problem statement, so we need to try to understand and look for some other, hidden, information*. For example, it sounds logical that N_M and N_A should be integers. With this in mind, we will do some algebra to get an equation to relate N_M and N_A.

From (5.15) we get that $V_M \times t_M = 400$, and substituting it into (5.13) we obtain $V_A \times t_A = 300$. Substituting both results into (5.11) and simplifying we get

$$4 \times N_M + 3 \times N_A = 20. \quad (5.16)$$

Because N_M and N_A should be integers and positive numbers, we will analyze the potential valid results from (5.16).

N_M	N_A
5	0
4	1.33
3	2.66
2	**4**
1	5.33
0	6.66

From the table we can infer that the only possible result is $N_M = 2$ masters and $N_A = 4$ apprentices. Then substituting N_M in (5.14) and dividing by (5.15) we obtain

$$t_A = \frac{5}{4} t_M.$$

That relates t_M to t_A, and solving together with (5.12) we obtain

$$t_A = 10 \text{ h and } t_M = 8 \text{ h}.$$

Finally, it is easy to obtain that $V_M = 50$ bricks/h and $V_M = 30$ bricks/h.

The lesson here is that sometimes certain equations, information, or constraints are somewhat hidden.

*Diophantine equations are a type of equation where variables are integers. Usually, they have fewer equations than variables, but the fact that the variables are integers makes it possible to find a single solution. The name of these equations comes from the Greek mathematician Diophantus of Alexandria, third century BC.

CANDLES

8. **Two candles [7].** A candle with height L_a burns in $\boldsymbol{a}$ hours; another candle with height L_b burns in $\boldsymbol{b}$ hours. Given that $L_a > L_b$ and $\boldsymbol{b} > \boldsymbol{a}$, if the candles are lit simultaneously, after how much time will they be the same height?

Solution

Step I

Reading and understanding. This problem combines two degrees of difficulty. First, it is very important to outline graphically the situation. Second, no numerical information is given, and

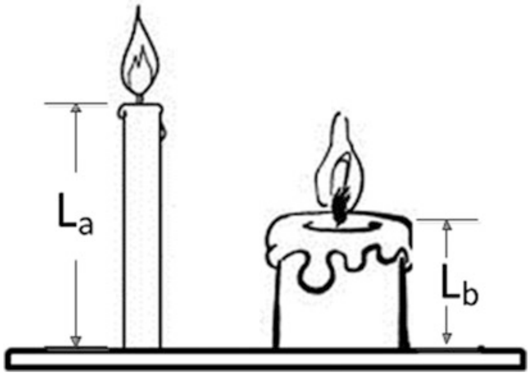

Fig. 5.10 Candles of height L_a and L_b

therefore the mathematical formulation is somewhat more difficult. As in the previous problems, it is important to conceptualize the rate of consumption of the candle. In this case, it could be generalized as

$v = \frac{L}{t}$, where v is the rate of consumption of the candle, L is the candle height, and t is the time it takes for a candle to be consumed completely.

Step II
Graphic representation and variable definition and codification (Fig. 5.10).

X: distance consumed by candle b
t: time for both candles to reach the same height

Step III
Mathematical formulation including all available data. It is possible to formulate two equations, one for each candle. We have two unknowns, so DF = 0.

According to the definition of speed, we may define

$$v_a = \frac{L_a}{a} \text{ and } v_b = \frac{L_b}{b}.$$

Once both candles reach the same height (Fig. 5.11), after a time t, we can write
$v_a = \frac{L_a - L_b + X}{t}$ and $v_b = \frac{X}{t}$, and replacing v_a and v_b we have
$\frac{L_a}{a} = \frac{L_a - L_b + X}{t}$ and $\frac{L_b}{b} = \frac{X}{t}$, and solving for X in both equations we obtain
$X = \frac{L_a}{a} t - L_a + L_b$ and $X = \frac{L_b}{b} t$, respectively, and matching both equations and solving for t we obtain

$$t = \frac{ab(L_a - L_b)}{bL_a - aL_b}. \tag{5.17}$$

Step IV
Resolution, results, analysis, and discussion. First, we can determine whether for all values of L_a, L_b, $\boldsymbol{a}$, and $\boldsymbol{b}$ the result obtained for time t is positive. Since $\boldsymbol{a}$ and $\boldsymbol{b}$ are positive, and the problem states that $L_a > L_b$, then the numerator in (5.17) is positive. In the case of the denominator, $\boldsymbol{b}L_a > \boldsymbol{a}L_b$ because $L_a > L_b$ and $\boldsymbol{b} > \boldsymbol{a}$; therefore, it is also positive. In conclusion, t is always positive. Another

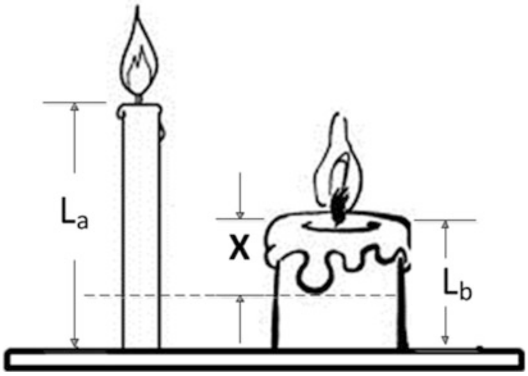

Fig. 5.11 Time t where both candles reach the same height

condition to be met is that t should be less than $\boldsymbol{a}$ before the candle with height L_a is consumed, and therefore rewriting (5.17), we obtain

$t = \frac{ab(L_a - L_b)}{b\left(L_a - \frac{a}{b}L_b\right)}$, simplifying and rewriting $t = a\frac{(L_a - L_b)}{\left(L_a - \frac{a}{b}L_b\right)}$, the multiplier for $\boldsymbol{a}$ is less than 1. Remember that $\boldsymbol{a} < \boldsymbol{b}$, and therefore $L_a - L_b < \left(L_a - \frac{a}{b}L_b\right)$, then $\boldsymbol{t} < \boldsymbol{a}$.

Flow

9. **Two valves [8]**. A tank is filled through valves A and B. Each separate valve fills the tank in $\boldsymbol{a}$ and $\boldsymbol{b}$ hours, respectively ($\boldsymbol{a} > \boldsymbol{b}$). After valve A is opened, when should valve B be opened so that the tank is filled in $\boldsymbol{b}$ hours?

Solution

Step I

Reading and understanding. As in the previous problems, first it is necessary to conceptualize the flow rate. In generic terms, the volumetric flow rate will be defined as

$C = \frac{V}{t}$, where C is the volumetric flow, and V is the volume to be filled in time t.

Step II

Graphical representation and variable definition and codification. Although in most cases it is useful to have a graphical representation, in this case it is unnecessary because a graphical representation would not help us in writing the equations.

V: volume of tank

C_a: volumetric flow rate of valve A

C_b: volumetric flow rate of valve B

Step III

Mathematical formulation including all available data. There is one unknown variable, and one equation can be formulated (DF = 0). Since valve A fills the tank in $\boldsymbol{a}$ hours and valve B fills it in $\boldsymbol{b}$ hours, the flow rate of each of the valves can be expressed as

$$C_a = \frac{V}{a}, \tag{5.18}$$

$$C_b = \frac{V}{b}. \tag{5.19}$$

The tank should be filled in ***b*** hours. The filling process is started by valve A, and after t hours valve B starts working. Therefore, the volume filled by valve A for b hours, plus the volume filled by valve B in $\boldsymbol{b} - \boldsymbol{t}$ hours must equal the volume of the tank, V:

$$C_a b + C_b(b - t) = V. \tag{5.20}$$

Step IV

Resolution, results, analysis, and discussion. Substituting C_a and C_b from (5.18) and (5.19) into (5.20) we obtain

$\frac{V}{a} b + \frac{V}{b}(b - t) = V$, solving for t,

$$t = \frac{b^2}{a}. \tag{5.21}$$

A condition that should be fulfilled is that t must be less than ***b***. If we rewrite (5.21), we obtain $t = \frac{b}{a} b$, and since $\boldsymbol{b} < \boldsymbol{a}$, then ***b***/a (multiplier of ***b***) is less than one. Therefore, $t < \boldsymbol{b}$.

10. Wine [3$^+$]. A vat of wine is filled by three valves, A, B, and C. To completely fill the vat of wine, each valve, operating alone, takes a, b, and c hours, respectively. How quickly will the vat of wine get filled if valves A and B operate simultaneously? How quickly will the vat get filled if valves A, B, and C operate simultaneously?

Solution

Step I

Reading and understanding. First, it is very important to define mathematically the flow rate:

$C = \frac{V}{t}$, where C is the volumetric flow rate, and V is the volume to be filled in a time t.

Step II

Graphical representation and variable definition and codification. In this case, a graphical or schematic representation is unnecessary.

V_{W}: volume of vat of wine (l)
C_{A}: flow rate of valve A L/h
C_{B}: flow rate of valve B L/h
C_{C}: flow rate of valve C L/h

Step III

Mathematical formulation including all available data. According to the hour variable codification, we will express mathematically the statement "*To completely fill the vat of wine, each valve, operating alone, takes a, b, and c hours, respectively*."

When valve A operates alone, it fills the vat of wine completely in a hours; thus,

$$C_{\mathrm{A}} = \frac{V_{\mathrm{W}}}{a}. \tag{5.22}$$

Analogously, for valves B and C we have

$$C_{\mathrm{B}} = \frac{V_{\mathrm{W}}}{b}, \tag{5.23}$$

$$C_{\mathrm{C}} = \frac{V_{\mathrm{W}}}{c}. \tag{5.24}$$

Now, if valves A and B operate together, the volume that valve A contributes plus the volume contribution of valve B in time t should completely fill the vat of wine. Therefore,

$$C_{\mathrm{A}} \times t + C_{\mathrm{B}} \times t = V_{\mathrm{W}}. \tag{5.25}$$

Step IV
Resolution, results, analysis, and discussion.

Substituting C_{A} and C_{B} in (5.25) we obtain

$$\frac{V_{\mathrm{W}}}{a} \times t + \frac{V_{\mathrm{W}}}{b} \times t = V_{\mathrm{W}}. \tag{5.26}$$

Simplifying V_{W} and rearranging we obtain

$$t = \frac{1}{\frac{1}{a} + \frac{1}{b}} = \frac{ab}{a + b}. \tag{5.27}$$

If the three valves operate simultaneously we get

$$C_{\mathrm{A}} \times t + C_{\mathrm{B}} \times t + C_{\mathrm{C}} \times t = V_{\mathrm{W}}. \tag{5.28}$$

As before, substituting C_{A}, C_{B} , and C_{C} into (5.28) we obtain

$$\frac{V_{\mathrm{W}}}{a} \times t + \frac{V_{\mathrm{W}}}{b} \times t + \frac{V_{\mathrm{W}}}{c} \times t = V_{\mathrm{W}}. \tag{5.29}$$

Simplifying V_{W} and rearranging (5.29) we obtain

$$t = \frac{1}{\frac{1}{a} + \frac{1}{b} + \frac{1}{c}}. \tag{5.30}$$

If valves A and B had equal flow rates, say C_{A}, then we would expect that both valves operating simultaneously will reduce the time to fill the vat of wine completely by half. In (5.27), if b equals a, then

$$t = \frac{1}{\frac{1}{a} + \frac{1}{b}} = \frac{ab}{a + b} = \frac{a^2}{2 \times a} = \frac{a}{2}.$$

In the same way, if the three valves had equal flow rates, then if the three valves operated simultaneously, it would take $a/3$ h to fill the vat of wine. If b and c are equal to a, then in (5.30) we obtain

$$t = \frac{1}{\frac{1}{a} + \frac{1}{b} + \frac{1}{c}} = \frac{1}{\frac{1}{a} + \frac{1}{a} + \frac{1}{a}} = \frac{a}{3}.$$

These results show that our solution is apparently correct.

11. *N* valves [8]. A tank is filled by a valve in 10 h. Another tank is filled by an unknown number of valves. Each valve of the second tank provides the same flow rate in liters per hour, and in addition, each valve of the second tank separately fills it in 100 h. First, an operator opens the valve of the first tank, and 1 h later the operator opens the set of valves on the second tank. Two hours later, the operator closes the set of valves on the second tank, and at this point, the volume of the second tank is 50 L greater than that of the first tank. Five hours later, the first tank has double the volume of the second tank and the valve of the first tank is closed. Finally, the operator pumps water from the first tank to the second until it is completely filled. The operator then checks that the remaining volume in the first tank is 200 L.

(a) What is the volume of the first tank?
(b) What is the volume of the second tank?
(c) What is the flow rate of the valves (first and second tank)?
(d) How many valves does the second tank have?

Solution

Step I

Reading and understanding. As was shown in the previous problem (Wine), the flow rate was defined as follows:

$C = \frac{V}{t}$, where C is the volumetric flow, and V is the volume to be filled in time t.

Step II

Graphical representation and variable definition and codification. In this case, it is not necessary to have a graphical or schematic representation. As explained earlier, a schematic representation would not help us to develop the necessary equations.

V_1: volume of first tank L
V_2: volume of second tank L
C_1: flow rate of valve on the first tank L/h
C_2: flow rate of each valve on second tank L/h
N: number of valves that feed the second tank

Step III

Mathematical formulation including all available data. We have five unknowns. From the problem statement we should be able to formulate five equations. If so, DF $= 0$, and we can solve them.

As we did in previous problems, we will be constructing the equations by translating the verbal expression into a mathematical expression.

If the first tank is filled in 10 h, then

$$10 \times C_1 = V_1. \tag{5.31}$$

If each valve of the second tank fills it in 100 h, then

$$100 \times C_2 = V_2. \tag{5.32}$$

The following statement indicates that the operator first opens the valve on the first tank, and 1 h later the valves on the second tank are opened. Then, 2 h after those valves are opened, the volume of the second tank is 50 L more than that of the first tank. Therefore,

$$3 \times C_1 + 50 = 2 \times N \times C_2. \tag{5.33}$$

The first tank took 3 h to fill and the second tank 2 h, but by N valves.

Our next statement says: "*Five hours later the first tank has double the volume of the second tank*"; mathematically this is

$$8 \times C_1 = 2(2 \times N \times C_2). \tag{5.34}$$

Finally, water is pumped in to fill the second tank completely, and 200 L remains in the first tank. Therefore, the amount pumped from the first tank, $8 \times C_1 - 200$ L, plus the amount already in the second tank, $2 \times N \times C_2$, should be equal to the volume of the second tank, V_2. Mathematically:

$$(8 \times C_1 - 200) + (2 \times N \times C_2) = V_2. \tag{5.35}$$

Thus, we have formulated five equations (5.31–5.35), and so we can solve the problem.

Step IV

Resolution, results, analysis, and discussion. From (5.33) to (5.34) we obtain

$$C_1 = 50 \text{ [L/h], and from (5.31) } V_1 = 500 \text{ [L]}.$$

We will leave to you the remaining algebra to obtain the following results:

$$C_2 = 4 \text{ [L/h]}, V_2 = 400 \text{ [L], and } N = 25 \text{ valves}.$$

5.3.2 Concentrations

12. **Wine barrel [10$^+$]**. A drunk man is standing next to a large wine barrel (a mixture of ethanol and water) of volume V. To conceal his actions and deceive the barrel owners, the drunk man does the following: he takes a cup of volume v ($v <<<< V$) from the barrel and, in return, pours a cup of water into the barrel, so that the barrel's volume is held constant.*If, initially, the barrel had pure wine, show that after the drunk man removes wine and pours in water n times the final ethanol concentration in the barrel will be

$$C_n = C_0\left(\frac{V - v}{V}\right)^n, \tag{5.36}$$

where C_0 is the initial concentration (unmixed wine) [liters of alcohol/liter solution], and C_n is the concentration after the nth operation (1, 2, 3, . . ., n) [liters of alcohol/liter solution].

*Suppose the volumes can be considered additive. If two liquids are mixed (for example, if you mix ethanol and water), then the volume of the mixture is not necessarily the sum of the volumes of the liquids since one of the liquids could occupy the empty spaces between the molecules of the other liquid. In this particular case, the wine has an alcohol concentration (ethanol) of about 12 %, and it is known that ethanol/water mixtures are not perfectly additive.

Solution

Step I

Reading and understanding. Initially, the alcohol volume is C_0V, and when the drunk man takes the first glass of wine and replaces it with water, the amount of alcohol left in the barrel is then C_0 $(V - v)$, which is equivalent to C_1V.

Step II

Graphical representation and variable definition and codification. Although in most cases it is useful to have a graphical representation, in this case is unnecessary. In addition, the variables were already defined in the problem statement.

Step III

Mathematical formulation of the problem

First operation

$$C_0(V - v) = C_1V, \text{then} \quad C_1 = C_0\left(\frac{V - v}{V}\right) \tag{5.37}$$

Second operation

$C_1(V - v) = C_2V$, replacing C_1, then $C_2 = C_0\left(\frac{V-v}{V}\right)^2$

Third operation

$C_2(V - v) = C_3V$, replacing C_2, then $C_3 = C_0\left(\frac{V-v}{V}\right)^3$

By induction, we can express that

$$C_n = C_0\left(\frac{V - v}{V}\right)^n. \tag{5.38}$$

Step IV

Resolution, results, analysis, and discussion. First, we observe in (5.38) that if the drunk man's cup volume tends to zero ($v = 0$), then the concentration in the barrel (C_n) will not change (virtually). On the other hand, if the drunk man's cup volume was similar to the volume concentration of the barrel (v tends to V), then this would approach zero quickly.

13. Two streams [9]. Two salt-free streams of unknown flow are mixed to form one stream. It is known that the mixed stream has a flow of ***a*** kg/h. Adding a soluble salt to the first process stream produces a solution of 4.76 % (w/w) of salt, and the salt content in the mixed stream is 0.62 % (w/w). What is the ratio of the two streams?

Solution

Step I

Reading and understanding. First, when the salt is added to the first stream, the resulting salt concentration is 4.76 % (w/w), but then when both streams are mixed, the salt concentration becomes significantly reduced. If both streams have the same flow rate, then we would expect the salt

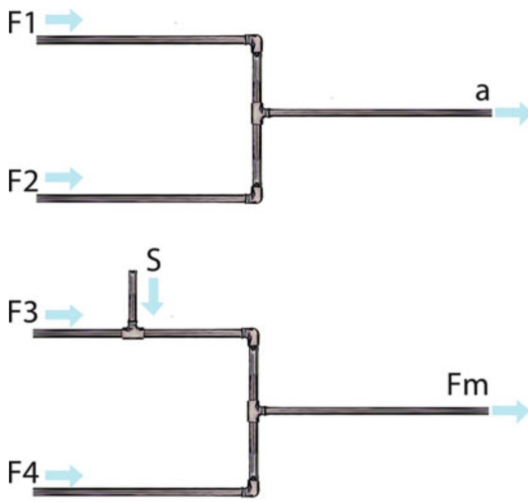

Fig. 5.12 Schematic representation of the mixture of two streams, firstly without adding salt and then with the addition of salt

concentration in the mixed stream to be cut in half. Given that the salt concentration is reduced several times (more than 7), we expect the second stream to have a much higher flow rate than the first one (on the order of 7 times).

Step II

Graphical representation and variable definition and codification.

According to Fig. 5.12, the variables are as follows:

F_1 : flow rate of stream (1 kg/h)

F_2: flow rate of stream (2 kg/h)

S: kilograms per hour of salt added to stream 1

F_M: flow rate of mixed stream with salt added kg/h

We assume that the flow rate of the mixed stream (with salt) is a known value (a kg/h).

Step III

Mathematical formulation including all available data. We have four unknowns (F_1, F_2, S, and F_M), and we can formulate four equations, as shown in the mathematical formulation, thus DF = 0. But do not forget that the problem is not to solve the equations for all unknowns, just to calculate the ratio of the streams 2 to 1, i.e., F_2/F_1.

Before the salt is added (Fig. 5.12), the addition of the flow rate of streams 1 and 2 must be equal to the flow rate of the mixed stream, thus:

$$F_1 + F_2 = a. \tag{5.39}$$

In the same way, when the salt is added (Fig. 5.12b) we can write

$$F_1 + F_2 + S = F_M. \tag{5.40}$$

When salt is added to stream 1, the salt concentration is 4.76 %, and thus

$$4.76 = 100\frac{S}{F_1 + S}. \tag{5.41}$$

And the salt concentration in the mixed stream is 0.62 %; therefore

$$0.62 = 100\frac{S}{F_M}. \tag{5.42}$$

Step IV

Resolution, results, analysis, and discussion. Performing some algebra with (5.39–5.42), we obtain

$$\frac{F_2}{F_1} = 7.$$

One approach is to work with (5.41) to obtain

$$S = \frac{1}{20}F_1.$$

Then with this result, working on (5.42), we get

$$F_M = 8.0645 \times F_1.$$

Substituting these two results into (5.40) we obtain

$$\frac{F_2}{F_1} = 7.$$

As we expected and almost correctly predicted, the flow rate of stream 2 was approximately 7 times the flow rate of stream 1. Normally, this is a really tough problem for students, but with the correct reading and understanding, and with the help of simple but vital graphical representation, the problem solution looks straightforward.

14. Calcium chloride solution [6⁺]. A student must prepare 15 L of calcium chloride with 6.0 % w/v concentration. She has 7.0 L of a 4.0 % w/v solution and 20 L of a 10 % w/v solution. The student carries out some calculations and discovers, to her surprise, that the solution of 4.0 % w/v is limiting. Then she decides to use the entire solution of 4.0 % w/v, an unspecified amount of 10 % w/v solution, and distilled water. (a) How much distilled water was used? (b) How much of the 10 % w/v solution was used?

Solution

Step I

Reading and understanding. First, we need to understand what is really meant by "the 4.0 % w/v solution is limiting." First, we want to know the lowest concentration that we can get in a 15 L solution if we use all of the 4.0 % w/v solution. A quick calculation shows us that if we use the entire solution of 4.0 % w/v (7.0 L) and 8.0 L of the 10 % w/v solution (to complete the required 15 L), then the final concentration will greater than 6.0 % w/v. This is why the student proposes to use some distilled water, to dilute the 10 % w/v solution, and then combine it with the 7.0 L of the 4.0 % w/v solution to get the required 6.0 % w/v.

Step II

Graphical representation and variable definition and codification. In this case, it is not necessary to use a graphical representation.

x: amount (L) of 10 % w/v solution used

y: amount (L) of distilled water used

Step III

Mathematical formulation including all available data.

We have two variables, and we can write two equations, one that takes into account that the final volume should be 15 L and the second one that complies with the required 6.0 % w/v solution. Thus, DF = 0. First, the total amount of solution should be 15 L, and taking into account that all 4.0 % w/v solution will be utilized (7.0 L)

$$x + y + 7 = 15 \tag{5.43}$$

And the final concentration in the 15 L solution should be 6.0 %. The mass of calcium chloride obtained from the 4.0 % w/v solution plus the mass of the calcium chloride obtained from the 10 % w/v solution divided by the total volume should be 6.0 % w/v; therefore

$$\frac{0.04 \times 7 + 0.1 \times x}{15} = 0.06 \tag{5.44a}$$

Step IV

Resolution, results, analysis, and discussion.

Then, from (5.44), $x = 6.2$ L of 10 % w/v solution, and substituting into (5.43) we obtain $y = 1.8$ L of distilled water.

As expected, some distilled water was needed to reach the 6.0 % w/v in the final solution of 15 L. As stated at the beginning, although we used the entire 4.0 % w/v solution, the lowest concentration in the 15 L solution was higher than 6.0 % w/v (7.2 %). Please check this result!

5.3.3 Percentages

15. **Concentrated product [10]**. A company offers two types of varnish, one with 25 % water and another, substantially dryer, with 4 % water. The product with the higher moisture content (25 % water) sells for $10 per 100 kg if you buy it at the company. The "dry" product (4 % water) sells for $12.5 per 100 kg if you buy it at the company. If the transportation cost to your company is $0.80 per 100 kg, which of the two products will it be more cost-effective to buy?

Solution

Step I

Reading and understanding. First, we must understand that we should be willing to pay more for the dry product. For example, if we buy 100 kg of the moist product (25 % water), we are paying for 75 kg of varnish, while in the case of the dry product, we are paying for 96 kg of varnish. In addition, the cost of transportation per kilogram of varnish is lower for the dry product. Therefore, we should be willing to pay more for the dry product, the one with a higher varnish concentration.

Step II

Graphical representation and variable definition and codification. Although in most cases it is useful to have a graphical representation, in this case it is unnecessary.

Y: kilogram of product with 25 % water to obtain 100 kg of varnish

Z: kilogram of product with 4 % water to obtain 100 kg of varnish

X: maximum price to be paid for dry product (4 % water)

Step III

Mathematical formulation of problem. We will calculate the most we should pay for the dry product and compare that to the price of \$12.5 per 100 kg of product. To do this, we calculate the cost of 100 kg of pure varnish with 25 % water delivered at your factory and compare it to the cost of 100 kg of pure varnish with 4 % water at a price of \$$X$ per 100 kg, also delivered at your factory.

Product with 25 % water. If we want to buy 100 kg of the pure varnish, then we must buy Y kg of the 25 % water (75 % varnish) and then pay for transportation. Therefore, we can write the following equation:

$Y \times 0.75 = 100$. Then $Y = 133.34$ kg of product with 25 % water will give 100 kg of pure varnish.

The cost of this 133.34 kg with 25 % water is the cost of the product plus the cost of transportation, as seen in the following equation:

$$133.34 \times \frac{10}{100} + 133.34 \times \frac{0.8}{100} = \text{US\$}14.40.$$

Product with 4 % water. First, we estimate how many kilograms of product (Z kg) must be purchased to obtain 100 kg of pure varnish and then calculate the total cost, including transportation:

$Z \times 0.96 = 100$. Then $Z = 104.167$ kg of the product with 4 % water will yield 100 kg of pure varnish.

If we are willing to pay \$$X$ per 100 kg of product with 4 % water, then the cost of this 104.167 kg of product with 4 % water, including transportation, should be equal to \$14.40 (and no more!):

$104.167 \times \frac{X}{100} + 104.167 \times \frac{0.8}{100} = \text{US\$}14.4$, then $\boldsymbol{X} = \$13.00$.

Since the dry product price is \$12.50, we should buy it. Moreover, we could even pay up to \$13.00, and it would still be more cost effective.

Step IV

Analysis and discussion. As we predicted, we can pay more for the product with the higher concentration of varnish. First of all, every kilogram of product has more varnish, and also transportation per kilogram of the dry varnish is cheaper because it carries less water per kilogram of product.

5.3.4 Ages

16. 50 years old [5]. Today my age is triple the age of my oldest son. Ten years ago, my age was seven times his age. Within how many years I will be 50 years old?

Solution

Step I

Reading and understanding. This is a middle school problem, but the idea is to show you how the method works with this easy problem and with intricate ones.

Step II

Graphical representation. Here there is a nice way to solve this kind of problem. The idea is to construct, in general, a table like the following one:

Person	Past	Present	Future
Person 1			
Person 2			
...			
Person N			

Variable definition and codification. Here we have the option to define X as my present age and Y as the present age of my son, or directly assume that at present my age is triple his age ($3X$ and X, respectively). Let us say that X is my present age and Y is his present age. Then the table will look like this:

Person	Past	Present	Future?
Me	$X - 10$	X	50
My oldest son	$Y - 10$	Y	

Step III

Mathematical formulation including all available data. We have two unknowns, and with the provided information, we can formulate two equations, one for the present and a second one for the past. Thus, DF = 0. Looking at the table and the problem statement we can write the following equations.

First, currently my age is triple the age of my son, therefore

$$X = 3 \times Y. \tag{5.44b}$$

Ten years ago, my age was seven times his age:

$$X - 10 = 7(Y - 10). \tag{5.45}$$

Step IV

Resolution, results, analysis, and discussion. Solving (5.44) and (5.45), we obtain $X = 45$ and $Y = 15$; therefore, within 5 years I will be 50 years old.

17. Marilyn's age [10$^+$]. N years ago, Marilyn's age was a fifth of her present age. Within M years her age will be double her present age. If Marilyn's age in the future (M years later) is ten times her age N years ago, what is Marilyn's present age?

Solution

We will skip step I.

Step II

Graphical representation and variable definition and codification.

As proposed in the previous problem (50 years old), we will build a table. In this case:

X: Marilyn's present age

Person	Past	Present	Future
Marilyn	$X - N$	X	$X + M$

Step III

Mathematical formulation of problem. We have three unknowns, and so it seems we can formulate three equations. If so, then DF = 0.

First, *N* years ago Marilyn's age was one-fifth of her present age:

$$(X - N) = \frac{X}{5}. \tag{5.46}$$

Second, within *M* years, Marilyn's age will be double her present age:

$$(X + M) = 2 \times X. \tag{5.47}$$

Finally, Marilyn's age in the future (*M* years later) will be ten times her age *N* years ago:

$$(X + M) = 10(X - N). \tag{5.48}$$

Step IV

Resolution, results, analysis, and discussion. Dividing (5.47) by (5.46) we obtain

$$\frac{X + M}{X - N} = 10. \tag{5.49}$$

This is identical to (5.48). Unfortunately, there is not enough information to know Marilyn's age. In fact, with the data provided in the problem statement, it is possible to write just two independent equations, not three. Equation (5.48) is not independent of (5.46) and (5.47). Therefore, DF = 1. This means that assigning a value to one unknown we can solve the problem. For example, if $M = 10$, then from (5.47) $X = 10$, and from (5.48), $N = 8$. You can verify these results with the problem statement.

5.3.5 Miscellaneous

18. Palindrome* [5]. The odometer on the family car shows 17,971 miles. The driver, driving on a highway in Columbus, Ohio, realizes that this number is palindromic. Curious, he thought, "It will be a long time before this happens again." But 2 h later, the odometer again shows a palindromic number. How fast did the car go during those 2 h?

*A palindrome is a phrase or number that reads identically from left to right and right to left. There are many famous palindromic phrases, e.g., "A man, a plan, a canal—Panama." In the case of numbers, there are not only palindromic numbers (131) but also palindromic equations, such as $11 \times 11 = 121$.

Solution

The problem is interesting, but because it is a somewhat different problem, we will not follow strictly the detailed method that we have employed in every problem so far.

Step I

Reading and understanding. Given that 2 h later the odometer on the family car showed a palindromic number, the distance that the car advanced will be approximately in the range of 100–130 miles if we assume the car travels at a rate of 50–65 miles/h on the highway. Therefore, we can estimate that the number that we are searching for is greater than 18,071 miles and less than 18,101 miles. In addition, to be palindromic, the last number should be 1 (because the first is 1). With the same reasoning, the preceding number should be 8 (because the second number is 8), and then the number is 18,081. The number cannot be 18,181 (also palindromic) because according to the speed limit, the number on the odometer 2 h later should not be much higher than 18,101.

Step II

Results, analysis, and discussion. If the searched for palindromic number is 18,081, then the speed of the car should be the distance traveled by the car divided by the time it took to go that distance. Therefore,

$$\text{Distance} = 18{,}081 - 17{,}971 = 110 \text{ miles and time} = 2.0 \text{ h}$$
$$\text{Then, } v = \frac{110}{2} = 55 \text{ miles/h}$$

19. The Lucas problem [10^{+}]. This problem was invented by Edouard Lucas, a nineteenth-century French mathematician. Every day at noon a ship leaves Le Havre (a port in France) to the port of New York, and at the same time another ship—every day at noon—leaves New York for Le Havre (assume the same time zone for both ports). The trip takes seven days and seven nights. How many ships will the ship that left Le Havre for New York encounter on its way?

Solution

Step I

Reading and understanding.

Again, as in the previous miscellaneous problem, this is an interesting challenge. Experience has shown us that almost everybody gets this one wrong, even bright graduate students and professors. As in the shoe problem in the introduction to this chapter, everybody is tempted to solve it mentally and fails. This problem really needs a graphical representation of the ship's journey. Then the solution is straightforward.

Step II

Graphical representation.

Step III

Results, analysis, and discussion. Looking at the diagram (Fig. 5.13), we see clearly that the ship encountered 15 ships on its travel. Why? For example, when a ship starting out from Le Havre completed 6 days of travel (noon of the sixth day), another ship was setting out from New York for Le Havre at the same time (Fig. 5.13). These two ships meet at midnight (x), from which you can infer that this ship encounters other ships not only at noon everyday but also at midnight. In addition, it will meet a ship from New York when the journey starts and also when it is ending its journey. In Fig. 5.13, you can count the encounters, which total 15. Surprised?

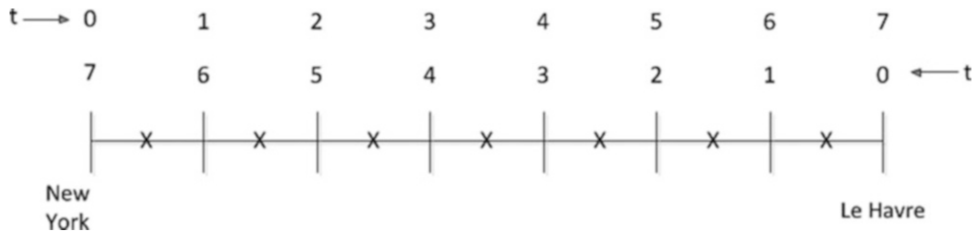

Fig. 5.13 Schematic representation of ships leaving Le Havre (a port in France) to the port of New York and simultaneously ships leaving New York to the port of Le Havre

As commented in the "Reading and Understanding" section of this problem, the key here is a graphical representation. For most of us, it is very hard to imagine this situation, and normally people underestimate the number of encounters.

5.4 Proposed Problems

VELOCITY

1. **Increasing the speed [7⁺]**. A automobile travels 20 km for a certain amount of time. If the automobile needs to cut the travel time by 1 h, the speed of the automobile should be increased by 10 km/h. What is the new speed?
   A: V = 20 km/h
2. **Sebastian Coe [6]**. In the early 1980s the world record in a 1,500 m race was 3 min 32 s. In his workouts, Sebastian Coe (considered by many to be one of the greatest athletes of the twentieth century) ran against his archrival Steve Ovett. In his last workout with Ovett, Ovett started the race 44.1 m over Coe, but they crossed the finish line together. If Ovett ran at a speed equivalent to 3 min 37.4 s for 1,500 m, what was Coe's time for the 1,500 m?
   A: 3 min 31 s
3. **Travel to Santiago [7]**. You are traveling from Viña del Mar to Santiago (120 km) on the same day as your cousin John. Your cousin departs a certain amount of time earlier than you, and that time is equal to 10 % of your travel time on your trip to Santiago. If your average speed is 20 % greater than that of your cousin John and his travel time to Santiago is 1 h and 35 min, (a) what is the speed of each driver? (b) what was your travel time from Viña del Mar to Santiago? and (c) who arrives first in Santiago?
   A: (a) 75.8 km/h (John) and 91.0 km/h (you). (b) 1.319 h or 1 h 19 min and 9 s. (c) As your travel time to Santiago is about 79 min and your cousin John departed just 7 min and 54 s before you, this is not enough time for your cousin to beat you to Santiago (remember his travel time was 1 h and 35 min), so you arrive first.
4. **Two trains [7]**. A train leaves New York on a nonstop trip to Washington, DC, at 60 km/h. At the same time, another train leaves Washington, DC, heading for New York at 40 km/h. How far are both trains half-hour before crossing each other?
   A: 50 km (30 + 20)
5. **A restless fly [8]**. Two cyclists begin biking simultaneously, each in the direction of the other. One sets out from Columbus, Ohio, the other from Cleveland, Ohio. When the two cyclists are 80 km away (from each other), a fly lands on the shoulder of one of the cyclists, flies to the other cyclist, and then flies back. The restless fly goes back and forth until the cyclists cross paths, then

lands on the nose of one of the cyclists. If the fly is traveling at a speed of 40 km/h and the cyclists are riding at 20 km/h, how many kilometers did the fly?

A: The problem is simpler than it appears. The cyclists' time to meet up was 2 h. Therefore, the fly traveled 80 km in 2 h.

6. **Harvest time [6^+]**. A farm must deliver her harvest to a packinghouse. The farm manager, in Greenville, has decided that the trucks must arrive in town at exactly 11 a.m. Traveling at 30 km/h, the trucks would reach the city at 10 a.m., i.e., 1 h early; and if they traveled at 20 km/h, they would arrive at noon, i.e., 1 h late. (a) How far is Greenville from the city? (b) How fast must the trucks travel to arrive exactly at 11 a.m.?

   A: (a) The Greenville farm is 120 km from the city. (b) The truck should travel at a speed of 24 km/h.
7. **Vacations [9]**. Two students are traveling by train from their home city to a summer house on the beach. One of the students says, "I have noticed that every 5 min, we pass an oncoming train. How many trains arrive in the home city in 1 h if trains travel at the same speed in both directions?" "12 trains, obviously," says the other student, because 60 divided by 5 equals 12. The first student disagrees. What do you think? **Hint**: Remember the Lucas problem.

   A: 6 trains per hour (in some ways, this problem is similar to the Lucas problem).
8. **Can John save time? [7]**. John is going home from Valparaiso, Indiana. He ride halfway. John rides 15 times faster than he walks. The second half of his trip John traveled by oxcart. He can walk twice as fast as that. (a) Would he have saved time if he had walked the entire way? (b) How much time would he have saved?

   A: (a) Yes, he would have saved time. (b) Assuming that the walking time is t, riding halfway and in the oxcart the other half would have a travel time of $(31/30) \times t$.
9. **Average speed [8^+]**. A horse runs half of its route with no load at **A** km/h, and the other half it does with a heavy load at **B** km/h (**A** > **B**). What is the horse's average speed in kilometers per hour?

   A: $2AB/(A + B)$
10. **More or less? [9^+]**. A rower travels 5 miles with the current and 5 miles against the current. On another occasion, the rower travels 10 miles in a lake where there is no wind. (a) Which one is longer? (b) Explain your answer in (a).

    A: (a) It takes longer in the first situation. (b) The rower requires less time rowing with the current. A good way to visualize this situation is to analyze an extreme case, such that the current is very strong. It The first leg will take a short time, but the second leg will take a very long time.
11. **A swimmer [10]**. A boat travels with the current. A swimmer jumps out and swims against the current for a certain amount of time then swims back to the boat. Did the swimmer swim for a longer time against the current or when he was going back to the boat? (Assume that the swimmer's strength does not depend one whether he is swimming with or against the current.)

    A: Both times were the same. The current carries the man and the boat downstream at the same speed. The distance between the swimmer and the boat is unaffected.
12. **How keen are you? [9]**. Two boats are in front of each bank of a river. One boat leaves to meet the other boat, which is leaving the other shore. Both boats are moving at a constant speed. They meet 400 m from the first shore. Each boat moves to the other side and returns to shore without stopping and crosses at 320 m off the second shore. How wide is the river and what is the relationship between the speeds of the two boats?

    A: 1,000 m; the speed ratio is 2:3
13. **Heavy truck [6^+]**. A heavily loaded truck sets off on a long trip from point A. When the truck has gone 200 km (now at point B), a motorcycle moving at a speed that is 6 times faster than the

loaded truck's speed begins traveling toward it. How far from point B does the truck get until the motorcycle meets up with it?

A: 40 km

14. **Fast, but with caution [3]**. A bus leaves New York headed for Boston at noon. An hour later, a cyclist sets out from Boston to New York, obviously moving much slower than the bus. Upon crossing, which of the two is farther from New York?

    A: Neither! They intersect, so they are the same distance from New York!
15. **Traveling to Cincinnati [9^+]**. Rodrigo and Elizabeth are traveling by car from Chicago to Cincinnati, and during the trip, Rodrigo notes that the telephone poles on the road were equally spaced. Elizabeth, a little bored with the long trip, times 1 min and tells Rodrigo the number of telephone poles she counted during that time. Rodrigo tells his traveling companion that if that number is multiplied by 6, it gives the car's speed in kilometers per hour. Considering that when Elizabeth started timing they were exactly between two telephone poles, and in the same way, when she finished counting the telephone poles, they were exactly between two of them, how far apart are the telephone poles?

    A: 100 m
16. **Train encounter [7^+]**. Two freight trains of the same length travel at the same speed, 90 km/h. Upon their encounter, they cross each other completely in 12 s. How long is each train?

    A: 300 m
17. **Changing tires [5]**. As always, I kept a spare tire in my car. During the year, I drove **A** km and I was continuously rotating the tires (including the spare), so that at the end of the year, each of the tires had traveled the same number of kilometers. How many kilometers did each tire travel?

    A: 4/5*A* km
18. **Going to San Francisco [8^+]**. An airplane is flying from Portland, Oregon, to San Francisco, California, and then back to Portland, maintaining constant speed on the outbound and return flights (no wind for or against). The next day, it makes the same flight at the same speed, but this time, there is a slight tailwind from Portland to San Francisco. That is, the plane flies downwind from Portland to San Francisco but then against the wind from San Francisco to Portland. On what day is the flight shorter, the first or the second? Explain!

    A: The first (similar analysis to problem 10 Sect. 5.4).
19. **Boat trip [6]**. A man takes his motorboat and travels downstream at a constant speed. Because he is going with the current, it takes only 6 min to go 3 km. When he returns, against the current, it takes 12 min to cover 3 km. How long would it have taken to travel 3 miles in still water?

    A: 8 min
20. **Heavy traffic [4^+]**. On a normal working day, I travel from home to office at an average speed of 60 km/h (round trip). Today, as usual, I drove at a speed of 60 km/h to work. Unfortunately, on the return trip, I got stuck in a traffic jam, and so my average speed was only 40 km/h. What was my average speed on the round trip?

    A: 48 km/h
21. **Long trains [7^+]**. Two trains of equal length travel by two parallel rails in the same direction. From the moment the fastest train, at a speed of 100 km/h, overtakes the slower train (60 km/h), it takes 3 min to overtake completely. (a) How long is each train? (b) If they travel in opposite directions, how much time will it take them to completely pass each other?

    A: (a) Each train is 1 km long. (b) 0.75 min
22. **Long journey [4^+]**. Let's take a long trip and figure out if having four new tires plus a spare is enough. I was told that each tire can be used for up to 35,000 km, and we estimate that our

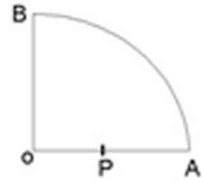

Fig. 5.14 Routes options for an individual who is at point B to get to point P

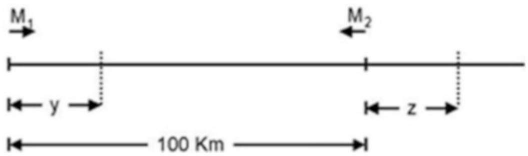

Fig. 5.15 Schematic representation for mobiles M1 and M2

journey, in total, will be 42,000 km. (a) Is it sufficient to have four tires plus the spare? If so, how many kilometers could I travel?
A: (a) Yes, it is sufficient. (b) 43,750 km

23. **Santiago to Talca [5]**. You have decided to take a trip from Santiago to Talca to participate in an important meeting. While you are planning your trip, you see that the distance between these two cities is 300 km. You start early in the morning, but unfortunately, after an hour, your car starts having problems and you must stop right away. This problem delays you for 30 min and forces you to continue the journey at a much slower (40 % slower) speed. If you arrive 1.5 h late to Talca, what was the speed at the beginning of the trip?
A: $V = 120$ km/h (maximum speed permitted on Chilean highways)
24. **By bus or on foot [5⁺]**. An individual at point B (Fig. 5.14 a quadrant) has two options to get to point P. The first is to jog from BO to OP. The second is to take a bus from point B to point A and then walk the remaining stretch AP. Knowing that BA is 400 m, the bus speed is 48 km/h, individual walking speed is 6 km/h, individual jogging speed is 18 km/h, and OP:PA = 2:3, what is the best alternative for this person?
A: The first (1 min 11 s: the second alternative is 2 min 2 s).
25. **Boat trip [10⁺]**. You travel by boat **a** km upstream and then return to the starting point. You know that you can go **b** km upstream, and in the same amount of time, you can go **c** km downstream. If the round trip takes you **e** hours, what is the speed of the current? **Hint**: start writing two equations for the first situation with both speeds as unknowns (boat and current), and also two times as unknowns, upstream and downstream times, but the total time should be **e** hours. Then you can add two more equations with the supplementary information.
A: The current speed is $a\frac{(c+b)(c-b)}{bce}$
26. **Two automobiles [10⁺]**. Two automobiles travel to meet each other, as shown in Fig. 5.15. These automobiles will pass each other at **y** km. With respect to the original position M1, in the case where automobile M1's speed is **n** times higher that of automobile M2. If automobile M2 travels in the same direction as automobile M1, the latter will reach M2 at point **z**. Knowing that the ratio y/z is 1:4: (a) What is the value of **n**? (b) What is the value of **z** and **y** if the distance between the two automobiles is 100 km?.
Hint: When formulating the equations, you will notice that there are more variables than equations, but be aware that it is not necessary to obtain the value of all variables, just three of them. Of course, if you have too few equations, you cannot calculate all the variables.
A: (a) **n** = 1.43, (b) **z** = 235 km and **y** = 58.8 km

WATCHES

27. **Overlapping [3^+]**. At what time exactly after 6 p.m. will the minute and hour hands overlap?
    A: 6 h 32 (8/11) min ~6 h 32 min 44 s.
28. **Right angle [3^+]**. At what time after 6 p.m. will the minute and hour hands form a right angle (90°)?
    A: 6 h 16 (4/11) min ~6 h 16 min 22 s.
29. **Extended angle [3^+]**. At what time after 12 am will the minute and hour hands form an extended angle (180°)?
    A: 12 h 32 (8/11) min ~12 h 32 min 44 s.
30. **Laboratory experiments [6]**. The next laboratory on heat transfer begins when the hour hand is between 8 and 9 and the minute hand between 1 and 2. The laboratory will end when the positions have been reversed. How did the laboratory experiments take?
    A: The lab began at 8 h 8.4 min and ended at 13 h 40.7 min, and so the lab experiments took 5 h 32.7 min.
31. **Overlapping [3^+]**. At what time after 3 p.m. do the minute and hour hands overlap?
    A: 3 h 16 min 22 s
32. **Right angle [3^+]**. At what time after 12 am do the minute and hour hands form a right angle?
    A: 12 h 16 min 22 s
33. **Watches [9^+]**. My wristwatch is fast by 1 s/h and Ricardo's wristwatch is 2 s/h slow. Right now, both show the same time, and there is no difference between a.m. and p.m. on either watch. (a) When will both watches show the same time again? (b) After how much time will my watch be a full day ahead and indicate the exact same time? Solve with equations!
    A: (a) After 600 days, (b) 3,600 days (almost 10 years).
34. **Long meeting [6]**. A meeting starts between 10 and 11 a.m. and ends between 1 and 2 p.m. The minute and hour hands have swapped places. When does the meeting begin and end?
    A: The meeting starts at 10 h 9 33/143 min a.m. and ends at 1 h 50 110/143 min p.m.

WORK

35. **The wall [6]**. A worker can build a wall in **a** days and another worker can do it in **b** days. (a) How long will take to complete the job if they work together? (b) How many days after the first worker should the second worker start to complete the job in **c** days?
    A: (a) $\frac{1}{\left(\frac{1}{a}+\frac{1}{b}\right)}$ days; (b) $\frac{c(a+b)-ab}{a}$ days
36. **Apprentice [5]**. A highly skilled worker is able to paint a wall in 2 h, a normal worker in 4 h, and an apprentice in 8 h. (a) How long will it take for the three workers to paint the wall if they work together? (b) What percentage of the wall did each one paint?
    A: (a) 8/7 h (1 h 8 min 34 s) (b) Skilled worker 57.1 %, normal worker 28.6 %, and apprentice 14.3 %
37. **Professional [4^+]**. A professional worker takes 20 h to build a wall, and an apprentice takes 40 h to complete the same job. Due to the high cost of a professional worker, the foreman decides to start the work with the apprentice; after 10 h he hires the professional to finish the job with him. How long did it take to finish the wall?
    A: 20 h
38. **Builder [6]**. A builder promises to build a set of houses in a certain number of days, which will require 50 workers. From the ninth day, he has a smaller crew and has only three-fifths of his workers; this forced a delay in delivery of 10 days. In how many days did they finish building the houses?
    A: 33 days.

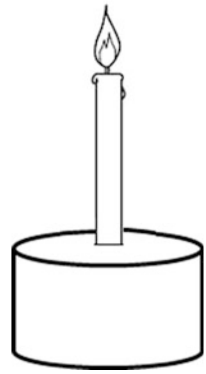

Fig. 5.16 Shape of the candle as proposed by your friend

39. **Harvesters [8]**. A group of harvesters should reap 2 meadows, where the large one is double the size of the smaller one. On the first half-day, the entire staff worked in the large meadow. Then half the group worked in the largest one, and the other half in the smaller one at the end of the day. Finally, in the small meadow, a certain section was not harvested, and the next day, one harvester finishes the harvesting. How many harvesters were in the group?
A: 8
40. **Extra workers [10⁺]**. A group of **a** workers is hired to do some work. After **m** days they do the **n**th part of the job. How many workers would have to be hired to complete the work in **b** days?
A: $\frac{a}{b}(mn - m - b)$
41. **Bridge [7⁺]**. In the construction of a bridge, an unknown number of workers work for 20 days. To finish the job in 10 days, 10 workers join them. In addition, it is known that the same job can be done by 20 workers in 20 days. How many workers are there?
A: 10 workers
42. **House [8⁺]**. 8 workers build a house in 210 days working in 2 shifts of 4 workers each. The first shift works from 8 a.m. to 4 p.m. and the second shift from 4 to 10 p.m. If a worker's performance decreases by 20 % after 10 h, how many workers are required to complete a house in no more than 180 days, with a single shift of 8–20 h?
A: 7 workers (6.1)

CANDLES

43. **Candles of different diameters [9⁺]**. Two candles of different diameters have lengths of *L* and *L*/2, respectively. The taller candle (smaller diameter) is consumed in 3 h, while the smaller candle (larger diameter) is consumed in 4 h. At 9 p.m., your best friend comes over and tells you that the date you have with two other girls is at 11:30 p.m., but unfortunately neither has a watch. You tell your friend, "Don't worry. If we light the candles right now, when both candles are the same height, that will be a good time to go meet them." Are you right?
A: Yes, you're right, the candles have the same height 2.4 h after being lit, which will make it 11:24 p.m.
44. **A single candle [8⁺]**. Days later, using the candles from the previous problem (Candles of different diameters), your friend suggests making a candle as depicted in Fig. 5.16, using part of the candle with height *L* (smaller diameter) and part of the larger diameter candle (*L*/2), so that the time to burn this composite candle is exactly 3.5 h. So your friend asks you, "Why not use half of each candle?" (a) Does that seem like a reasonable suggestion, or would you come up with a simpler solution to get a candle that burned out after 3.5 h?

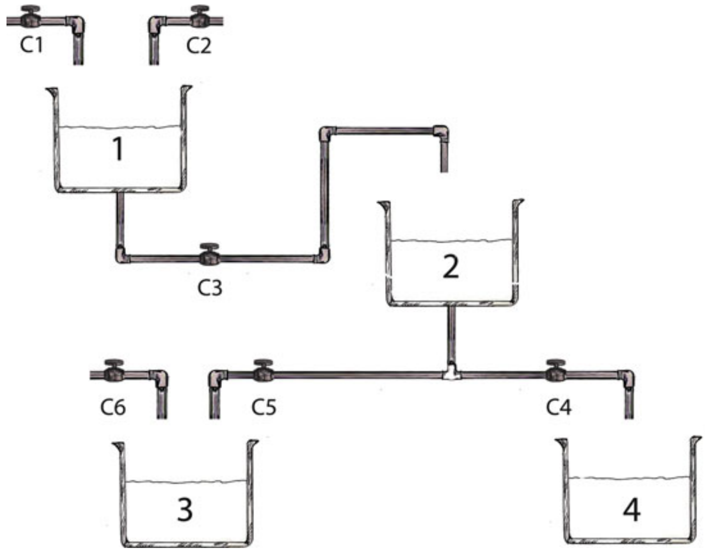

Fig. 5.17 Bioreactor system

A: (a) Yes, but a simpler solution would be to use only the large diameter candle, leaving it with a height of 7/8*L*.

45. **Two candles [8⁺]**. Two candles have different lengths and diameters. The taller one lasts 3.5 h and the shorter one 5 h. After 2 h of being lit, the candles are the same height. What is the ratio of the lengths of the two candles?
A: 7:5

FLOW

46. **Tank [6]**. A tank has two valves. To fill the tank, each valve operating individually takes **a** and **b** hours. How soon will the tank be filled if both valves operate simultaneously?
A: $\frac{1}{\left(\frac{1}{a}+\frac{1}{b}\right)}$ h

47. **Three valves [6]**. A tank has three valves. To fill the tank, each valve operating individually takes **a**, **b**, and **c** hours. How soon will the tank be filled if the three valves operate simultaneously?
A: $\frac{1}{\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)}$ h

48. **Bioreactor system [8]**. Figure 5.17 shows a bioreactor system. Initially, all bioreactors are empty and the valves closed. Then, valves 1 and 2 are opened simultaneously. How many minutes later into the operation should valves 3–6 be opened so that all bioreactors are full after 100 min?
Data: The flow per valve and the volume of each bioreactor are as follows**:** $C_1 = 4$ L/min; $C_2 = 6$ L/min; $C_3 = 10$ L/min; $C_4 = 5.5$ L/min; $C_5 = 4$ L/min; $C_6 = 20$ L/min; $V_1 = 100$ L; $V_2 = 220$ L; $V_3 = 1{,}240$ L; $V_4 = 440$ L.
A: Valve 3 opens 10 min later, valve 4 opens 20 min later, valve 5 opens 40 min later, and valve 6 opens 50 min later

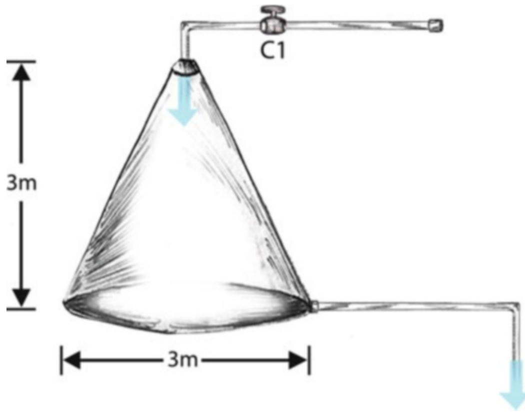

Fig. 5.18 Conical tank

49. **Conical tank [5]**. A conical tank (Fig. 5.18) is fed with a valve that has a flow rate of C_1 L/h. The bottom of the tank has a drain that allows 600 drps/min to flow through it, and 5 drops have a volume of 2 cm^3. If the tank is filled in 50 h: (a) What is the flow rate of the valve? (b) How long after the operation begins will the liquid height be 40 % of the total height?
    A: (a) $C_1 = 155.8$ L/h. (b) 39.2 h
50. **Cylindrical tank [10^+]**. A cylindrical tank with a radius of 5.0 m and height of 1.5 m is fed by a valve whose flow rate is 2.0 m^3/h. If you later add a second valve and it is observed that the linear velocity of the liquid height has increased by 20 %, then: (a) What is the filling time with valve 1? (b) What is the filling time with 2 valves?, (c) What is the flow rate of valve 2?. **Hint:** If the second valve increases the linear speed of the liquid level by 20 %, then valve 2 has one-fifth the flow rate of valve 1.
    A: (a) 1.8×10^1 h, (b) 1.5×10^1 h, (c) 0.4 m^3/h
51. **Spherical tank [8]**. A spherical tank ($r = 2$ m), initially empty, is fed by a valve that delivers a flow rate of 5 m^3/h. (a) What is the liquid height after 3.35 h? (b) What is the liquid height after 4 h?
    A: (a) 2 m (b) 2.26 m

NUMBERS

52. **Two-digit number [7]**. First, we switch the numbers of a two-digit number. Then we perform subtraction (the first minus the second number) to obtain 9. If we add those numbers, we obtain 55. What is the number? **Hint**: A two-digit number says *XY* can be defined as $10 \times X + Y$, e.g., 54 can be written as $5 \times 10 + 4 = 54$.
    A: 32
53. **How many times? [4]**. Given two numbers, if we subtract half of the smaller number from each number, the result is that the larger number is three times the smaller number. What is the ratio of the original numbers?
    A: 2

AGES

54. **John and his cousins [8⁺]**. John has two little cousins. If the age of the oldest cousin squared plus the age of the younger cousin add up to 234, how old are John's cousins? **Hint:** Review solved problem 7 Sect. 5.3.1 (Diophantine equations).
    A: 15 and 9 years old
55. **John and Mary [3]**. John is 5 years older than Mary. If 15 years ago John was twice Mary's age, how old are John and Mary?
    A: John is 25, Mary is 20
56. **Natalie Portman [6]**. A reporter indiscreetly asks Natalie Portman her age. Trying to avoid the question, she answers that her age 2 years ago was half of the sum of her age and the age of her close friend Scarlett Johansson. In addition, 24 years ago, she was twice the age of Scarlett. Undeterred, Natalie tells the reporter: (a) "So, in how many years will I be 40? (b) Also, how old is Scarlett Johansson?"
    A: (a) 8 years and (b) 28 years

PERCENTAGES, PROPORTIONS, AND MIXTURES

57. **Wine [5]**. A jar of undiluted wine is mixed with another jar that has three times the first jar's volume, but the second one is diluted. If the second jar contains two parts water and one part wine, what is the proportion of wine in the final mix?
    A. Half.
58. **Milk [7]**. Milk sold in stores contains **a%** protein, **b%** fat, and **c%** carbohydrates, with the rest being mostly water. In a laboratory experiment, milk is concentrated by evaporating the water, so **d%** of the total mass is evaporated. Due to the high temperatures used in the evaporation process, **e%** of the proteins precipitate (denatured). What is the concentration of protein, fat, and carbohydrates following the evaporation process?
    A: $\%\text{protein} = a\left(\frac{1-\frac{e}{100}}{1-\frac{d}{100}}\right)$; $\%\text{fat} = b\left(\frac{100}{100-d}\right)$; $\%\text{carbohydrates} = c\left(\frac{100}{100-d}\right)$
59. **Big sale [7⁺]**. At a sale, we sell two items for \$20 each. On the first article we earn 25 % profit and in the second article, we lose 20 %. (a) Can you tell whether the total sale produces a net loss or profit? (b) How much is earned or lost?
    A: (a) There is a net loss. (b) The first item cost us \$16, the second was \$25, totaling \$41, so we lost \$1.
60. **Mixed class [8⁺]**. In a co-ed class, a group of 80 men was selected, and the total number of men from this group who failed the test represented 15 % of the total number of men in the course. The nonselected men represented 80 % of the total women. If 40 % of these women failed and 90 of them passed, while 75 % of nonselected men were approved, then answer the following questions: (a) How many men failed? (b) How many women failed? (c) What percentage of men is in the co-ed class (d) What percentage of the total number of people who failed are women?
    A: (a) 60, (b) 60, (c) 57.14 %, (d) 50 %
61. **Insured vehicles [8]**. Of all city vehicles, **a%** have insurance. Due to a wave of accidents in recent months, the mayor estimated that of the total vehicles, **b%** had accidents in that period. If you also know that all vehicles that are not insured were in accidents (collision), then: (a) What percentage of insured cars were in an accident? (b) What percentage of the cars that were in accidents was insured? (**a** + **b** > 100).
    A: (a) $100\frac{(a+b-100)}{a}$, (b) $100\frac{(a+b-100)}{b}$

62. **Cone [7⁺]**. Increasing the radius of a cone by **a%**, by how much should the height be reduced to maintain the same volume?
A: $100 \times \frac{200a+a^2}{(100+a)^2}\%$
63. **Iron rod [8⁺]**. When an iron rod is heated, the length L is expanded by a%. By what percentage should it be contracted to reach the original length?
A: $\frac{100a}{100+a}\%$
64. **Fuel consumption [9]**. In a small town, there are 30,000 cars and 1,500 trucks. Fuel efficiencies are 8 km/L for cars in the city, increasing by 50 % on the highway, and 7 km/L for trucks on the highway and 10 % less in the city.
The average trip for trucks is 22 times longer than that of cars, and 1/11 of it is urban driving, while cars spend 80 % of their driving time on urban streets. The cost of car fuel is 50 % more expensive than truck fuel. By how much is the total cost of fuel consumption raised, as a percentage, if the cost per liter of fuel rises 10 % for cars and 15 % for trucks?
A: 12.4 %
65. **Alloys [5⁺]**. We have three alloys with 70, 80, and 90 % purity. In a mixture, we use 10 kg of the first alloy, 15 kg of the second alloy, and X kg of the third alloy. If the final purity of the mixture is 10 % greater than the purity of the second alloy, then: (a) How much is used for the third alloy? (b) If we use twice that amount, what is the final purity?
A: (a) 150 kg (b) 88.9 %
66. **Quantity to exchange [6]**. Two tanks with volumes of V_1 and V_2 liters are full. The first tank has an $X_1\%$ of solution **A** and the second tank an $X_2\%$ of solution **A** (both v/v). How much should be interchanged between tanks so that the concentrations in both tanks are the same? (Assume that the volumes are additive).
A: $\frac{V_1V_2}{V_1+V_2}$
67. **NaOH and NaCl [7]**. A solution containing 10 % (w/w) NaOH and 20 % (w/w) NaCl is concentrated by evaporating 200 kg H_2O. Then an additional 200 kg solution is added containing 60 % NaCl (w/w) and an unknown percentage of NaOH (w/w). If the resulting mixture is 20 % NaOH (w/w) and 50 % (w/w) NaCl, then: (a) What are the NaOH and NaCl concentrations in the first solution after the evaporation of H_2O? (b) What are the NaOH and NaCl concentrations in the second solution?
A: (a) 20 % NaOH and 40 % NaCl, (b) 60 % NaCl and 20 % NaOH
68. **Copper [7⁺]**. If copper having a purity of 70 % is mixed with the same amount of another metal with $X\%$ purity, the final mixture will have a purity of **a%**. (a) Obtain an expression for X as a function of **a**. (b) What should the purity of the second metal be if a final purity of 83 % is desired?
A: (a) $x = 2a - 70$, (b) 96 %
69. **Blue and red balls [8⁺]**. A bag contains blue and red balls. If the probability of drawing two red balls (without replacement) in a row is 1/10 and the blue balls exceed the red balls by one unit, how many balls of each color are in the bag?
A: 2 red balls and 3 blue balls
70. **Birthday [9⁺]**. Today, when I went to classes, I noted that more than two of my fellow students had the same birthday. If 40 students are in the class, what is the probability that two classmates have the same birthday?
A: ~0.891 (~89.1 %)

71. **Chemical company [7]**. Two sales representatives at two different chemical manufacturers are talking about sales at their respective companies. The first one says that 4 years ago his sales were only 80.0 % of the sales of his friend (at the second manufacturer). If the second sales rep has increased his sales at a rate of 10.0 % a year and the first salesman has increased his at a rate of 8 % for the first 3 years, then at what rate did the first manufacturer increase its sales in the fourth year if, at present, both manufacturers have the same level of sales?
    A: 45.3 %
72. **Survey [9+]**. A survey has yielded the following information about a group of people: 40 % have light eyes and 70 % have dark hair. 5 % of the light-haired people have dark eyes. (a) What percentage of the dark-eyed people have light hair? (b) What percentage of those with dark hair have light eyes?
    A: (a) 2.5 % (b) 16.43 %

MISCELLANEOUS

73. **Cows [8]**. If 10 cows eat 10 packages of grass in 10 days, how many packages of grass will 5 cows eat in 5 days?
    A: 2.5 packages of grass
74. **Student scholarship [9]**. A university has decided to award scholarships in the amount of $2,000 to every medical student and $1,000 for each engineering student. If the university's total number of engineering and medical students is 1,000, and half of the medical students reject the scholarship, what is the total dollar amount allocated to scholarships?
    A: $1 million.
75. **Number of balls [6+]**. In a bag, I have an unknown number of balls. First, I retire half of the balls plus 1, then I retire the remaining half plus 2 and subsequently withdraw the remaining half plus 4. If, finally, there is 1 ball in the bag, how many balls were there at the beginning?
    A: 50 balls.
76. **Euler's problem [7]**. A problem attributed to *Leonhard Euler* states that: "In a bed and breakfast are housed 20 people, both men and women. Each man pays 8 coins for his lodgings, and each woman pays 7 coins. If the total bill comes to 144 coins, how many men and women are staying at the bed and breakfast?"
    A: 4 men and 16 women.
77. **Valentine's Day [8]**. As Valentine's Day approaches, I rush out to buy roses. Fortunately, this time, for $50 I get 5 more roses than before. Indeed, the price of a dozen roses fell $10. How much was a rose originally worth?
    A: $3.34
78. **Notebooks and iPads [7]**. A large purchase was made of televisions, notebooks, and iPads, a total of 100 units. The TVs cost $600, the notebooks $1,000, and the iPads $800. If the total bill came to $90,000, what is the difference between the total number of notebooks and TVs bought?
    A: 50
79. **Foot length, mathematics, and logic [5]**. A statistician gives a math test in a town of 60,000 inhabitants and, at the same time, measures the length of all the villagers' feet. Surprisingly, among the inhabitants 5–22 years of age, he finds that mathematical ability in this town is very well correlated with the length of a person's foot. Of course, the same study for people between 22 and 50 years old does not yield a good correlation. Explain this.

A: Clearly foot size is directly related to age, and math skills are also correlated with age. Of course, beyond a certain age, this correlation is lost.

80. **Party [10]**. Rodrigo is excited to go a party this Saturday at the home of Macarena. But first he must get permission from his father. As usual, the father gives him a challenge. Dad tells Rodrigo that he will let him go to Macarena's party, but he has to play three games of chess and win at least two games in a row. Rodrigo must choose one of the following two sequences of play and win two games in a row to get permission to go to the party: play his mother, then his father, then again his mother; or starting with his father, he then plays his mother and finishes by playing his father again. Rodrigo knows that he rarely beats his dad and that most of the time he beats his mom. Which of the two sequences is more likely to end up in Rodrigo's going to Macarena's party?
 A: The sequence that improves his chances is: Dad, Mom, Dad.
81. **Preparing for the GRE [4]**. Your father is tired of your lack of preparation for the GRE. Therefore, he offers to pay \$8 per solved problem and fine you \$5 per unsolved problem. If you submit to him 26 solved problems and neither owes the other any money, how many problems did you solve correctly?
 A: 10 correct and 16 incorrect
82. **New Year's Eve party [9$^+$]**. You have a family party to go to on New Year's Eve. When everyone starts giving out hugs, you're curious to know how many hugs are given, so you count them. You come up with 36 hugs. How many people were at the party?
 A: Including you, 9
83. **Canning plant [7]**. A canning company has two processing plants, one where seafood is canned and another that cans fruits. In the seafood canning plant there are three production lines, one with 60 % more capacity than each of the other two lines, expressed in cans processed per hour. This plant operates with two shifts of 8 h each. On the other hand, the fruit processing plant has two production lines of equal capacity and operates with three 8-h shifts per day. If both plants process the same amount of cans per day, what is the ratio of production between the first production line of the seafood plant and either of the fruit production lines.
 A: First seafood processing line: Fruit processing line = 4:3
84. **Three substances [9]**. If we have 84 kg of **A**, 66 kg of **B**, and 18 kg of **C,** and their densities are in the proportion 1: 2: 3, what is the v/v percentage of each substance?
 A: 68.3 %, 26.8 %, and 4.9 %, respectively
85. **Alcohol and water [10$^+$]**. Two vessels whose volumes are in a ratio of **a**:**b** are full of alcohol and water, respectively. Swapping **C** liters equalizes the concentrations in both vessels. What is the total volume of both vessels?
 A: $V_{\mathrm{T}} = \frac{C(a+b)^2}{ab}$

Solution to the shoe problem (Sect. 5.2, Warm-Up Example1)

From the scam Rodrigo is out **one pair of shoes and \$20**. Surprised? First, if your answer was \$200 you were close but not strictly correct. One easy way to solve the problem is to analyze all the transactions made by Rodrigo (Fig. 5.19).

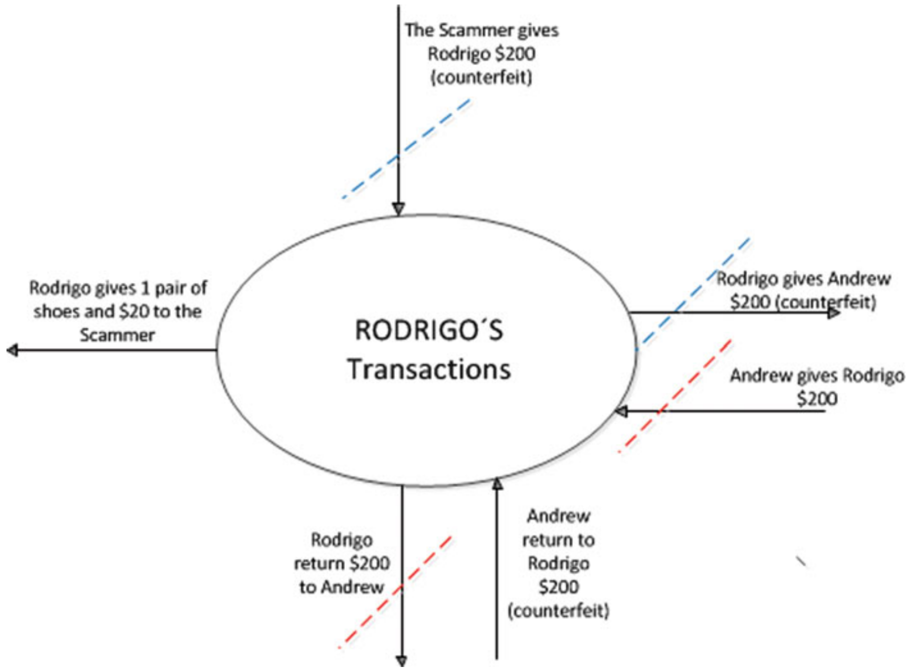

Fig. 5.19 Schematic representation of the transactions made by Rodrigo

From Fig. 5.19 it is clear that Rodrigo is out one pair of shoes and $20. The blue and red dotted lines cancel out, and in the end, Rodrigo is out one pair of shoes and $20 and retains two counterfeit $100 bills ($200).

The problem is easy to solve and has a difficulty level of **2,** but student answers would suggest a level of difficulty of **10**$^{+}$.

Additional Web References

Integration of Math and Life http://www.youtube.com/watch?v=ZXq993ZLmg8
Planning a Party by Julia Pollak http://www.youtube.com/watch?v=GHSlVN8N82A